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# Statistical properties of random banded matrices with strongly fluctuating diagonal elements

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Random banded matrices (RBM's) whose diagonal elements fluctuate more than the off-diagonal elements were introduced recently by Shepelyansky as a convenient means to model the coherent propagation of two interacting particles in a random potential. We treat the problem analytically by using a mapping onto the same supersymmetric nonlinear $\sigma$ model that was used earlier when considering standard RBM ensemble, but with renormalized parameters. A Lorentzian form of the local density of states and a two-scale spatial structure of the eigenfunctions presented recently by Jacquod and Shepelyansky are reproduced by direct calculation of the distribution of eigenfunction components.

Random banded matrices (RBM's) can be generally described as large $N\times N$ matrices that have nonzero elements effectively within some wide band of the width $b\gg 1$ around the main diagonal. Such a structure naturally appears in various physical contexts, and serves as a useful model in quantum chaos,[1] atomic physics,[2] and solid-state physics.[3] Because of this varied utility, much effort has been spent to study different kinds of RBM's, both numerically[4,5] and analytically.[6,7] In particular, it was found that the problem can be mapped onto a supersymmetric one-dimensional nonlinear $\sigma$ model, introduced in Ref. 8, provided all matrix elements within the band are independent and distributed around zero. More precisely, the mapping was shown to exist for those matrices whose variance $\langle |H_{ij}|^2\rangle$ was dependent on the distance $|i-j|$ from the main diagonal: $\langle |H_{ij}|^2\rangle = b^{-1}f(|i-j|/b)$, where the function $f(r)$ is of the order of unity when $r\lesssim 1$ and decreases exponentially (or faster) at $r\gg 1$.

Quite recently, Shepelyansky[9] argued that a very interesting problem of two interacting particles propagating in a quenched random potential can be effectively mapped onto a class of RBM's whose diagonal elements $H_{ii}$ fluctuate more than off-diagonal ones: $\langle |H_{ii}|^2\rangle/\langle |H_{ij}|^2\rangle \propto b\gg 1$. Using this kind of mapping, Shepelyansky predicted a considerable interaction-assistant enhancement of the two-particle localization length as compared with the localization length of one particle in the same random potential. This conclusion was confirmed later on by Imry,[10] who employed the Thouless scaling block picture bypassing the mapping to RBM's. Subsequent numerical studies[11] also confirmed the main qualitative result by Shepelyansky, but revealed some deviations from the predicted behavior of the two-particle localization length that were attributed to oversimplified statistical assumptions concerning RBM elements in the Shepelyansky construction. Nevertheless, it is clear that the Shepelyansky RBM model (SRBM) captures adequately at least some of the important features of the original physical problem and thus deserves more detailed study.

In a very recent paper,[12] Jacquod and Shepelyansky presented their detailed numerical results on statistical properties of SRBM's. They revealed a peculiar structure of eigenfunctions $\Psi_\alpha$ consisting of a set of large spikes separated by regions of relatively small amplitude. Such a "sparse" spatial arrangement shows up in a difference between the localization length $l$ related to the rate of a spatial decay of an eigenfunction envelope, $l=(1/n)\lim_{n\to\infty}\ln|\Psi_\alpha(0)\Psi_\alpha(n)|$, and the length $\xi$ defined as the participation ratio, $\xi=[\sum_n|\Psi_\alpha(n)|^4]^{-1}$. For a conventional "dense" eigenfunction these two lengths are expected to be of the same order of magnitude, whereas for the SRBM it was found that $l\gg\xi$. Another interesting feature making the SRBM different from earlier studied cases is that in any given realization of the disorder the local density of states (LDOS) defined as

$$\rho(E,n)=\sum_\alpha |\Psi_\alpha(n)|^2\delta(E-E_\alpha) \tag{1}$$

was found to follow the simple Lorentzian form with a width $\Gamma\propto W_b^{-1}$ independent of the parameter $b$, where $W_b\gg 1$ determines the scale of fluctuations of the diagonal elements $H_{nn}$. To this end it is appropriate to mention that the Lorentzian form of LDOS was earlier found to be typical for RBM's with linearly increasing mean value of the diagonal elements: $\langle H_{nn}\rangle=\beta n$.[13,2]

In the present paper we show that the Shepelyansky RBM model can again be mapped onto the standard one-dimensional nonlinear $\sigma$ model with modified parameters. This fact allows us to reproduce analytically most of the peculiar features of the SRBM discussed above.

