Variational approach to superconductive networks

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An approach to superconductive micronetworks is presented that makes use of the currents in the loops and the order parameter along branches as fundamental variables. Fluxoid quantization is introduced as a constraint and inductive effects are explicitly taken into account. The theory is made the starting point of a variational formulation which can use any physically sound guess for the order parameter as a trial function to minimize the free energy. For second-order transitions the zeroth-order approximation of de Gennes and Alexander can be used as a trial function. The formalism allows for the amplitude of the order parameter to be determined as a function of temperature and field. A different choice of ansatz allows the theory to describe transitions taking place when external currents are fed. In this paper we apply the new method to some systems, including a superconducting interferometer without Josephson junctions. The results compare quite well with experiments as well as with exact numerical calculations, giving a fair description of these systems.

I. INTRODUCTION

The field of superconductive micronetworks[1] has evolved steadily in the last ten years and the development of preparation techniques has allowed for different geometries to be considered. Simple loop structures have been studied as model systems both theoretically and experimentally.[1–4] Sierpinsky gasket and other networks have received attention.[5] The experimentally determined phase transition lines have been shown to be in agreement with calculations based on the theory put forward by de Gennes[1] and Alexander.[1]

The characteristics of the superconducting state in these systems is less accurately known, partly due to the lack of a complete solution to the coupled Ginzburg-Landau (GL) equations. Since this is not possible to obtain in general, one must resort to numerical calculations[6] or to approximation schemes. First- and second-order perturbation theories have been developed in Ref. 7. Wang *et al.*[8] have devised an approximation which gives the same results as a first-order perturbation without inductive effects.

Experiments analyzing the vortex distribution in a square lattice have been reported,[9] and other experiments will surely come up soon, presenting details of the superconductive state of micronetworks. For this reason, a more complete description is needed which would give a good qualitative understanding of the superconductive state for the wide range of fields and temperatures where the GL theory applies. To this end we devised an approach to micronetworks, applicable to two- or three-dimensional systems, which uses the modulus of the order parameter and the currents in the different loops as independent variables, linked via the fluxoid quantization condition.

This approach, when applied to the GL free energy, gives an alternative way to look at the same problem. It has the advantage of bringing to light from the beginning the physical variables relevant to the network geometry.

As we shall see below, the formulation is very adequate as a starting point for a variational calculation assuming a given form for the trial order parameter. Variational approximations are known to provide manageable and accurate results in many physical situations. In superconductivity the problem of the surface sheath and Abrikosov's solution for the mixed state have been dealt with by this method. It can be applied whenever complete solutions of the Ginzburg-Landau (GL) equations lead to complex numerical calculations. In this paper we apply the method to one- and two-loop systems. As a particular example of the latter we consider a quantum interferometer *without Josephson junctions*, a system that attracted attention when Moshchalkov *et al.*[10] reported very interesting experimental results which confirmed the theoretical calculations by Fink, Grünfeld, and López.[11] The variational approach describes quite well the superconducting state of these systems and also transitions taking place in the presence of an externally fed current, as in the interferometer.

II. MODULUS AND CURRENT FORMULATION FOR NETWORKS

We begin our formulation by writing the GL free energy difference between superconducting and normal states, using the same normalization as in Ref. 7,

$$\Delta G = \frac{1}{\sigma}\int_v \left[-|\Delta|^2 + \frac{|\Delta|^4}{2} + \xi^2 \left| \left(i\vec{\nabla} - \frac{\vec{A}}{\xi} \right) \Delta \right|^2 \right] dv + \frac{1}{\sigma}\int |\vec{B} - \vec{H}|^2 \, dv$$
$$= \Delta G_S + \Delta G_M,$$

where Δ is the normalized order parameter, $\Delta = \psi/\psi_\infty = |\Delta| e^{i\theta}$; $\vec{A}=\vec{A}_{\rm conv}/\sqrt{2}\lambda H_c$, where "conv" means the ordinary cgs quantity, and λ and H_c are the penetration depth and the thermodynamic critical field of the bulk material. $\vec{B}$ is the normalized total magnetic field, $\vec{B}=\vec{B}_{\rm conv}/\sqrt{2}H_c$, and similarly for the external applied field $\vec{H}$; all geometrical quantities (length, areas, and volumes) are normalized by the temperature-dependent coherence length at the second-order phase transition boundary $\xi(T_0)$ as determined by the de Gennes–Alexander theory;[1–3] σ is the cross section of the wires. The first integral ΔG_S extends over the network material, and the second one ΔG_M over all space.

We consider the network divided into μ elementary loops, which can be adjacent or intertwined. The number of independent loops can be seen to be $\mu = r - n + 1$, where r is the number of branches and n the number of nodes; each loop m carries a current σj_m. The division of the network into loops is arbitrary. This stems from the fact that the physically important quantities are the net currents in the branches, which can be expressed as a sum of arbitrarily chosen loop currents.[12] The magnetic energy term can be written as

$$\Delta G_M = \frac{1}{\sigma}\int |\vec{B}-\vec{H}|^2\,dv = \frac{\sigma}{2\lambda^2}\left[\sum_m \Lambda_m j_m^2 + \sum_{m\neq m'} M_{mm'} j_m j_{m'}\right], \quad (1)$$

where $M_{mm'}$ are the mutual induction coefficients, $\Lambda_m = M_{mm}$ are the self-induction coefficients, and j_m are the currents in each loop. This can be simplified further introducing a $\mu \times \mu$ matrix $\mathcal{M}$, such that we can write

$$\frac{1}{\sigma}\int |\vec{B}-\vec{H}|^2\,dv = \frac{\sigma}{2\lambda^2}\,(\tilde{\mathcal{J}},\mathcal{M}\,\tilde{\mathcal{J}}).$$

