RAPID COMMUNICATIONS

PHYSICAL REVIEW B VOLUME 52, NUMBER 24 15 DECEMBER 1995-II

Transport in an inhomogeneous interacting one-dimensional system

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(Received 18 May 1995)

Transport through a clean one-dimensional wire of interacting electrons connected to semi-infinite leads is investigated using a bosonization approach. An incident electron is transmitted as a sequence of partial charges. The dc conductance is found to be entirely determined by the properties of the leads. The dynamic nonlocal conductivity is rigorously expressed in terms of the transmission. For abrupt variations of the interaction parameters at the junctions, the central wire acts as a Fabry-Pérot resonator. When one of the connected wires has a tendency towards superconducting order, partial Andreev reflection of an incident electron occurs.

Due to advances in semiconductor microtechnology it is now possible to fabricate high-mobility quantum wires that are very close to being ideally one dimensional.[1,2] In one-dimensional electron systems interactions play a crucial role, giving rise to the so-called Luttinger liquid behavior.[3,4] Transport measurements are a possible method to test theoretical predictions. For example, applying an electric field over a finite segment of a homogeneous infinitely long wire, Kane and Fisher[5] find (for spinless electrons) a conductance $g = g_0 K$, where $g_0 = e^2/h$ is the conductance quantum and K is a nonuniversal number depending on the interactions in the wire. However, it seems more reasonable to describe a two-probe measurement by connecting the interacting wire to two long leads intended to mimic the role of reservoirs where the chemical potential is fixed. This type of idealization was often used in theoretical work on transport in mesoscopic devices.[6–8] This work continues on the path paved by Landauer,[9] who relates the conductance of a coherent device to the transmission T of an incident electron on the device viewed as a scattering entity. Depending on assumptions about the measuring procedure[8] one finds either[6,7]

$$g = g_0 T \tag{1}$$

or Landauer's original result,[9,10] $g = g_0 T/(1-T)$ (T is the transmission coefficient). Equation (1) agrees with the conductance of a one-channel ballistic constriction for which $T = 1$.[11] In Landauer-type theories, interactions are mostly ignored,[12] or at least are accounted for by a phenomenological time larger than the characteristic time to cross the sample. It is our purpose to include interactions explicitly and to investigate the transmission processes through a finite wire as well as the effect of the contacts on its transport properties. Confined to one-dimensional sample and leads, our electrons are interacting everywhere, but in general with different strength in the leads and in the sample. We will not consider disorder, and deal with the nonlocal time or frequency-dependent conductivity, containing much more information than the bulk conductance. Also, due to the space-dependent interactions, phenomena familiar from metal-superconductor contacts like Andreev reflection[13] and the proximity effect will be seen.

Momentum-conserving interactions between spinless electrons can be described by the bosonized Hamiltonian,[3,4]

$$H = \int_{-L}^{L} \frac{dx}{2\pi} \left[uK(\partial_x \Theta)^2 + \frac{u}{K}(\partial_x \Phi)^2 \right], \tag{2}$$

where the boson field Φ is related to the particle density by $\rho - \rho_0 = -\partial_x \Phi/\pi$, and $\partial_x \Theta/\pi$ is the momentum conjugate field to Φ. The interaction-dependent parameter u determines the velocity of the elementary excitations, and K (also interaction dependent) determines the algebraic decay of correlation functions, indicating a tendency of the Luttinger liquid towards either superconducting or charge-density-wave order, depending on whether $K > 1$ or $K < 1$. A "Fermi liquid" is found for the noninteracting case, $K = 1$, $u = v_F$. We now consider a finite interacting wire perfectly connected to two identical leads at its end points $\pm a$. We shall label the quantities pertaining to the leads (central wire) by the subscript 1 (2). In H the parameters u, K then vary from u_2, K_2 in the wire to u_1, K_1 outside. We use periodic boundary conditions; i.e., we join the exterior wires to form a ring of length $2L$. This simulates two semi-infinite perfect leads if we set $K_1 = 1$ and $L \gg a$. Thus the electrons coming out of the central wire cannot come back to it, the interactions being absent and the time to go around the ring being too long. We can therefore define properly the reflection and transmission coefficients of an electron incident on the central wire. Creating an electron amounts to introducing a kink of height π in Φ, and the problem of reflection and transmission then reduces to a solution of the equations of motion for Φ. This leads us naturally to diagonalize the Hamiltonian (2) with space-dependent u and K: we expand the field Φ in terms of a discrete set of boson creation and annihilation operators and eigenfunctions.

