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Weak localization coexisting with a magnetic field in a normal-metal–superconductor microbridge

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A random-matrix theory is presented which shows that breaking time-reversal symmetry by itself does *not* suppress the weak-localization correction to the conductance of a disordered metal wire attached to a superconductor. Suppression of weak localization requires applying a magnetic field as well as raising the voltage, to break both time-reversal symmetry and electron-hole degeneracy. A magnetic-field-dependent contact resistance obscured this anomaly in previous numerical simulations.

Weak localization is a quantum correction of order e^2/h to the classical conductance of a metal.[1] The word "localization" refers to the negative sign of the correction,[2] while the adjective "weak" indicates its smallness. In a wire geometry the weak-localization correction takes on the universal value[3] $\delta G=-\frac{2}{3}e^2/h$ at zero temperature, independent of the wire length L or mean free path ℓ.[4] The classical (Drude) conductance $G_0\simeq(N\ell/L)e^2/h$ is much greater than δG in the metallic regime, where the number of scattering channels $N\gg L/\ell$. Theoretically, the weak-localization correction is the term of order N^0 in an expansion of the average conductance $\langle G\rangle=G_0+\delta G+\mathcal{O}(N^{-1})$ in powers of N. Experimentally, δG is measured by application of a weak magnetic field B, which suppresses the weak-localization correction but leaves the classical conductance unaffected.[5] The suppression occurs because weak localization requires time-reversal symmetry ($\mathcal{T}$). In the absence of $\mathcal{T}$, quantum corrections to G_0 are of order N^{-1} and not of order N^0. As a consequence, the magnetoconductance has a dip around $B=0$ of magnitude δG and width of order B_c (being the field at which one flux quantum penetrates the conductor).

What happens to weak localization if the normal-metal wire is attached at one end to a superconductor? This problem has been the subject of active research.[6–12] The term G_0 of order N is unaffected by the presence of the superconductor.[6] The $\mathcal{O}(N^0)$ correction δG is increased but remains universal,[9,10]

$$\delta G=-(2-8\pi^{-2})e^2/h\approx-1.19e^2/h. \tag{1}$$

In all previous analytical work zero magnetic field was assumed. It was surmised, either implicitly or explicitly,[7] that $\delta G=0$ in the absence of $\mathcal{T}$ — but this was never actually calculated analytically. We have now succeeded in doing this calculation and would like to report the result, which was entirely unexpected.

We find that a magnetic field by itself is not sufficient to suppress the weak-localization correction, but only reduces δG by about a factor of 2. To achieve $\delta G=0$ requires in addition the application of a sufficiently large voltage V to break the degeneracy in energy between the electrons (at energy eV above the Fermi energy E_F) and the Andreev-reflected holes (at energy eV below E_F). The electron-hole degeneracy ($\mathcal{D}$) is effectively broken when eV exceeds the Thouless energy $E_c=\hbar v_F\ell/L^2$ (with v_F the Fermi velocity). Weak localization coexists with a magnetic field as long as $eV\ll E_c$. Our analytical results are summarized in Table I. These results disagree with the conclusions drawn in Ref. 7 on the basis of numerical simulations. We have found that the numerical data on the weak-localization effect were misinterpreted due to the presence of a magnetic-field-dependent contact resistance, which was not understood at that time.

The starting point of our calculation is the general relation between the differential conductance $G=dI/dV$ of the normal-metal–superconductor (NS) junction and the transmission and reflection matrices of the normal region,[6]

$$G=(4e^2/h)\mathrm{tr}\, m(eV)m^\dagger(eV), \tag{2a}$$

$$m(\varepsilon)=t'(\varepsilon)[1-\alpha(\varepsilon)r^*(-\varepsilon)r(\varepsilon)]^{-1}t^*(-\varepsilon), \tag{2b}$$

