Effective parameters of a statistically homogeneous fluid with strong density and compressibility fluctuations

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A multiple-scattering theory for the mean acoustic field in a linear statistically homogeneous fluid with strong fluctuations is presented. Previous strong-fluctuation theories for acoustic wave propagation deal exclusively with random media of constant density that experience compressibility fluctuations only. Here a random medium with strong fluctuations of both density and compressibility is considered using a renormalization method. Analytic expressions for the effective random-medium parameters for long wavelengths are calculated within the framework of the conventional bilocal approximation as well as its nonlinear generalization. Through the proper choice of the renormalization constant, the inadequacy of the previous multiple-scattering approaches in the case of strong density fluctuations is explicitly shown, and the validity of our results is proved. Possible extensions of the proposed technique are discussed.

I. INTRODUCTION

One of the most effective methods to treat wave propagation in random media is based on the multiple-scattering theory (see, e.g., Refs. 1 and 2). This theory applies a two-step procedure which consists first of deriving equations for the statistical moments of a random field and then solving them through the appropriate analytic techniques. In particular, the problem for the mean field assumes the form of equations pertinent to some deterministic effective medium with the nonlocal constitutive parameters.

For random composite media formed by embedding a random collection of scatterers in a homogenous host material, the early attempts to calculate the effective macroscopic response can be traced back to Maxwell Garnett.[3] A new line of work was initiated by Foldy[4] where the effective field approximation for a dilute ensemble of scatterers has been proposed. In the context of solid-state physics, multiple-scattering effects in dense systems have been dealt with by using the quasicrystalline approximation in Ref. 5, the coherent potential approximation in Ref. 6, and the effective medium method in Ref. 7. They have also been applied to acoustic[8] and electromagnetic[9,10] problems. Alternative strategies used for the calculation of mechanical properties of solid composites invoke a stochastic variational principle[11,12] or a self-consistent embedding procedure.[13,14,12] Finally, one may view a composite medium as a fluctuating continuum with random constitutive parameters and utilize the multiple-scattering theory for fluctuating media (see, e.g., Ref. 15). As is well known,[1,2] the effective properties of disordered systems are sensitive to the topology of its microstructure described by the correlation function of random perturbations (or by the two-particle distribution function in a discrete-medium model). A beneficial feature of the fluctuating medium approach is that it easily accounts for topology effects, even within the framework of a simplest—bilocal—approximation. This bilocal approximation was introduced in connection with the study of the elastic properties of polycrystals in Ref. 16 and was later utilized in the problem of electromagnetic[17] and acoustic[18] wave propagation in random media. It should be noted that, as regards the electromagnetic problems, the results of Ref. 17 and subsequent works which employ similar techniques are applicable to weakly fluctuating media only. This limitation has been removed in Refs. 19–22 with the help of a renormalization method.

The present paper is concerned with the model of a linear stationary motionless fluid with strong fluctuating constitutive parameters. The acoustic wave propagation in a medium with a fluctuating refractive index $n_r(x)$ (and a unit reference acoustic velocity) is governed by the equation

$$[\nabla^2+\omega^2 n_r^2(x)]p_r(x)=0 \tag{1}$$

for an acoustic pressure $p_r(x)$. The effective refractive index referring to this case has been calculated, e.g., in Refs. 1, 2, and 23. When limited to long wavelengths, these solutions are valid for both weak and strong fluctuations of the refractive index. Note that Eq. (1) and the related results are restricted to the case where only compressibility fluctuations occur while the medium's density remains constant.

When a medium experiences both density and compressibility fluctuations described by random functions $\mu_r(x),\beta_r(x)$, the equation for the acoustic pressure becomes[24,25]

$$\left[\nabla\cdot\frac{1}{\mu_r(x)}\nabla+\omega^2\beta_r(x)\right]p_r(x)=0\ . \tag{2}$$

The effective parameters for such a case have been calculated, e.g., in Refs. 26 and 25. [Note that Ref. 25 treats stochastic equations for a stationary moving medium of

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which Eq. (2) is a particular case.] However, the bulk of the work referring to Eq. (2) is restricted by the requirement of weak density fluctuations. This is due to the presence of secular terms in the expressions for the effective parameters which are generated by the strong singularity of the respective acoustic Green's function at the origin.[27] Similar difficulty was characteristic of the early versions of multiple-scattering theories for electromagnetic waves.[17]

In the present paper, we consider an acoustic problem in a statistically homogeneous medium with density and compressibility fluctuations described by Eq. (2) or subsequent Eqs. (3) and (4). Our goal here is to develop a multiple-scattering theory for a mean acoustic field which incorporates the case of strong fluctuations of both density and compressibility. For the electromagnetic case, similar investigations have been published by Ryzhov, Tamoikin, and Tatarskii,[19] Ryzhov and Tamoikin,[20] Tsang and Kong,[21] and Stogryn.[22]

To achieve our goal, in Sec. III we make use of the renormalized integral equations involving Green's operators for a deterministic background medium. On this basis, the effective perturbation operators are introduced and calculated for long wavelengths in the conventional bilocal approximation, as well as in the nonlinear bilocal approximation which includes the former as a limiting case. A nonlinear approximation of this type does not seem to be available in the literature on the renormalization approach. In the electromagnetic case, the same method for refining the bilocal approximation has been proposed in Ref. 28. In Sec. IV, we analyze several variants of choosing the renormalization constant and their implications on the validity range of the final results, including the case of strong fluctuations of density. Possible applications and extensions of the proposed approach are discussed in Sec. V.

Throughout the present paper, a time dependence $\exp(-i\omega t)$ is assumed, with corresponding time factor omitted. Algebraic vectors $x=(x_1,x_2,x_3)$, $k=(k_1,k_2,k_3)$ refer to the points in the **x** and **k** spaces characterized by the respective position vectors; then $d^3x=dx_1dx_2dx_3$, $d^3k=dk_1dk_2dk_3$. Angular brackets $\langle\cdots\rangle$ stand for statistical averaging, the caret ^ signifies a dyad, and $\cdot$ denotes the dot product referring to geometrical vectors or dyads.

