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Vortex dynamics in a model of superflow: The role of acoustic excitations

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In this paper we present numerical simulations for the dynamics of a vortex pair described by the two-dimensional nonlinear Schrödinger equation. The simulations reveal significant nonvanishing fluctuations around the classical velocities of vortices in an ideal incompressible fluid due to excitation of the acoustic (or sound) field. The reason for the fluctuations is the potentiality of the superflow, which introduces significant compressibility to the superfluid. We discuss some implications of this observation, especially on existing discrepancies between theories of superfluid turbulence. We indicate a possible mechanism for the spontaneous nucleation of a small vortex ring in the three-dimensional nonlinear Schrödinger equation.

The nonlinear Schrödinger equation (NLSE) is a popular qualitative model for superfluidity, nonlinear optics, pattern formation, and more. The vortices with nonzero topological charge play an important role in the dynamics. For example the turbulent state is described as a vortex tangle.[1] Using the Madelung transformation, the NLSE may be transformed into the well known Euler equation for inviscid fluids having the same continuity equation. This analogy has led some researchers to describe the dynamics of the vortices in NLSE exactly in the same way as the dynamics of vortex lines in classical ideal fluids, with the only constraint that the vorticity in the former case is quantized. In the turbulent state for example, using this analogy, one describes the interaction between vortices in a classical manner. In this paper we show that this description suffers from serious deficiencies, and try to clarify an existing discrepancy in the theory of superfluid turbulence.

The corresponding NLSE reads as follows:

$$-i\hbar\partial_t\psi=(\hbar^2/2M)\Delta\psi+g(1-|\psi|^2)\psi\ , \tag{1}$$

where Δ is the Laplacian operator, g is the strength of the short-range interparticle potential, $\hbar$ is Planck's constant, and M is the ^4H_e atom mass. In normalized units we obtain

$$-i\partial_t\psi=\Delta\psi+(1-|\psi|^2)\psi\ , \tag{2}$$

where we rescaled time as $t\rightarrow tg/\hbar$, space as $r\rightarrow r/a$, and $a=\hbar/\sqrt{2Mg}$ is the so-called healing length which for ^4H_e superfluid in the low-temperature limit is of the order of a few Å. The Madelung transformation for this equation is simply $\psi=Re^{iS}$ where R and S are real functions. This gives the following Euler equation:[2]

$$D_tV=\nabla P\ , \tag{3}$$

where D_t is the material derivative $(D_t=\partial_t+V\cdot\nabla)$, the velocity V is defined as $V=2\nabla S$, and the pressure P is

$$P=-\Delta R/R+R^2-1\ , \tag{4}$$

where the first term in the right-hand side of this equation is the so-called "quantum pressure."

In the limit $g\rightarrow 0$ of Eq. (1) we obtain the free Schrödinger equation which in its hydrodynamic form is the same, except the pressure which contains only the quantum pressure term. In this case one can calculate the vortex dynamics using the exact propagator. The resulting dynamics in most cases is not at all compatible with the incompressible vortex line dynamics. In the Euler equation the incompressibility approximation is reliable for velocities much smaller than the sound velocity. In our case normalized sound velocity is unity (dimensionally it goes like $\sqrt{g}$), and thus this approximation is more credible for intervortex distances much larger than a. In the opposite case, $g=0$, the sound velocity vanishes, and clearly the incompressible regime does not exist at all. Therefore one could expect the analogy with classical incompressible thin vortex tubes to hold for velocities much smaller than the sound velocity, or alternatively, when the quantum term is not the dominant one in the above defined pressure P. A notorious example to an incompressible simple dynamics is an antiparallel vortex pair at a distance $d\gg 1$, which for incompressible classical Eulerian fluid drifts with velocity of order $2/d+O(d^{-2})$ in the direction normal to the line connecting the cores of the vortices.

