Statistics and superfluidity

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Using finite-size scaling, anyons on a triangular lattice at zero temperature with statistics $\nu=\frac{1}{3}$, $\frac{2}{3}$, $\frac{1}{4}$, and $\frac{3}{4}$ are tested for flux quantization. It is found that all anyons, except those with $\nu=\frac{1}{3}$, form a superfluid for some range of densities. These results compare favorably with previous mean-field investigations of lattice anyons, although the $\nu=\frac{1}{4}$ case is not predicted to be a superfluid by the usual fermion-based mean-field theories.

I. INTRODUCTION

The quasiparticle excitations of two-dimensional strongly correlated electronic systems may obey fractional statistics. These quasiparticles, called anyons,[1,2] play a crucial role in the fractional quantum Hall effect,[3,4] and have been invoked in theories of frustrated spin systems[5–7] as well as high-temperature superconductivity.[8]

Mean-field studies which incorporate the effect of local fluctuations indicate that anyons on a lattice can form a superfluid at zero temperature. Fermion-based[9,10] mean-field calculations yield superfluid states for lattice anyons with statistics $\nu=1-1/n$ (n integer), but with a number of lattice-dependent density restrictions.[9,11] Boson-based[12] mean-field calculations also predict superfluidity but for lattice anyons with statistics $\nu=\frac{m}{n}$ (mn even). While there is considerable numerical evidence in support of the mean-field predictions for superfluidity of anyons with statistics $\nu=\frac{1}{2}$ (semions),[10,11,13–15] other statistics have not been investigated to date.

Previous numerical studies of lattice semions[11,15] indicate that the torus is the ideal geometry for an investigation of flux quantization since edge effects are eliminated. The braid group[16] for anyons with statistics $\nu=m/n$ on the torus stipulates that the entire energy spectrum must be ϕ_0/n periodic in applied flux (where $\phi_0=hc/e$ is the flux quantum).[17,18] Flux quantization, a signature of superfluidity, can be monitored by investigating the change in the energy barrier, the difference between the maximum and adjacent minimum of the ground state energy, with system size.

In the present work, anyons at zero temperature with statistics $\nu=\frac{1}{3}$, $\frac{2}{3}$, $\frac{1}{4}$, and $\frac{3}{4}$, on the triangular lattice with periodic boundary conditions in both directions, are tested for flux quantization in order to address the validity of different mean-field predictions for superfluidity driven by quantum statistics. While both the Fermi and Bose-based mean-field theories predict superfluid states for anyons with the statistics $\nu=\frac{2}{3}$ and $\frac{3}{4}$ and an insulating state for anyons with $\nu=\frac{1}{3}$, only the Bose-based mean-field analysis[12] yields a superfluid for anyons with $\nu=\frac{1}{4}$. The triangular lattice is chosen since this lattice was found previously[11] to yield the most robust signature of superfluidity, compared to the square or *kagomé*. The fact that the Hilbert space for an anyon system with $\nu=m/n$ is n-fold larger than for a corresponding fermion or boson system[17] makes it unfeasible to extend these studies to statistics with $n\geq 5$.

In Sec. II, the Hamiltonian employed in the numerical investigation of lattice anyons on a torus is explicitly given. The flux quantization results are given in Sec. III and are compared with the corresponding mean-field predictions.

II. ANYONS ON A TORUS

Anyons with statistics $\nu=m/n$ on the triangular lattice with periodic boundary conditions in both directions are considered. Anyons are constructed by coupling hard-core bosons to gauge fields.[2] The triangular lattice can be represented by a square lattice with additional nearest-neighbor links in the $\hat{x}+\hat{y}$ direction.[11] In the tight-binding approximation, the Hamiltonian is written

$$H=-\sum_{\langle ij\rangle}T_{ij}b_j^{\dagger}b_i+\text{H.c.} \tag{2.1}$$

The operator $b_j^{\dagger}$ (b_j) creates (destroys) a boson-gauge field composite at site j, and the sum is over nearest neighbors. The hopping term T_{ij} includes the energy scale (chosen to be unity), the Aharonov-Bohm phases due to other particles, and the $n\times n$ matrices T_x and T_y imposed by considerations of the braid group for anyons on the torus. The matrices T_x and T_y, included whenever an anyon crosses the lattice boundary in the $\hat{x}$ and $\hat{y}$ directions, respectively, satisfy