Here $\tilde{\mathcal{J}}$ is a row vector with μ components. Since, as we said, the division of the network into loops is arbitrary, there is not a unique way of writing *each term* in Eq. (1). The total sum, however, being identical to the total magnetic energy, is uniquely determined.[12,13] The induction coefficients needed are those associated with the particular set of loops chosen. The loop currents are associated with the selected division of the network into loops.[14] The current in branch b is j_b which is obtained from the loop currents through $j_b = j_m - j_{m'}$, m and m' being contiguous loops. The minimization procedure will determine the loop currents corresponding to a given physical situation; once these currents are known, the total current in each branch is fixed. The remaining part of the free energy can be written in terms of the modulus of the order parameter $|\Delta|$ and of the auxiliary quantity

$$\vec{s} = -\xi|\Delta|^2\left(\vec{\nabla}\theta + \frac{\vec{A}}{\xi}\right).$$

It will be shown below that these $\vec{s}$ are equal to the real physical currents $\vec{j}$ if we impose the fluxoid quantization condition. The first term in the free energy, ΔG_S, becomes

$$\Delta G_S = \sum_b \int_0^{L_b}\left[-|\Delta|^2 + \frac{|\Delta|^4}{2} + \xi^2\left(\frac{d|\Delta|}{dl}\right)^2 + \frac{s^2}{|\Delta|^2}\right]dl.$$

The total free energy difference is the sum $\Delta G = \Delta G_S + \Delta G_M$. In our formulation we look for an extremum of ΔG subject to the μ constraints given by the fluxoid quantization conditions on the loops

$$F_m = \oint_m \frac{s}{|\Delta|^2}dl + \left(\frac{\sigma}{2\lambda^2}\right)\sum_{m'} M_{mm'} j_{m'} - \xi(2\pi n_m - \phi_m) = 0.$$

We can define a row vector $\tilde{\mathcal{F}}$ with the μ components F_m which are well defined on each loop; the constrains imply that we have to minimize the "extended" GL free energy,

$$\mathcal{L} = \Delta G + (\tilde{\beta},\tilde{\mathcal{F}}),$$

where $\tilde{\beta}$ is a row vector representing μ Lagrange multipliers, one for each loop. From the corrresponding Euler-Lagrange equations it turns out that $\vec{s}=\vec{j}$ which allows us to identify the quantities $\vec{s}$ as equal to the physical currents $\vec{j}$ on the branches, uniform on each branch. The remaining Euler-Lagrange equations can now, after some algebra which we omit for the sake of clarity,[14] be written, for the loops μ,

$$\tilde{\beta} = -2\,\tilde{\mathcal{J}}, \quad (2)$$

$$\left(\mathcal{T} + \frac{\sigma}{2\lambda^2}\mathcal{M}\right)\tilde{\mathcal{J}} - \xi\,\tilde{\phi} = 0, \quad (3)$$

for the branches b,

$$\xi^2\frac{d^2|\Delta|}{dl^2} + \left[1 - |\Delta|^2 - \frac{j_b^2}{|\Delta|^4}\right]|\Delta| = 0, \quad (4)$$

and for the nodes n,

$$\sum_b\left[\frac{d|\Delta|}{dl}\right] = 0, \quad (5)$$

where the summation is over all branches b joining at node n. $\mathcal{T}$ is a $\mu \times \mu$ matrix with the elements

$$T_{mm} = \oint_m \frac{dl}{|\Delta|^2}, \qquad T_{mm'} = -C_{mm'}\int_0^{L_{mm'}}\frac{dl}{|\Delta|^2},$$

$C_{mm'}$ being the connectivity matrix for the dual lattice;[14] its elements are 1 or 0 according to whether two given nodes are connected or not. We note that for the three-dimensional lattices the dual lattice can also be defined

if the original one is planar (can be deformed to two dimensions without intersection of the branches).[15] Finally $L_{mm'}$ is the length of the branch common to loops m and m', and the μ components of the row vector $\tilde{\phi}$ are the quantities $2\pi n_m - \phi_m$ for each loop, n_m being an integer and ϕ_m the applied flux.

These equations give us the following information: Equation (2) shows that the Lagrange multipliers are proportional to the currents associated to the assumed division of the network into loops; Eq. (3) tells us that the physical solution satisfies the fluxoid quantization condition in each loop; Eq. (4) implies that the modulus of the order parameter satisfies the nonlinear GL equation.[11] Finally Eq. (5) means that the slopes of $|\Delta|$ at the nodes satisfy a boundary condition which is already well known.[1-8]

Equations (2), (3), (4), and (5) above show that this formulation of the GL theory for micronetworks partially satisfies the need for a discrete system of equations, one for each loop. The coefficients in these equations depend, however, on the solutions of (4) with the conditions (5). As will be shown below this formulation is a very adequate starting point for a variational approximation.