We start with the case where u and K jump from $u_1, K_1 = 1$ to u_2, K_2 at $\pm a$. We call t_y the time it takes for an electron to go from y on the lead to the closest contact, i.e., $u_1 t_y = |y| - a$, and $t_2 = 2a/u_2$ is the traversal time of the central wire. Φ obeys simply a wave equation with velocity u_1 outside and u_2 inside. The propagating solutions have to be joined at $\pm a$. The system of cou-

pled equations of motion for Φ,Θ require their continuity, and so is $\frac{u}{K}\partial_x\Phi = \partial_t\Theta$. Indeed, a discontinuous Φ would lead to an unphysical singularity: a charge accumulation, thus a nonconservation of the current $j = \partial_t\Phi/\pi$, as can be derived from the continuity equation. Note that the continuity of Φ and Θ guarantees that of the fermion field. In the leads, the propagating solutions can be simply related to the original right- and left-going electrons' density $\rho_\pm$. In the interacting wire we instead define their counterpart by

$$\widetilde{\rho}_\pm = \tfrac{1}{2}(\rho \pm j/u). \tag{3}$$

For the following, it is useful to write the time evolution of their corresponding current $j_\pm(x) = \pm u\widetilde{\rho}_\pm(x)$ (defined for any x) as

$$J(x,t) = \int_{-L}^{L} \frac{dy}{u} M(x,y,t)J(y,0), \tag{4}$$

where $J = (j_+, j_-)$ and M is a 2×2 integral kernel matrix that can be simply interpreted for $a \leq |x|, |y| < L$: $M_{rr'}(x,y,t)$ $(r,r' = \pm 1)$ is the electronic charge with velocity $r'u_1$ reaching x at time t if an electron initially emanates at y with velocity ru_1 [i.e., if $\langle\rho(x)\rangle = r\langle j(x)\rangle/u_1 = \delta(x-y)$ at $t=0$]. M verifies time-reversal symmetry: $M_{rr'}(t) = M^*_{-r-r'}(-t) = M_{-r-r'}(-t)$; we will henceforth restrict t to $[0,t_L]$. For points on opposite leads verifying $a \leq -y, x < (L-a)/2$, all the entries of M but M_{++} are zero: this expresses the fact that only the direct path allows a particle to travel from y to x, the other trips would take more than t_L. Therefore, there is no ambiguity in identifying $M_{++}(x,y,t)$ with the charge transmitted from y to x at time t. For points on the same lead, e.g., the left one, $M_{+-}(-L < y', y \leq -a, t) = 0$: a left-going electron originating at y will travel counterclockwise and takes at least $4t_L - t_{y'} - t_y$ to generate a right-going charge at y' after its reflection on a. Besides, $M_{rr}(y',y,t) = \delta[(y'-y)/u_1 - rt]$ (direct propagation) while $M_{-+}(y',y,t)$ can be identified with the reflected charge appearing at (y',t) for an initial right-going electron at y.

Our simple model allows us to express all the $M_{rr'}$ (for any x,y and for times less than t_L) solely in terms of the "descending Dirac comb" $\Delta(t) = \sum_{0\leq p<t_L/t_2} \gamma^{2p}\delta(t-2pt_2)$, where $\gamma = (1-K_2)/(1+K_2)$ is the reflection coefficient of an electron incident on the contact between wires 1 and 2. $\Delta(t)$ is nothing but $M_{++}(x,x,t)$ for $|x| \leq a$; it reveals the cyclic motion inside the finite wire. A peak in $\langle\widetilde{\rho}_+\rangle$ initially localized at $x \in [-a,a]$ reappears at x after each time elapse $2t_2$, with its charge reduced by γ^2 due to two successive reflections at the end points.

