Helical instability of a magnetic flux line in a current-carrying superconducting film

Yuri A. Genenko
*Donetsk Physical & Technical Institute of the National Academy of Sciences of Ukraine, 340114 Donetsk, Ukraine** *and Metal Physics Institute of the University of Göttingen, Windausweg 2, D-37073, Göttingen, Germany*
(Received 28 July 1995)

An exact solution for helical magnetic vortex structure in a type-II superconductor strip subjected to a magnetic field parallel to a transport current is obtained. A field-dependent characteristic current of flux-line instability with respect to helical distortions is found to be comparable to experimental values of the critical current of dissipation onset in thin films. A resistive picture following from the above results includes a regime of magnetic moment and voltage oscillations that may be controlled by the current, field, and sample dimensions.

I. INTRODUCTION

Both type-I and type-II superconductors (SC's) subjected to a magnetic field parallel to a transport current are known to demonstrate rather specific resistive behavior.[1,2] The most unusual features of the latter are a nonmonotonous field dependence of the critical current observed eventually,[1,3] onset of a considerable magnetic moment in a small (or even zero) magnetic field at a current close to the critical one,[4,5] and regular[5,6] or stochastic[7] oscillations of the magnetic moment and voltage in certain regions of fields and currents.

The nature of the resistivity in a parallel field remains still controversial. Some authors discussed the possibility of a nonconventional ''Lorentz-independent'' mechanism of energy dissipation;[8] others suggested unusual ''current-carrying'' magnetic vortices to explain the observed phenomena.[9] However, the major part of the above-mentioned experimental features has found long ago a quite plausible qualitative explanation in terms of helical magnetic vortices, arising naturally due to superimposing of the external field and current self-field.[1,10,11]

A key factor of the resistive and magnetic behavior of a SC in a parallel field is the helical instability of the linear flux line (FL), considered for a single vortex by Clem[10] and for a flux-line lattice (FLL) by Brandt.[11] The origin of this instability is in the (Lorentz) driving forces exerted on the FL elements by the transport current that makes the left-handed spiral grow, while the right-handed ones are damped. If the Lorentz force (proportional to the current) is large enough, the random left-handed distortions of the linear vortex are enhanced and the vortex then leaves the sample, contributing to the voltage. In a certain region of fields and currents the entry of right-handed spirals followed by the exit of left-handed spirals may form a cycle behavior with magnetic moment and voltage oscillations.[10,12]

Since inside a bulk SC the density of the transport current is exponentially small, the critical current of instability turned out to be exponentially large: $j_{\rm in}\cong j_L\exp(R/2\lambda)$,[10] where R is the characteristic transverse size of SC's, which is much larger than the London penetration depth λ, and j_L is the London value of the critical current.[1,13] Thus, for the bulk SC, the effect of single flux line instability is not actual (contrary to the FLL instability[11]), though, in low-dimensional samples with one or two dimensions less than λ, i.e., in thin filaments or films, the critical current of instability acquires a physical meaning and the phenomenon of instability itself may dominantly affect the resistive picture in a field parallel to the current. But in the cases of samples of small dimensions, the effect of the surface on the vortices becomes crucial and should be accounted for. This was not done in Refs. 10,11, which considered bulk samples. Recently, an exact solution for the helical vortex problem in a SC cylinder of arbitrary radius was obtained in the London approximation.[12] The critical current of helical instability in thin filaments was found to be comparable to the critical currents of dissipation onset observed on such samples.

In this paper the phenomenon of spiral instability is considered in the case of thin SC films that is of much more practical interest. In the London approximation, an exact solution for the helical vortex structure inside a plate of arbitrary thickness is found. The Gibbs free energy is calculated for the system loaded with a transport current in parallel to the current magnetic field. The critical current of left-handed spiral instability turns out to be a quite observable value for a wide thin film that is reasonable since the transport current in the latter case is not exponentially small in any region, contrary to the bulk case.

II. SPIRAL VORTEX STRUCTURE INSIDE A SUPERCONDUCTING PLATE

Let us consider the SC plate filling the space $|z|\leq d$ loaded with a transport current flowing in the positive y direction (Fig. 1) and subjected to a magnetic field $\mathbf{H}$ applied in the same direction. To study the stability of a linear magnetic FL lying along the y axis $(x=z=0)$ with respect to helical distortions let us consider a helical vortex of arbitrary pitch length $2\pi L$ lying on an imaginary cylinder of arbitrary radius $r<d$ so that at $r\to 0$ it transforms to the above linear vortex. A full magnetic field in the system $\mathbf{H}_f=\mathbf{H}_M+\mathbf{h}$ consists of the Meissner solution