where $\alpha(\varepsilon)\equiv\exp[-2i\arccos(\varepsilon/\Delta)]$. Equation (2) holds for subgap voltages $V\lesssim\Delta/e$, and requires also $\Delta\ll E_F$ (Δ being the excitation gap in S). We assume that the length L of the disordered normal region is much greater than the superconducting coherence length $\xi=(\hbar v_F\ell/\Delta)^{1/2}$. This implies that the Thouless energy $E_c\ll\Delta$. In the voltage range $V\lesssim E_c/e$ we may therefore assume that $eV\ll\Delta$, hence $\alpha=-1$. The $N\times N$ transmission and reflection matrices t, t', r, and r' form the scattering matrix $S(\epsilon)$ of the disordered normal region (N being the number of propagating modes at the Fermi level, which corresponds to $\varepsilon=0$). It is convenient to use the polar decomposition

$$\begin{pmatrix} r' & t' \\ t & r \end{pmatrix}=\begin{pmatrix} v_1 & 0 \\ 0 & w_1 \end{pmatrix}\begin{pmatrix} i\sqrt{R} & \sqrt{T} \\ \sqrt{T} & i\sqrt{R} \end{pmatrix}\begin{pmatrix} v_2 & 0 \\ 0 & w_2 \end{pmatrix}.$$

TABLE I. Dependence of the weak-localization correction δG of a normal-metal wire attached to a superconductor on the presence or absence of time-reversal symmetry ($\mathcal{T}$) and electron hole-degeneracy ($\mathcal{D}$). The entry in the upper-left corner was computed in Refs. 9 and 10.

$-\delta G[e^2/h]$	$\mathcal{T}$	no $\mathcal{T}$
$\mathcal{D}$	$2-8/\pi^2$	2/3
no $\mathcal{D}$	4/3	0

Here v_1, v_2, w_1, and w_2 are $N\times N$ unitary matrices, T is a diagonal matrix with the N transmission eigenvalues $T_i\in[0,1]$ on the diagonal, and $R=1-T$. Using this decomposition, and substituting $\alpha=-1$, Eq. (2b) can be replaced by

$$m(\varepsilon)=\sqrt{T(\varepsilon)}[1+u(\varepsilon)\sqrt{R(-\varepsilon)}u^*(-\varepsilon)\sqrt{R(\varepsilon)}]^{-1}\times u(\varepsilon)\sqrt{T(-\varepsilon)},\quad u(\varepsilon)\equiv w_2(\varepsilon)w_1^*(-\varepsilon). \tag{2c}$$

We perform our calculations in the general framework of random-matrix theory. The only assumption about the distribution of the scattering matrix that we make is that it is isotropic, i.e., that it depends only on the transmission eigenvalues.[13] In the presence of $\mathscr{T}$ (for $B\ll B_c$), $S=S^T$, hence $w_1=w_2^T$. (The superscript T denotes the transpose of a matrix.) If $\mathscr{T}$ is broken, w_1 and w_2 are independent. In the presence of $\mathscr{D}$ (for $eV\ll E_c$), the difference between $S(eV)$ and $S(-eV)$ may be neglected. If $\mathscr{D}$ is broken, $S(eV)$ and $S(-eV)$ are independent. Of the four entries in Table I, the case that both $\mathscr{T}$ and $\mathscr{D}$ are present is the easiest, because then $u=1$ and Eq. (2a) simplifies to[6]

$$G=(4e^2/h)\sum_n T_n^2(2-T_n)^{-2}. \tag{3}$$

The conductance is of the form $G=\Sigma_n f(T_n)$, known as a linear statistic on the transmission eigenvalues. General formulas[9,10] for the weak-localization correction to the average of a linear statistic lead directly to Eq. (1). The three other entries in Table I, where either $\mathscr{T}$ or $\mathscr{D}$ (or both) are broken, are more difficult because G is no longer a linear statistic. We consider these three cases in separate paragraphs.

(1) $\mathscr{D}$, *no* $\mathscr{T}$. Because of the isotropy assumption, w_1 and w_2, and hence u, are uniformly distributed in the unitary group $\mathscr{U}(N)$. We may perform the average $\langle\cdots\rangle$ over the ensemble of scattering matrices in two steps: $\langle\cdots\rangle=\langle\langle\cdots\rangle_u\rangle_T$, where $\langle\cdots\rangle_u$ and $\langle\cdots\rangle_T$ are, respectively, the average over the unitary matrix u and over the transmission eigenvalues T_i. We compute $\langle\cdots\rangle_u$ by an expansion in powers of N^{-1}. To integrate the rational function (2) of u over $\mathscr{U}(N)$, we first expand it into a geometric series and then use the general rules for the integration of polynomials of u.[14,15] The polynomials we need are

$$\langle G\rangle_u=\frac{4e^2}{h}\sum_{p,q=0}^{\infty}M_{pq}, \tag{4a}$$

$$M_{pq}=\langle \mathrm{tr}\ T(u\sqrt{R}u^*\sqrt{R})^p uTu^\dagger(\sqrt{R}u^{\mathrm{T}}\sqrt{R}u^\dagger)^q\rangle_u. \tag{4b}$$