$$T_xT_y=T_yT_xe^{i2\pi\nu}, \tag{2.2}$$

and have nonzero elements[18]

$$\begin{aligned}T_x^{k,k}&=e^{i\phi_1}e^{-i2\pi\nu k}\ \ (1\leq k\leq n),\\ T_y^{n,1}&=T_y^{k,k+1}=e^{i\phi_2}\ \ (1\leq k<n),\end{aligned} \tag{2.3}$$

where the phases ϕ_1 and ϕ_2 are chosen so that the ground state energy is a minimum with zero applied flux.

Individual N-anyon states can be written

$$|\psi\rangle=|S\rangle\sigma, \tag{2.4}$$

where $|S\rangle$ labels one of the ways N anyons can occupy the sites of a lattice with R rows and C columns, and the nonzero element of the n-component spinor σ labels the Dirac sheet that is occupied. Notice from Eq. (2.3) that the sheet index can change only if the anyon crosses a boundary in the $\hat{y}$ direction.

As in previous studies,[11,15] the "string gauge" is employed, and periodic boundary conditions are applied in both directions such that the lattice is bounded by the cuts A and B corresponding to hops across the boundary in the $\hat{y}$ and $\hat{x}$ directions, respectively. The matrices $T_{ij}^{(h)\alpha\beta}$, where the code α (β) is a "$\times$" if the hop h crosses cut B (A) and a "$\circ$" otherwise, are given below. For hops in the $\hat{x}$ direction,

$$T_{ij}^{(\hat{x})\circ\circ} = \exp\left[i\pi\nu\left(N_i^{>} - N_j^{<}\right)\right] I, \tag{2.5}$$

where $N_i^{>}$ ($N_i^{<}$) is the number of particles with the same x coordinate as the hopping particle but with a larger (smaller) y coordinate, and I is the identity matrix of order n,

$$T_{ij}^{(\hat{x})\times\circ} = \exp\left[i\pi\nu\left(N_i^{>} - N_j^{<}\right) + i2\pi\Phi_x/\phi_0\right] T_x, \tag{2.6}$$

where Φ_x is a flux coincident with cut B included in order to test for flux quantization. For hops in the $\hat{y}$ direction,

$$T_{ij}^{(\hat{y})\circ\circ} = I, \tag{2.7}$$

$$T_{ij}^{(\hat{y})\circ\times} = \exp\left[i\pi\nu\left(2N_i^{-} + N_i^{\circ}\right) + i2\pi\Phi_y/\phi_0\right] T_y, \tag{2.8}$$

where N_i^{-} is the number of particles with a smaller x coordinate than that of the hopping particle, independent of their y coordinate, and N_i° is the number of particles in the same column as, but not including, the hopping particle. The external flux Φ_y, felt by particles crossing cut A, is also included in order to find the absolute minimum of energy as well as to check the rotational invariance of the results. For hops in the $\hat{x}+\hat{y}$ direction,

$$T_{ij}^{(\hat{x}+\hat{y})\circ\circ} = \exp\left[i\pi\nu\left(N_i^{>} - N_j^{<}\right)\right] I, \tag{2.9}$$

$$T_{ij}^{(\hat{x}+\hat{y})\times\circ} = \exp\left[i\pi\nu\left(N_i^{>} - N_j^{<}\right) + i2\pi\Phi_x/\phi_0\right] T_x, \tag{2.10}$$

$$T_{ij}^{(\hat{x}+\hat{y})\circ\times} = \exp\left[i2\pi\left(\nu N_j^{-} + \Phi_y/\phi_0\right)\right] T_y, \tag{2.11}$$

$$T_{ij}^{(\hat{x}+\hat{y})\times\times} = \exp[i2\pi(\Phi_x + \Phi_y)/\phi_0] T_y T_x \tag{2.12}$$

$$= \exp[i2\pi\nu(N-1) + i2\pi(\Phi_x + \Phi_y)/\phi_0] T_x T_y, \tag{2.13}$$

where the equivalence of Eqs. (2.12) and (2.13) is proven by making use of Eq. (2.2) and the braid group condition that N must be an integer multiple of n.