$$\mathbf{H}_M=\mathbf{H}\frac{\cosh z/\lambda}{\cosh d/\lambda}, \qquad (1)$$

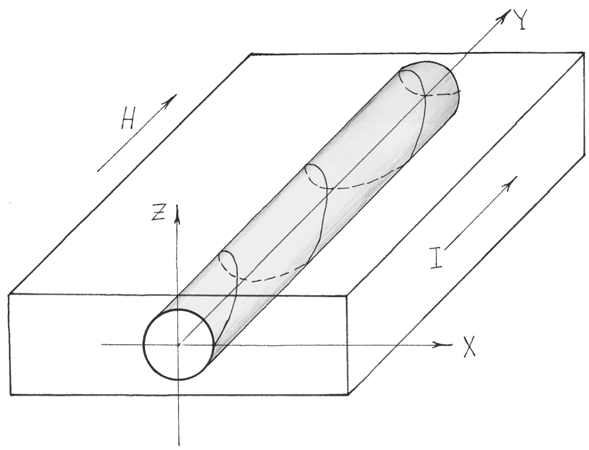

FIG. 1. Helical magnetic vortex lying on an imaginary cylinder inside a current-carrying superconducting plate in a external magnetic field **H** parallel to the current I.

satisfying the boundary condition $\mathbf{H}_M = \mathbf{H}$ at the plate surfaces $z = \pm d$ and the field $\mathbf{h}$ of the vortex helix itself. The latter may be described inside the plate by the London equation with a special right-hand side,[12]

$$\lambda^2 \mathrm{curlcurl}\mathbf{h} + \mathbf{h} = \Phi, \tag{2}$$

where

$$\Phi = \Phi_0 \frac{\mathbf{e}_l}{\mathbf{e}_l \mathbf{e}_\varphi} \delta(\rho - r)\, \delta(y - L\varphi) \tag{3}$$

is presented in cylindrical coordinates $\rho = \sqrt{x^2+z^2}, \varphi = \arctan z/x$ and y. $\mathbf{e}_\varphi$ is the unit azimuthal vector in xz plane, $\mathbf{e}_l$ the unit vector tangential to helical core of the vortex, and Φ_0 is the unit flux quantum. Equation (2) should be supplemented by the Maxwell equations

$$\mathrm{curl}\mathbf{h} = 0, \quad |z| \geqslant d,$$

$$\mathrm{div}\mathbf{h} = 0, \tag{4}$$

the first of which is valid outside the SC plate while the second one holds in the whole space. The resulting field $\mathbf{h}$ must be continual on the boundaries $z = \pm d$ and vanish at infinity, $x(z) \to \infty$.

Taking into account the periodicity of the problem along the y axis the solution of Eq. (2) may be presented in Fourier amplitudes as follows:

$$h^i(x,y,z) = \frac{1}{2\pi} \sum_k \exp(iky/L) \int dq\ h_k^i(q,z) \exp iqx. \tag{5}$$

Then, inside the plate ($|z| \leqslant d$) one finds for amplitudes

$$h_k^i = C_+^i \exp Qz + C_-^i \exp(-Qz) - \frac{\Phi_0 \theta(z-r)}{4\pi Q \lambda^2} [I_-^i(\pi/2) \exp Qz - I_+^i(\pi/2) \exp(-Qz)]$$
$$- \frac{\Phi_0 \theta(r - |z|)}{4\pi Q \lambda^2} [I_-^i(T) \exp Qz - I_+^i(T) \exp(-Qz)], \tag{6}$$

where

$$I_\pm^i(T) = \int_{-\pi-T}^{T} dt\ \phi_i(t) \exp(-ikt - iqr\ \cos t \pm Qr\ \sin t) \tag{7}$$

and $Q^2 = \lambda^{-2} + q^2 + k^2/L^2, \phi_x = -(r/L)\sin t$, $\phi_y = 1$, $\phi_z = (r/L)\cos t$, $T = \arcsin z/r$. Outside the plate one finds for $z \geqslant d$ ($z \leqslant -d$)

$$h_k^i(q,z) = h_k^i(q, \pm d) \exp p(d \mp z), \tag{8}$$

where $p^2 = q^2 + k^2/L^2$.

The unknown constants $C_\pm^i$ and $h_k^i(q, \pm d)$ may be easily found from the boundary conditions and Eqs. (4):

$$C_\pm^i = \frac{\Phi_0}{4\pi Q \lambda^2} [I_-^i(\pi/2) \exp Qz - I_+^i(\pi/2) \exp(-Qz)] + g_\pm^i - \frac{\Phi_0 r}{4\pi \lambda^2 L} \frac{[Q\ \cosh 2Qd + p\ \sinh 2Qd] I_+^z(\pi/2) + Q I_-^z(\pi/2)}{[(Q^2+p^2)\sinh 2Qd + 2Qp\ \cosh 2Qd]\sinh 2Qd} f_\pm^i, \tag{9}$$