Neglecting terms of order N^{-1}, we find

$$M_{pq}=\begin{cases} N\tau_1^2(1-\tau_1)^{2p} & \text{if } p=q,\\ \tau_1(\tau_1^2+\tau_1-2\tau_2)(1-\tau_1)^{p+q-1}-2\min(p,q) & \\ \times\tau_1^2(\tau_1^2-\tau_2)(1-\tau_1)^{p+q-2} & \text{if } |p-q| \text{ odd},\\ 0 \quad \text{else}, & \end{cases}$$

where we have defined the moment $\tau_k=N^{-1}\Sigma_i T_i^k$. The summation over p and q leads to

$$\frac{h}{4e^2}\langle G\rangle_u=\frac{N\tau_1}{2-\tau_1}-\frac{4\tau_1-2\tau_1^2+2\tau_1^3-4\tau_2}{\tau_1(2-\tau_1)^3}. \tag{5}$$

It remains to average over the transmission eigenvalues. Since τ_k is a linear statistic, we know that its sample-to-sample fluctuations $\delta\tau_k\equiv\tau_k-\langle\tau_k\rangle$ are an order $1/N$ smaller than the average.[13] Hence

$$\langle f(\tau_k)\rangle_T=f(\langle\tau_k\rangle)[1+\mathscr{O}(N^{-2})], \tag{6}$$

which implies that we may replace the average of the rational function (5) of the τ_k's by the rational function of the average $\langle\tau_k\rangle$. This average has the $1/N$ expansion

$$\langle\tau_k\rangle=\langle\tau_k\rangle_0+\mathscr{O}(N^{-2}), \tag{7}$$

where $\langle\tau_k\rangle_0$ is $\mathscr{O}(N^0)$. There is no term of order N^{-1} in the absence of $\mathscr{T}$. From Eqs. (5)–(7) we obtain the $1/N$ expansion of the average conductance,

$$\frac{h}{4e^2}\langle G\rangle=\frac{N\langle\tau_1\rangle_0}{2-\langle\tau_1\rangle_0}-\frac{4\langle\tau_1\rangle_0-2\langle\tau_1\rangle_0^2+2\langle\tau_1\rangle_0^3-4\langle\tau_2\rangle_0}{\langle\tau_1\rangle_0(2-\langle\tau_1\rangle_0)^3}+\mathscr{O}(N^{-1}). \tag{8}$$

Equation (8) is generally valid for any isotropic distribution of the scattering matrix. We apply it to the case of a disordered wire in the limit $N\to\infty$, $\ell/L\to 0$ at constant $N\ell/L$. The moments $\langle\tau_k\rangle_0$ are given by[3]

$$\langle\tau_1\rangle_0=\ell/L,\quad \langle\tau_2\rangle_0=\tfrac{2}{3}\ell/L. \tag{9}$$

Substitution into Eq. (8) yields the weak-localization correction $\delta G=-\frac{2}{3}e^2/h$, cf. Table I.