II. PROBLEM STATEMENT

In this section, the nonlocal effective constitutive parameters are introduced that determine the properties of a stationary motionless fluid with density and compressibility fluctuations with respect to the mean acoustic pressure and velocity. For statistically homogeneous media, limited to long wavelengths, they convert to the local effective parameters (23) which will be calculated in Sec. III.

We start with the excitation problem for a random acoustic field in an unbounded inhomogeneous fluid characterized by the random density $\mu_r(x)$ and compressibility $\beta_r(x)$. The said functions are allowed to take complex values to account for small dissipative losses. The acoustic pressure $p_r(x)$ and velocity $\mathbf{v}_r(x)$ due to the deterministic impressed sources $\mathbf{f}(x),s(x)$ are governed by the equations[24,25]

$$\nabla p_r(x)-i\omega\mu_r(x)\mathbf{v}_r(x)=\mathbf{f}(x)\ , \tag{3}$$

$$-i\omega\beta_r(x)p_r(x)+\nabla\cdot\mathbf{v}_r(x)=s(x)\ , \tag{4}$$

provided the amplitude is low enough that nonlinear effects are negligible. In addition, the radiation condition at infinity, as well as the condition of continuity of acoustic pressure and the normal component of acoustic velocity at every (random) interface in the medium, must be satisfied. Since the acoustic velocity is given by

$$\mathbf{v}_r(x)=\frac{1}{i\omega\mu_r(x)}[\nabla p_r(x)-\mathbf{f}(x)]\ , \tag{5}$$

the problem is expressible in the form of Eq. (2) where zero in the right-hand side should be replaced with the random source term

$$f_r(x)=i\omega s(x)+\nabla\cdot\frac{\mathbf{f}(x)}{\mu_r(x)}\ . \tag{6}$$

Owing to the linearity of the problem, the quantities $\langle\beta_r p_r\rangle,\langle\mu_r\mathbf{v}_r\rangle$ allow the following representation in terms of the mean field quantities $\langle p_r\rangle,\langle\mathbf{v}_r\rangle$:

$$\langle\beta_r(x)p_r(x)\rangle\equiv b\langle p_r(x)\rangle+\mathbf{c}\cdot\langle\mathbf{v}_r(x)\rangle\ , \tag{7}$$

$$\langle\mu_r(x)\mathbf{v}_r(x)\rangle\equiv \mathbf{l}\langle p_r(x)\rangle+\hat{m}\cdot\langle\mathbf{v}_r(x)\rangle\ , \tag{8}$$

where b, $\mathbf{c}$, $\mathbf{l}$, and $\hat{m}$ are the effective constitutive operators acting on x. This representation can be viewed as an acoustic counterpart to the effective relations in Ref. 12 for composite materials. Averaging of Eqs. (3) and (4) leads, through Eqs. (7) and (8), to a system of equations

$$\nabla\langle p_r(x)\rangle-i\omega[\mathbf{l}\langle p_r(x)\rangle+\hat{m}\cdot\langle\mathbf{v}_r(x)\rangle]=\mathbf{f}(x)\ , \tag{9}$$

$$-i\omega[b\langle p_r(x)\rangle+\mathbf{c}\cdot\langle\mathbf{v}_r(x)\rangle]+\nabla\cdot\langle\mathbf{v}_r(x)\rangle=s(x)\ , \tag{10}$$

with operator coefficients. The use of inverse operators $b^{-1},\hat{m}^{-1}$ eliminates $\langle\mathbf{v}_r(x)\rangle$ and $\langle p_r(x)\rangle$ from Eqs. (9) and (10), respectively. The expressions arising can be inserted into the corresponding complementary equation (10) or (9). Hence, equations for the mean acoustic field are obtained

$$\nabla\langle p_r(x)\rangle-i\omega\hat{\mu}_e\cdot\langle\mathbf{v}_r(x)\rangle=\mathbf{f}_e(x)\ , \tag{11}$$

$$-i\omega\beta_e\langle p_r(x)\rangle+\nabla\cdot\langle\mathbf{v}_r(x)\rangle=s_e(x)\ , \tag{12}$$

characteristic of a spatially dispersive medium with the nonlocal "compressibility" β_e and dyadic "density" $\hat{\mu}_e$,

$$\beta_e=b+\mathbf{c}\cdot\hat{m}^{-1}\cdot\left[\frac{\nabla}{i\omega}-\mathbf{l}\right]\ , \tag{13}$$

$$\hat{\mu}_e=\hat{m}+\mathbf{l}b^{-1}\left[\frac{\nabla}{i\omega}-\mathbf{c}\right]\ . \tag{14}$$

Note that the right-hand side in Eqs. (11) and (12) involves the renormalized source terms

$$\mathbf{f}_e(x)=\mathbf{f}(x)-\mathbf{l}b^{-1}s(x) , \quad (15)$$

$$s_e(x)=s(x)-\mathbf{c}\cdot\hat{m}^{-1}\cdot\mathbf{f}(x) . \quad (16)$$

As a consequence of the expression

$$\langle \mathbf{v}_r(x)\rangle=\frac{\hat{\mu}_e}{i\omega}\cdot[\nabla\langle p_r(x)\rangle-\mathbf{f}_e(x)] , \quad (17)$$

Eqs. (13) and (14) reduce to an equation

$$[\nabla\cdot\hat{\mu}_e^{-1}\cdot\nabla+\omega^2\beta_e]\langle p_r(x)\rangle=i\omega s_e(x)+\nabla\cdot\hat{\mu}_e^{-1}\cdot\mathbf{f}_e(x) , \quad (18)$$

which constitutes, in a source-free case, the averaged version of Eq. (2). It is clearly seen from Eq. (18) that averaging of Eq. (2) necessitates the knowledge of two effective parameter operators β_e and $\hat{\mu}_e$. This contrasts with Eq. (1) which can be averaged by knowledge of only one effective parameter operator.[1,2,23]