where $g_\pm^z = (\Phi_0 r / 4\pi Q \lambda^2 L) I_\pm^z(\pi/2)/\sinh 2Qd$, $g_\pm^x = g_\pm^y = 0$, $f_\pm^x = \pm iq/p$, $f_\pm^y = \pm ik/pL$, $f_\pm^z = p/Q$. The values of the integrals $I_\pm^i(\pi/2)$ may be found analytically[14] and are equal to

$$I_\pm^y = i^{-k} \frac{2}{\pi} \left(\frac{Q \pm q\ \ \mathrm{sgn} k}{\sqrt{Q^2 - q^2}} \right)^{|k|} I_{|k|}(r\sqrt{Q^2-q^2}), \tag{10}$$

$$I_\pm^x = \pm \frac{1}{L} \frac{\partial}{\partial Q} I_\pm^y,$$

$$I_\pm^z = \frac{i}{L} \frac{\partial}{\partial q} I_\pm^y.$$

Equations (6)–(10) define completely the magnetic field induced inside the plate and outside due to the presence of a helical vortex.

The latter transforms to a linear FL (Ref. 15) when $r \to 0$. Really, in this case the set of equations for the coefficients has only a trivial solution $C^i_\pm = h^i_\pm = 0$ for all $k \neq 0$. For $k=0$, $C^x_\pm = C^z_\pm = h^i_\pm = 0$ too, but

$$C^y_\pm = \pm \frac{\Phi_0}{4Q\lambda^2} \frac{\exp(\pm Qd)}{\cosh Qd}, \tag{11}$$

which coincides with the solution of Shmidt.[15]

Let us note that the vortex field (6) is not equal to zero on the surface $z = \pm d$ as in the case of a linear vortex (11), but the difference vanishes when $L \to \infty$ or $r \to 0$.

To analyze the stability of a linear FL with respect to spiral pertubations one should consider the dependence of the energy of a helical vortex against its radius r, which will be done in the next section.

III. GIBBS FREE ENERGY OF THE VORTEX SPIRAL

To consider the process of helical instability nucleation one should use the Gibbs free energy of the system accounting for the self-energy of the vortex F, the work done by the source of the transport current I, ΔW_I, and the work done by the source of external magnetic field, ΔW_H,[16]

$$G = F - \Delta W_H - \Delta W_I. \tag{12}$$

The self-energy of the system F may be calculated using the usual definition[2,13] and reduced to the form (for the details see the Appendix)

$$F = \frac{1}{8\pi}\int dV\, \mathbf{h}\cdot\mathbf{\Phi} = F_0 + \Delta F, \tag{13}$$

where the part dependent on the boundary conditions equals

$$F_0 = \frac{\Phi_0}{(4\pi)^2} \sum_{k,i,\sigma} \int dq\, C^i_\sigma I^{i*}_\sigma, \tag{14}$$

where $\sigma = \pm$. The part conditioned by the source function (3) only and independent of boundary conditions equals

$$\Delta F = -\sum_k \frac{2}{Q}\left(\frac{\Phi_0}{2\pi\lambda}\right)^2 \int_0^\pi d\tau_+ \int_0^\pi d\tau_- \left(1 + \frac{r^2}{\lambda^2}\cos 2\tau_-\right)\cosh[2Qr\,\sin\tau_+ \sin\tau_-]\exp[2ik\tau_- + 2iqr\,\cos\tau_+ \sin\tau_-]. \tag{15}$$

The expression for the free energy (13) with the above ΔF and F_0 is valid for a plate of any thickness. For a thick one the current of instability will naturally be exponentially large as well as in the case of a bulk cylinder.[10,12] The most interesting is the case of a thin film of thickness $d \leqslant \lambda$. Let us suppose also that the nucleating spiral is twisted softly, i.e., $L \gg d, \lambda$. For the study of a helical distortion nucleation the expansion of energy in small r up to r^2 is sufficient. Then one finds for the self-energy of helical vortex inside a thin film (see the Appendix)

$$F = \left[\frac{\Phi_0}{4\pi\lambda}\right]^2 \left[\ln\frac{d}{\xi} + \frac{r^2}{2L^2}\ln\frac{\lambda}{\xi} + \frac{r^2}{\lambda^2} - \frac{r^2}{2d^2}\right]. \tag{16}$$

The magnetic field contribution per unit length along the y axis in the simple strip geometry can be easily evaluated as

$$\Delta W_H = \frac{1}{2\pi L}\frac{1}{4\pi}\int dV\, \mathbf{h}\cdot\mathbf{H} = \frac{H\Phi_y(r)}{4\pi}, \tag{17}$$

where integration is over one flight of the helix along the y axis and $\Phi_y(r)$ is the flux flowing through the vortex in a positive y direction. This contribution is in favor of entry of the vortex directed along the external field. The flux $\Phi_y(r)$ may be calculated using Eq. (6) and equals

$$\Phi_y(r) = \Phi_0\left[1 - \frac{I_0(r/\lambda)}{\cosh d/\lambda}\right], \tag{18}$$

where $I_0(x)$ is the modified Bessel function of zeroth order.[14] It is seen to have a proper limit at $r \to 0$.[15]