(2) $\mathscr{T}$, *no* $\mathscr{D}$. In this case one has $u^\dagger(-eV)=u(eV)$ and $u(eV)$ is uniformly distributed in $\mathscr{U}(N)$. A calculation similar to that in the previous paragraph yields for the average over u:

$$\frac{h}{4e^2}\langle G\rangle_u=N\tau_{1+}\tau_{1-}(\tau_{1+}+\tau_{1-}-\tau_{1+}\tau_{1-})^{-1}+(\tau_{1+}+\tau_{1-}-\tau_{1+}\tau_{1-})^{-3}[2\tau_{1+}^2\tau_{1-}^2-\tau_{1+}^3\tau_{1-}^2-\tau_{1+}^2\tau_{1-}^3-\tau_{2+}\tau_{1-}^2-\tau_{1+}^2\tau_{2-}+\tau_{2+}\tau_{1-}^3+\tau_{1+}^3\tau_{2-}], \tag{10}$$

where we have abbreviated $\tau_{k\pm}=\tau_k(\pm eV)$. The next step is the average over the transmission eigenvalues. We may still use Eq. (6), and we note that $\langle\tau_k(\varepsilon)\rangle\equiv\langle\tau_k\rangle$ is independent of ε. [The energy scale for variations in $\langle\tau_k(\varepsilon)\rangle$ is E_F, which is much greater than the energy scale of interest E_c.] Instead of Eq. (7) we now have the $1/N$ expansion

$$\langle\tau_k\rangle=\langle\tau_k\rangle_0+N^{-1}\delta\tau_k+\mathscr{O}(N^{-2}), \tag{11}$$

which contains also a term of order N^{-1} because of the presence of $\mathscr{T}$. The $1/N$ expansion of $\langle G\rangle$ becomes

$$\frac{h}{4e^2}\langle G\rangle=\frac{N\langle\tau_1\rangle_0}{2-\langle\tau_1\rangle_0}+\frac{2\delta\tau_1}{(2-\langle\tau_1\rangle_0)^2}+\frac{2\langle\tau_1\rangle_0^2-2\langle\tau_1\rangle_0^3-2\langle\tau_2\rangle_0+2\langle\tau_1\rangle_0\langle\tau_2\rangle_0}{\langle\tau_1\rangle_0(2-\langle\tau_1\rangle_0)^3}+\mathscr{O}(N^{-1}). \tag{12}$$

For the application to a disordered wire we use again Eq. (9) for the moments $\langle \tau_k \rangle_0$, which do not depend on whether $\mathscr{T}$ is broken or not. We also need $\delta\tau_1$, which in the presence of $\mathscr{T}$ is given by[3] $\delta\tau_1 = -\frac{1}{3}$. Substitution into Eq. (12) yields $\delta G = -\frac{4}{3}e^2/h$, cf. Table I.

(3) *No $\mathscr{T}$, no $\mathscr{D}$.* Now $u(eV)$ and $u(-eV)$ are independent, each with a uniform distribution in $\mathscr{U}(N)$. Carrying out the two averages over $\mathscr{U}(N)$ we find

$$\frac{h}{4e^2}\langle G \rangle_u = \frac{N\tau_{1+}\tau_{1-}}{\tau_{1+}+\tau_{1-}-\tau_{1+}\tau_{1-}}. \tag{13}$$

The average over the transmission eigenvalues becomes

$$\frac{h}{4e^2}\langle G \rangle = \frac{N\langle \tau_1 \rangle_0}{2-\langle \tau_1 \rangle_0} + \mathscr{O}(N^{-1}), \tag{14}$$

where we have used that $\delta\tau_1 = 0$ because of the absence of $\mathscr{T}$. We conclude that $\delta G = 0$ in this case, as indicated in Table I.

This completes the calculation of the weak-localization correction to the average conductance. Our results, summarized in Table I, imply a universal B and V dependence of the conductance of an NS microbridge. Raising first B and then V leads to two subsequent increases of the conductance, while raising first V and then B leads first to a decrease and then to an increase.

So far we have only considered the $\mathscr{O}(N^0)$ correction δG to $\langle G \rangle = G_0 + \delta G$. What about the $\mathscr{O}(N)$ term G_0? From Eqs. (8), (12), and (14) we see that if either $\mathscr{T}$ or $\mathscr{D}$ (or both) are broken,

$$G_0 = \frac{4e^2}{h}\frac{N\langle \tau_1 \rangle_0}{2-\langle \tau_1 \rangle_0} = (2e^2/h)N(\tfrac{1}{2}+L/\ell)^{-1}. \tag{15}$$