We consider the random Hermitian[14] matrix $H_{ij}=W_i\delta_{ij}+H_{ij}^{(0)}$, where the matrix $H_{ij}^{(0)}$ is a standard RBM characterized via the variances: $J_{ij}=\langle H_{ij}^{(0)*}H_{ij}^{(0)}\rangle =(1/b)f(|i-j|/b)$, normalized in such a way that $\sum_{r=-\infty}^{\infty}(1/b)f(r/b)=1$. This normalization ensures that the width of the energy spectrum of the matrix $H_{ij}^{(0)}$ is of order unity in the limit $b\to\infty$. The parameters $W_i$ are assumed to be independently distributed around zero according to the probability density $\mathscr{P}(W)=(1/W_b)h(W/W_b)$ where $h(\tau\sim 1)\sim 1$ and $\int_{-\infty}^{\infty}h(\tau)d\tau=1$.

 

Depending on the value of $W_b$, the following three regimes should be distinguished.

(i) $W_b \ll 1$. The ensemble is completely equivalent to the conventional RBM ensemble; diagonal matrix elements do not play an essential role.

(ii) $W_b \gg \sqrt{b}$. Perturbative regime. The eigenstates can be approximated by the eigenstates of the diagonal matrix $W_i\delta_{ij}$, which are localized on single sites. The nondiagonal term $H_{ij}^{(0)}$ in the Hamiltonian can be then treated via the perturbation theory.

(iii) $1 \ll W_b \ll \sqrt{b}$. Intermediate regime. It is just this regime which is shown to be relevant for the problem of two interacting particles in random potential.[9] This case is our main concern in the present paper.

We are going to characterize eigenfunction statistics via the following correlation function, see Refs. 3 and 7:

$$\begin{aligned}\mathscr{K}_{l,m} &= \langle n|(E+i\eta-\hat{H})^{-1}|n\rangle^l \langle n|(E-i\eta-\hat{H})^{-1}|n\rangle^m \\ &= \frac{i^{m-l}}{l!m!}\int \prod_i d\Phi_i\,(S_{n,1}^*S_{n,1})^l (S_{n,2}^*S_{n,2})^m \exp\left[i\sum_i \Phi_i^\dagger(E-W_i+i\eta\Lambda)L\Phi_i - i\sum_{\langle ij\rangle} H_{ij}\Phi_i^\dagger L\Phi_j\right],\end{aligned} \tag{2}$$

where $\Phi_i^\dagger = (S_{i,1}^*, \chi_{i,1}^*, S_{i,2}^*, \chi_{i,2}^*)$, with $S_{i,p}$ and $\chi_{i,p}$ being complex commuting and Grassmannian variables, respectively. The $4\times 4$ matrices $\hat{\Lambda}, \hat{L}$ are diagonal and have the following structure: $\hat{\Lambda} = \mathrm{diag}(1,1,-1,-1)$; $\hat{L} = \mathrm{diag}(1,1,-1,1)$.

Let us first calculate such a correlation function for an arbitrary *fixed* value of the potential $W_n$ in the observation point $n$, performing both averaging over $H_{ij}^{(0)}$ and over all $W_j$ with $j \neq n$ (the latter averaging we denote as $\langle\cdots\rangle_W$ henceforth). Repeating all the necessary steps outlined in Ref. 7 and presented in more detail in Ref. 3 one expresses the correlation function in terms of the integral over the set of supermatrices $R_i = T_i^{-1}P_iT_i$, where the supermatrices $P_i$ are $4\times 4$ block-diagonal ones and $T_i$ belong to the graded co-set space $U(1,1/2)/U(1/1)\times U(1/1)$. The resulting expression is as follows:

$$\begin{aligned}\mathscr{K}_{l,m}(E,n;\eta) &= \frac{i^{l-m}}{l!m!}\int \prod dR_i \mathscr{F}(R_n)\exp\left[-i\eta\sum_i \mathrm{Str}R_i\Lambda - \mathscr{L}(R)\right], \\ \mathscr{L}(R) &= \tfrac{1}{2}\sum_{ij}(J^{-1})_{ij}\,\mathrm{Str}\,R_iR_j - \sum_{i=1}^N \ln\langle \mathrm{Sdet}^{-1}(E-W-R_i)\rangle_W, \\ \mathscr{F}(R_n) &= \sum_{k=0}\binom{l}{k}\binom{m}{k} G_{n,11}^{l-k}G_{n,33}^{m-k}G_{n,13}^kG_{n,31}^k \frac{\mathrm{Sdet}G_n}{\langle \mathrm{Sdet}(E-W-R_n)^{-1}\rangle_W},\end{aligned} \tag{3}$$

where $G_n = (E-W_n-R_n)^{-1}$. The notations Sdet and Str stand for the graded determinant and graded trace, respectively.