If an electron emanates initially at $y \leq -a$, the charge transmitted to $x \geq a$ at time $t < t_L$ is

$$M_{++}(x,y,t) = (1-\gamma^2)\Delta[t-(t_x+t_y+t_2)]. \tag{5}$$

The first Dirac peak, occurring at $t_0 = t_x + t_y + t_2$, accounts for the first transmission to x of the partial charge $1-\gamma^2$, after two reflections at $-a,a$ on the way. Each subsequent transmission at $t = t_0 + 2pt_2$ is reduced by γ^{2p}, i.e., *the incident electron is composed into a series of noninteger charged maxima in the charge density.* The series sums up to unity in the limit $t_L \gg t_2$: *the transmission is perfect.* It would be cumbersome to write out M for other locations. Let us just add that the expression of $M_{+-}(y' \leq -a, y \leq -a, t)$ reveals a first reflection with coefficient γ, while the subsequent ones are of opposite sign and sum up to $-\gamma$, leading to a vanishing total reflection.

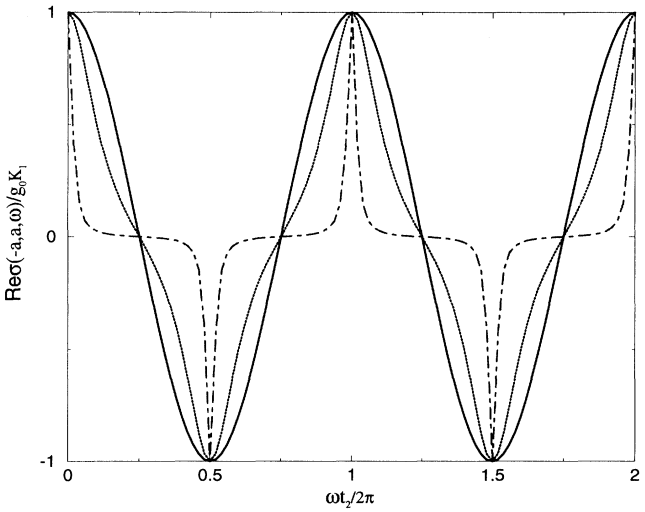

FIG. 1. The real part of the nonlocal conductivity through the contacts versus the ratio of the external frequency to the proper frequency of the wire $2\pi/t_2$ for $|\gamma| = 0.1$ (solid line), $|\gamma| = 0.5$ (dotted line), and $|\gamma| = 0.9$ (dash-dotted line). If $K_1 \neq 1$, σ is multiplied by K_1 and γ is replaced by γ'. The resonances (antiresonances) at integer (half-integer) values correspond to symmetric (antisymmetric) modes of the central wire.

The nonlocal dynamic conductivity $\sigma(x,y,t)$ is given by

$$\sigma(x,y,t) = g_0K(y)\theta(t)\sum_{r,r'} M_{rr'}(x,y,t). \tag{6}$$

To obtain this result one has only to realize that $\sigma(x,y,t)$ yields exactly $\langle j(x,t)\rangle$ in response to an electric-field pulse $\delta(t)E$ applied at y. This field generates in turn an initial current peak at y so that $\langle j(x,t)\rangle$ develops in accordance with the right-hand side of Eq. (6) [see Eqs. (3) and (4)]. For points on opposite leads $(x \geq a, y \leq -a)$ and $t < t_L$, Eq. (6) reduces to

$$\sigma(x,y,t) = g_0M_{++}(x,y,t). \tag{7}$$

This is a generalization of the Landauer formula, Eq. (1), to a dynamic situation in the sense that it relates transport properties to transmission. To infer the frequency dependence of the conductivity, caution is needed in the

order of limits: t_L is taken to infinity before δ^{-1}, the adiabatic turn-on time of the dc electric field. We neglect $e^{-\delta t_L} \ll 1$, thus preventing the electrons and holes generated by the electric field to go all around. With $\overline{\omega} = \omega + i\delta$, from Eq. (7) we find σ for points on opposite sides as

$$\sigma(x, y, \overline{\omega}) = g_0 \exp i\overline{\omega}(t_x + t_y) \times \frac{1-\gamma^2}{\exp(-i\overline{\omega}t_2) - \gamma^2 \exp(i\overline{\omega}t_2)}. \quad (8)$$

For points inside $[-a, a]$, all the entries of M contribute and we obtain

$$\sigma(x, y, \overline{\omega}) = g_0 K_2 \left\{ \exp \frac{i\overline{\omega}}{u_2} |x-y| + \frac{\gamma}{\exp(-2i\overline{\omega}t_2) - \gamma^2} \sum_{r=\pm 1} \left[\gamma \exp\left(\frac{i\overline{\omega} r}{u_2}(x-y) \right) + \exp\left(\frac{i\overline{\omega}}{u_2}[r(x+y) - 2a] \right) \right] \right\}. \quad (9)$$