Anticipating further needs, let us introduce the kernel $\mathcal{B}(x,x')$ and the symbol $\mathcal{B}(k,x)$ of an arbitrary x-acting linear operator $\mathcal{B}$ by the identities

$$\begin{aligned}&\mathcal{B}\delta(x-x')\equiv\mathcal{B}(x,x') ,\\&\mathcal{B}\exp(i\mathbf{k}\cdot\mathbf{x})\equiv\exp(i\mathbf{k}\cdot\mathbf{x})\mathcal{B}(k,x) ,\end{aligned} \quad (19)$$

where $\delta(x-x')$ is the three-dimensional Dirac δ function and $\mathbf{k}$ is a spectral parameter. Since any arbitrary function of the variable x can be represented by a linear superposition of δ functions, we may regard $\mathcal{B}$ as an integral operator $\mathcal{B}=\int d^3x'\mathcal{B}(x,x')\cdots$ whose kernel $\mathcal{B}(x,x')$ belongs, in general, to the class of distributions.

For a statistically homogeneous medium the operators $\mathcal{B}=b$, $\mathbf{c}$, $\mathbf{l}$, $\hat{m}$, and $\beta_e,\hat{\mu}_e$, the Green's operators G, $\mathbf{G}$, and $\hat{G}$ from Eqs. (24) and (25), and the effective perturbation operators α, $\boldsymbol{\gamma}$, $\boldsymbol{\nu}$, and $\hat{\rho}$ in the right-hand side of Eqs. (42) and (43) will have an x-independent symbol $\mathcal{B}(k)$, and the kernel which depends on x,x' through the difference variable $x-x'$ only: $\mathcal{B}(x,x')\equiv\mathcal{B}(x-x')$. In this case there exists a simple relationship between the symbol and kernel of an operator given by the distributional Fourier transform:

$$\mathcal{B}(k)=\int d^3x\ \mathcal{B}(x-x')\exp[-i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')] . \quad (20)$$

Obviously, $\beta_e(k),\hat{\mu}_e(k)$ are obtained by replacing operators with their symbols in Eqs. (13) and (14), e.g., $\nabla\to i\mathbf{k}$, etc.

The important physical feature that distinguishes the statistically homogeneous media is that the mean acoustic field created by spatially harmonic impressed sources

$$\mathbf{f}(x)=\mathbf{f}(k)\exp(i\mathbf{k}\cdot\mathbf{x}),\quad s(x)=s(x)\exp(i\mathbf{k}\cdot\mathbf{x}) , \quad (21)$$

will be spatially harmonic as well:

$$\begin{aligned}&\langle p_r(x)\rangle=p(k)\exp(i\mathbf{k}\cdot\mathbf{x}) ,\\&\langle \mathbf{v}_r(x)\rangle=\mathbf{v}(k)\exp(i\mathbf{k}\cdot\mathbf{x}) .\end{aligned} \quad (22)$$

This can be easily checked through Eqs. (9), (10), and (19).

In this paper, we want to calculate approximately the effective constitutive parameters in the long-wavelength limit:

$$\begin{aligned}&b_0=\lim b(k),\quad \mathbf{c}_0=\lim\mathbf{c}(k),\quad \mathbf{l}_0=\lim\mathbf{l}(k) ,\\&\hat{m}_0=\lim\hat{m}(k),\quad \beta_{e0}=\lim\beta_e(k) ,\\&\hat{\mu}_{e0}=\lim\hat{\mu}_e(k)\quad (k\to 0) .\end{aligned} \quad (23)$$

III. RENORMALIZATION METHOD

We shall here calculate the quantities of interest listed in Eqs. (23) using the conventional bilocal approximation as well as the nonlinear bilocal approximation. We base the calculation on the renormalized integral formulation (36) and (37) of the original acoustic problem (3) and (4) which involves an arbitrary renormalization constant S. The latter will be specified in the next section to ensure applicability of the final results to the case of strong density fluctuations.

The renormalization approach requires the introduction of an unbounded background medium with constant density μ and compressibility β which will be specified later by Eq. (50). Let us consider the acoustic field $p(x)$, $\mathbf{v}(x)$ created by the impressed sources $\mathbf{f}(x),s(x)$ in the background medium. The solution for $p(x),\mathbf{v}(x)$ can be expressed in terms of sources with the aid of the Green's operators[27,29] G, $\mathbf{G}$, and $\hat{G}$:

$$p(x)=Gs(x)+\mathbf{G}\cdot\mathbf{f}(x) , \quad (24)$$

$$\mathbf{v}(x)=\mathbf{G}s(x)+\hat{G}\cdot\mathbf{f}(x) , \quad (25)$$

where[27]

$$G=i\omega\mu g,\quad \mathbf{G}=\nabla g,\quad \hat{G}=\frac{1}{i\omega}[\nabla\nabla g-\hat{I}] , \quad (26)$$

$$g(x-x')=-\exp(ik_0R)/4\pi R,\quad R=|\mathbf{x}-\mathbf{x}'| , \quad (27)$$

$\hat{I}$ is the identity dyad, and k_0 is the wave number: $k_0=\omega\sqrt{\mu\beta}$ $(0\le\arg\sqrt{X}<\pi)$. From Eq. (26) and the expression $g(k)=1/(k_0^2-k^2)$ the spectral representation for operator $\hat{G}$ is obtained, namely,

$$\hat{G}(k)=\frac{1}{i\omega\mu}(\mathbf{nn}-\hat{I})+\hat{G}^{(1)}(k) , \quad (28)$$