III. APPROXIMATE VARIATIONAL METHODS

As is known, de Gennes, Alexander, and others[1-4] have studied the second-order phase transition boundary using the linearized form of the GL equations. The results thus obtained follow at once from the present approach after letting $\Delta \to 0$, $j \to 0$, $j/|\Delta|^2$ finite, $\tilde{\phi} \to \tilde{\phi}_0$, $\xi \to 1$.

In this approximation, the modulus of the order parameter in a branch carrying a current j_0 can be obtained from the expressions given in Ref. 1:

$$|\Delta_0(l)|^2 = C\left[1 + \left(1 - \frac{j_0^2}{C^2}\right)^{1/2} \sin(2l + \delta)\right], \tag{6}$$

where C and δ are determined by the order parameter at the nodes.

This zero-order expression for the order parameter is strictly valid only at the second-order phase transition boundary, where its amplitude is vanishingly small. It can be chosen as a trial function for a variational approximation, writing it in the form $|\Delta(l)| = \alpha\,|\Delta_0(l)|$ where α is an undetermined amplitude to be fixed by the condition of minimum $\mathcal{L}$. This means that we "guessed" an approximate order parameter whose spatial variation is that of the zero-order approximation and whose amplitude is to be determined by the condition of minimum free energy. This is similar to Abrikosov's approach to the mixed state for bulk materials, except that in such a case, because of the degeneracy of the zero-order solution, a linear combination is taken;[16] in that case the variational parameters are the coefficients of the linear combination.

Within a variational approach, other trial functions can be used. For a London type of approach, for example, $|\Delta(l)|$ could be taken to be uniform throughout the network.

To apply the variational calculation we must find an extremum of $\mathcal{L}$ for the assumed form of the trial order parameter; to this end we must rewrite ΔG using the given form for $\Delta(l)$. If we use expression (6), we obtain

$$\Delta G = \alpha^2 X^{(2)} + \alpha^4 X^{(4)} + (1/\alpha^2)(\tilde{\mathcal{J}}, \mathcal{T}_0\,\tilde{\mathcal{J}}) + \frac{\sigma}{2\lambda^2}(\tilde{\mathcal{J}}, \mathcal{M}\,\tilde{\mathcal{J}}), \tag{7}$$

where

$$X^{(2)} = \xi^2 \sum_b \int_0^{L_b} \left[\frac{d\,|\Delta_0|}{dl}\right]^2 dl - \sum_b \int_0^{L_b} |\Delta_0|^2\, dl,$$

$$X^{(4)} = \frac{1}{2}\sum_b \int_0^{L_b} |\Delta_0|^4\, dl.$$

The constrain, Eq. (3), becomes $\tilde{\mathcal{F}} = \frac{1}{\alpha^2}\mathcal{T}_0\tilde{\mathcal{J}} + \frac{\sigma}{2\lambda^2}\mathcal{M}\,\tilde{\mathcal{J}} - \xi\,\tilde{\phi} = 0$. The quantities $X^{(2)}$ and $X^{(4)}$ as well as $\mathcal{T}_0$ are evaluated using the trial function $\Delta_0(l)$ at a given point on the phase transition boundary; $X^{(2)}$ carries an additional temperature dependence through ξ.

Carrying on the variation of $\mathcal{L} = \Delta G + (\tilde{\beta}, \tilde{\mathcal{F}})$ with respect to α^2, $\tilde{\mathcal{J}}$, and $\tilde{\beta}$ it follows that

$$X^{(2)}\alpha^4 + 2X^{(4)}\alpha^6 + (\tilde{\mathcal{J}}, \mathcal{T}_0\,\tilde{\mathcal{J}}) = 0,$$
$$\left(\frac{1}{\alpha^2}\mathcal{T}_0 + \frac{\sigma}{2\lambda^2}\mathcal{M}\right)\tilde{\mathcal{J}} = \xi\,\tilde{\phi}\,. \tag{8}$$

Substituting back into the free energy it can be shown that the approximate equilibrium value for ΔG is

$$\Delta G_{\text{eq}} = -\alpha^4 X^{(4)} + \frac{\sigma}{2\lambda^2}(\tilde{\mathcal{J}}, \mathcal{M}\,\tilde{\mathcal{J}}), \tag{9}$$

which clearly shows, within the variational approximation, the interplay between condensation energy and magnetic energy.

The unknowns in Eqs. (8) are α^2 and the μ loop currents $\tilde{\mathcal{J}}$. We can formally give an implicit relation containing only α^2,

$$\alpha^2 = \frac{-X^{(2)} - \xi^2\left[\mathcal{Z}_1(\mathcal{T}_0^{-1}\tilde{\phi}), \mathcal{Z}_2\,\tilde{\phi})\right]}{2X^{(4)}}, \tag{10}$$

where

$$\mathcal{Z}_1 = \left[1 + \frac{\sigma}{2\lambda^2}\alpha^2\mathcal{T}_0^{-1}\mathcal{M}\right]^{-1},$$

$$\mathcal{Z}_2 = \left[1 + \frac{\sigma}{2\lambda^2}\alpha^2\mathcal{M}\mathcal{T}_0^{-1}\right]^{-1}.$$

This is a self-consistent relation for α^2. It can be easily solved if one neglects inductive effects ($\mathcal{M} = 0$). For $\tilde{\mathcal{J}}$

we obtain $\tilde{\mathcal{J}} = \xi\alpha^2 \mathcal{Z}_1(\mathcal{T}_0^{-1}\tilde{\phi})$.