We can deduce $\sigma(-a, a, \overline{\omega})$ from both expressions (the current being continuous at the junctions), and it is easier to see from Eq. (8) that it reaches its maximum g_0 at each eigenfrequency of the central wire $\omega_n = 2n\pi/t_2$ (this is so if we keep t_2 finite, while $\delta \to 0$). This is the resonant absorption of an electric field by a finite system. The usual Dirac peaks are broadened by the connection to infinite leads as shown in the figure (see Fig. 1).

In summary, the finite wire behaves as a Fabry-Pérot resonator with the junctions playing the role of symmetric mirrors. One can ask which results persist if we consider the more general situation where u and K vary inside the wire and can be asymmetric. We also allow for interactions in the leads with constant u_1 and K_1. Following the previous steps, we can still show that, for points on opposite leads, only the matrix element M_{rr} is nonzero, r being the sign of the direct way to go from one point to the other. Even if the leads are interacting, we can continue to view $M_{++}(x, y, t)$ as the transmitted charge at $(x > a, t)$ when a unit flux was incident at $(y \leq -a, 0)$. This is because wherever $\langle \widetilde{\rho}_- \rangle$ vanishes, $\langle j \rangle = u_1 \langle \rho \rangle = u_1 \langle \widetilde{\rho}_+ \rangle$ so that a peak in the total density propagates at velocity u_1. Similarly, the reflection is expressed through M_{-+}. Denoting $\lim_{\omega \to 0} F(x, y, \omega + i\delta) \equiv F(x, y)$ where F is any function, we get $M_{++}(x \geq a, y \leq -a) = 1$ and $M_{-+}(y', y) = 0$. As before, the total transmission through the central wire is perfect, even if u, K are not symmetric on $[-a, a]$. This fact is not surprising; both j and $u/K\rho$ are uniform in the steady state. But u/K is the same on the opposite leads, and so is $\langle \widetilde{\rho}_+(x, \omega = 0) \rangle$. A partial transmission can occur only if the leads are asymmetric.

We can also generalize the decomposition (6), which relies merely on the definition of σ and M and may also be checked from their eigenfunction expansion. If $y < -a$, one has only to replace $K(y)$ by K_1 instead of 1, so that the identity (6) yields, for points on the leads, $\sigma(x \geq a, y \leq -a, t) = g_0 K_1 M_{++}(x, y, t)$ and $\sigma(y' \leq -a, y \leq -a, t) = g_0 K_1 [\delta(t - |y' - y| / u_1) + M_{-+}(y', y, t)]$. In the zero-frequency limit the transmission is perfect: $\sigma(x, y) = \sigma(y', y) \equiv g_0 K_1$. This is also true if one or both of x and y are in the central wire because $\sigma(x, y)$ is independent on its arguments as we can check from its mode expansion. Actually, this is a constraint to be obeyed by any sensible transport theory for any one-dimensional system exhibiting time-reversal symmetry.[14]

We can verify explicitly this constraint in the special model we solved ($K = K_2$ on the central wire) by taking the limit $\omega \to 0$ leaving t_2 finite. Our results [Eqs. (5) and (8)] concern $K_1 = 1$, but we can rewrite them for $K_1 \neq 1$ provided the reflection coefficient γ is replaced by $\gamma' = (K_1 - K_2)/(K_1 + K_2)$. We thus recover $\sigma(x, y) = g_0 K_1$ for any x, y, even if these are on different wires.

Let us discuss the conductance one can measure. If we impose a current through a nondissipative system, no voltage drop is measured and the conductance would be infinite. Conversely, we can connect it to perfect leads intended to simulate reservoirs where dissipation takes place and where the chemical potential is not affected by the current through the sample.[8] We define the conductance g of the central wire as the ratio of the current to the potential drop between the junctions imposed by the leads. The general relation of the current to the electric field then shows that g is given by the uniform value of $\sigma(x, y) = g$, where x, y can be inside or outside $[-a, a]$. Therefore, $g = g_0 K_1$ is the conductance of the central wire, irrespective of the form of its own parameters.

