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Coulomb-gas approach to quantum motion in random magnetic fields

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We study the motion in the plane of a spinless particle subject to a random magnetic field, in terms of Feynman's influence functional. We show that with δ-correlated fields the Gaussian regime is dominated by one-dimensional quantum motion over classical walks. In the case of long-range field correlations the system is described as a two-dimensional Coulomb gas, where the plasma phase is dominated by free-particle propagation and the low-temperature phase by a regime of strong coupling between advanced and retarded paths. The semiclassical regime is also discussed.

The quantum mechanics of a spinless particle in the plane under the influence of a random magnetic field models various condensed-matter situations, ranging from propagation in chiral spin liquids[1,2] to quantum Hall systems near even-denominator filling fractions.[3–5] In particular, at filling factor $\nu=\frac{1}{2}$ the flux lines carried by the particles screen the external field, so that the effective dynamics is described by a zero-average static random field. Various numerical studies on the lattice and analytical works have been devoted to the spectral properties of the problem.[6–11] A mobility edge has been found,[12,13] while *a priori* one would expect only localized states, since the system belongs to the unitary class of localization theory.[14] More recently the critical exponent of the localization length has been obtained,[15] apparently with the character of universality, also in the presence of an impurity potential. By generalizing the localization theory in the framework of the unitary ensemble to the case of a random magnetic field with zero average, Zhang and Arovas[16] pointed out that, although the topological term is absent, field fluctuations generate a logarithmic coupling between topological densities. This leads in a natural way to a sine-Gordon model for the topological phase so that a localization-delocalization transition is expected with the characters of the Kosterlitz-Thouless class.

We discuss the system in the absence of an impurity potential, using the quantum description of Feynman's influence functional,[17] which allows one to deal with classical averages of quantum probabilities: in this approach one directly considers two-particle Green's functions, and interference effects between advanced and retarded contributions are made transparent. We determine the effective action with δ-correlated and long-range field fluctuations: in the latter case we map the problem to a Coulomb gas system. We also study the Gaussian regime, where a zero mode is found corresponding to area-preserving deformations of the paths.

As announced, we deal, rather than with amplitudes, with quantum-mechanical probabilities. The time evolution of the density operator

$$\langle \mathbf{x}_1|\rho(T)|\mathbf{x}_2\rangle=\int d^2x_1' d^2x_2' \langle \mathbf{x}_1'|\rho(0)|\mathbf{x}_2'\rangle \mathscr{F}(\mathbf{x}_1,\mathbf{x}_1';\mathbf{x}_2,\mathbf{x}_2') \tag{1}$$

involves a two-path integral,

$$\mathscr{F}(\mathbf{x}_1,\mathbf{x}_1';\mathbf{x}_2,\mathbf{x}_2')=\int \mathscr{D}\mathbf{x}_1(\tau)\mathscr{D}\mathbf{x}_2(\tau)e^{(i/\hbar)(S_0[\mathbf{x}_1]-S_0[\mathbf{x}_2])}, \tag{2}$$

where $\mathbf{x}_1(\tau)$ and $\mathbf{x}_2(\tau)$ represent retarded and advanced paths, respectively, satisfying the boundary conditions $\mathbf{x}_1(T)=\mathbf{x}_1$, $\mathbf{x}_1(0)=\mathbf{x}_1'$, $\mathbf{x}_2(T)=\mathbf{x}_2$, and $\mathbf{x}_2(0)=\mathbf{x}_2'$. In our context the action $S_0[\mathbf{x}]$ describes the two-dimensional spinless particle in a magnetic field $B(\mathbf{x})$ orthogonal to the plane:

$$S_0=S_{\mathrm{kin}}+S_{\mathrm{magn}}=\int_0^T d\tau\left[\frac{M}{2}\,\dot{\mathbf{x}}^2(\tau)+\frac{e}{c}\mathbf{A}[\mathbf{x}(\tau)]\cdot\dot{\mathbf{x}}(\tau)\right]. \tag{3}$$