with $\mathbf{n}=\mathbf{k}/k$ and

$$\hat{G}^{(1)}(k)=i\omega\beta\mathbf{nn}g(k) . \quad (29)$$

We now introduce a spectral domain function $\hat{G}^{(2)}(k)$,

$$\hat{G}^{(2)}(k)=\hat{G}(k)+\frac{S}{i\omega}\hat{I} , \quad (30)$$

$$=\hat{G}^{(1)}(k)+\frac{\mathbf{nn}}{i\omega\mu}+\frac{\hat{I}}{i\omega}\left[S-\frac{1}{\mu}\right] , \quad (31)$$

and the corresponding dyadic operator acting on x

$$\hat{G}^{(2)}=\hat{G}+\frac{S}{i\omega}\hat{I} , \quad (32)$$

where S plays the part of renormalization constant. On resorting to a well-known technique, an integral equation substitute for differential equations (3) and (4) can be readily derived, namely,

$$p_r(x)=p(x)+i\omega G(\beta_r-\beta)p_r(x)+i\omega\mathbf{G}\cdot(\mu_r-\mu)\mathbf{v}_r(x)\ , \quad (33)$$

$$\mathbf{v}_r(x)=\mathbf{v}(x)+i\omega\mathbf{G}(\beta_r-\beta)p_r(x)+i\omega\hat{G}\cdot(\mu_r-\mu)\mathbf{v}_r(x)\ . \quad (34)$$

After making use of representation (32) in Eqs. (33) and (34) and replacing $\mathbf{v}_r$ by another field variable,

$$\mathbf{w}_r(x)\equiv\{1+S[\mu_r(x)-\mu]\}^{-1}\mathbf{v}_r(x)\ , \quad (35)$$

there results a system of renormalized equations

$$p_r(x)=p(x)+i\omega G\beta_f p_r(x)+i\omega\mathbf{G}\cdot\xi\mathbf{w}_r(x)\ , \quad (36)$$

$$\mathbf{w}_r(x)=\mathbf{v}(x)+i\omega\mathbf{G}\beta_f p_r(x)+i\omega\hat{G}^{(2)}\cdot\xi\mathbf{w}_r(x)\ , \quad (37)$$

where $\beta_f(x),\xi(x)$ stand for random perturbation functions

$$\beta_f(x)=\beta_r(x)-\beta\ ,\quad \xi(x)=[\mu_r(x)-\mu]\{1+S[\mu_r(x)-\mu]\}^{-1}\ . \quad (38)$$

For the sake of procedural advantage, we rewrite Eqs. (36) and (37) in matrix form as

$$\psi_r(x)=\psi(x)+i\omega\Gamma\Pi\psi_r(x)\ . \quad (39)$$

In this relation

$$\psi_r=\mathrm{col}[p_r,\mathbf{w}_r],\quad \psi=\mathrm{col}[p,\mathbf{v}],\quad \Pi=\mathrm{diag}[\beta_f,\xi]\ , \quad (40)$$

and a 2×2 matrix $\Gamma=[\Gamma_{mn}]$ comprises Green's operators from Eq. (26):

$$\Gamma_{11}=G,\quad \Gamma_{12}=\mathbf{G}\cdot,\quad \Gamma_{21}=\mathbf{G},\quad \Gamma_{22}=\hat{G}^{(2)}\cdot\ . \quad (41)$$

The averaged version of Eqs. (36) and (37) is simply arrived at, if the effective perturbation operators α, γ, ν, and $\hat{\rho}$ that satisfy the identities

$$\langle\beta_f(x)p_r(x)\rangle\equiv\alpha\langle p_r(x)\rangle+\gamma\cdot\langle\mathbf{w}_r(x)\rangle\ , \quad (42)$$

$$\langle\xi(x)\mathbf{w}_r(x)\rangle\equiv\nu\langle p_r(x)\rangle+\hat{\rho}\cdot\langle\mathbf{w}_r(x)\rangle\ , \quad (43)$$

are available. With these operators at hand, one can introduce a 2×2 matrix $\Pi_e=[\Pi^e_{mn}]$,

$$\Pi^e_{11}=\alpha,\quad \Pi^e_{12}=\gamma\cdot,\quad \Pi^e_{21}=\nu,\quad \Pi^e_{22}=\hat{\rho}\cdot\ , \quad (44)$$

which possesses the property

$$\langle\Pi(x)\psi_r(x)\rangle\equiv\Pi_e\langle\psi_r(x)\rangle\ . \quad (45)$$

On substituting the solution to Eq. (39) $\psi_r=(1-i\omega\Gamma\Pi)^{-1}\psi$ into Eq. (45) it is an easy matter to verify the following representation for Π_e:

$$\Pi_e=\langle\Pi(1-i\omega\Gamma\Pi)^{-1}\rangle\langle(1-i\omega\Gamma\Pi)^{-1}\rangle^{-1}\ . \quad (46)$$

Once the matrix Π_e is known, one can establish, through Eqs. (7), (8), (42), (43), and the intermediate equalities,

$$\langle\mathbf{w}_r(x)\rangle=S\langle p_r(x)\rangle+[\hat{I}+S(\hat{m}-\mu\hat{I})]\cdot\langle\mathbf{v}_r(x)\rangle\ , \quad (47)$$

$$\langle\xi(x)\mathbf{w}_r(x)\rangle=\mathbf{l}\langle p_r(x)\rangle+(\hat{m}-\mu\hat{I})\cdot\langle\mathbf{v}_r(x)\rangle\ , \quad (48)$$

the relations which connect the quantities of interest figuring in Eqs. (7), (8) and the effective perturbation operators:

$$b-\beta=\alpha+S\gamma\cdot(\hat{I}-S\hat{\rho})^{-1}\cdot\nu,\quad \mathbf{c}=\gamma\cdot(\hat{I}-S\hat{\rho})^{-1}\ ,\quad \mathbf{l}=(\hat{I}-S\hat{\rho})^{-1}\cdot\nu,\quad \hat{m}-\mu\hat{I}=(\hat{I}-S\hat{\rho})^{-1}\cdot\hat{\rho}\ . \quad (49)$$

They convert to simple algebraic expressions in the spectral domain.