The current source contribution per one flight of the spiral, calculated in the spirit of Clem's work,[10] is simply the work done by the Lorentz driving force $\mathbf{f}_L = [\mathbf{j}\cdot\Phi]/c$ exerted upon the FL element dl where $\mathbf{j}$ is the local density of transport current. Taking into account that the angle between $\mathbf{j}$ and Φ remains equal to the pitch angle α while integration along the FL and $dl\,\sin\alpha = \rho d\varphi$ one finds for this work per unit length

$$\Delta W_I = \frac{1}{2\pi L}\int_0^r d\rho\, \rho \int_0^{2\pi} d\varphi \frac{j\Phi_0}{c} = \frac{I(r)\Phi_0}{2\pi Lc}, \tag{19}$$

where $I(r)$ is the current flowing through the cross section πr^2 of the imaginary cylinder on which the helix lies.

We considered so far a plate of infinite width. Now we shall turn to applying our results to the most interesting case of a thin film ($d < \lambda$) of finite width ($|x| \leqslant W$) which recently attracted considerable attention[17,18] because of its practical significance and model properties. In such a film the transport current density does not depend on z and its distribution over the film is given by.[16,19]

$$j(x) = \frac{I}{\pi d\sqrt{W^2 - x^2}}, \tag{20}$$

where I is the transport current applied. Then for the current contribution to the energy (per unit length) one gets

$$\Delta W_I = \frac{I\Phi_0 r^2}{2\pi c d W L}. \tag{21}$$

The essential point is that for the magnetic field contribution in energy of a wide film ($W \gg d,\lambda$) one can still use expressions (17), (18) derived for a film infinite in the x direction. Really, the vortex lying parallel to a thin film surface is strongly coupled to its images in the $z=\pm d$ planes but undergoes an exponentially weak effect of finite edges, $x=\pm W$, contrary to the vortex perpendicular to a film which undergoes a long-range effect of film edges.[19] The negligible correction to the value of the flux, Eq. (18), following from a finite width W is as small as $\exp[-W\sqrt{(\pi/d)^2+(2/\lambda)^2}]$.

Upon susbstitution of Eqs. (16), (17), (21) into Eq. (12) one can obtain the Gibbs free energy of the small radius spiral,

$$G=\left(\frac{\Phi_0}{4\pi\lambda}\right)^2\left[\ln\frac{d}{\xi}+\frac{r^2}{2L^2}\ln\frac{\lambda}{\xi}+\frac{r^2}{\lambda^2}\ln\frac{d}{\lambda}-\frac{r^2}{2d^2}\right] -\frac{H\Phi_0}{4\pi}\left[\frac{d^2}{2\lambda^2}-\frac{r^2}{4\lambda^2}\right]-\frac{I\Phi_0 r^2}{2\pi c d W L}. \tag{22}$$

Following the notation of Ref. 10, we introduce the force exerted upon the unit length of the FL as $f=-\partial G/\partial r = rK(r,y)$, where $y=\lambda/L$. In what follows we consider for simplicity the case of a weak magnetic field and suggest that essentially $y \ll 1$ (in the opposite limit the result does not change significantly[12]). Then one finds

$$K(0,y)=\frac{\Phi_0}{8\pi\lambda^2}\left[H_d-H+\frac{4\lambda}{d}H_I y-2H_{c1}y^2\right], \tag{23}$$

where $H_d=\Phi_0/2\pi d^2-(\Phi_0/\pi\lambda^2)\ln\lambda/d$ is the characteristic field of an order of a lower critical field for the plate,[15] H_{c1} is the lower critical field for the bulk material, and $H_I=2I/Wc$ is the current self-field magnitude above the center of the film ($x=0,\ z=d$).

The vortices for which $K(0,y)>0$ are instable. The FL with L defined by this inequality grows as a left-handed spiral until $r\approx d$ and then transforms to a chain of tilted vortex pairs, which eventually leave the film through its edges. Let us find the conditions for this expansion to occur. At $H<H_d$ the vortex is absolutely unstable even in the absence of a current. At $H>H_d$ the vortex is stable at a small current but becomes unstable at a larger one. $K(0,y)$ reaches its maximum at $y_0=(\lambda/d)H_I/H_{c1}$ and first attains zero at $H_{Ic}=H_{c1}(d/\lambda)\sqrt{(H-H_d)/2H_{c1}}$, which corresponds to the critical current of instability

$$I_{\rm in}=\frac{cWH_{c1}}{2}=\frac{cWdH_{c1}}{2\lambda}\sqrt{\frac{H-H_d}{2H_{c1}}}. \tag{24}$$

The maximum current density $j_{\max}$ that is achieved at the edge of the film, $|W-x|\approx d$,[18] is then of the order of $j_{c1}\sqrt{(W/d)(H-H_d)/2H_{c1}}$. Here $j_{c1}=cH_{c1}/4\pi\lambda$ is some orders less than the London critical value $j_L=cH_c/4\pi\lambda$ close to the depairing current,[2,13] where H_c is the thermodynamic critical field.