The Coulomb gauge, $\mathrm{div}\mathbf{A}=0$, can be solved in terms of a scalar field $\varphi(\mathbf{x})$: $A_x=\partial_y\varphi$, $A_y=-\partial_x\varphi$ with equation $-\nabla^2\varphi(\mathbf{x})=B(\mathbf{x})$. The action is then written in terms of the magnetic field:

$$\begin{aligned} S_{\mathrm{magn}}[\mathbf{x}]&=-\frac{e}{2\pi c}\int_0^T d\tau\left(\dot{x}(\tau)\frac{\partial}{\partial y(\tau)}\right.\\ &\qquad\left.-\dot{y}(\tau)\frac{\partial}{\partial x(\tau)}\right)\int d^2x' B(\mathbf{x}')\ln|\mathbf{x}'-\mathbf{x}(\tau)|\\ &=\frac{\hbar}{\Phi_0}\int d^2x' B(\mathbf{x}')\Theta(\mathbf{x}';[\mathbf{x}]), \end{aligned} \tag{4}$$

where $\Phi_0=hc/e$ is the elementary flux quantum and, using the cross product $\mathbf{a}\times\mathbf{b}=a_xb_y-a_yb_x$,

$$\Theta(\mathbf{x}';[\mathbf{x}])=-\int_0^T d\tau\,\frac{\dot{\mathbf{x}}(\tau)\times[\mathbf{x}(\tau)-\mathbf{x}']}{|\mathbf{x}(\tau)-\mathbf{x}'|^2}. \tag{5}$$

The global phase $\Theta(\mathbf{x}';[\mathbf{x}])$ of the path $\mathbf{x}(\tau)$ only depends on its geometry, that will herefrom be called walk $[\mathbf{x}]$, and is the sum of two contributions: the first is given by 2π times the winding number of $[\mathbf{x}]$ around the point $\mathbf{x}'$; the second is a boundary term, being the angular width of the vector $\mathbf{x}(T)-\mathbf{x}(0)$ viewed from $\mathbf{x}'$. For closed paths it can be nonzero only in the interior of [x], where it takes the values $\pm 2\pi n$: in this case the magnetic action is proportional to the flux of the magnetic field through [x]. Notice that $\Theta(\mathbf{x}';[\mathbf{x}])$ is discontinuous across the walk: a natural prescription is to fix its value there as the average among the values taken in contiguous domains. In this way $\Theta(\mathbf{x}';[\mathbf{x}])$ is piecewise constant along [x], where its discontinuities correspond to self-intesections. Being a topological index, its value in $\mathbf{x}'$ is not affected by deformations of the walk, unless it is crossed by it. Thus the first variation is nonzero only on the walk itself:

$$\Theta(\mathbf{x}';[\mathbf{x}+\xi])\simeq\Theta(\mathbf{x}';[\mathbf{x}])-2\pi\int_0^T d\tau\,\dot{\mathbf{x}}(\tau)\times\xi(\tau)\,\delta_2(\mathbf{x}'-\mathbf{x}(\tau)). \tag{6}$$

The second variation is

$$-\pi\int_0^T d\tau\{\dot{\xi}(\tau)\times\xi(\tau)+[\dot{\mathbf{x}}(\tau)\times\xi(\tau)]\xi(\tau)\cdot\nabla_{x'}\}\times\delta_2(\mathbf{x}'-\mathbf{x}(\tau)). \tag{7}$$

While studying the probability of return one restricts the integration in the functional (2) to couples of paths $\mathbf{x}_1(\tau)$ and $\mathbf{x}_2(\tau)$ joining the same points: $\mathbf{x}_1(T)=\mathbf{x}_2(T)=\mathbf{x}$, $\mathbf{x}_1(0)=\mathbf{x}_2(0)=\mathbf{y}$. Therefore, the walks $[\mathbf{x}_1]$ and $[\mathbf{x}_2]$ together form a closed walk $[\mathbf{x}]=[\mathbf{x}_1]\cup[\mathbf{x}_2]$ with geometric index

$$Q(\mathbf{x}';[\mathbf{x}])=\frac{1}{2\pi}\{\Theta(\mathbf{x}';[\mathbf{x}_2])-\Theta(\mathbf{x}';[\mathbf{x}_1])\}. \tag{8}$$

The time stationarity of the problem allows one to reformulate the path integration in terms of a single closed path $\mathbf{x}(\tau)$, $-T<\tau<T$, subject to the conditions $\mathbf{x}(-T)=\mathbf{x}(T)=\mathbf{x}$, $\mathbf{x}(0)=\mathbf{y}$:

$$\mathscr{F}(\mathbf{x},\mathbf{y};B)=\int\mathscr{D}\mathbf{x}(\tau)e^{-(i/\hbar)S_{\rm kin}[\mathbf{x}]-(i/\hbar)S_{\rm magn}[\mathbf{x}]}, \tag{9}$$

where $S_{\rm magn}[\mathbf{x}]$ has the form (4) and the kinetic term is

$$S_{\rm kin}[\mathbf{x}]=\frac{M}{2}\int_0^T d\tau\{\dot{\mathbf{x}}^2(\tau)-\dot{\mathbf{x}}^2(-\tau)\}. \tag{10}$$

Let us assume that the magnetic field is random, with a δ-correlated distribution:

$$P[B]\propto\exp\left[-\frac{1}{2D}\int d^2x\,B^2(\mathbf{x})\right]. \tag{11}$$