At the phase transition boundary $\alpha^2 = 0$. If we use this condition in (10), we obtain the relation

$$X_0^{(2)} + (\mathcal{T}_0^{-1}\tilde{\phi}_0, \tilde{\phi}_0) = 0. \tag{11}$$

This gives a relation between L_b, the length of the branches normalized by ξ_0, the temperature-dependent coherence length at the phase boundary, and the applied magnetic fluxes $\tilde{\phi}_0$. Equation (11) is an alternative way of determining the phase transition boundary. Looking at Eq. (7) for ΔG, we see that this condition is equivalent to that obtained in the linearized theory of de Gennes[1] and Alexander.[1] Here it appears as a condition for the vanishing of the free energy difference, to lowest order in the order parameter, a condition also satisfied by the lowest-order solutions.

The present variational approach can be simply related to the perturbative approach previously derived.[7] To do this we expand the matrices $\mathcal{Z}_1$ and $\mathcal{Z}_2$ in powers of α, and insert back in Eq. (10) obtaining to lowest order

$$\alpha^2 = \frac{-X^{(2)} - \xi^2(\mathcal{T}_0^{-1}\tilde{\phi}, \tilde{\phi})}{2X^{(4)} - \frac{\sigma}{\kappa^2}(\mathcal{T}_0^{-1}\tilde{\phi}, \mathcal{M}\mathcal{T}_0^{-1}\tilde{\phi})}, \tag{12}$$

where κ is the GL constant. For the currents we obtain

$$\tilde{\mathcal{J}} = \xi\alpha^2 \mathcal{T}_0^{-1}\tilde{\phi}\,. \tag{13}$$

Near the phase transition boundary we can expand $\tilde{\phi} = \tilde{\phi}_0 - \tilde{\mathcal{R}}\,\delta\phi$, where $\tilde{\mathcal{R}}$ is the row vector containing the ratios of each loop area to a reference loop used to measure the fluxes[7] and $\delta\phi$ is the departure of the applied flux from the phase boundary. We also expand the temperature-dependent coherence length: $\xi = 1 - \frac{1}{2}\delta\tau/\tau_0$. Inserting these expansions in Eq. (10) it follows that

$$\alpha^2 = \frac{\left[\sum_b \int_0^{Lb} |\,\Delta_0\,|^2\, d\ell\right]\delta\tau/\tau_0 + 2(\tilde{\mathcal{J}}_0, \tilde{\mathcal{R}})\delta\phi}{2X^{(4)} - \sigma/\kappa^2(\tilde{\mathcal{J}}_0, \mathcal{M}\,\tilde{\mathcal{J}}_0)}$$

and the currents become

$$\tilde{\mathcal{J}} = \alpha^2\,\tilde{\mathcal{J}}_0,$$

where $\tilde{\mathcal{J}}_0 = \mathcal{T}_0^{-1}\tilde{\phi}_0$.

These relations coincide with the first-order perturbation results obtained in Ref. 7. To this order, the free energy becomes

$$\Delta G_{\rm eq} = -\alpha^4\left[X^{(4)} - \frac{\sigma}{2\kappa^2}(\tilde{J}_0, \mathcal{M}\,\tilde{J}_0)\right].$$

As pointed out in Ref. 7, to this order one obtains the following simple expressions for the specific heat jump, the thermal magnetization coefficient, and the magnetic susceptance of the lattice (which are the second derivatives of the free energy with respect to temperature and field) in cgs units:

$$\Delta C_H = \frac{T_0\Omega^2}{4\pi\beta'(2\,\tilde{\kappa}^2 - 1)},$$

$$\Delta\varepsilon = \frac{\Omega}{4\pi\beta''(2\,\tilde{\kappa}^2 - 1)},$$

$$\Delta\eta = \frac{1}{4\pi\beta'''(2\,\tilde{\kappa}^2 - 1)},$$

where $\Omega = \Phi_0/2\pi[\xi(0)]^2 Tc$, $\beta', \beta'', \beta'''$, and $\tilde{\kappa}$ are defined in the previously quoted Ref. 7, Φ_0 is the flux quantum, and T_c the bulk transition temperature. The parameters β are related to the topology and the geometry of the lattice through the ratio of the magnetic energy of the induced currents to the other forms of energy. $\tilde{\kappa}$ is κ multiplied by the ratio (condensation energy/magnetic energy). This renormalization of κ through specific factors associated with the topology is one of the most important results in Ref. 7, and the theory shows that a lattice behaves like a bulk material with renormalized constants. From this it is possible to calculate an "effective London penetration depth."

IV. APPLICATION TO SIMPLE SYSTEMS

In order to test the variational approach, we have applied it to some systems, in which comparison with exact results and with the perturbation theory of Ref. 7 is possible. We report in this section the results for a bare ring and for the symmetric two-loop system depicted in the inset of Fig. 1, where the central branch carries no current. In the next section we shall deal with an interferometer *without Josephson junctions*, a system for which some recent experimental results are available.