Obviously, it is impossible to calculate the effective perturbation operators in accordance with formula (46) both rigorously and in closed form. A formal solution can be obtained by expanding the right-hand side in Eq. (46) in powers of Π. To ensure the fastest rate of convergence, it is assumed that the condition $\langle\Pi(x)\rangle=0$ is satisfied, or equivalently, $\langle\beta_f(x)\rangle=0$, $\langle\xi(x)\rangle=0$. In view of Eq. (38), these requirements can be formulated as

$$\beta=\langle\beta_r(x)\rangle\ ,\quad \langle[\mu_r(x)-\mu]\{1+S[\mu_r(x)-\mu]\}^{-1}\rangle=0\ . \quad (50)$$

The first of these two relations yields an explicit expression for the compressibility of the background medium, while the second one constitutes an equation to determine parameter μ, provided S is specified.

Retaining the first nonvanishing term in the aforementioned perturbation series solution yields the bilocal approximation for Π_e: $\Pi_e\cong i\omega\langle\Pi\Gamma\Pi\rangle$. Recast in terms of symbols of the effective perturbation operators, the bilocal approximation reads as

$$\alpha(k)=-\omega^2\mu\int d^3k'B_\beta(k-k')g(k')\ , \quad (51)$$

$$\gamma(k)=-\omega\int d^3k'B_{\beta\xi}(k-k')\mathbf{k}'g(k')=-\nu(-k)\ , \quad (52)$$

$$\hat{\rho}(k)=-\omega^2\beta\int d^3k'B_\xi(k-k')\mathbf{n}'\mathbf{n}'g(k')+\int d^3k'B_\xi(k-k')\left[\hat{I}\left[S-\frac{1}{\mu}\right]-\frac{\mathbf{n}'\mathbf{n}'}{\mu}\right]\ . \quad (53)$$

We encounter in Eqs. (51)–(53) the spectra $B_\beta(k)$, $B_{B\xi}(k)$, and $B_\xi(k)$ of random perturbations which are expressible through correlation functions

$$B_\beta(x-x')=\langle\beta(x)\beta(x')\rangle\ ,\quad B_{\beta\xi}(x-x')=\langle\beta(x)\xi(x')\rangle\ ,\quad B_\xi(x-x')=\langle\xi(x)\xi(x')\rangle \quad (54)$$

via integral Fourier transformation

$$(2\pi)^3B_\beta(k)=\int d^3xB_\beta(x-x')\exp[-i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')]\ . \quad (55)$$

Similar expressions for $B_{\beta\xi}(k),B_\xi(k)$ are omitted for brevity.

On performing operation $k\to0$, Eqs. (51) and (52) take the form

$$\alpha_0=-\omega^2\mu\int d^3k'B_\beta(k')g(k')\ , \quad (56)$$

$$\gamma_0=\omega\int d^3k'B_{\beta\xi}(k')\mathbf{k}'g(k')=-\nu_0\ , \quad (57)$$

where subscript "0" signifies the long-wavelength limit.

It should be noted that, in general, $\gamma_0 \neq 0$, $\nu_0 \neq 0$ and, consequently, $c_0 \neq 0$, $l_0 \neq 0$.

From now on we restrict our attention to the case in which the cross-correlation function $B_{\beta\xi}(x-x')$ is an even function of difference variable $x-x'$:

$$B_{\beta\xi}(x-x')=B_{\beta\xi}(x'-x) , \tag{58}$$

and the random perturbation $\xi(x)$ is characterized with the spherically symmetric correlation function:

$$B_{\xi}(x-x')\equiv B_{\xi}(R) . \tag{59}$$

In this case $B_{\beta\xi}(k)=B_{\beta\xi}(-k)$, and spectrum $B_\xi(k)$ turns into a function of the magnitude of vector $\mathbf{k}$: $B_\xi(k)\equiv B_\xi(k)$. Then it can be shown without difficulty that

$$\gamma_0=\nu_0=0 , \tag{60}$$

and $\hat{\rho}(k\to 0)$ is a multiple of the identity dyad:

$$\hat{\rho}(k\to 0)=\rho_0 \hat{I} , \tag{61}$$

$$\begin{aligned} \rho_0 &= -(4\pi/3)\omega^2\beta\int_0^{+\infty}k^2B_\xi(k)g(k)dk \\ &\quad +4\pi\left[S-\frac{2}{3\mu}\right]\int_0^{+\infty}k^2B_\xi(k)dk \end{aligned} \tag{62}$$

$$\begin{aligned} &= -(4\pi/3)\omega^2\beta\int_0^{+\infty}k^2B_\xi(k)g(k)dk \\ &\quad +\sigma_\xi^2\left[S-\frac{2}{3\mu}\right] , \end{aligned} \tag{63}$$

where σ_ξ^2 is the value of $B_\xi(x-x')$ at $x'\to x$. For lossless media which are characterized by real positive k_0, the expression for ρ_0 can be given a simpler structure by performing a limiting operation $\mathrm{Im}k_0 \to +0$ in Eqs. (62) and (63). This is tantamount to replacing, in the aforementioned equations, $g(k)$ with $[\mathrm{PV}]g(k)-(\pi i/2k_0)\delta(k-k_0)$. Setting $k_0\cong 0$ in the arising principal value (PV) integral in compliance with the long-wavelength approximation, followed by taking note of the relations

$$\begin{aligned} &4\pi\int_0^{+\infty}B_\xi(k)dk=\int_0^{+\infty}RB_\xi(R)dR , \\ &2\pi^2B_\xi(k_0)\cong\int_0^{+\infty}R^2B_\xi(R)dR \end{aligned} \tag{64}$$

leads to a characterization of ρ_0 of the form

$$\begin{aligned} \rho_0 &\cong (1/3)\omega^2\beta\int_0^{+\infty}RB_\xi(R)dR \\ &\quad +i(1/3)\omega^2k_0\int_0^{+\infty}R^2B_\xi(R)dR+\sigma_\xi^2\left[S-\frac{2}{3\mu}\right] . \end{aligned} \tag{65}$$

Likewise, if the correlation function B_β is spherically symmetric,

$$B_\beta(x-x')\equiv B_\beta(R) , \tag{66}$$

it follows from Eq. (56) that for lossless media

$$\alpha_0\cong\omega^2\mu\int_0^{+\infty}RB_\beta(R)dR+i\omega^2\mu k_0\int_0^{+\infty}R^2B_\beta(R)dR . \tag{67}$$

Note that conditions (58), (59), and (66) imply, in the wider sense, the statistical isotropy of each of the random functions $\xi(x),\beta_r(x)$ but do not presuppose the statistically isotropic correlation between these functions.