The influence functional, by means of which average quantum probabilities are computed, is easily obtained:

$$\overline{\mathscr{F}}(\mathbf{x},\mathbf{y})=\int\mathscr{D}B(\mathbf{x})P[B]\mathscr{F}(\mathbf{x},\mathbf{y};B)=\int\mathscr{D}\mathbf{x}(\tau)e^{-(i/\hbar)S_{\rm eff}[\mathbf{x}]}, \tag{12}$$

with effective action

$$S_{\rm eff}[\mathbf{x}]=S_{\rm kin}[\mathbf{x}]-i\frac{g}{2}\int d^2x'Q^2(\mathbf{x}';[\mathbf{x}]),\qquad g=4\pi^2\hbar D\Phi_0^{-2}. \tag{13}$$

If A_i are the areas, with winding numbers n_i, into which the interior of the walk [x] is partitioned, the topological term in the effective action is

$$\int d^2x'Q^2(\mathbf{x}';[\mathbf{x}])=\sum_i n_i^2A_i. \tag{14}$$

For large disorder D, the main contribution comes from closed paths $\mathbf{x}(\tau)$ with zero enclosed area, that is, in the time interval $(0,T)$ the particle retraces the geometry covered in the time interval $(-T,0)$, although not necessarily with the same time law. If one naively restricts to such trajectories, he is left with a sum over retarded and advanced histories along each walk, together with a sum over walks. This corresponds to a classical average over one-dimensional quantum motions, with scattering at the self-intersections.

By making the effective action (13) stationary one obtains the equations

$$\begin{aligned}M\ddot{x}(\tau)+\dot{y}(\tau)\mathrm{sgn}(\tau)\{b-igQ(\mathbf{x}(\tau);[\mathbf{x}])\}=0,\\ M\ddot{y}(\tau)-\dot{x}(\tau)\mathrm{sgn}(\tau)\{b-igQ(\mathbf{x}(\tau);[\mathbf{x}])\}=0,\end{aligned} \tag{15}$$

where a nonzero average value $b=(e/c)\bar{B}$ for the $B(\mathbf{x})$ field has been assumed, and the effective interaction appears as an imaginary contribution to the magnetic field. Since we are dealing with a quantum open system, complex-valued solutions should also *a priori* be taken into account;[18] notice that the kinetic energy is always conserved. A solution that minimizes the effective action is clearly given by the classical motion in the absence of disorder joining x to y and backwards.

The effective action in the Gaussian regime, around solutions $\mathbf{x}(\tau)$ of (15), is affected by a δ-like singularity in the term

$$-i\frac{g}{2}\int_{-T}^T d\tau_1\int_{-T}^T d\tau_2[\dot{\mathbf{x}}(\tau_1)\times\xi(\tau_1)][\dot{\mathbf{x}}(\tau_2)\times\xi(\tau_2)]\times\delta_2(\mathbf{x}(\tau_1)-\mathbf{x}(\tau_2)), \tag{16}$$

which fully inhibits the fluctuations $\xi(\tau)$ normal to the orbit

$$\dot{\mathbf{x}}(\tau)\times\xi(\tau)=0; \tag{17}$$

hence the quantum fluctuations are strictly one dimensional. While studying the semiclassical regime around a high-energy rectilinear path, Altshuler and Ioffe[7] pointed out that the transverse variations are exponentially suppressed. Contrary to the present approach, where we deal with quantum probabilities and compute variations of the exact effective action, they averaged the single-particle Green's function and put by hand a reference retarded trajectory in the Feynman path integral, ensuring gauge invariance.

Let us examine the case of smooth fluctuations, where we average over the measure

$$P[B]\propto\exp\left[-\frac{1}{2\gamma}\int d^2x[|\nabla B(\mathbf{x})|^2+m^2B^2(\mathbf{x})]\right]. \tag{18}$$

For small m, the effective action becomes

$$S_{\text{eff}}[\mathbf{x}]=S_{\text{kin}}[\mathbf{x}]-i\hbar\pi\gamma\Phi_0^{-2}\int d^2x'\int d^2y' Q(\mathbf{x}';[\mathbf{x}])\times Q(\mathbf{y}';[\mathbf{x}])\ln(m|\mathbf{x}'-\mathbf{y}'|). \quad (19)$$