In the second equality we substituted[3] $\langle \tau_1 \rangle_0 = (1+L/\ell)^{-1}$, which in the limit $\ell/L \to 0$ reduces to Eq. (9). If both $\mathscr{T}$ and $\mathscr{D}$ are unbroken, then we have instead the result[16]

$$G_0 = (2e^2/h)N[1+L/\ell+\mathscr{O}(\ell/L)]^{-1}. \tag{16}$$

The difference between Eqs. (15) and (16) is a contact resistance, which equals $h/4Ne^2$ in Eq. (15) but is twice as large in Eq. (16). In contrast, in a normal-metal wire the contact resistance is $h/2Ne^2$, independent of B or V. The B- and V-dependent contact resistance in an NS junction is superimposed on the B- and V-dependent weak-localization correction. Since the contribution to $\langle G \rangle$ from the contact resistance is of order $(e^2/h)N(\ell/L)^2$, while the weak-localization correction is of order e^2/h, the former can only be ignored if $N(\ell/L)^2 \ll 1$. This is an effective restriction to the diffusive metallic regime, where $\ell/L \ll 1$ and $N\ell/L \gg 1$. To measure the weak-localization effect without contamination from the contact resistance if $N(\ell/L)^2$ is not $\ll 1$, one has two options: (1) measure the B dependence at fixed $V \gg E_c/e$; (2) measure the V dependence at fixed $B \gg B_c$. In both cases we predict an increase of the conductance, by an amount $\frac{4}{3}e^2/h$ and $\frac{2}{3}e^2/h$, respectively. In contrast, in the normal-state weak localization leads to a B dependence, but not to a V dependence.

We performed numerical simulations similar to those of Ref. 7 in order to test the analytical predictions. The disordered normal region was modeled by a tight-binding Hamiltonian on a square lattice (lattice constant a), with a random impurity potential at each site (uniformly distributed between $\pm\frac{1}{2}U_d$). The Fermi energy was chosen at $E_F = 1.57U_0$ from the band bottom ($U_0 = \hbar^2/2ma^2$). The length L and width W of the disordered region are $L = 167a$, $W = 35a$, corresponding to $N = 15$ propagating modes at E_F. The mean free path is obtained from the conductance $G = (2e^2/h)N(1+L/\ell)^{-1}$ of the normal region in the absence of $\mathscr{T}$. The scattering matrix of the normal region was computed numerically at $\varepsilon = \pm eV$, and then substituted into Eq. (2a) to obtain the differential conductance.

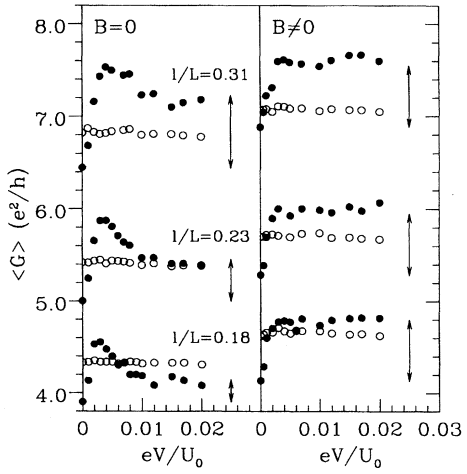

FIG. 1. Numerical simulation of the voltage dependence of the average differential conductance for $B = 0$ (left panel) and for a flux $6\,h/e$ through the disordered normal region (right panel). The filled circles are for an NS junction; the open circles represent the V-independent conductance in the normal state. The three sets of data points correspond, from top to bottom, to $\ell/L = 0.31$, 0.23, and 0.18, respectively. The arrows indicate the theoretically predicted net increase of $\langle G \rangle$ between $V = 0$ and $V \gg E_c/e$.

In Fig. 1 we show the V dependence of G (averaged over some 10^3 impurity configurations) for three values of ℓ. The left panel is for $B = 0$ and the right panel for a flux of $6\,h/e$ through the disordered region. The V dependence for $B = 0$ is mainly due to the contact resistance effect of order $N(\ell/L)^2$, and indeed one sees that the amount by which G increases depends significantly on ℓ.[17] The V dependence in a $\mathscr{T}$-violating magnetic field is entirely due to the weak-localization effect, which should be insensitive to ℓ (as long as $\ell/L \ll 1 \ll N\ell/L$). This is indeed observed in the simulation. Quantitatively, we would expect that application of a voltage increases $\langle G \rangle$ by an amount $\frac{2}{3}e^2/h$ for the three