The integral over the matrices $P_i$ can be calculated in the limit $b \gg 1$ by the saddle-point method. The saddle-point solution $P_i \equiv P_s$ is diagonal and independent of the index $i$. The diagonal matrix elements $d$ satisfy the following equation:

$$d = \int \frac{dW}{W_b} h(W/W_b)\frac{1}{E-W-d}. \tag{4}$$

An equation of similar type, known as the Pastur equation, appeared in earlier studies of full random matrices with preferential diagonal[17] and more recently in Ref. 15 and 16. The subsequent analysis depends on the value of the parameter $W_b$ characterizing the strength of the diagonal disorder. If $W_b \ll 1$, we can neglect $W$ in the denominator in the right-hand side of Eq. (4). Then the diagonal matrix elements distribution $h(\tau)$ drops out from the formulas, and the results are precisely the same as for the conventional RBM ensemble. In the present paper we are interested in the opposite case, $W_b \gg 1$. Then one obviously has $|d| \propto W_b^{-1} \ll W_b$ and to the leading order in $W_b^{-1}$ one finds

$$\begin{aligned}d &= \frac{1}{W_b}\mathscr{P}\int \frac{dW}{E-W}h(W/W_b) \pm i\frac{\pi}{W_b}h(E/W_b) \\ &\equiv \mathrm{Re}\,d \pm i\mathrm{Im}\,d\ ,\end{aligned} \tag{5}$$

where $\mathscr{P}$ stands for the principal value of the integral.

As usual, the correct saddle-point solution is equal to $P_s = (\mathrm{Re}\,d)\,\hat{I} + i(\mathrm{Im}\,d)\,\Lambda$. In order to find the region of applicability of the saddle-point method we expand the functional $\mathscr{L}(R)$ around the saddle-point value and calculate the corrections due to Gaussian fluctuations. The latter turn out to be of the order of $\overline{(\delta P_i)^2} \propto b^{-1}$. Comparing this value with the saddle point value $P_s^2 \sim d^2 \sim W_b^{-2}$ we conclude that corrections are small as long as $W_b^2 \ll b$. Thus, our calculation is completely legitimate everywhere in the *nonperturbative* regime $1 \ll W_b \ll b^{1/2}$, which is just the case relevant to the physical applications of the SRBM model.[9]

Introducing the set of matrices $\hat{Q}_i = -iT_i^{-1}\Lambda T_i$ and using the identity

$$(E-W_n-R_n)^{-1}|_{P=P_s} = \frac{(E-W-\mathrm{Re}\,d)\,\hat{I} - (\mathrm{Im}\,d)\,\hat{Q}}{(E-W-\mathrm{Re}\,d)^2 + (\mathrm{Im}\,d)^2}$$

one arrives at the following expression for the correlator (2):

$$\mathscr{K}_{l,m}(E,n;\eta)=\int d\mu(Q)\mathscr{F}_{l,m}(Q)e^{-S(Q)}, \tag{6}$$

where the action

$$S(Q)=-\frac{\gamma}{2}\mathrm{Str}\sum_{i=1}^{N-1} Q_iQ_{i+1}+i\epsilon\sum_{i=1}^{N} \mathrm{Str}(Q_i\Lambda) \tag{7}$$

defines the standard one-dimensional nonlinear graded $\sigma$ model on a lattice characterized by the coupling constant $\gamma=(\Gamma/2)^2\Sigma_rJ(r)r^2$ and the effective level broadening $\epsilon=(\Gamma/2)\eta$, the parameter $\Gamma/2$ being equal to Im $d\propto W_b^{-1}$.

The nonlinear $\sigma$ model defined in Eqs. (6,7) was studied in great detail in Refs. 3, 6, and 7. In particular, one can immediately extract the value of the localization length $l$, which is known to be proportional to the coupling constant, $l=4\gamma\propto b^2/W_b^2$, in full agreement with the results of Ref. 9. Another quantity that can be most easily calculated is the mean local density of states (DOS) defined in Eq. (1) and given by

$$\begin{aligned}\rho(E,n)&=\frac{1}{\pi}\mathrm{Im}\langle n|(E-i\eta-\hat{H})^{-1}|n\rangle|_{\eta\to0}\\ &\equiv\frac{1}{\pi}\mathrm{Im}\ \mathscr{K}_{l=0,m=1}(n,E;\eta\to0)\\ &=\frac{1}{\pi}\frac{\mathrm{Im}\ d}{(E-W_n-\mathrm{Re}\ d)^2+(\mathrm{Im}\ d)^2}.\end{aligned} \tag{8}$$

We conclude, therefore, that typically the local DOS is a Lorentzian centered around $E=W_n+\mathrm{Re}\ d$ with the width

$$\Gamma/2=\mathrm{Im}\ d=\frac{\pi}{W_b}h\left(\frac{E}{W_b}\right)\propto\frac{1}{W_b}, \tag{9}$$

as was indeed found in the numerical studies.[12] The center of this Lorentzian is shifted from the local value of the random potential $W_n$ by the amount Re $d\sim W_b^{-1}$. The shift is small compared to the typical values of $W_n\sim W_b$.