In order to test the validity of the incompressibility approximation, we have performed numerical simulations of the vortex pair dynamics in the two-dimensional NLSE. We used a pseudospectral implicit split-step method based on fast Fourier transform (FFT). This method is conservative and conserves also some additional integrals of NLSE.[3] We used typically 256×256 Fourier modes in a domain of 200×200 units; the time step was chosen to be 0.05. The results were verified for different resolutions. The boundary conditions were periodic in both the x direction and the y

direction, which are equivalent to a periodic array of vortices in both directions. The stability of such an array in the Eulerian context is discussed in the classical Lamb's book.[4] Although symmetric double row vortex array, according to the Ref. 4, is known to be unstable in an infinity system, our periodic array is stable because the periodic boundary conditions, which we impose in our simulations, admit only stable wave vectors ($\phi=0$, 2π in Lamb's notation).

The method of finding the center of the vortices is comprised of two stages. A linear approximation is applied to find a rough position of the vortex core, and then a Newton-Raphson method for finding the global zero[5] is used to refine the position found in the first stage.

It is believed that an exact two-vortex solution cannot be written analytically. Moreover, there is no simple method to generate it numerically from specific initial conditions. For initial conditions which are close enough to two single stationary vortices ψ_0 such as $\psi=\psi_0(r-d/2)\psi_0(r+d/2)$, Fetter[6] obtained analytically for the leading order the classical $2/d$ result. This result later was reproduced by other researchers (see, e.g., Ref. 7). It was shown that $2/d$ law is consistent from the point of view of matching the far field asymptotics with the perturbed core solution. A more suitable initial condition to our investigation than that of Fetter[6] in the sense that it is closer to the classical dynamics may be obtained by solving numerically the real dissipative Ginzburg-Landau (GL) equation [no $-i$ in the left-hand side in Eq. (2)].[8] This equation has the same stationary behavior as the NLSE. Although the GL has no stationary solution representing a vortex pair, we expect that for sufficiently long time evolution of a well separated pair of the GL equation, the obtained solution is sufficiently close to the vortex pair solution of the NLSE case.

This argument led us to use initial conditions as follows: we take an initial generic function ψ having two single zeros of opposite charge in the domain of integration, then iterate it with the GL equation until it converges to a regularized enough function. This function is used as the initial condition for the NLS equation. In Fig. 1 we present typical results for the dynamics of a vortex pair. One can see that indeed the mean velocity is $2/d$ in the x direction [see Fig. 1(a)] and zero in the y direction [Fig. 1(b)] as one would expect, but in both directions there are significant fluctuations around this mean value. These fluctuations are caused by scattering of sound waves on the vortices. Moreover, those fluctuations do not decay with time which is an obvious result of the Hamiltonian structure of the NLSE. In order to evaluate the effect of the fluctuations as a function of the box size L, we performed numerical experiments for different values of the box size L and the intervortex distance d.

Our results can be characterized by defining the normalized standard deviation of the velocity $\bar{w}_i$ namely, $\bar{w}_i=\sqrt{\langle v_i^2\rangle-\langle v_i\rangle^2}/\langle v_x\rangle$ where v_i is the vortex core velocity in the direction x or y (note that we normalize by the velocity in the x direction only, because this is the only typical velocity of the system, and $\langle v_y\rangle=0$).

The relative average fluctuations $\bar{w}_x$ against the inverse box length is shown in Figs. 2 and 3. In Fig. 2 we keep constant the intervortex distance $d=40$, and in Fig. 3 the ratio L/d is kept constant and equal to 10.