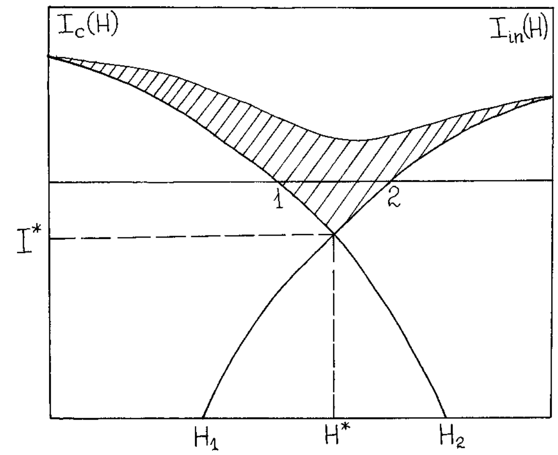

FIG. 2. Sketch of a diagram of a resistive state of superconductor in a magnetic field parallel to transport current presented in coordinates of current and field. In the dashed region the oscillation regime of dissipation is possible (discussed in the text). When raising the magnetic field at a fixed value of the current, $I>I^*$, the oscillation of the magnetic moment and voltage should take place between points 1 and 2.

IV. DISCUSSION

The value of $j_{\max}$ turns out to be comparable to a critical current of the magnetic self-field entry against the geometrical edge barrier $j_c=cH_{c1}/4\pi d$ close to a typical experimental value of 10^6 A/cm^2.[18] It is important that the current of instability is not exponentially large as in the bulk case.[10] That means that the resistive behavior of wide thin SC films may be primarily determined by the interplay of processes of a single vortex entry and exit in a wide range of currents and fields.

Further conclusions are somewhat speculative and concerned with the general regularities of an overcritical behavior that may follow from the above results. Since the critical current of dissipation onset $I_c(H)$ (of any nature, for example, pinning mediated) is normally a decreasing function of a parallel magnetic field (sometimes with a maximum[1]) and the critical current of helical instability, $I_{\rm in}(H)$, is an increasing one, there is always a region of the I-H diagram where transport current is less then $I_c(H=0)$ but more than both $I_c(H)$ and $I_{\rm in}(H)$ (dashed region in Fig. 2). In this region a nondissipative state is unstable against the entry of a magnetic vortex, but the latter in its turn is unstable against the left-handed helical expansion. Therefore, a dissipation cycle consisting of a vortex entry and subsequent exit should be formed above some characteristic magnitudes I^* and H^* (see Fig. 2) as discussed in Ref. 12. The frequency of the corresponding magnetic moment and voltage oscillations is material dependent and is also controlled by the field, current, and film width.

This cycle is not unique. First, such a scenario was considered by Clem for a pair of vortices.[10] It is easy to see that it should take place at a somewhat higher magnetic field and should have another (lower) frequency. It is reasonable to expect the nonstationary regime of dissipation with a few dominant oscillation modes in the closest over critical region (dashed region in Fig. 2). Then, with the increase of field and current the number of modes should grow and, perhaps, the whole dissipation scenario may become stochastic at some conditions. At high fields, of course, one should address rather the problem of FLL instability[11] than the single-vortex scenario.

The validity of the above picture may be easily checked by means of spectral analysis of the voltage noise in parallel to the current magnetic field just above the critical current curve $I_c(H)$. It is interesting that a behavior very similar to that suggested above was found by Landau on type-I SC's (Ref. 5) where, at fixed current I, in the region of fields $H_-(I)<H<H_+(I)$ low-frequency oscillations of the longitudinal magnetic moment and voltage were observed (compare to the region between points 1 and 2 in Fig. 2). A similar effect was clearly observed on type-II SC's by Walmsley and Timms.[6]

In conclusion, we have performed a rigorous study of the stability of a single magnetic FL inside a current-carrying SC plate (strip) subjected to a magnetic field parallel to the current. Following from the exact solution for a helical vortex structure the critical current of helical instability was evaluated and shown to be comparable to the observed magnitudes of the critical current of dissipation onset. Possible oscillating scenarios of dissipation resulting from the latter fact were discussed.