III. RESULTS AND DISCUSSION

As discussed above, superfluid states can be monitored by investigating the scaling of the energy barrier, the difference between the maximum and adjacent minimum of the ground state energy, with system size. The energy barrier for a finite system of N anyons with statistics $\nu = m/n$ on an R rows by C columns lattice with periodic boundary conditions in both directions can be written[19]

$$E\left(\Phi = \frac{\phi_0}{2n}\right) - E(\Phi = 0) = \rho_s \frac{N}{R^2}\left(\frac{\pi}{n}\right)^2, \tag{3.1}$$

where ρ_s is the superfluid fraction. This equation assumes that the external flux is along a single direction; for all the results presented below, $\Phi_x = 0$ and $\Phi_y = \Phi$.

The largest Hilbert space considered is 16 anyons with $\nu = \frac{x}{4}$ ($x = 1,3$) on 25 sites. The 8 171 900 states are reduced to blocks of 326 876 or less by making use of the translational symmetry of the lattice.[15] A modified Lanczos algorithm is used in order to calculate the ground state energies.

A. Anyons with $\nu = \frac{1}{3}$

The system studied with the largest Hilbert space was 9 (or 12) anyons on a 3×7 (or 7×3) lattice; this has 881 790 states, reduced to a maximum of 41 985 by Bloch diagonalization. The actual number is dependent on the value of $\mathbf{k}$ for the block. Anyons with $\nu = \frac{1}{3}$ statistics were not found to favor any particular $\mathbf{k}$ state. Due to the considerable computational time required to find the true minimum of energy, the largest system studied was comparatively small.

In the above respect, anyons with $\nu = \frac{1}{3}$ behave much like fermions; by contrast, bosons and semions always minimize their ground state energy when $\mathbf{k} = 0$,[20] as do anyons with statistics $\nu = \frac{2}{3}$, $\frac{1}{4}$, and $\frac{3}{4}$ (discussed below). The implication that anyons with $\nu = \frac{1}{3}$ do not form a superfluid is supported by the numerical calculations. The flux quantization results for these anyons on the triangular lattice are shown in Fig. 1. No evidence of flux quantization is found for any density. Indeed, the energy barrier is as likely to be positive or negative, a behavior identical to that found for fermions.[11] These results corroborate the mean-field prediction that only $\nu = \frac{m}{n}$ anyons with mn even can form a superfluid.

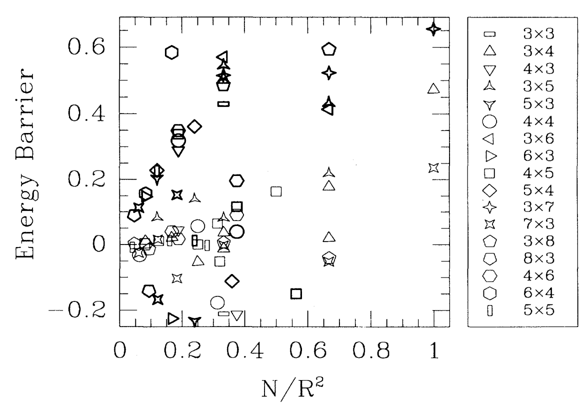

FIG. 1. The energy barrier $E(\phi_0/6) - E(0)$ is given as a function of N/R^2 for anyons with statistics $\nu = \frac{1}{3}$. Darkened points correspond to $\rho < \frac{1}{2}$. These anyons are predicted by mean-field theory to form an insulating ground state.

B. Anyons with $\nu = \frac{2}{3}$

The flux quantization results for anyons with statistics $\nu = \frac{2}{3}$ are shown in Fig. 2. Note that anyons do not have particle-hole symmetry. Indeed, anyons at a density ρ are equivalent in mean-field theory to anyons at a density $1-\rho$ in a uniform field of $\phi_0\nu$ per unit cell (where anyon holes have negative statistics).[21] For both $\rho < \frac{1}{2}$ and $\rho > \frac{1}{2}$ it is found that the energy barrier scales more or less linearly with N/R^2, indicating that anyons with statistics $\nu = \frac{2}{3}$ form a zero-temperature superfluid at most densities.