Knowing all the correlators $\mathscr{K}_{lm}(E,n;\eta)$, one can extract the full set of the eigenfunction moments $P_q(E)=\Sigma_n|\Psi_\alpha(n)|^{2q}$ and, finally, the whole probability distribution of the eigenfunction amplitude $|\Psi_\alpha(n)|^2$.[7] Straightforwardly repeating all the necessary steps one finds that all moments $P_q(E)$ for the SRBM are proportional to the corresponding moments for the standard RBM at the same values of the parameters $N$ and $\gamma$. Setting the energy $E$ to zero for the sake of simplicity, one obtains

$$P_q(E=0)|_{\mathrm{SRBM}}=\left\langle\frac{1}{[W^2+(\Gamma/2)^2]^q}\right\rangle_W P_q(E=0)|_{\text{standard RBM}}. \tag{10}$$

It is well known that for the standard RBM in the localized regime $N\gg\gamma$ one has $P_q\propto\gamma^{1-q}$.[7] The relation in Eq. (10) tells us that $P_q|_{\mathrm{SRBM}}\propto(\Gamma^2\gamma)^{1-q}\propto(W_b^2/l)^{q-1}$. In particular, for the participation ratio $\xi=P_2^{-1}$ one has $\xi\propto l/W_b^2\propto b^2/W_b^4$, thus proving the above-mentioned difference between $\xi$ and $l$ discovered by Jacquod and Shepelyansky.[12]

The relation between the moments in Eq. (10) allows one to express the distribution function of the normalized eigenfunction amplitude $y=N|\Psi^2|$ for the Shepelyansky RBM model in terms of that for the standard RBM:

$$\mathscr{P}^{\mathrm{SRBM}}(y)=\frac{\Gamma^3}{4W_b}\int_1^\infty\frac{duu}{(u-1)^{1/2}}h\left(\frac{\Gamma}{2W_b}\sqrt{u-1}\right)\times\mathscr{P}^{\mathrm{RBM}}\left(y\frac{\Gamma^2}{4}u\right). \tag{11}$$

The actual form of the function $\mathscr{P}^{\mathrm{RBM}}(y)$ depends on the scaling ratio $N/\gamma$ and can be found in Refs. 3 and 7. It takes a simple form in both the localized limit $N\gg\gamma$ and the delocalized limit $N\ll\gamma$. For example, for the latter case $\mathscr{P}^{\mathrm{RBM}}(y)=e^{-y}$ and therefore one gets

$$\mathscr{P}^{\mathrm{SRBM}}(y)=-W_b^{-1}\frac{\partial}{\partial y}\left[\frac{e^{-y\Gamma^2/4}}{\sqrt{y}}\int_{-\infty}^{\infty}dze^{-z^2}h\left(\frac{z}{\sqrt{y}W_b}\right)\right]. \tag{12}$$

This distribution clearly displays the presence of two scales. All moments $\langle y^q\rangle\equiv P_q$ with $q\geq1$ are dominated by the region where $y\Gamma^2\sim1$ (correspondingly, $|\Psi^2|\sim W_b^2/N$), whereas the normalization integral $\int\mathscr{P}(y)dy$ is dominated by the values $yW_b^2\sim1$, where eigenfunction amplitude is small: $|\Psi^2|\sim(W_b^2N)^{-1}$. This result corresponds to the following picture of a typical delocalized eigenstate: the eigenfunction consists of isolated peaks with typical amplitude $|\Psi^2|\sim W_b^2/N$ separated by regions of a typical spatial extent $L\sim W_b^2$ filled in with low-amplitude components $|\Psi^2|\sim(W_b^2N)^{-1}$.

Essentially the same picture holds for the regime of strong localization $l\ll N$. Here any eigenstate has a profile that is exponentially small outside the spatial region of size $l$. However, within this region there are isolated spikes of amplitude $|\Psi^2|\sim W_b^2/l$ separated by the low-amplitude regions with a typical extent $L\sim W_b^2$, where the wave-function amplitude is small: $|\Psi^2|\sim(W_b^2l)^{-1}\sim1/b^2$.