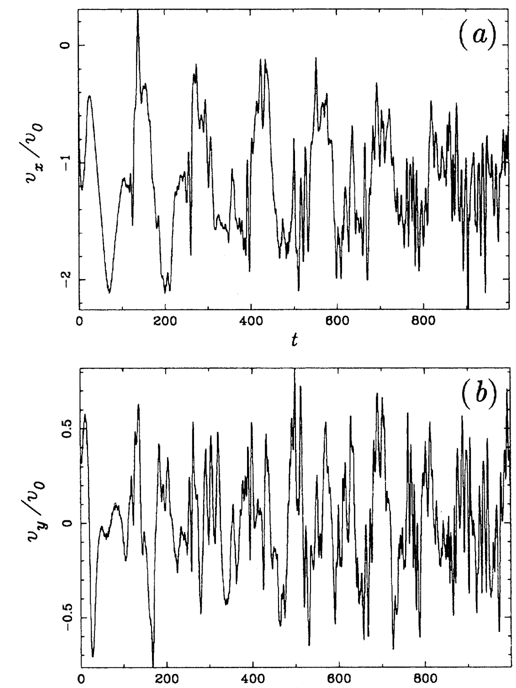

FIG. 1. Normalized components of the velocity of a vortex pair. $d=40$, $L=200$. Here $v_0=2/d$ is the mean classical drift velocity in x direction.

From Fig. 2 one clearly sees that the standard deviation is of the same order of magnitude as the mean velocity for small enough box length L. This means that even for the incompressible regime there are large velocity fluctuations around the mean value $2/d$. In this regime the magnitude of the fluctuations is basically box independent, but the frequency decreases with L. This frequency dependence on L is a strong evidence to our interpretation, that those fluctuations are caused by sound waves which reflected back from the boundaries. The time of flow to the boundaries and back is proportional to L with a sound velocity 1. This was concluded from the analysis of the full field pattern (see, e.g., Ref. 9).

The ratio of fluctuations to mean velocity would be even worse when any other initial condition is taken, as explained before. We checked this statement by simulating the same scenarios as in Fig. 1, but with a less regularized initial con-

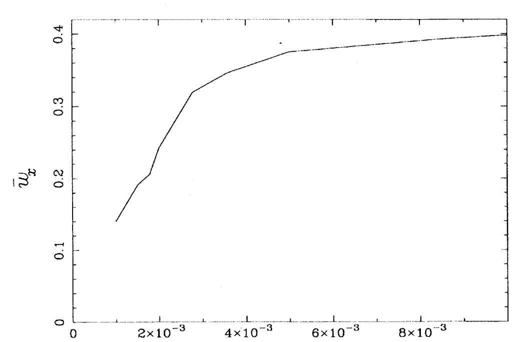

FIG. 2. The normalized standard deviation versus $1/L$ for fixed $d=40$.

ditions. We found that *for a small enough number of GL steps we can get much larger $\bar{w}_i$ of up to 10 or more*. Without any regularization we got a very erratic dynamics.

For box lengths larger than 250 we see from Fig. 2 a significant decrease in these fluctuations. This figure could bring one to argue that in real systems L is much higher than 250 healing lengths a (a typical value of L/a is about 10^6). This fact means that for real boxes there are no significant fluctuations around the incompressible velocity. This argument seems reasonable for an isolated vortex pair, but for typical cases (especially in the turbulent state) there are many other vortices around. For those cases one can think of other vortices as representing effective boundaries which interact with this specific pair by scattering of waves. In those cases the effective boxes are not so big, and could be considered to have the same order of magnitude as d. To check the implications of such a picture we performed the second calculation, in which we varied L and kept $L/d=10$. As one can see in Fig. 3 the fluctuations increase with L. This means that the mean velocity decreases as $1/d$, but the fluctuations stay with the same order of magnitude, so the normalized standard deviation increases. We conclude that the incompressibility approximation is not appropriate for dense enough arrangement of the vortices even for very large systems.