For the ring and for the two-loop system, the squared

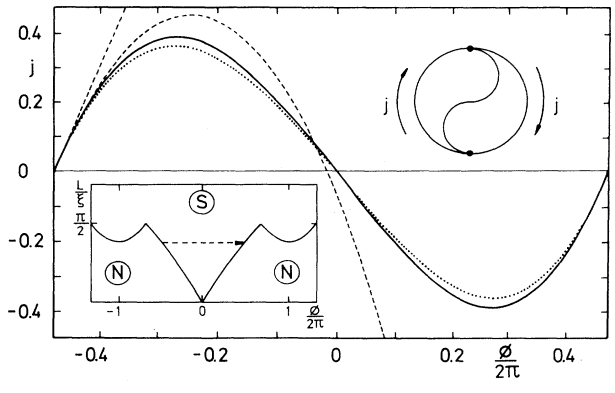

FIG. 1. Comparison of exact and variational results for the two-loop system depicted in the inset at the top right. The path in the phase diagram is shown in the inset at the bottom left. Exact results for the current in the ring as a function of flux correspond to the solid line; the dotted line shows the variational results. The dashed lines correspond to first- and second-order perturbation theory, as in Ref. 7.

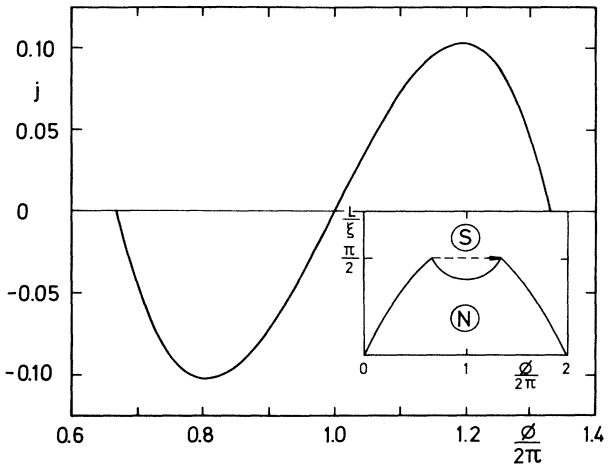

FIG. 2. Shown is the current in the ring as a function of flux, for the same system as in Fig. 1. The path in the phase diagram is shown in the inset. Here the agreement of exact and variational results is so close that the curves are indistinguishable.

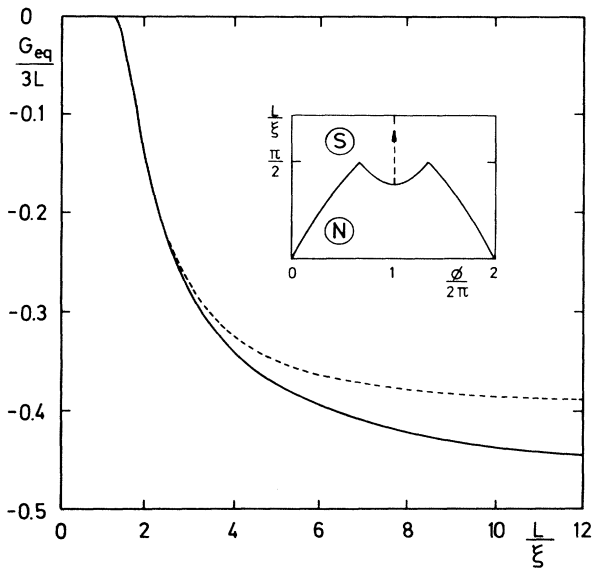

FIG. 3. Equilibrium free energy as a function of temperature, along the path indicated in the inset. The variational results (dotted line) fit the exact ones (solid line) up to $L/\xi \sim 3$; from there on there is a departure of at most 15.

amplitude α^2 is obtained from a cubic equation $(\alpha^2 + U)^2(\alpha^2 - V) + W^2 = 0$, where[17]

$$U = \frac{2\lambda^2}{\sigma\Lambda}T_0, \quad V = -\frac{X^{(2)}}{2X^{(4)}},$$

$$W = \frac{2\lambda^2}{\sigma\Lambda}\xi \mid 2\pi n - \phi \mid \left[\frac{T_0}{2X^{(4)}}\right]^{1/2}.$$

Here $T_0 = \oint \mid \Delta_0 \mid^{-2} dl$, and Λ is the self-induction coefficient of the ring. The physical quantities of interest are the modulus of the order parameter, the current in the circumferential branches, and the equilibrium free energy:

$$\mid \Delta(l) \mid = \alpha \mid \Delta_0(l) \mid , \quad j = \pm\alpha^2\sqrt{V - \alpha^2}\left[\frac{2X^{(4)}}{T_0}\right]^{1/2}, \tag{14}$$

$$\Delta G_{eq} = -\alpha^4\left[1 - 2\left(\frac{V - \alpha^2}{U}\right)\right]X^{(4)}.$$

For the bare ring, a London-type order parameter $|\Delta(l)| = \alpha$ gives the exact solution, previously obtained by Fink and Grünfeld[17] by solving the GL equations.

For the symmetric two-loop system we resort, as an approximation, to the de Gennes–Alexander form (6), and use Eq. (12) to determine the amplitude. Figure 1 shows the current in the external loop as a function of flux as one moves through a portion of the phase diagram accross the superconducting region. The graph shows how close the variational approximation lies to the exact result numerically obtained from a Runge-Kutta calculation. For comparison the results of perturbation theory are also shown. It should be noted that the variational calculation for this case uses the values of $\Delta_0(l)$ at one of the points in the phase boundary where the path into the superconducting region starts. Figure 2 shows another comparison of the approximation methods to exact results. Here the degree of agreement of the variational result is so close that the curves are indistinguishable.

