Andreev reflections and resonance tunneling in Josephson junctions

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The current-voltage charcteristics (I-V) and Josephson critical current for the tunneling junctions with the resonant levels in the weak link are calculated. It seems likely that the current in these structures occurred due to resonant tunneling through a localized state. This resonant state effects the formation of long-range proximity effect. The subgap structure that appears on I-V characteristic of such junctions is caused by multiple Andreev reflections of quasiparticles at the interfaces. The I-V dependence is calculated for different values of the resonance parameter. This parameter is proportional to the ratio of the escape of the quasiparticle from the localized state into the superconducting electrodes. It is then shown that the larger subgap current corresponds to a smaller value of the resonant parameter.

I. INTRODUCTION

Experimental studies of tunneling processes in the normal-metal–insulator–normal-metal (NIN) junctions with amorphous αSi as a tunnel barrier[1–3] clearly reveal the existence of localized states through which the tunneling of electrons occurs. At the several nm thickness of αSi the resonance tunneling processes become important in these structures. The resonance tunneling can be also important in recently studied Josephson junctions[4] with Ge or InO film as a barrier. This interesting situation also occurs with the high-temperature oxide superconductors and their compounds. Cuprate barriers like $Bi_2Sr_2CuO_6$ (2201) placed along C-axes direction with 10 layers thickness and sandwiched between the low-temperature superconductors are the other systems where (as was suggested in Ref. 5) the resonant tunneling via localized states is important. Glazman and Matveev[6] and Larkin and Matveev[7] have calculated resonance conductance. They have analyzed the impact of one site Coulomb energy on simultaneous resonant tunneling of two electrons through a single site.

The resonance states increase the transparency of the barrier. In the case of the superconductor-insulator-superconductor (SIS) Josephson junction, this opens the way for the multiple Andreev reflections. Such multiple reflections are typical for superconductor–normal-metal–superconductor (SNS) junctions[8–12] as well as for the SIS point junctions when the barrier transparency increases.[13–15] These reflections lead to the formation of the subgap structure on I-V characteristics of Josephson junctions. Many aspects of resonance tunneling in Josephson SIS and S|Sm|S junctions were analyzed before.[16–19] Nevertheless the problem of combined Andreev and resonance scattering remains unresolved. This work aims to investigate the current-voltage (I-V) characteristic of Josephson junctions which have barriers with resonant scatters (such as αSi), in particular with regard to the possibility of observing the subgap current caused by multiple Andreev reflections. Here we consider the simple case of a single resonant state localized in the insulator (weak link). When resonance tunneling is the most pronounced effect, usual tunneling probability is small and can be considered a small parameter. The resonant state in the insulator of the SIS junction renders the perturbation theory inapplicable. Because all the tunneling processes should be retained, the whole problem appears highly nonlinear. Therefore, we have to find precisely the value of Green's function in the system. Due to the time dependence of the phase of the order parameter in the superconductors, the situation becomes even more complicated. For two limits—the normal NIN junctions and zero-biased Josephson junction—we achieved close analytical results. Generally, to resolve this nonlinear problem we applied the technique of nonequilibrium Green's functions (Keldish method) and calculated the I-V characteristics of Josephson junctions numerically for the various values of the resonant parameter.

This paper is organized in the following way. In Sec. II we provide the basic equations of our essentially three-dimensional problem for Green's functions. Here we obtain the exact barrier Green's function and find the resolvent operator of a quasiparticle which tunnels through a weak link resonant state. Keldish Green's function and its derivative at the interfaces are expressed in terms of the barrier functions, electrode Green's functions, and the resolvent Green's functions. In Sec. III we apply this formalism and Green's functions which were calculated in Sec. II to obtain the current for a general nonequilibrium case. We rewrite the I-V characteristic in the form convenient for numerical calculations. The fact that the energy difference in the matrix elements of the resolvent propagator is an integer multiple of 2 eV opens the way to construct a discrete basis in energy representation. This we then truncate according to the total number of Andreev reflections. Section IV represents the analytical results for NIN and zero-biased Josephson junctions. Here we also state the results of numerical calculations for a simplified model when the localized state is situated in the middle of the weak link. Moreover, calculations are provided only for the zero component of the current that reflects the whole I-V curves. The estimations for observing the subgap current structure for the αSi|SiO_x barriers are given. Section V, our conclusion,

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critically examines our results and notes the limitations of approximating the voltage distribution in the contact region.