It is useful to view the geometric index $Q(\mathbf{x}';[\mathbf{x}])$ as a (integer-valued) charge density. Each walk $[\mathbf{x}]=[\mathbf{x}_1]\cup[\mathbf{x}_2]$ determines in the plane a pattern of uniformly charged domains undergoing a screened Coulomb interaction; in analogy with Eq. (12), we can write the Coulomb term as a sum over the domains S_i enclosed by the loops in $[\mathbf{x}]$ with uniform charge densities n_i:

$$\sum_i n_i^2\int_{S_i} d^2x\, d^2y\, \ln(m|\mathbf{x}-\mathbf{y}|)+2\sum_{i<j} n_i n_j\int_{S_i} d^2x\int_{S_j} d^2y\, \ln(m|\mathbf{x}-\mathbf{y}|). \quad (20)$$

From now on we will limit ourselves to the case of long-range fluctuations $m\to 0$, where only globally neutral charge configurations contribute to the functional integral, $\Sigma n_i A_i=0$. In the Gaussian regime around a fixed trajectory $\mathbf{x}(\tau)=\mathbf{x}_1(\tau)=\mathbf{x}_2(\tau)$, $0<\tau<T$, the transversal fluctuations are no longer inhibited, but a zero mode is found satisfying the condition

$$\dot{\mathbf{x}}(\tau)\times[\boldsymbol{\xi}_1(\tau)-\boldsymbol{\xi}_2(\tau)]=0. \quad (21)$$

In other words the domains can undergo deformations provided the total charge is conserved: the mode spans a "charged-liquid sector" of the theory.

Notice that since $Q(\mathbf{x}';[\mathbf{x}])$ is adimensional, the coupling constant $\gamma\Phi_0^{-2}$ in (19) is dimensional. A connection with known results of the Coulomb gas is made by introducing a coarse-graining of the charge boundary on a cutoff scale λ, defined by fixing a minimal de Broglie wavelength $\lambda=h(2ME_{\max})^{-1/2}$. By rewriting the magnetic action term in (19) as sums over plaquettes of area λ^2, one obtains a Coulomb gas at the inverse temperature $\beta=\gamma\lambda^4\Phi_0^{-2}$. The identification allows us to estimate the plasma-dielectric transition at

$$\beta_c=\frac{\gamma_c\lambda^4}{\Phi_0^2}=\frac{2}{\pi}. \quad (22)$$

This result is consistent with the analysis performed in Ref. 16 in that we also find a critical behavior belonging to the Kosterlitz-Thouless class.

According to this picture, at weak disorder ($\gamma\ll\gamma_c$), the extended wave function is enhanced over islands of the typical size of the plasma screening length $\lambda_s\approx\lambda\ \ln^{-1}\Phi_0^2/\gamma\lambda^4$; with the inclusion of an impurity potential this phase corresponds to the localized phase of Ref. 16.

In the strong disorder phase ($\gamma>\gamma_c$) there is infinite correlation among charges, and, accordingly, the advanced and retarded paths are strongly coupled. Indeed, in this regime, for each field configuration, mainly walks lying close to the network $B(\mathbf{x})=0$ do contribute to the conductivity, since particles in regions of strong magnetic field tend to drift along orbits with small cyclotron radius. The single-particle spectrum in a smoothly varying magnetic field has been studied in Refs. 19 and 20: together with states confined within a narrow one-dimensional region near the line of zero field, also states propagating in the opposite direction, but spatially separated, do occur. As a result, finite longitudinal conductivity was obtained along the network $B(\mathbf{x})=0$. If the field has a nonzero average the network does not percolate, and a positive magnetoresistance is expected. These arguments suggest that strongly coupled paths enhance the conductivity when the field fluctuates over large scales, but only by a careful analysis of backscattering effects can one determine whether the quantum phase coherence survives over an infinite length.

Let us now summarize our results: in the case of δ correlations the strong disorder regime forces the advanced and retarded histories to lie along the same walk. In the Gaussian regime the transversal fluctuations are completely inhibited.

Smooth correlations lead to a Coulomb interaction between domains enclosed by advanced and retarded paths. More precisely, the real (interface) part of the action gives the kinetic phase along the boundaries of the domains, and the imaginary (bulk) part is the Coulomb term, which depends on the walks but not on the histories. The problem is then to determine the effect of the bulk transition both on the boundaries (walks) and on quantum motion (histories) over them. On qualitative grounds the following picture can be drawn. If the charges are in the plasma phase ($\gamma<\gamma_c$), we have a regime of weak perturbation of the free motion, where the extended wave function is enhanced in islands of the order of the plasma screening length. In the dielectric regime, at strong disorder ($\gamma>\gamma_c$), the charges form bound states with infinite correlation length and oppositely oriented channels are strongly coupled. The question arises as to whether this implies the conservation of quantum phase coherence. In this respect our picture is in agreement with Ref. 16, but we stress that the proof of the existence of extended states requires an analysis of the effect of backscattering at various energies.

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