If one explores the phase diagram following the path indicated in Fig. 3, one finds the current to be zero. The free energy difference ΔG in this case becomes simply $\Delta G_{eq} = -\alpha^4 X^{(4)}$ where α^2 is determined by the cu-

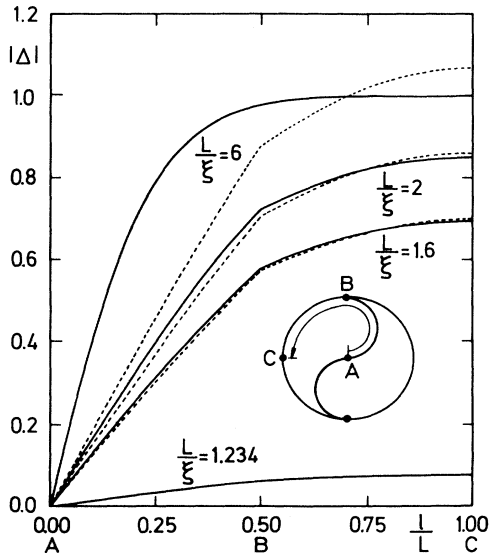

FIG. 4. Modulus of the order parameter for the odd solution ($n = 1$) in the two-loop system, shown as a function of position, starting at point A and ending at point C. In this drawing ϕ and L/ξ take four different values. The agreement of exact (solid lines) and variational (dotted lines) results is quite good up to $L/\xi \sim 3$; for $L/\xi = 1.234$ both results are indistinguishable.

bic, with $W = 0$. The result for ΔG_{eq} as a function of temperature is shown in Fig. 3. In Fig. 4 we show the modulus of the order parameter for the odd solution ($n = 1$) in the two-loop system, beginning at the central point of the network and ending half-way on an external branch.

V. APPLICATION TO AN INTERFEROMETER

As an example of a different kind, we will discuss in this section a quantum interferometer *without Josephson junctions*. This system has been studied theoretically by Fink *et al.*[11] and experimentally by Moshchalkov *et al.*[10] The conclusion of these works is that a simple loop with two branches that allow an *external current* to be fed behaves similarly to a superconducting quantum interference device (SQUID) with two Josephson junctions. In order to treat this system within the present formalism, we take a network composed of two loops like that of Fig. 5. The small loop is the system of interest; the large loop is a device to feed the "external" current into the small one. In the present case and according to the experimental results, we expect first-order transitions to the normal state for most of the range of currents; we shall not use as a reference the zero-order solutions of de Gennes and Alexander. For this reason we shall normalize lengths with the coherence length at the temperature of interest $\xi(\tau)$, and not the coherence length at the second-order phase boundary as above. With this convention, our previous formulas have to be modified replacing ξ by 1 and λ by κ. The free energy for the whole circuit is given by

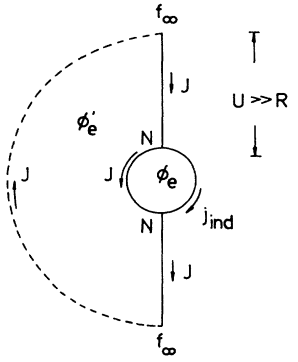

FIG. 5. Micronetwork corresponding to the quantum interferometer *without Josephson junctions*. Primed quantities correspond to the large loop, which will eventually be made to close itself at infinity to obtain a small loop with an "external" current fed into it.

$$\Delta G = \int_0^L \left[-\Delta_1^2 - \Delta_2^2 + \frac{1}{2}\Delta_1^4 + \frac{1}{2}\Delta_2^4 + \left(\frac{d\Delta_1}{dl}\right)^2 + \left(\frac{d\Delta_2}{dl}\right)^2 \right] dl + 2\int_0^u \left[-\Delta_3^2 + \frac{1}{2}\Delta_3^4 + \left(\frac{d\Delta_3}{dl}\right)^2 \right] dl + \left(\tilde{\mathcal{J}}, \left[\mathcal{T} + \frac{\sigma}{2\kappa^2}\mathcal{M}\right]\tilde{\mathcal{J}}\right), \tag{15}$$

where

$$\tilde{\mathcal{J}} = \begin{bmatrix} j \\ J \end{bmatrix}$$

and

$$\mathcal{T} = \begin{bmatrix} \int_0^L \frac{dl}{\Delta_1^2} + \int_0^L \frac{dl}{\Delta_2^2} & -\int_0^L \frac{dl}{\Delta_1^2} \\ -\int_0^L \frac{dl}{\Delta_1^2} & \int_0^L \frac{dl}{\Delta_1^2} + 2\int_0^\mu \frac{dl}{\Delta_3^2} \end{bmatrix}.$$

Furthermore,

$$\mathcal{M} = \begin{bmatrix} \Lambda_0 & -M \\ -M & \Lambda \end{bmatrix},$$

where the self-induction for the small loop Λ_0, the large loop Λ, and the mutual coefficient are[18] $\Lambda_0 = \frac{L}{\pi}[\ln(\frac{4L}{\sqrt{\pi}\sigma}) - 7/4]$, $\Lambda = \frac{2u}{\pi}\ln(2u\sqrt{\frac{\pi}{\sigma}})$, and $M = \frac{\Lambda_0}{2}$. In these expressions we have taken the origin of coordinates at node N (see Fig. 5). Fluxoid quantization implies here two equations, which can be written in matrix form as

$$\left[\mathcal{T} + \frac{\sigma}{2\kappa^2}\mathcal{M}\right]\tilde{\mathcal{J}} = \tilde{\phi}, \tag{16}$$

with

$$\tilde{\phi} = \begin{bmatrix} 2\pi n - \phi_e \\ 2\pi n' - \phi_e' \end{bmatrix}.$$