It is notable that these dynamical fluctuations are not caused by temperature but originated by the internal dynamics of superfluid. The analogy with the Euler equation has broken down because there is another condition, that the velocity field is a potential flow (it is a gradient of the scalar function S).[10] We see that even for low velocities the dynamics of vortices depends on the entire field and cannot be written in a local universal form as it is usually done. In an infinite system these fluctuations are not relevant because the waves are radiated away and play the role of an "effective dissipation." Certainly, the waves propagate due to the compressibility of the NLSE. The potentiality of the velocity field in the quantum case presumes significant compressibility, because the vortices are not just thin vortex tubes as in the classical Euler equation, but they are topological defects in the complex scalar field ψ. For these defects the order parameter (the superfluid density) vanishes at their axis. This means that the density has large gradients, breaking the compressibility condition.

To summarize our main observations from the numerical experiments, we get a persistent nontrivial dynamics which obeys the $2/d$ law only in a very restricted statistical sense. Moreover, the vortices do not play a crucial role in the dynamics, because an infinite (or very high) number of degrees of freedom, related to the acoustic field, are effectively excited. The same consideration holds also for arrays of vortices as one can conclude from our Figs. 2 and 3.

This observation has many possible physical implications. The first concerns the widely used method to describe a superfluid with vortex Hamiltonian dynamics (see, e.g., Ref. 11). This model assumes the vortices to interact as Coulomb particles, and the superfluid dynamics depends on a discrete number of degrees of freedom (effective many-body dynamics). This is true only on the average in the quantum case, as shown before, and we think that one has to use a more proper (but much more difficult) way to describe superfluid dynamics in the full field theory. In contrast, for dissipative systems, the dynamics is well presented by a few localized structures (vortices) because the radiation dies out exponentially fast (see, e.g., Ref. 12).

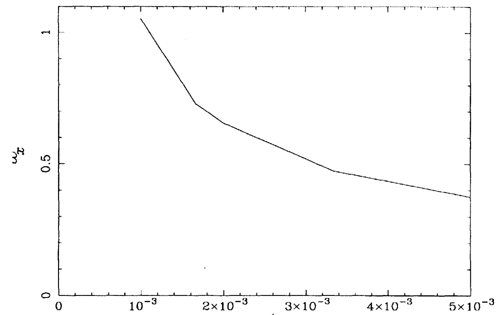

FIG. 3. The normalized standard deviation versus $1/L$ for fixed $L/d=10$.

Now we discuss other implications of these results. Two general theoretical approaches to superfluid turbulence are the scaling argument due to Hall and Vinen,[13] and the numerical method of Schwarz.[14] In both cases the parameter which is used to characterize the turbulent state is the total length of vortex lines per unit volume l. The scaling argument describes the turbulent state as a balance between generation and decay terms. In this argument, as Vinen himself pointed out, there is a problem of initiation of turbulence, because if $l=0$ there is no way to get a finite generation rate. This is due to the form of the generation term which is proportional to $l^{3/2}$. The only mechanism he could think of is thermal generation of vortex rings which will expand in time. But the radius of such a vortex ring has to be of order 10^{-4} cm, which is unreasonably large. He concludes that this is a "serious fundamental difficulty."

According to our observations, if there are significant velocity fluctuations (sound waves), then it might occur that even smaller rings will expand because of a local instability (it is commonly believed that the two-dimensional NLSE is not integrable in contrast to the one-dimensional one). In the real turbulent state we would expect that thermal fluctuations would create "seed" vortices which are much less "optimal" than those we used in our calculations, so we expect the fluctuations to be much larger than those we have presented. In this sense we expect that NLSE with "GL initial conditions" as we used represents a lower bound on more realistic cases as stated before. Those fluctuations might be the source of the phenomenological term presented by Vinen.