ACKNOWLEDGMENTS

The author is grateful for most stimulating discussions to Professor J. R. Clem and Dr. E. H. Brandt who drew his attention to the problem of helical instability and to Professor H. C. Freyhardt for helpful discussions and hospitality during a research visit to the University of Göttingen supported by the Deutscher Akademischer Austauschdienst. Numerous valuable remarks of Professor V. M. Pan, Dr. K. Heinemann, Dr. W. Mexner, and Dr. J. Hoffmann are gratefully acknowledged.

APPENDIX: FREE ENERGY CALCULATION

The formula (13) for the vortex free energy was shown so far to be valid only in the case of a linear FL parallel to the flat sample surface[15] since in this case the vortex field on the surface equals zero. In this appendix, we will show that formula (13) is valid for any FL configuration located inside the SC plate regardless of the field value on the surface.

The full free energy of the system[2,13] equals

$$\mathcal{F}=\int_{|z|\leqslant d}dV\frac{\mathbf{H}_f^2+\lambda^2(\mathrm{curl}\mathbf{H}_f)^2}{8\pi}+\int_{|z|\geqslant d}dV\frac{\mathbf{H_f}^2}{8\pi},\quad (A1)$$

where $\mathbf{H}_f$ is the full magnetic field defined before Eq. (1). Making use of the vector identity

$$\mathrm{div}[\mathbf{a}\times\mathbf{b}]=\mathbf{b}\ \mathrm{curl}\mathbf{a}-\mathbf{a}\ \mathrm{curl}\mathbf{b},$$

one can present Eq. (A1) in the form

$$\mathcal{F}=F+\Delta\mathcal{F},\quad (A2)$$

where the energy F is defined by the formula (13) and energy correction

$$\Delta\mathcal{F}=\frac{\lambda^2}{4\pi}\int_{|z|\leqslant d}dV\ \mathrm{div}[\mathbf{h}\times\mathrm{curl}\mathbf{H}_M]+\frac{1}{4\pi}\int_{|z|\geqslant d}dV\ \mathbf{h}\cdot\mathbf{H}_M+\frac{1}{8\pi}\int_{|z|\geqslant d}dV\ \mathbf{h}^2+\frac{\lambda^2}{8\pi}\int_{|z|\leqslant d}dV\ \mathrm{div}[\mathbf{h}\times\mathrm{curl}\mathbf{h}].\quad (A3)$$

To deal with the above expression we need in some relations valid for an arbitrary FL form. In this case the problem is no longer periodical; therefore one should substitute the discrete wave vector $(q,k/L)$ used in the Fourier transformation (5) by a contitunal one $\mathbf{q}=(q_x,q_y)$ and write

$$h^i(x,y,z)=\int\frac{d^2q}{(2\pi)^2}h^i_{\mathbf{q}}\exp i\mathbf{q}\cdot\mathbf{s},\quad (A4)$$

where $\mathbf{s}=(x,y)$.

The London equation (2) takes a form

$$\left(Q^2-\frac{\partial^2}{\partial z^2}\right)h^i_{\mathbf{q}}(z)=\lambda^{-2}\Phi^i_{\mathbf{q}}(z),\quad |z|\leqslant d,$$
$$\left(q^2-\frac{\partial^2}{\partial z^2}\right)h^i_{\mathbf{q}}(z)=0,\quad |z|\geqslant d,\quad (A5)$$

where $q^2=q_x^2+q_y^2$ and $Q^2=q^2+\lambda^{-2}$. Let us suggest that the FL is located as a whole inside a plate so that $\Phi^i_{\mathbf{q}}(z)\equiv 0$ outside the region $|z|\leqslant r$, where $r<d$.

First equation of Maxwell equations (4) is then reduced to the equalities

$$iq_xh^z_{\mathbf{q}}=\mp qh^x_{\mathbf{q}}(\pm d),\quad iq_yh^z_{\mathbf{q}}=\mp qh^y_{\mathbf{q}}(\pm d),$$
$$q_yh^x_{\mathbf{q}}=q_xh^y_{\mathbf{q}}(\pm d).\quad (A6)$$

The latter of Eqs. (A6) means the vanishing of the perpendicular current component at the surface. The second equation of Eqs. (4) is reduced to the condition

$$iq_xh^x_{\mathbf{q}}+iq_yh^y_{\mathbf{q}}+\frac{\partial h^z_{\mathbf{q}}}{\partial z}\equiv 0.\quad (A7)$$

Now we are in a position to consider the energy correction expression. With an account of the explicit form of the Meissner solution (1), the first integral in Eq. (A3) is reduced to

$$L_1=\frac{\lambda H}{4\pi}\tanh\frac{d}{\lambda}[h^y_{\mathbf{q}=0}(d)+h^y_{\mathbf{q}=0}(-d)].\quad (A8)$$

The flux of the vortex magnetic field through the surfaces $z=\pm d$ equals zero since the vortex is located as a whole inside the SC plate. That means that

$$h^z_{\mathbf{q}=0}(\pm d)=\int\int dx\ dy\ h^z(x,y,z)=0.$$

Then, from the second equation of Eqs. (A6) one obtains

$$h^y_q(\pm d)=\mp\frac{iq_y}{q}h^z_q(\pm d)_{q\to 0}\to 0, \qquad \text{(A9)}$$

regardless of the indefinite value of q_y/q at $q_x,q_y\to 0$. Hence, $L_1=0$.