One can also calculate for the SRBM other quantities known for the standard RBM. For example, one can be interested in level-to-level fluctuation of the participation ratio $\xi$. Performing such a calculation one finds that if one normalizes the inverse participation ratio by its mean value $\langle P_2\rangle\propto W^4/b^2$, then the distribution of the quantity $z=P/\langle P_2\rangle$ coincides exactly with the distribution found for the standard RBM in Ref. 6. This fact suggests that envelopes of high-amplitude peaks in the SRBM are typically quite similar to envelopes of eigenfunctions in the conventional RBM, after appropriate rescaling.

Our last comment concerns the spectral correlator $Y_2(\omega)=\langle\rho(E)\rho(E+\omega)\rangle$ for the SRBM. It is easy to satisfy oneself that everywhere in the region $W_b\ll b^{1/2}$ the function $Y_2(\omega)$ for the SRBM coincides with that known for the RBM, as long as two-level separation $\omega$ is small in comparison with the spectral width $\Gamma$ of the LDS: $\omega\ll W_b^{-1}$. Recently Prus[16] addressed the same question for the case of full matrices $b=N$. He found that the function $Y_2(\omega)$ is given by the same expression as that for the standard Gaussian matrices everywhere in the region $\omega\ll1/W_b$ as long as

$W_b \ll N^{1/2}$. The latter result is also in agreement with numerical studies by Lenz *et al.*[18] who found that the crossover from Wigner-Dyson to Poissonian statistics occurs at the scale $W_b^2 \sim N$. This scale is much larger than the scale $W_b \sim 1$ necessary to induce changes in the form of the mean density.[18,19] Thus, the sparse structure of the eigenstates discussed above has no effect on the spectral statistics at relatively low frequency $\omega$, as long as the system stays well within the *nonperturbative* regime.

*Note added in proof.* Recently, we learned about an unpublished paper by K. Frahm and A. Müller-Groeling who considered the same model and arrived at very similar results.

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[1] F.M. Izrailev, Phys. Rep. **129**, 299 (1990).
[2] V.V. Flambaum *et al.*, Phys. Rev. A **50**, 267 (1994).
[3] Y.V. Fyodorov and A.D. Mirlin, Int. J. Mod. Phys. **8**, 3795 (1994).
[4] T.H. Seligman *et al.*, J. Phys. A **18**, 2751 (1985); G. Casati, L. Molinari, and F. Izrailev, Phys. Rev. Lett. **64**, 16 (1990); J. Phys. A **24**, 4755 (1991); K. Zyczkowski *et al.*, Phys. Rev. A **45**, 811 (1992).
[5] M. Feingold, D. Leitner, and M. Wilkinson, Phys. Rev. Lett. **66**, 986 (1991); J. Phys. A **24**, 1751 (1991); G. Casati *et al.*, Phys. Rev. E **48**, 1613 (1993); M. Feingold *et al.*, Phys. Rev. Lett. **70**, 2936 (1993).
[6] Y.V. Fyodorov and A.D. Mirlin, Phys. Rev. Lett. **67**, 2405 (1991); **69**, 1093 (1992); **71**, 412 (1993).
[7] A.D. Mirlin and Y.V. Fyodorov, J. Phys. A **26**, L551 (1993).
[8] K.B. Efetov, Adv. Phys. **32**, 53 (1983).
[9] D.L. Shepelyansky, Phys. Rev. Lett. **73**, 2607 (1994).
[10] Y. Imry, Europhys. Lett. **30**, 405 (1995).
[11] K. Frahm, A. Müller-Groeling, J.-L. Pichard, and D. Weinmann (unpublished); F. von Oppen, T. Wettig, and J. Müller (unpublished).
[12] Ph. Jacquod and D. Shepelyansky (unpublished).
[13] E. Wigner, Ann. Math. **62**, 548 (1955); **65**, 203 (1957); Y.V. Fyodorov, O. Chubykalo, F.M. Izrailev, and G. Casati (unpublished).
[14] For the sake of convenience we present our results for Hermitian matrices. All results can be straightforwardly generalized to the symmetric case as well.
[15] B. Lauritzen *et al.*, Phys. Rev. Lett. **74**, 5190 (1995).
[16] O. Prus (unpublished).
[17] L.A. Pastur, Theor. Math. Phys. **10**, 67 (1972).
[18] G. Lenz, K. Zyczkowski, and D. Saher, Phys. Rev. A **44**, 8043 (1991).
[19] M. Kreynin and B. Shapiro, Phys. Rev. Lett. **74**, 4122 (1995).