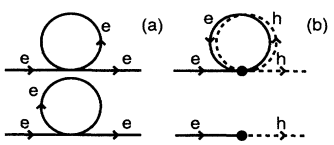

FIG. 2. (a) Two Feynman paths interfering constructively in the presence of $\mathscr{T}$. (b) Two paths involving Andreev reflection (solid dot), which interfere destructively both in the presence and absence of $\mathscr{T}$.

curves in the right panel, which agrees very well with what is observed. In the absence of a magnetic field the analytical calculation predicts a net increase in $\langle G \rangle$ by 0.79, 0.46, and $0.25 \times e^2/h$ (from top to bottom), which is again in good agreement with the simulation.

In normal metals, the weak-localization correction δG is explained in terms of constructive interference of pairs of time-reversed Feynman paths [Fig. 2(a)].[5] This interference is destroyed by a magnetic field. One might wonder what kind of interfering paths are responsible for δG in an NS junction without $\mathscr{T}$. Although our theory is not formulated in terms of Feynman paths, an interpretation of the quantity M_{pq} in Eq. (4b) using Feynman paths is possible. The two simplest interfering paths are shown in Fig. 2(b). Regardless of whether $\mathscr{T}$ is broken or not, there is an exact cancellation of the phase shifts accumulated by the electron and the hole which traverse the loop in the same direction. What remains is a phase shift of π due to the double Andreev reflection. As a consequence, the path with the double loop interferes destructively with the path without a loop, giving rise to a negative δG.

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[1] P. W. Anderson, E. Abrahams, and T. V. Ramakrishnan, Phys. Rev. Lett. **43**, 718 (1979); L. P. Gor'kov, A. I. Larkin, and D. E. Khmel'nitskiĭ, Pis'ma Zh. Eksp. Teor. Fiz. **30**, 248 (1979) [JETP Lett. **30**, 228 1979].

[2] Spin-orbit scattering is assumed to be negligible, otherwise a positive quantum correction appears.

[3] P. A. Mello and A. D. Stone, Phys. Rev. B **44**, 3559 (1991).

[4] The restriction to a wire geometry is essential for the universality. In a square or cube geometry, the weak-localization correction is L and ℓ dependent.

[5] G. Bergmann, Phys. Rep. **107**, 1 (1984).

[6] C. W. J. Beenakker, Phys. Rev. B **46**, 12 841 (1992).

[7] I. K. Marmorkos, C. W. J. Beenakker, and R. A. Jalabert, Phys. Rev. B **48**, 2811 (1993).

[8] Y. Takane and H. Otani, J. Phys. Soc. Jpn. **63**, 3361 (1994).

[9] C. W. J. Beenakker, Phys. Rev. B **49**, 2205 (1994).

[10] A. M. S. Macêdo and J. T. Chalker, Phys. Rev. B **49**, 4695 (1994).

[11] K.-M. H. Lenssen *et al.*, in *Coulomb and Interference Effects in Small Electronic Structures*, edited by D. C. Glattli, M. Sanquer, and J. Trân Thanh Vân (Editions Frontières, Gif-sur-Yvette, 1994).

[12] Yu. V. Nazarov (unpublished).

[13] A. D. Stone, P. A. Mello, K. A. Muttalib, and J.-L. Pichard, in *Mesoscopic Phenomena in Solids*, edited by B. L. Al'tshuler, P. A. Lee, and R. A. Webb (North-Holland, Amsterdam, 1991).

[14] M. Creutz, J. Math. Phys. **19**, 2043 (1978).

[15] W. A. Friedman and P. A. Mello, Ann. Phys. (N.Y.) **161**, 276 (1985); P. A. Mello, J. Phys. A **23**, 4061 (1990).

[16] C. W. J. Beenakker, B. Rejaei, and J. A. Melsen, Phys. Rev. Lett. **72**, 2470 (1994).

[17] Our analytical calculation refers to the net increase of $\langle G \rangle$ between $V=0$ and $V \gg E_c/e$. We cannot describe the nonmonotonic V dependence at intermediate V, observed in the simulation for $B=0$.