Second integral in Eq. (A3) is reduced to

$$L_2=\frac{H}{4\pi}\int_d^\infty dz[h^y_{\mathbf{q}=0}(z)+h^y_{\mathbf{q}=0}(-z)]. \qquad \text{(A10)}$$

As follows from the second equation in Eqs. (A5), $h^y_{\mathbf{q}}(z)=h^y_{\mathbf{q}}(\pm d)\exp q(d\mp z)$ for any $z\geqslant d$ $(z\leqslant -d)$. Then, from Eq. (A9) one finds $L_2=0$.

Third integral in Eq. (A3) is reduced to

$$L_3=\frac{1}{8\pi}\sum_i\int\frac{d^2q}{(2\pi)^2}\frac{h^i_{\mathbf{q}}(d)h^i_{-\mathbf{q}}(d)+h^i_{\mathbf{q}}(-d)h^i_{-\mathbf{q}}(-d)}{2q}. \qquad \text{(A11)}$$

Let us consider this expression together with the last integral in Eq. (A3), which may be rewritten as

$$L_4=\frac{1}{8\pi}\int\frac{d^2q}{(2\pi)^2}\left[iq_xh^x_{\mathbf{q}}h^z_{-\mathbf{q}}+iq_yh^y_{\mathbf{q}}h^z_{-\mathbf{q}}+h^x_{\mathbf{q}}\frac{\partial h^x_{-\mathbf{q}}}{\partial z}+h^y_{\mathbf{q}}\frac{\partial h^y_{-\mathbf{q}}}{\partial z}\right]\Bigg|_{z=-d}^{z=d}. \qquad \text{(A12)}$$

To evaluate the derivatives in Eq. (A12) we take a derivative of Eq. (A7) in the region $r<|z|\leqslant d$ and use the first equation of Eqs. (A5) as follows:

$$iq_x\frac{\partial h^x_{\mathbf{q}}}{\partial z}+iq_y\frac{\partial h^y_{\mathbf{q}}}{\partial z}=-\frac{\partial^2 h^z_{\mathbf{q}}}{\partial z^2}=-Q^2h^z_{\mathbf{q}}. \qquad \text{(A13)}$$

Upon substituting the derivative $\partial h^y_{\mathbf{q}}/\partial z$ from Eq. (A13) into Eq. (A12) one finds the exact compensation of the two contributions due to Eqs. (A6): $L_3+L_4=0$. Finally, we get

$$\Delta\mathscr{F}=L_1+L_2+L_3+L_4=0. \qquad \text{(A14)}$$

Thus, the self-energy of the vortex lying completely inside a SC plate (or cylinder[12]) $\mathscr{F}=F$; i.e., it is expressed by formula (13) as if the vortex field on the SC surface and outside were equal to zero. Probably, this statement is valid for the SC cylinder of a general cross section.

Now we return to the periodical problem of a vortex spiral. The self-energy F per unit length along the y axis equals

$$F=\frac{1}{2\pi L}\sum_i\int_{-\pi L}^{\pi L}dy\int_{-\infty}^{\infty}dx\int_{-r}^{r}dz\ h^i\Phi^i=r\sum_i\sum_{k=-\infty}^{\infty}\int_{-\infty}^{\infty}dq\int_{-\pi/2}^{\pi/2}dT\ h^i_k(q,r\ \sin T)\Phi^i_{-k}(q,r\ \sin T)\cos T. \qquad \text{(A15)}$$

Upon substituting the field $h^i_k(q,z)$ from Eq. (6) into formula (A15) one finds the expression for the energy, Eq. (13).

Let us note that for the study of helical distortion nucleation the expansion of energy in small r up to r^2 is sufficient. For the finite but small $r\ll\lambda,d,L$ (softly twisted FL) at $|k|>1$ all the values $C^i_\pm,I^i_\pm\propto r^k$ and make a negligible contribution to the energy, Eq. (13). At $k=0$ $I^{x,z}_\pm\propto r^2$. Since, at $k=0$, $C^{x,z}_\pm$ are infinitesimal at $r\to 0$, they do not contribute to F either. Thus, it is sufficient at $k=0$ to retain the $C^y_\pm$ and $I^y_\pm$ up to order r^2. At $|k|=1$ all the $C^i_\pm,I^i_\pm\propto r$ and, so all the terms in Eq. (14) contribute to the same order of r^2. Finally, to the order of r^2

$$F_0=\left(\frac{\Phi_0}{4\pi\lambda}\right)^2\int_0^\infty\frac{dq}{2\pi}\left\{\left[\frac{\tanh Qd}{Q}\left(1+\frac{r^2}{2\lambda^2}\right)\right]\Bigg|_{k=0}+\frac{r^2}{\lambda^2}\left[\frac{\tanh Qd}{2Q}+\frac{1}{Q\ \sinh 2Qd}\left(\frac{L^{-2}}{p(Q\ \cosh Qd+p\ \sinh Qd)}-q^2\lambda^2\right)\right]\Bigg|_{k=1}\right\}. \qquad \text{(A16)}$$