The linear system of equations (16) allows us to obtain the currents in each loop $\tilde{J}$ in terms of the external fluxes and of the induction coefficients. In order to make a variational calculation we use simple functions which qualitatively describe the order parameter in each branch, and assume the length of the large loop u to be much bigger than the length of the small loop L. For this we choose

$$\Delta_b(l) = f\left[\alpha_b + \beta_b\left(l - \frac{L}{2}\right)^2\right] \quad (b = 1, 2). \tag{17}$$

For Δ_3 we choose two possible behaviors:

$$\Delta_3(l) = \begin{cases} f\ \tanh(l + \gamma), \\ f\ [2 - \tanh(l + \gamma)]. \end{cases} \tag{18}$$

These correspond to the order parameter in the small ring being in average smaller or larger than the order parameter in the large loop. The continuity equations for the order parameter and its derivative, Eq. (5), allow for the reduction of the number of variational parameters to three: α_1, α_2, and f. Using Eqs. (16), (17), and (18) we can express j, J, and ΔG. These func-

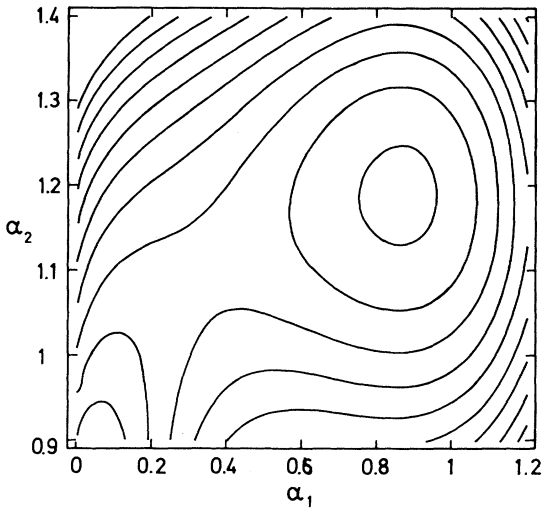

FIG. 6. Variational free energy plotted as a function of the variational parameters α_1 and α_2 for $L = 3.14$, $J = 0.3849$, and $\phi_e = 0.4$. This function normally has a minimum and a saddle point within the physically interesting range of parameters. The phase boundary of the interferometer is determined when these two features merge.

tions can be expanded in powers of u. For ΔG we obtain $\Delta G = 2u\Delta G_u(f) + \Delta G_L(\alpha_1, \alpha_2) + \vartheta(u^{-1})$ where $\Delta G_u(f)$ is the free energy per unit length for the wire in the large loop, $\Delta G_L(\alpha_1, \alpha_2)$ corresponds to the small loop and depends only on α_1 and α_2, and $\vartheta(u^{-1})$ represents terms of order $(1/u)$ or smaller which are negligible for $u \gg L$. Minimizing ΔG_u with respect to f it follows that $J = f^2\sqrt{1-f^2}$, which determines f for a given J. The values for α_1 and α_2 corresponding to the extrema of ΔG can be determined searching for minima of $G_L(\alpha_1, \alpha_2)$. Figure 6 shows the result of a contour

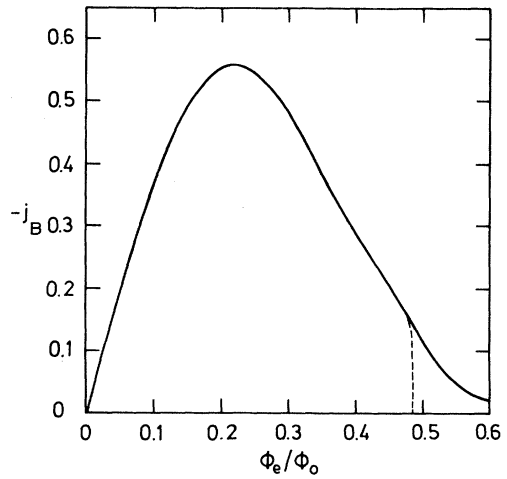

FIG. 7. Shown is $j_{\rm ind}$, the current induced by the external field, as a function of the applied flux in the interferometer, for $L = \pi/4$ and $J = 0.01$. The solid line corresponds to the parabolic approximation for the order parameter; the dotted line corresponds to the hyperbolas, which give a better description of the self-generated weak link. Compare with the corresponding curve in Ref. 11.

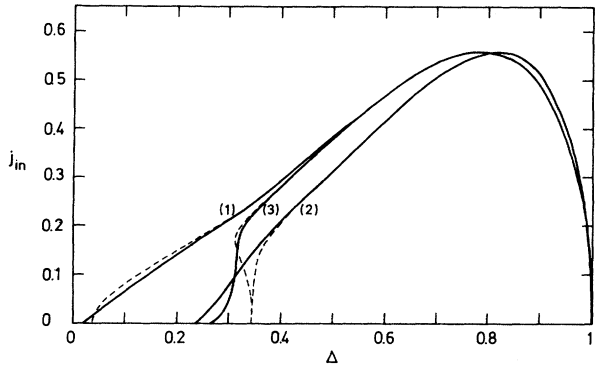

FIG. 8. The induced current in the interferometer as a function of the order parameter at the middle point in each branch, curves (1) and (2), and at the node, curve (3). The solid line corresponds to the parabolic approximation, the dotted line to the hyperbolic one. L and J take on the same values as in Fig. 7. Compare with Ref. 11.

plot of the free energy in a typical case; the transition to the normal state takes place when the minimum and the saddle point merge.