Here we discuss another mechanism that can be responsible for initiation of a turbulent state by the dynamical creation of vortex rings in the context of NLSE. One has to rescale the dynamic variables in Eq. (2) in the following manner: $x\rightarrow\epsilon x$, $t\rightarrow\epsilon t$, and $S\rightarrow\epsilon S$. This specific type of the scaling describes a slow dynamics of the sound waves with a very large period. Then one gets the Euler Eq. (3) with the pressure as in Eq. (4), and no quantum term. This form of the equation is equivalent to the classical Euler equation with the specific equation of the state. One easily sees that the pressure $\sim\rho^2$[after multiplying the Eq. (3) by $\rho=R^2$ in both sides]. For such an equation it is known that in $3d$ there occurs a finite time singularity in $V=\nabla S$ (Ref. 15) for a generic class of initial conditions. This phenomenon occurs also for the potential flow which is our case.[16] A singularity in V means divergence of the gradient of the phase S. This mean a spontaneous creation of a multivalueness of the order parameter. The phenomenon can be presumably treated as a creation of the vortex ring for which the velocity in the core diverges. If our interpretation to this phenomena is right, one can expect a creation of small vortex rings. We point out here that this divergence of the phase has nothing to do with the well-known finite time singularities in the focusing case of the NLSE (opposite sign of the interaction g), which is the blowup of the density (R). This argument involves only the first order in ϵ equation, but it is nevertheless a very strong indication of spontaneous nucleation of small vortex rings in three-dimensional NLSE. This mechanism may be the source to the initiation term of Vinen described before. Of course, this hypothesis needs detailed numerical verification.

Now we consider Schwarz's approach. He constructs a model of vortex dynamics consisting of two components. The first is the dynamics of the vortex due to the superfluid field effects which he treats with the self-induction approximation, and the second is due to the mutual friction with the normal field. Nonlocal effects are treated through the ansatz that when vortices are close enough they reconnect. He obtains the same scaling relations as the above mentioned approach. He obtained also a total vortex length per unit volume l which is compatible with the experimental values. In 1987 Buttke[17] raised a question about the validity of the numerical scheme that Schwarz used, and it was argued, and to the best of our knowledge proved, that careful enough calculations with his method gives another result for l which is not compatible with the above mentioned experimental results. Buttke concludes that the self-induction approximation is not valid for quantum vortex dynamics, and a more reliable model to describe superfluid turbulence "has to include more of the nonlocal character of Euler equation than self-induction does." Schwarz's basic argument as we understand it is that for thin enough vortices in the Euler equation this approximation is shown to be valid.

We think that Buttke's numerical argument was correct and Schwarz's answer also is correct for the Euler classical case. The problem lies in the fact that we deal with quantum vortices and not only very thin ones. For this reason, as we pointed out before, the analogy between superfluid vortex dynamics and classical (even thin) vortex tubes breaks down. This is the reason why one has to consider the nonlocal character of the superfluid field even for thin vortices where the regular Euler dynamics is of local character. For the same reason it might occur that vortex rings can expand even below the classical critical radius as we argued before.

In the end we consider one more general question that could be raised to oppose the significance of our results. In a real case where there is dissipation with the normal component of the helium, these fluctuations would be depressed much faster than the pair annihilates, and the mean dynamics seems to be correct. This is the case if the friction is described by small real addition ϵ to the imaginary unit i before the time derivative in the NLSE. Even in this case the decay of fluctuations depends on the ratio of $1/\epsilon$ and L, which determines the time scale of dissipation versus the time scale of reflection of waves from the boundaries back to the vortex pair. However, in the helium context for low enough temperature this is not the form of the interactions with the normal. In the turbulent state which we discussed in this paper there is a constant pumping of the superfluid by the normal. For example, in Schwarz calculations this pumping is done by keeping the velocity of the normal constant. In that case, there is persistent creation of waves which keeps our argument relevant even for very long time scales.

To summarize, we have shown that quantum vortices behave differently from classical ones even for low velocities. We observed that the dynamics is much less local for the vortices in NLSE due to compressibility effects, and discussed possible implications on superfluid dynamics and especially on the initialization and characterization of the homogeneous turbulent state. We pointed out that the above mentioned nonlocality gives a clue that the basic building block for the theory of superfluid turbulence may not be the quantum vortices "many body" dynamics, which cannot be separated from the whole field dynamics, but one must treat the field itself as the basic dynamical variable.

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