The scatter of the data points for the $\rho \leq \frac{1}{2}$ case, particularly near $N/R^2 = 0.4$ and at $N/R^2 = \frac{2}{3}$, is most likely a finite-size effect. This artificial scaling of the energy barrier with increased C at constant N/R^2 decreases rapidly with increasing lattice size, providing a clear indication of the dependence on system size of the flux quantization signature. It should be emphasized, however, that the existence of off-diagonal long-range order (ODLRO) is made manifest by the scaling of the points for the smallest lattice sizes considered. A crude estimate of the superfluid density from Eq. (3.1) yields $\rho_s = 0.71 \pm 0.05$, where the uncertainty is due to the scatter.

The reduced slope of the data points corresponding to anyons at half-filling is a lattice effect seen previously in studies of lattice bosons[14] and semions.[11,15] The lattice is most capable of suppressing superfluidity as the density approaches half-filling. The linear scaling of the points corresponding to $\rho = \frac{1}{2}$ indicates that anyons with $\nu = \frac{2}{3}$ nevertheless form a superfluid at this density, in agreement with mean-field theory.

The scaling of the energy barrier with N/R^2 for densities $\rho \geq \frac{1}{2}$ provides clear evidence of ODLRO for most lattice fillings. The data points corresponding to $\rho = \frac{3}{4}$ account for the majority of the scatter, indicating that anyons with $\nu = \frac{2}{3}$ may not form a superfluid on the triangle at this density. This observation is confirmed by a fermion-based mean-field theory of lattice anyons

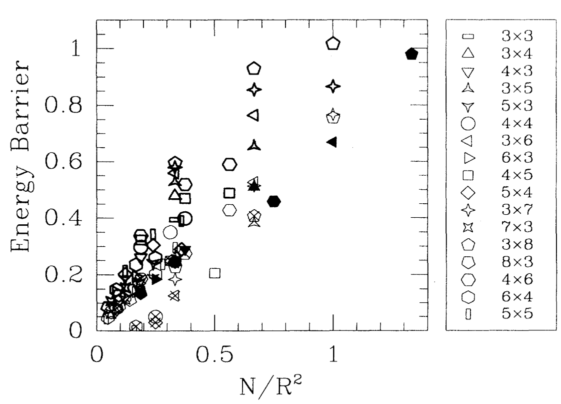

FIG. 2. The energy barrier $E(\phi_0/6) - E(0)$ is given as a function of N/R^2 for anyons with statistics $\nu = \frac{2}{3}$. Filled points correspond to $\rho = \frac{1}{2}$. Mean-field theory predicts an insulating state for $\rho = \frac{3}{4}$; points corresponding to this lattice filling are marked with crosses.

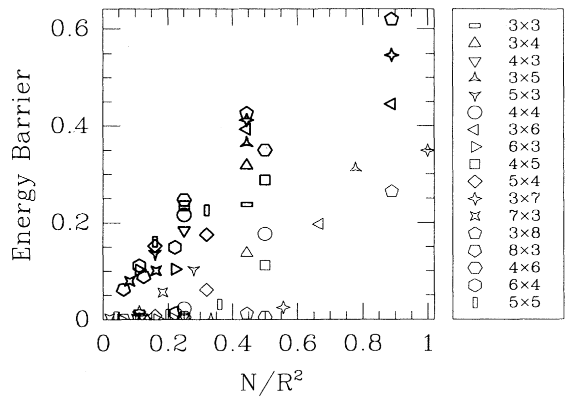

FIG. 3. The energy barrier $E(\phi_0/8) - E(0)$ is given as a function of N/R^2 for anyons with statistics $\nu = \frac{1}{4}$.

which incorporates the effect of Gaussian fluctuations;[9,10] anyons on the triangular lattice with statistics $\nu = 1 - \frac{1}{n}$ form a zero-temperature superfluid for all densities except $\rho = \frac{n}{n+1}$.[20] Neglecting the points corresponding to $\rho = \frac{3}{4}$, the superfluid fraction is roughly calculated to be $\rho_s = 0.64 \pm 0.03$. It is possible that the weaker superfluidity is due to frustration caused by the effective field induced in the transformation from particles to holes.