Since, as noted by Moshchalkov *et al.*,[10] the transition takes place by the effect of a self-generated weak link in one of the branches of the loop, we have also tried variational functions which more accurately describe this situation. What is needed is a function with an accute

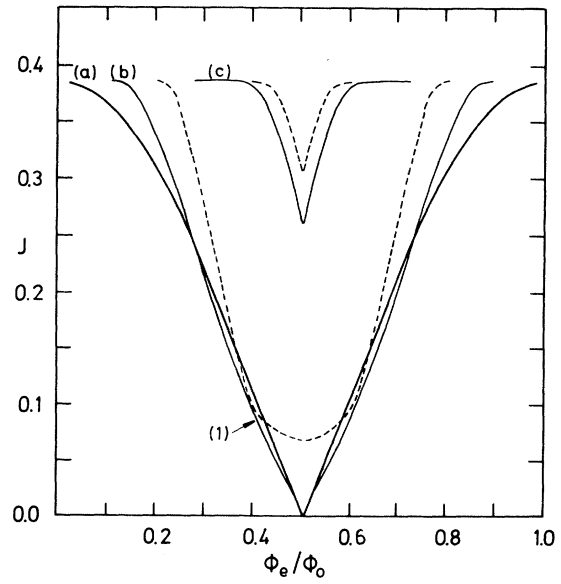

FIG. 9. Phase diagram for the interferometer in the J-ϕ plane. $J_c(\phi)$ is the boundary between normal and superconducting regions, for different values of $L = \pi R$. Shown are (a) phase boundary for the ordinary Josephson junction SQUID, (b) boundary for $R = 0.25$, and (c) boundary for $R = 1$. The heavy line is the exact numerical calculation of Ref. 11 and the dotted line corresponds to the variational calculation of this work. The hyperbolic approximation follows the dashed line up to the region marked (1) and from there on confounds itself with the exact calculation. Compare with the experimental results of Ref. 10.

minimum. It can be seen that a hyperbola of the form $\Delta_b(l) = f\{\alpha_b[\alpha_b+\beta_b(l-L/2)^2]\}^{1/2}$ satisfies this requirement. For small values of β_b, this function goes over into the parabolas previously used. Figure 7 shows the result for $j_{\rm ind} = j - J/2$ which is the current induced by the external field in the small loop, as a function of the applied flux ϕ_e for $n = 0$, for both parabolas and hyperbolas. In Fig. 8 we show the relation between $j_{\rm ind}$ and $\Delta_1(L/2) = f\alpha_1$, $\Delta_2(L/2) = f\alpha_2$, and $\Delta_3(0)$ also for both cases. These results should be compared with those of Ref. 11. Although both approximations give qualitatively good results, the hyperbolas are a better approximation near the transition point. In Fig. 9 we show the critical current through the system as a function of applied flux, for $R/\xi = 0.25$ and 1. The presence of the weak link in one of the branches can be seen through the fact that at the transition, one of the α's becomes much smaller than the other. This is observed at the left hand corner of Fig. 8, which corresponds to the transition region. It should be noted that in spite of the crude approximations made in the variational order parameter, the agreement with both experiment and exact calculations is quite acceptable.

VI. CONCLUSIONS AND PERSPECTIVES

We have developed a theory of superconductive networks that takes the structure of these systems explicitly into account. Based on this, we have shown that the approach is particularly suitable as a starting point for a variational formulation.

The approximate results fit very well with experiments as well as with exact numerical calculations. We see at least three reasons for this agreement: (a) The formulation takes into account all the terms of the GL free energy, including self-induction effects; (b) the network topology is explicity described by the connectivity matrix of the dual lattice; and (c) the most relevant phenomenon of multiple connected superconducting networks, i.e., fluxoid quantization, is explicitly introduced as a condition to be satisfied by the solutions in the complete range of field and temperature.

There are several advantages in the variational approach: (1) It gives rise to expressions which are analytically manageable up to a certain point and, if necessary, can be solved numerically by simple methods; (2) it allows for nonuniform external fields (the fluxes ϕ_m can be produced by arbitrary fields); (3) the analysis of metastable states and of stationary states with trapped flux can be easily done, by preselecting the number of fluxons in each loop n_m; (4) as shown in Sec. VI, it can be also used to describe transitions in the presence of an "external" current; (5) it can be extended to Josephson junctions networks, because the Hamiltonian of these systems can be written in terms of currents and modulus of the order parameter; and (6) the fact that the variational method requires evaluation of the free energy provides an—*albeit approximate*—explicit form for this functional, which can be used to study properties that depend on the shape of the free energy surface in parameter space, as for instance, the effect of fluctuations (see Fig. 6).

The results obtained above give us confidence that our method can be applied to more complex structures, provided reasonable approximations to the order parameter are made at the outset. Work in this direction is in progress and the results will be published soon.

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