Returning to the initial model with abrupt variations of u and K, it is quite instructive to take a different limit where t_2 is of the same order as t_L. In particular, $t_2 \gg \delta^{-1}$ so that we can neglect $e^{-\delta t_2}$ in the expressions of σ. For instance, if x, y are in $[-a, a]$, Eq. (9) becomes (after $\gamma \to \gamma'$)

$$\sigma(x, y, \overline{\omega}) = g_0 K_2 \left\{ \exp[i\overline{\omega} |x-y| / u_2] + \sum_{r=\pm} \gamma' \exp[i\overline{\omega}\{t_2 + r(x+y)/u_2\}] \right\}. \quad (10)$$

We focus on three regions, corresponding to the neighborhood of the origin (the "bulk"), denoted by N_b ($|x| \ll a$), and to that of each contact, denoted by $N_\pm$ ($|x \pm a| \ll a$). The second exponentials are of order $e^{-\delta t_2}$ in N_b, thus σ regains its bulk form in a homogeneous infinite wire. In $N_\pm$, σ depends on both x and y, but the zero-frequency limit is uniform on $N_\pm$: $\sigma(x, y) \equiv 2g_0/(K_1^{-1} + K_2^{-1})$.[15]

If we can do a measurement that does not introduce any additional scattering mechanism and restrict an electric field to a segment in one of those neighbor-

hoods, their respective conductances are $g_b = g_0 K_2$ and $g_\pm = 2g_0/(K_1^{-1} + K_2^{-1})$. g_b is the value of the bulk conductance as derived in Ref. 5. $g_\pm$ can be thought of as a conductance of one junction: It is worth noting that $g_\pm$ coincides with the transmission through $\pm a$, multiplied by the ratio u_1/u_2.

In the same limit of large t_2, the local correlation functions for superconducting pairing decay asymptotically in time as $\tau^{-2\kappa}$ with $\kappa_b = 1/K_2$ near the origin (as is known for a homogeneous wire), while $\kappa_\pm = 2/(K_1+K_2)$ near the junction points. If $K_1 = 1$ and $K_2 > 1$, $\kappa_b < \kappa_\pm < 1$: the pairing fluctuations in the central wire extend to the external leads. This is the analogue of a proximity effect for the situation where there are only power-law pairing correlations but no long-range order. On the other hand, an electron injected in one lead and incident on the junction is reflected with coefficient $\gamma = (1 - K_2)/(1 + K_2) < 0$, i.e., *a partial hole is reflected back.* We recognize the phenomenon of Andreev reflection:[13] When an electron is incident on a normal-metal-superconductor interface, it needs to make a pair with an electron from the normal metal to enter the superconductor. According to whether its energy is lower or higher than the superconducting gap, an entire or partial hole is reflected back. In the present case there is no gap, so we only get a partial hole reflected. However, in the limit $K_2 \to \infty$, we get exactly one hole reflected.

To summarize, we have investigated transmission and transport through a finite clean interacting wire perfectly connected to semi-infinite one-dimensional leads, all being treated on the same footing. The total transmission of an incident flux on the finite wire is perfect; this is related to the momentum-conserving interactions and the symmetry of the external leads. Accordingly, the conductance one measures is just $g_0 K_1$, regardless of the internal u, K. If u, K are also constant on the central wire, the ac conductivity linking the end points shows a resonance at the eigenfrequencies of the finite wire. The multiple reflections processes at the contacts, as in a Fabry-Pérot resonator, shed light on the perfect total transmission and the departure of the conductance from its bulk value $g_0 K_2$ to $g_0 K_1$. This is another manifestation of the fact that the measured conductance is sensitive to the geometry and may be governed by the scattering at the contact probes. In real experiments, inhomogeneities at the contacts are unavoidable and the device opens into wide two-dimensional electrodes. The effects of this deserve further study. Nevertheless, the experimental results[2] obtained in quantum wires tend to confirm the relevance of the $g = g_0$ result.

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[15] In the limit $t_2 \sim t_L$, not only are we not allowing the excitations to go around the ring, but neither do they have time to travel along all the central wire. Even in the stationary state, $\omega \gg t_L^{-1} \sim t_2^{-1}$ and the current is not uniform on scales of order t_2. It is normal that $\sigma(x, y)$ is uniform only on each neighborhood separately. Note that $\sigma(-a, a) = 0$.