A more refined recipe for calculating the effective perturbation operators is furnished by the nonlinear bilocal approximation. Let Λ be the inverse of Γ. When $\Lambda=\Gamma^{-1}$ is applied to both sides of Eq. (39), we find

$$(\Lambda-i\omega\Pi)\langle\psi_r(x)\rangle=\Lambda\psi . \tag{68}$$

Taking account of Eq. (45), we may readily establish that

$$\Lambda_e\langle\psi_r(x)\rangle\equiv(\Lambda-i\omega\Pi_e)\langle\psi_r(x)\rangle=\Lambda\psi . \tag{69}$$

We may observe that the conventional bilocal approximation can be displayed in the form $\Pi_e\cong i\omega\langle\Pi\Lambda^{-1}\Pi\rangle$. Guided by well-known ideas (see, e.g., Ref. 2), we infer a nonlinear extension of said approximation by the replacement Λ^{-1} with $\Lambda_e^{-1}=(1-i\omega\Gamma\Pi_e)^{-1}\Gamma$, thus obtaining a nonlinear equation $\Pi_e\cong i\omega\langle\Pi(1-i\omega\Gamma\Pi_e)^{-1}\Gamma\Pi\rangle$ to determine Π_e. A solution to this equation comprises an infinite subsequence of terms in the perturbation series expansion of the right-hand member in Eq. (46).

When the nonlinear equation in Π_e is recast in terms of symbols of the effective perturbation operators and we assume that $k\to 0$, it appears that, in the circumstances described by Eqs. (58) and (59), the quantities γ_0,ν_0 equal zero [recall Eq. (60)], $\hat{\rho}(k\to 0)$ reduces to scalar constant ρ_0 in agreement with Eq. (61), and α_0,ρ_0 satisfy a system of nonlinear equations

$$\alpha_0=-\omega^2\mu\left[1+\rho_0\left[S-\frac{1}{\mu}\right]\right]\int d^3kB_\beta(k)g_e(k) , \tag{70}$$

$$\begin{aligned} \rho_0 &= -\frac{4\pi\omega^2(\alpha_0+\beta)}{3(1-\rho_0S)}\int_0^{+\infty}k^2B_\xi(k)g_e(k)dk \\ &\quad +\sigma_\xi^2\frac{[(S-1/\mu)(1-\rho_0S)+1/3\mu]}{(1-\rho_0S)[1+\rho_0(1/\mu-S)]} , \end{aligned} \tag{71}$$

where

$$\begin{aligned} g_e(k)=\Bigg\{&[\omega^2\mu(\alpha_0+\beta)-k^2] \\ &\times\left[1+\rho_0\left[\frac{1}{\mu}-S\right]\right]+\frac{k^2\rho_0}{\mu}\Bigg\}^{-1} , \end{aligned} \tag{72}$$

and the notations of Eqs. (56), (62), and (63) are in effect. If we concentrate on the situation implied by Eq. (66), a simplified version of Eq. (70) is obtained, namely,

$$\begin{aligned} \alpha_0 &= -4\pi\omega^2\mu\left[1+\rho_0\left[S-\frac{1}{\mu}\right]\right] \\ &\quad \times\int_0^{+\infty}k^2B_\beta(k)g_e(k)dk . \end{aligned} \tag{73}$$

After neglecting α_0,ρ_0 in the right-hand side of Eqs. (70)–(73) the respective formulas of the bilocal approximation derived before are recovered.

With the above approximations for the effective perturbation operators at hand, we are now in a positive to develop the effective random-medium parameters at long wavelengths. Reference to Eq. (49) furnishes the result that we are seeking:

$$\mathbf{c}_0 = \mathbf{l}_0 = 0 \ , \tag{74}$$

$$b_0 = \beta + \alpha_0 \ , \tag{75}$$

$$\hat{m}_0 = m_0 \hat{I}, \quad m_0 = \mu + \rho_0/(1 - S\rho_0) \ . \tag{76}$$

Consequently Eqs. (13) and (14) yield

$$\beta_{e0} = b_0 = \beta + \alpha_0 \ , \tag{77}$$

$$\hat{\mu}_{e0} = \mu_{e0}\hat{I}, \quad \mu_e = m_0 = \mu + \rho_0/(1 - S\rho_0) \ . \tag{78}$$

Equations (74)–(78) show that, under conditions (58) and (59), the quantities α_0, ρ_0 provide an exhaustive characterization of the effective random-medium properties for long wavelengths.

IV. CONSIDERATION OF THE STRONG-FLUCTUATION CASE

In this section, we shall analyze the scope of the results of the preceding section. The aim is to make these results suitable for the case of strong density fluctuations through an appropriate choice of the renormalization constant.

The criterion which legitimizes Eqs. (75)–(78) demands[21] that the ratio of the two consecutive terms in these relations be small as compared with unity, namely,

$$|\alpha_0/\beta| \ll 1 \ , \tag{79}$$

$$\left| \frac{m_0}{\mu} - 1 \right| = \left| \frac{\rho_0/\mu}{1 - \rho_0 S} \right| \ll 1 \ . \tag{80}$$

Here the absolute values of the respective quantities are used which, in general, may be complex valued due to slight dissipation in the random medium.

