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Random-matrix-theory approach to the intensity distributions of waves propagating in a random medium

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Statistical properties of coherent radiation propagating in a quasi-one-dimensional random medium are studied in the framework of random-matrix theory. Distribution functions for the total transmission coefficient and the angular transmission coefficient are obtained.

The discovery of universal conductance fluctuations[1,2] (UCF's) has induced a partial shift of the main interest in the studies of electronic properties from the averaged values of physical quantities to their variance and then to the whole distribution functions (see Altshuler *et al.*[3] and references therein). Later it was demonstrated that UCF's exist also for the propagation of classical waves (e.g., light) through disordered systems.[4] In contrast to electronic measurements which can measure only the conductance of a system, light experiments have the advantage of being able to measure the angular and the total transmission coefficients for an experimental realization.

In our previous publication[5] we analyzed the problem of statistics of radiation using diagrammatic techniques. It was rigorously shown that the distribution function can be represented through the contribution of connected diagrams only. This representation allowed to develop a perturbation theory; in the framework of this theory it was found that only for moderate values of the angular transmission coefficient the distribution function is a simple exponential, as predicted by Rayleigh statistics. For larger values of intensity, the distribution function differs drastically from a simple exponential and its asymptotical behavior is a stretched exponential decay. Also for the total transmission coefficient the Gaussian distribution function was obtained.

An important step was made by Nieuwenhuizen and van Rossum.[6] While in Ref. 5 the perturbation series was truncated after the second term, Nieuwenhuizen and van Rossum using diagrammatic techniques combined with random-matrix theory managed to sum up the whole perturbation series, obtaining in particular a more precise stretched exponent for the angular transmission coefficient distribution function and deviations from the simple Gaussian for the total transmission coefficient.

In this paper we reproduce the results of Ref. 6 in the framework of the random-matrix theory. The approach is based on the analysis of the transfer matrix R (see Stone *et al.*[7] and references therein). Under the restrictions of flux conservation and time-reversal invariance, this matrix can be represented in the form

$$R=\begin{pmatrix} u & 0 \\ 0 & u^* \end{pmatrix}\begin{pmatrix} \sqrt{1+\lambda} & \sqrt{\lambda} \\ \sqrt{\lambda} & \sqrt{1+\lambda} \end{pmatrix}\begin{pmatrix} v & 0 \\ 0 & v^* \end{pmatrix}, \tag{1}$$

where u and v are arbitrary $N\times N$ unitary matrices and λ is a real, diagonal matrix with N positive elements $\lambda_1,\ldots,\lambda_N$, where $N=W^2k^2$ is the number of transverse channels (W^2 is the area of the sample). The $N\times N$ transmission matrix is given by

$$t=u\tau^{1/2}v, \tag{2}$$

where $\tau\equiv(1+\lambda)^{-1}$.

In the isotropic approximation[8] an ensemble of R matrices is described by the differential probability $dP(R)=P(\{\tau\})\Pi_a d\tau_a d\mu(u)d\mu(v)$, where $d\mu(u)[d\mu(v)]$ is the invariant measure of the unitary group $U(N)$. This isotropic approximation is a rather strong assumption implying the perfect mode mixing but for a quasi-one-dimensional system it is known to be good.[7]

The angular transmission coefficient T_{ab}, defined as the ratio of the energy carried away by the transmitted wave with the transverse wave vector $\vec{q}_b$ to the energy of the incident wave with the transverse wave vector $\vec{q}_a$, is given by $|t_{ab}|^2$, t_{ab} being the ab matrix element of Eq. (2). The nth moment of T_{ab} can be written down in the following way:

$$\begin{aligned}\langle T_{ab}^n\rangle = \sum_{\{\alpha\},\{\beta\}} &\langle (u_{a\alpha_1}\cdots u_{a\alpha_n})(u_{a\beta_1}\cdots u_{a\beta_n})^*\rangle_0 \\ &\times\langle(\tau_{\alpha_1}\cdots\tau_{\alpha_n}\tau_{\beta_1}\cdots\tau_{\beta_n})^{1/2}\rangle_\tau \\ &\times\langle(v_{b\alpha_1}\cdots v_{b\alpha_n})(v_{b\beta_1}\cdots v_{b\beta_n})^*\rangle_0, \end{aligned} \tag{3}$$

where the average indicated by the index 0 is performed with the invariant measure of the unitary group and $\langle X\rangle_\tau \equiv \int d\{\tau\}P(\{\tau\})X$.