II. BASIC EQUATIONS

For our purposes, the most adequate approach for studying the dynamic properties of the Josephson junction follows the method suggested Feuchtwang[20] and Arnold.[13,21] These authors derived the tunneling current through an insulating barrier beyond the lowest order in the transmission probability. The main impact of the theory[13,20,21] is an expression for the tunneling current which is given by the second derivative of the exact Green's function for a three layer system. This derivative is taken at the interfaces between insulator and superconductors. The equation for the exact Green's function of the three layer system in the particle-hole (Nambu) space assumes the form

$$\left[i\hbar\frac{\partial}{\partial t} - H_0(\mathbf{r}) \right] G(\mathbf{r}t,\mathbf{r}'t) - \int dt_1 \int d\mathbf{r}_1 \Sigma(\mathbf{r}t,\mathbf{r}_1 t_1) G(\mathbf{r}_1 t_1,\mathbf{r}'t') = \delta(\mathbf{r}-\mathbf{r}')\delta(t-t') , \quad (1)$$

where

$$H_0(\mathbf{r}) = \left[-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + \frac{\hat{p}_\parallel^2}{2m} - \mu \right] \tau_3 . \quad (2)$$

x is the coordinate normal to the plane of interfaces, $\hat{p}_\parallel^2 = -i(\partial^2/\partial y^2 + \partial^2/\partial z^2)$ is the Laplacian which acts in the superconductor-insulator (SI) boundary, and μ stands for the chemical potential. Furthermore, here and below τ_1,τ_2,τ_3 denote Pauli matrices and $\tau_\pm = (\tau_1 \pm i\tau_2)/2$.

Self-energy $\Sigma(\mathbf{r}t,\mathbf{r}'t')$ is approximated by a spatially local form

$$\Sigma(\mathbf{r}t,\mathbf{r}'t') = \delta(\mathbf{r}-\mathbf{r}')\Sigma(\mathbf{r}tt') . \quad (3)$$

The method which was used in Refs. 13, 20, and 21 relates the exact Green's function for a three layer system to the Green's functions of three isolated layers (we denote them by small character letter g). In each isolated layer, $g_i(\mathbf{r}t,\mathbf{r}'t')$ satisfies the same equation (1), but only in the restricted ith region with H_0, Σ replaced by H_{0i},Σ_i, where $i=0$ defines the left-hand superconductor, $i=1$ the barrier, and $i=2$ the right-hand region. As in Refs. 13, 20, and 21 we choose the boundary conditions such that the first derivatives of $g_i(\mathbf{r}t,\mathbf{r}'t')$ with respect to x,x' vanish at the interfaces. In the case of a single localized state in the weak link, the self-energy Σ is

$$\Sigma_i(\mathbf{r}tt') = \delta(t-t')[V_B(r) + V_{\text{res}}(r)]\tau_3 \text{ if } i=0 ,$$
$$= \delta(t-t')\hat{\Delta}(rt) = \Delta\tau_1 \exp[i\Phi(xt)\tau_3] \quad \text{for } i=1 \text{ and } 2 , \quad (4)$$

where the barrier potential $V_B(r) = V_B$ is taken from the rectangular form. $V_{\text{res}}(r)$ stands for the short-range resonance level potential situated in the insulating region, Δ is the constant module of the order parameter, and $\Phi(xt)$ represents its phase. Due to the low transparency of the barrier, in the weak link domain the Gor'kov amplitude Δ is small. Therefore we ignore the renormalization of the self-energy Σ and in each region keep it equal to the Σ_i. To achieve the expression for the exact Green's function we generalize the theory[13,20,21] on the three-dimensional case (due to the localized level in the weak link, the homogeneity in the z,y directions is lost, and the system cannot be considered as one dimensional). The equation which relates the exact Green's functions (retarded or advanced) to Green's functions of separate layers g_i assumes the form

$$G(\mathbf{r}t,\mathbf{r}'t') = g(\mathbf{r}t,\mathbf{r}'t') + \int dt_1 \int d\rho_1 \int dx_1 g(\mathbf{r}t,\mathbf{r}_1 t_1) H_T(\mathbf{r}_1) G(\mathbf{r}_1 t_1,\mathbf{r}'t') . \quad (5)$$

Here the explicit integration in the plane of the Josephson junction (in the plane of SI interface) ($\boldsymbol{\rho}$) and normal to it (x_1) is shown $\mathbf{r}_1 = (\boldsymbol{\rho},x_1)$. Also, the quasitunneling Hamiltonian H_T is introduced

$$H_T(\mathbf{r}_1) = -\frac{\hbar^2}{2m}\tau_3 \left\{ [-\delta(x_1-L^-) + \delta(x_1-L^+) - \delta(x_1-R^-) + \delta(x_1-R^+)], \left[\hat{x}_1 \frac{\partial}{\partial \mathbf{r}_1} \right] \right\} = -H_T^+(\mathbf{r}_1) . \quad (6)$$