C. Anyons with $\nu = \frac{1}{4}$

The results of the flux quantization investigation for anyons with statistics $\nu = \frac{1}{4}$ are given in Fig. 3. The energy barrier generally scales linearly with N/R^2 for $\rho < \frac{1}{2}$, indicating a superfluid state. The scatter of the points, particularly near $N/R^2 = 0.4$, is likely a finite-size effect similar to that found for anyons with $\nu = \frac{2}{3}$.

The flux quantization results for anyons with $\nu = \frac{1}{4}$ differ markedly for densities $\rho \geq \frac{1}{2}$. Approximately half of the data points appear to scale in a manner appropriate to superfluidity. The remainder of the points, which are associated with a near-zero energy barrier, correspond

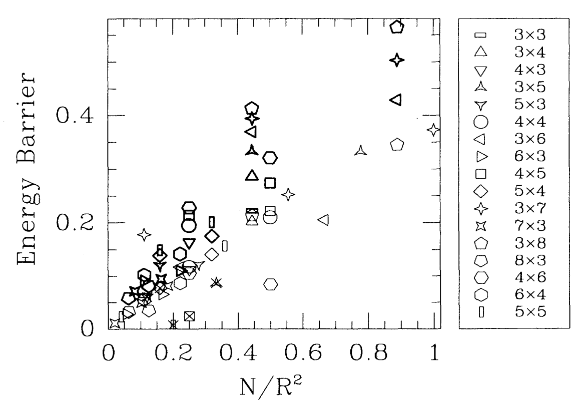

FIG. 4. The energy barrier $E(\phi_0/8) - E(0)$ is given as a function of N/R^2 for anyons with statistics $\nu = \frac{3}{4}$. Points marked with crosses correspond to $\rho = \frac{4}{5}$; mean-field theory predicts that these anyons form an insulator at this density.

to densities approaching unity (low hole concentrations). There is, however, no lattice mean-field theory for anyons with these statistics that has yielded explicit density restrictions. It is possible that the relatively high field ($\frac{3}{4}\phi_0$ in the same direction as the statistical gauge field) induced in the transformation from particles to holes is sufficient to destroy superfluidity.

D. Anyons with $\nu = \frac{3}{4}$

The numerical results for anyons with statistics $\nu = \frac{3}{4}$ and densities $\rho \leq \frac{1}{2}$ (Fig. 4) are virtually identical to those found for anyons with $\nu = \frac{1}{4}$ in the same density regime. The general scaling of the energy barrier with N/R^2 indicates that anyons with $\nu = \frac{3}{4}$ form a superfluid for densities less than half-filling, albeit with some frustration manifested by the scatter of the data points. The superfluid density estimated from the slope of the points is 0.89 ± 0.07.

While the majority of the data points for anyons with $\nu = \frac{3}{4}$ and densities $\rho \geq \frac{1}{2}$ (Fig. 4) scale in a manner consistent with superfluid states, the deviation from linear behavior for data points corresponding to $\rho = \frac{4}{5}$ indicates that this filling is special. Indeed, mean-field theory predicts that anyons with these statistics form an insulator at this density. Neglecting the $\rho = \frac{4}{5}$ data points, the superfluid fraction is estimated to be 0.58 ± 0.04.

In summary, the numerical investigation of flux quantization for small systems of lattice anyons supports the existence of a hierarchy of superfluid states for anyons with statistics $\nu = \frac{m}{n}$, with mn even. Such a hierarchy, predicted by boson-based mean-field theory, also includes the $\nu = 1 - \frac{1}{n}$ superfluid states resulting from a fermion-based mean-field analysis. It appears that the boson mean-field theory has more general validity, in that the fermion-based mean-field theory does not appear to be a valid predictor of superfluidity for statistics $\nu < \frac{1}{2}$. On the other hand, the numerical results presented here and previously[11] do corroborate specific density restrictions on superfluid states ascertained within the fermion mean-field treatment of lattice anyons. To date, the boson mean-field theory has not been extended to include possible lattice effects.

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