It is known that to leading order in $1/N$ both real and imaginary components of $u_{a\alpha}$ and $v_{\beta b}$ are independently distributed Gaussian variables with zero mean and variance $1/2N$.[9,10] Then we can write down correlator $\langle(v_{b\alpha_1}\cdots v_{b\alpha_n})(v_{b\beta_1}\cdots v_{b\beta_n})^*\rangle_0$ as the product of correlators $\langle v_{b\alpha}v_{b\beta}^*\rangle_0 = \delta_{\alpha\beta}/N$ summed up with respect to all $n!$ possible pairings between $\{\alpha\}$ and $\{\beta\}$. So from Eq. (3) we get

$$\begin{aligned}\langle T_{ab}^n\rangle &= \frac{n!}{N^n}\sum_{\{\alpha\}} \langle |u_{a\alpha_1}|^2\cdots|u_{a\alpha_n}|^2\rangle_0 \times\langle\tau_{\alpha_1}\cdots\tau_{\alpha_n}\rangle_\tau \\ &= \frac{n!}{N^n}\langle T_a^n\rangle, \end{aligned} \tag{4}$$

where

$$T_a = \sum_\alpha |u_{a\alpha}|^2\tau_\alpha . \tag{5}$$

It can be easily seen that T_a is just the total transmission coefficient: $T_a = \Sigma_b T_{ab}$. In fact the nth moment of the total transmission coefficient $\Sigma_b T_{ab}$ is

$$\begin{aligned}\left\langle\left(\sum_b T_{ab}\right)^n\right\rangle = \sum_{\{\alpha\},\{\beta\},\{b\}} &\langle (u_{a\alpha_1}\cdots u_{a\alpha_n})(u_{a\beta_1}\cdots u_{a\beta_n})^*\rangle_0 \\ &\times\langle(\tau_{\alpha_1}\cdots\tau_{\alpha_n}\tau_{\beta_1}\cdots\tau_{\beta_n})^{1/2}\rangle_\tau \\ &\times\langle(v_{b_1\alpha_1}\cdots v_{b_n\alpha_n})(v_{b_1\beta_1}\cdots v_{b_n\beta_n})^*\rangle_0 . \end{aligned} \tag{6}$$

To leading order in $1/N$

$$\sum_{\{b\}}\langle(v_{b_1\alpha_1}\cdots v_{b_n\alpha_n})(v_{b_1\beta_1}\cdots v_{b_n\beta_n})^*\rangle_0 = \delta_{\alpha_1\beta_1}\cdots\delta_{\alpha_n\beta_n} \tag{7}$$

(because the b indexes are different we should take into account only one pairing), and the right-hand part of Eq. (6) is exactly $\langle T_a^n\rangle$.

Returning to Eq. (5) we see that the distribution function $P(T_a)$ can be written as an integration over eigenvalues and eigenvectors:

$$P(T_a) = \int d\tau_1\cdots\int d\tau_n\int dU P(\{\tau\}) \times\delta\left(T_a - \sum_\alpha |u_{a\alpha}|^2\tau_\alpha\right). \tag{8}$$

It is convenient to work with the Laplace transform (we also measure T_a in units of $\langle T_a\rangle = g/N$, where g is classical conductance):

$$P(T_a) = \int_{-i\infty}^{i\infty}\frac{ds}{2\pi i}\exp(sT_a)F(s/g). \tag{9}$$

Then easily carrying out the integration with respect to dU, we find

$$F(s) = \left\langle \prod_{\alpha=1}^N \frac{1}{1+s\tau_\alpha}\right\rangle_\tau . \tag{10}$$