$L=-d$ and $R=d$ ($2d$ is the thickness of the weak link) stand for left or right SI interfaces; superscripts $+(-)$ indicate an infinitesimal distance to the right (left) from corresponding superconductor-insulator boundaries, and curly brackets indicate an anticommutator. The quasitunneling Hamiltonian constitutes the essential point of the method. It absorbs the boundary conditions at the SI interfaces and is the time independent. To find the current, we use momentum and energy representations. For this it is convenient to introduce Dirac notations for the bra and cat vectors:

$$\langle \rho t | \rho' t' \rangle = \delta(\rho-\rho')\delta(t-t') ,$$
$$\langle \rho t | G(x,x') | \rho' t' \rangle = G(x\rho t, x'\rho' t') , \quad (7)$$
$$\langle \rho t | F(x,x_1) G(x_1,x') | \rho' t' \rangle = \int dt_1 \int d\rho_1 F(x\rho t, x_1 \rho_1 t_1) G(x_1 \rho_1 t_1, x'\rho' t') .$$

The current through the Josephson junction is defined by nonequilibrium Green's function $G<(x,x')$ which is directly proportional to the distribution function. Equivalently, we can express the current by the Keldish function $G^K(x,x')$. Similar to Eq. (5), these Green's functions may be written symbolically into two different forms as

$$G^{<}=g^{<}+g^{<}H_T G^a+g^r H_T G^{<} \tag{8}$$

or

$$G^{<}=g^{<}+G^{<}H_T^{+}g^a+G^r H_T^{+}g^{<} . \tag{8$'$}$$

Superscripts a and r stand for advanced and retarded Green's functions, respectively.

To resolve Eq. (8) or (8′) with respect to $G^{<}(x,x')$, two additional relations are useful:

$$\vec{g}^{\,-1}=\overleftarrow{G}^{-1}+H_T, \overleftarrow{g}^{\,-1}=\vec{G}^{-1}+H_T^{+} , \tag{9}$$

where arrows above operators indicate the direction in which they act.

Thus with the help of Eqs. (9) and (5) for $G^{<}$ or G^K we get

$$G^{<}=g^{<}+g^{<}H_T G^a+G^r H_T^{+}g^{<}+G^r H_T^{+}g^{<}H_T G^a . \tag{10}$$

The tunneling current in x direction yields

$$J(t)=\int d\rho\, \hat{x} j(\mathbf{r}) =\frac{e\hbar^2}{m}\int d\rho\, \hat{x}\left[\frac{\partial}{\partial \mathbf{r}}-\frac{\partial}{\partial \mathbf{r}'}\right] \times \mathrm{Sp}\left[\frac{1+\tau_3}{2}G^{<}(\mathbf{r}t,\mathbf{r}'t')\right]_{\mathbf{r}\to \mathbf{r}', t'=t+0} . \tag{11}$$

Using the momentum representation

$$\langle \rho|\mathbf{p}_\parallel\rangle=\frac{1}{2\pi}\exp(i\mathbf{p}_\parallel \rho) , \quad \langle p_\parallel|\mathbf{p}'_\parallel\rangle=\delta(p_\parallel-\mathbf{p}'_\parallel) ,$$

we evaluate the x,x' derivations of $G^{<}(x,x'')$ or $G^K(x,x'')$. After the differentiation at the left interface, the current reduces to the form

$$J(t)=\frac{e}{\hbar m}\sum_{\mathbf{p}_\parallel}\langle \mathbf{p}_\parallel t|\mathrm{Sp}\left[\frac{1+\tau_3}{2}(S_L^K g_L^a-g_L^r S_L^K+S_L^r g_L^K -g_L^K S_L^a)\right]|\mathbf{p}_\parallel t\rangle , \tag{12}$$

where $g_L=g_L(LL)$ is the layer Green's function of the left superconductor at $x=x'=L=-d$ and the S functions are directly related to the second derivatives of exact operator [operator in the sense of Eq. (7)] Green's functions $G^{<}$, G^K, G^a, and G^r, respectively,

$$S_L^{<}=S^{<}(LL) =-\frac{\hbar^2}{2m}\tau_3\left[\hat{x}\frac{\partial}{\partial \mathbf{r}}\right]\left[\hat{x}'\frac{\partial}{\partial \mathbf{r}'}\right]G^{<}(x,x')_{x=x'=L}\tau_3 . \tag{13}$$