The evaluation of ΔF represents quite a problem even at small r. Fortunately, it may be done without direct calculation of Eq. (15). Really, this part of the free energy is independent of the plate thickness d and remains unchanged in the limit of an infinite bulk SC at $d\to\infty$. The boundary-sensitive part of the energy, Eq. (A16), may be easily found in this limit, and so we obtain

$$F=\left(\frac{\Phi_0}{4\pi\lambda}\right)^2\left[\left(1+\frac{r^2}{2\lambda^2}\right)\ln\frac{\lambda}{\xi}+\frac{r^2}{2\lambda^2}\ln\frac{\Lambda}{\xi}\right]+\Delta F, \qquad \text{(A17)}$$

where $\Lambda^{-2}=\lambda^{-2}+L^{-2}$. On the other hand this limit may be reached by setting $R\to\infty$ starting from the finite cylinder of the radius R considered in Ref. 12. For this case the same free energy reads

$$F=\left(\frac{\Phi_0}{4\pi\lambda}\right)^2\left(1+\frac{r^2}{2L^2}\right)\ln\frac{\lambda}{\xi}. \tag{A18}$$

Comparing Eq. (A17) with Eq. (A18) one can easily find ΔF. In the case of a thin film $d \leqslant \lambda$, one then easily finds from (A16)–(A18) the expression (16).

*Permanent address.

[1] A.M. Campbell and J.E. Evetts, *Critical Currents in Superconductors* (Taylor & Francis, London, 1972).

[2] R.P. Huebener, *Magnetic Flux Structures in Superconductors* (Springer-Verlag, New York, 1979).

[3] B.C. Belanger and M.A.R. LeBlanc, Appl. Phys. Lett. **10**, 298 (1967).

[4] D. Dew-Hughes, Rep. Progr. Phys. **34**, 821 (1971).

[5] I.L. Landau, Zh. Éksp. Teor. Fiz. **64**, 557 (1973) [Sov. Phys. JETP **37**, 2851 (1973)].

[6] D.G. Walmsley and W.E. Timms, J. Phys. F **7**, 2373 (1977).

[7] T.S. Teasdale and H.E. Rorschach, Phys. Rev. **90**, 709 (1953).

[8] K. Kadowaki, Y. Songliu, K. Kitazawa, Supercond. Sci. Technol. **7**, 519 (1994).

[9] G.E. Marsh, Phys. Rev. B **49**, 571 (1994).

[10] J.R. Clem, Phys. Rev. Lett. **38**, 1425 (1977).

[11] E.H. Brandt, Phys. Lett. **79A**, 207 (1980); Phys. Rev. B **25**, 5756 (1982).

[12] Yu.A. Genenko, Pis'ma Zh. Éksp. Teor. Fiz. **59**, 807 (1994) [JETP Lett. **59**, 841 (1994)]; Phys. Rev. B **51**, 3686 (1995).

[13] M. Tinkham, *Introduction to Superconductivity* (McGraw-Hill, New York, 1975).

[14] I.S. Gradshteyn and I.M. Ryzhik, *Tables of Integrals, Series and Products* (Academic, New York, 1980).

[15] V.V. Shmidt, Zh. Éksp. Teor. Fiz. **61**, 398 (1971) [Sov. Phys. JETP **34**, 211 (1972)].

[16] J.R. Clem, R.P. Huebener, and D.E. Gallus, J. Low Temp. Phys. **12**, 449 (1973).

[17] A. Gurevich and E.H. Brandt, Phys. Rev. Lett. **73**, 178 (1994); Th. Schuster *et al.*, *ibid.* **73**, 1424 (1994); E.H. Brandt, Phys. Rev. B **48**, 12 893 (1993); **49**, 9024 (1994); **50**, 4034 (1994); E. Zeldov *et al.*, *ibid.* **49**, 9802 (1994); I. Aranson, M. Gittermann, and B.Ya. Shapiro, *ibid.* **51**, 3092 (1995).

[18] E. Zeldov *et al.*, Phys. Rev. Lett. **73**, 1428 (1994).

[19] A.I. Larkin and Yu.N. Ovchinnikov, Zh. Éksp. Teor. Fiz. **61**, 1221 (1971) [Sov. Phys. JETP **34**, 651 (1972)].