We are going to use an approximation of the uniform distribution of the "charges" ν_α, which are defined by the relation: $\tau_\alpha = 1/\cosh^2(\nu_\alpha/2)$.[7] That is, knowing that the distribution of "charges" is statistically homogeneous,[7] instead of averaging with respect to all possible configurations of "charges" we take into account only one configuration–crystal lattice, which leads to the following relation:[6]

$$\sum_{\alpha=1}^N f(\tau_\alpha) = g\int_0^1 \frac{d\tau}{2\tau\sqrt{1-\tau}} f(\tau) \tag{11}$$

for any $f(\tau)$ which goes to zero when τ goes to zero. Then from Eq. (10) we get

$$\begin{aligned}F(s) &= \exp\left[-g\int_0^1\frac{d\tau}{2\tau\sqrt{1-\tau}}\ln(1+s\tau)\right] \\ &= \exp[-g\ln^2(\sqrt{1+s}+\sqrt{s})], \end{aligned} \tag{12}$$

which exactly coincides with the result of Ref. 6. Equation (12) gives, in particular, Gaussian behavior for $T_a \approx 1$:

$$P(T_a) \approx \sqrt{\frac{3g}{4\pi}}\exp\left[-\frac{3g}{4}(T_a-1)^2\right] \tag{13}$$

and simple exponential decay for large T_a:

$$P(T_a) \sim \exp(-gT_a). \tag{14}$$

Now let us return to Eq. (4). As is known[5] it means

$$P(T_{ab}) = \int_0^\infty dT_a\ P(T_a)\ \frac{1}{T_a}\exp\left(-\frac{T_{ab}}{T_a}\right) \tag{15}$$

(we measure T_{ab} in units of $\langle T_{ab}\rangle = g/N^2$, where g is classical conductance). This distribution function can be described as the Rayleigh distribution function for the angular transmission coefficient but with some effective averaged value which in turn fluctuates around the real averaged value, and the latter fluctuations are described by the total transmission coefficient distribution function.[5,6] Equation (15) gives, in particular, Rayleigh statistics

$$P(T_{ab}) \approx \exp(-T_{ab}) \tag{16}$$

for $T_{ab} \ll \sqrt{g}$,[5] and stretched exponential tail

$$P(T_{ab}) \sim \exp(-2\sqrt{gT_{ab}}) \tag{17}$$

for $T_{ab} \gg g$.[6]

Having in mind the comparison of the theoretical result with an experiment it is convenient to express g through the first two moments either of the total or of the angular transmission coefficient distribution function. Calculating the coefficient before s^2 in the expansion of the exponent in the right-hand part of Eq. (12) we get

$$\frac{\langle T_a^2 \rangle}{\langle T_a \rangle^2} - 1 = \frac{2}{3g} , \tag{18}$$

and

$$\frac{\langle T_{ab}^2 \rangle}{\langle T_{ab} \rangle^2} - 2 = \frac{4}{3g} , \tag{19}$$

which exactly coincides with the result of Ref. 8.

In conclusion we want to discuss the difference between the statistics of the total transmission coefficient and the statistics of conductance $g = \Sigma_m \tau_m$. Taking into account the bimodal distribution of τ we may say, at least qualitatively, that the conductance is simply the number of "open" channels:[11,12] $g = N_{\text{eff}}$. The total transmission coefficient is also the sum with respect to "open" channels but each channel comes with a random weight. So the Gaussian law for the total transmission coefficient distribution function is just the manifestation of the central limit theorem, which is true when $N_{\text{eff}} \to \infty$. In the paper we are taking into account the finiteness of the parameter N_{eff}, which is important, in particular, for obtaining correct asymptotics. On the other hand, the conductance fluctuations are determined by the strongly suppressed fluctuations of the number of open channels, which in our case can be neglected. This principal difference between the two statistics would also manifest itself if one tries to go beyond quasi-one-dimensionality. While the eigenvalue distribution (and hence the conductance distribution function) can be not very sensitive to the dimensionality and stay bimodal as long as we are in a diffusive regime,[13] the isotropic approximation which was essential in obtaining Eq. (12) ceases to be valid beyond quasi-one-dimensionality.[14]

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