For purposes of deriving an estimate of α_0, ρ_0, the case where the random medium is characterized by spherically symmetric correlation functions given by Eqs. (59) and (66) is considered. It is convenient to express correlation functions in the scaled form: $B_\xi(R) = \sigma_\xi^2 \Phi_\xi(R/L_\xi)$, $B_\beta(R) = \sigma_\beta^2 \Phi_\beta(R/L_\beta)$. Here $L_{\xi,\beta}$ are correlation lengths, functions $\Phi_{\xi,\beta}(t)$ approach zero sufficiently fast as t exceeds unity, and $\Phi_{\xi,\beta}(0) = 1$. The constants σ_ξ, σ_β may be viewed as the measure of the strength of the random-medium fluctuations. By direct inspection of any of the Eqs. (56), (67), (70), or (73) one can form the following estimate: $\alpha_0 \sim (k_0 L_\beta)^2 \sigma_\beta^2/\beta$. It can thus be concluded that the condition (79) is met when

$$(k_0 L_\beta)^2 |\sigma_\beta/\beta|^2 \ll 1 \ . \tag{81}$$

This condition ensures the validity of the approximate expressions for parameter α_0, which arises due to compressibility fluctuations. Note that inequality (81) is satisfied in the case of strong compressibility fluctuations ($|\sigma_\beta/\beta|^2 \gg 1$) provided their characteristic scale is small enough, $(k_0 L_\beta)^2 \ll 1$.

We now proceed with the analysis of the approximate expressions for ρ_0 developed in the preceding section. They involve a renormalization constant S which has remained unspecified. It can be easily shown that the integral terms in Eqs. (63), (65), and (71) have the order of $(k_0 L_\xi)^2 \sigma_\xi^2/\mu$. The off-integral terms do not explicitly involve the correlation length of random perturbations and are proportional to σ_ξ^2. Thus, they may be readily identified as secular terms.[21] If one selects $S = 0$, the aforementioned equations reproduce the results of the conventional multiple-scattering theory.[25,26] In particular, it follows from Eqs. (76) and (78) that

$$m_0 = \mu_{e0} = \mu + \rho_0 \ . \tag{82}$$

On substituting the arising expressions for ρ_0 in Eq. (80), the requirements

$$(k_0 L_\xi)^2 |\sigma_\xi/\mu|^2 \ll 1, \quad |\sigma_\xi/\mu|^2 \ll 1 \ , \tag{83}$$

are obtained, the second of which results from the presence of secular terms and clearly implies, through the corresponding identity $\xi(x) \equiv \mu_r(x) - \mu$ from Eq. (38), the weakness of density fluctuations. This explicitly reveals the incompleteness of previous multiple scattering theories for scalar wave propagation described by Eq. (2).

To lift the weak density fluctuation restriction, one has to eliminate the secular terms through a suitable choice of S. This is made by equating to zero the off-integral terms in the expressions (63), (65), or (71) which refer to the conventional and nonlinear bilocal approximations, respectively. In the first case, we arrange for the elimination of the secular term via equation

$$S - (2/3\mu) = 0 \ . \tag{84}$$

It gives the value $S = 2/3\mu$, so, through the second half of Eqs. (38) and (50),

$$\xi(x) = 3\mu \frac{\mu - \mu_r(x)}{\mu + 2\mu_r(x)} \ , \tag{85}$$

$$\left\langle \frac{1}{\mu + 2\mu_r(x)} \right\rangle = \frac{1}{3\mu} \ , \tag{86}$$

and, in view of Eqs. (76) and (78),

$$m_0 = \mu_{e0} = \mu \frac{3\mu + \rho_0}{3\mu - 2\rho_0} \ . \tag{87}$$

The relevant version of Eq. (63) turns out to be

$$3\rho_0 = -4\pi\omega^2 \beta \int_0^{+\infty} k^2 B_\xi(k) g(k) dk \ , \tag{88}$$

or in the case of a lossless medium,

$$3\rho_0 \cong \omega^2 \beta \int_0^{+\infty} R B_\xi(R) dR + i\omega^2 k_0 \int_0^{+\infty} R^2 B_\xi(R) dR \ . \tag{89}$$

When the nonlinear bilocal approximation is considered, an equation which expresses the declared choice of S takes the form

$$\rho_0 S^2 - S \left[1 + \frac{\rho_0}{\mu} \right] + \frac{2}{\mu} = 0 \ . \tag{90}$$

The resulting formula for S which is consistent with Eq. (84) in the limit $\rho_0 \rightarrow 0$, proves to be

$$S=\frac{1}{2\rho_0}\left\{1+\frac{\rho_0}{\mu}-\left[1+\frac{\rho_0}{\mu}\left[\frac{\rho_0}{\mu}-\frac{2}{3}\right]\right]^{1/2}\right\} \quad (91)$$

$$\cong\frac{2}{3\mu}\left[1-\frac{5\rho_0}{12\mu}\right]+\mathcal{O}(\rho_0^2) , \quad (92)$$

where $0\leq\arg(\cdots)^{1/2}<\pi$. Taking note of Eq. (90) in Eq. (71), there follows the representation for ρ_0 that we are seeking:

$$\rho_0=-\frac{4\pi\omega^2(\alpha_0+\beta)}{3(1-\rho_0 S)}\int_0^{+\infty}k^2B_\xi(k)g_e(k)dk . \quad (93)$$

Equations (93) and (70), joined by an equation that follows from the second half of Eq. (50), after inserting expressions (91) or (92), constitute a system of three nonlinear equations for the unknowns α_0, ρ_0, and μ. It pays to note that, apart from the explicit nonlinearity in α_0,ρ_0 manifested in Eqs. (70), (93), or (73), the nonlinearity enters this problem implicitly through the dependence of $B_\xi(k)$ on ρ_0,μ, as is evidenced by expression (38) for $\xi(x)$ and Eqs. (91) and (93) for the renormalization constant. Numerical solution to this problem can easily be generated with the help of Newton's iteration method.

The criterion (80) acquires, in the present circumstances, the form

$$(k_0L_\xi)^2|\sigma_\xi/\mu|^2\ll 1 , \quad (94)$$

which clearly indicates the applicability of our results to the case of strong ($|\sigma_\xi/\mu|^2\gg 1$) but small-scale $[(k_0L_\xi)^2\ll 1]$ fluctuations of density.

V. CONCLUSIONS

In this paper, we have calculated the long-wavelength effective parameters b_0, $\mathbf{c}_0$, $\mathbf{l}_0$, m_0, and β_{e0}, μ_{e0} of a linear statistically homogeneous fluid with density and compressibility fluctuations characterized by spherically symmetric correlation function $B_\xi(R)$ and shift-invariant correlation functions $B_\beta(x-x')$, $B_{\beta\xi}(x-x')$. Since all the previous work in this field has been done for weak density fluctuations and, possibly, strong compressibility fluctuations, the main purpose here was to extend the mean-field theory by taking account of the strong fluctuations of both density and compressibility.

The analysis was grounded on the renormalization method (method of changing the field variable). We introduced a set of the effective perturbation operators, established their relationship with the effective constitutive operators, and calculated the former for long wavelengths within the framework of the conventional bilocal approximation, as well as the nonlinear bilocal approximation. By choosing appropriate values of renormalization constant, we explicitly demonstrated the inadequacy of previous multiple-scattering theories and showed the validity of our results in the case of strong density fluctuations.

The knowledge of the effective random-medium parameters at long wavelengths enables the solution of the mean-field problems, fully analogous with the acoustic problems in homogeneous deterministic media. For instance, the excitation of the mean acoustic field can be described in terms of impressed sources as

$$\langle p_r(x)\rangle=G_m s(x)+\mathbf{G}_m\cdot\mathbf{f}(x) , \quad (95)$$

$$\langle \mathbf{v}_r(x)\rangle=\mathbf{G}_m s(x)+\hat{G}_m\cdot\mathbf{f}(x) , \quad (96)$$

where the mean-field Green's operators, signified by the appropriate subscript, are obtained from their counterparts referring to the background medium via replacement of μ with μ_{e0} in expression (26) for G, and k_0 with the effective wave number $k_e=\omega(\mu_{e0}\beta_{e0})^{1/2}$ in Eq. (27) for $g(x-x')$ $(0\leq\arg k_e<\pi)$.

The theory of the present paper can be improved, of course, in several important respects. We shall list only two of them. First, it can be extended to cover the case of statistically anisotropic fluctuations. This refers mainly to density fluctuations, since the fluctuations of compressibility, in the present context, may be anisotropic. Such an extension can be achieved through the proper choice of a background medium and the related Green's operators. Second, we have deliberately excluded the discussion of an opportunity to select the renormalization constant in accordance with the requirement that $\rho_0^{\rm app}\equiv\rho_0^{\rm app}(S)$ as given by Eqs. (63), (65), or Eq. (71) be zero value. This recipe can be expressed, in the case of bilocal approximation and its nonlinear extension, by corresponding equations which result from Eqs. (63) and (71):

$$\sigma_\xi^2\left[S-\frac{2}{3\mu}\right]-(4\pi/3)\omega^2\beta\int_0^{+\infty}k^2B_\xi(k)g(k)dk=0 , \quad (97)$$

$$\sigma_\xi^2\left[S-\frac{2}{3\mu}\right]-(4\pi/3)\omega^2(\alpha_0+\beta)\int_0^{+\infty}k^2B_\xi(k)g_e(k)dk=0 . \quad (98)$$

The respective representations for α_0 are given by Eq. (56) and the formula

$$\alpha_0=-\omega^2\mu\int d^3kB_\beta(k)g_e(k) , \quad (99)$$

which stems from Eq. (70). Here

$$g_e(k)=1/[\omega^2\mu(\alpha_0+\beta)-k^2] . \quad (100)$$

Once $\rho_0^{\rm app}$ is a null value, there follows from Eqs. (76) and (78) the consequence that the effective parameters m_0,μ_{e0} coincide with the density of the background medium μ, and the second half of Eq. (50) acquires the status of an equation for the effective random-medium density. The emerging expressions for α_0 are compatible with condition (79) if, as before, the inequality (81) is fulfilled. As to Eq. (80), the left-hand side is, in view of the equalities $m_0=\mu$, $\rho_0^{\rm app}=0$, unconditionally equal to zero. This prompts the conclusion that the proposed choice of S enables the escape from any restriction on the strength of density fluctuations. However, a more careful analysis reveals that this is not the case. Concentrating, for simplicity, on the version of the proposed scheme which invokes the bilocal approximation for ρ_0, Eq. (97) makes only the first term $\rho_0^{(2)}(S) \equiv \rho_0^{\text{app}}(S)$ vanish in the full asymptotic expansion of $\rho_0(S)$ in powers of Π,

$$\rho_0(S) = \sum_{n=2}^{+\infty} \rho_0^{(n)}(S) , \qquad (101)$$

where $\rho_0^{(n)}(S)$ has the order of Π^n [recall Eq. (46)]. Retaining in Eq. (101) the first nonvanishing term $\rho_0^{(q)}(S)$ leads, through expression (49) for $\hat{m}$, to an improved estimate,

$$m_0 - \mu \cong \frac{\rho_0^{(q)}(S)}{1 - S\rho_0^{(q)}(S)} \sim \rho_0^{(q)}(S) . \qquad (102)$$

Taking note of this estimate in Eq. (80) furnishes a condition

$$|\rho_0^{(q)}(S)/\mu| \ll 1 , \qquad (103)$$

which determines the validity range of the proposed scheme. It would be useful to investigate the applicability of said scheme or that of a related scheme which removes the prescribed number of terms in the perturbation series (101) through an appropriately chosen S for the case of strong density fluctuations.

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