Formulas (12) and (13) are the three-dimensional generalization of the corresponding one-dimensional, homogeneous in the y,z plane result for the current. We have to average $J(t)$ over the energy and coordinates of the localized state. This will be done later. Here we note that according to Eq. (10) all derivatives of G^K at the interface are defined by corresponding derivatives of retarded and advanced Green's functions G^a and G^r. In the case of equal effective masses of weak link and of superconductor electrodes, $S^{a,r}$ are related to the Green's functions $g^{a,r}$ at the interfaces by a matrix equation which is similar to that obtained in Refs. 13, 20, and 21. Unlike the one-dimensional limit, however, in our case the matrix equation acts not only in time space but in the momentum space as well. This equation follows from exact Eq. (1) applied in the vicinity of SI interfaces and from boundary conditions on $G^{a,r}$ and $g^{a,r}$. As a result we obtain an equation for $S^{a,r}$ in an operator matrix form

$$\begin{bmatrix} S_L & S(LR) \\ S(RL) & S(RR)\end{bmatrix} \times \begin{bmatrix} g_L(LL)+g_B(LL) & -g_B(LR) \\ -g_B(RL) & g_R(RR)+g_B(RR)\end{bmatrix} =\frac{2m}{\hbar^2} . \tag{14}$$

Here $g_B(x,x')$ is the matrix Green's function, retarded or advanced [see Eq. (7) for the matrix elements] of the separate layers in the barrier region taken at the points of the SI interfaces.

Now we outline the procedure for obtaining the current through the junction. At the first step we get from Eq. (14) the function $S_L^{a,r}$ in terms of $g^{a,r}$; at the next step with the help of (10) we find S^K. Finally including a resonant level, we calculate $g^{a,r}$ for our system. To consider the nonequilibrium situation, we assume that the entire voltage drops across the barrier so that the chemical potential in the left electrode differs from that of the right on eV (thus $\mu_{\text{left}}=\text{eV}+\mu$, $\mu_{\text{right}}=\mu$). We ignore the influence of the voltage on Green's functions inside the insulator, considering that the barrier height $V_B-\mu$ satisfies inequality

$$e(V_B-\mu)\gg \text{eV}, E . \tag{15}$$

This holds for all characteristic energies E and voltages V. For such barriers, Green's functions in the weak link are affected negligibly by the applied voltage. On the other hand, the electrodes Green's functions are strongly time dependent due to their phase $\Phi(t)$. Therefore, we express nonequilibrium Green's functions in the banks in terms of the equilibrate ones as follows:

$$\langle t|g_L|t'\rangle = g_L(t,t') = \exp\left[-\frac{i}{2}\Phi(t)\tau_3\right]\bar{g}_L(t-t') \times\exp\left[\frac{i}{2}\Phi(t')\tau_3\right] . \quad (16)$$

Here we take[13]

$$\Phi(t)=\Phi_0+2eVt/\hbar$$

and Φ_0 is equilibrium phase at $V=0$.

Due to the short range of the $V_{\rm res}$, the electrode Green's functions $g_i(t-t')$ ($i=1,2$) with resonant level in the weak link are the same as they would be without such a level. They were obtained earlier.[13] In the energy and momentum representation we have $[\bar{g}_L(LL)\equiv\bar{g}_L]$:

$$\langle \mathbf{p}_\parallel E|\bar{g}_L^r|\mathbf{p}_\parallel' E'\rangle = \frac{2m}{\hbar^2 k(\mathbf{p}_\parallel)}\delta_{\mathbf{p}_\parallel,\mathbf{p}_\parallel'}\delta(E-E')\bar{g}_L^r(E) ,$$
$$\bar{g}_L^r(E)=i[\varepsilon_1^r(\varepsilon)+\eta^r(\varepsilon)\tau_1] , \quad (17)$$
$$\varepsilon_1^r(\varepsilon)=\frac{\theta(|\varepsilon|-1)|\varepsilon|}{\sqrt{\varepsilon^2-1}}-i\frac{\theta(1-|\varepsilon|)\varepsilon}{\sqrt{1-\varepsilon^2}} ,$$

where ε is the dimensionless quasiparticle energy in the units of order parameter: $\varepsilon=E/\Delta$ and $k(\mathbf{p}_\parallel)=\sqrt{p_F^2-\mathbf{p}_\parallel^2}$, $\eta^r(E)=\varepsilon_1^r/\varepsilon$, p_F stands for Fermi momentum in superconductors, $\theta(x)$ is the Heaviside step function: $\theta(x)=1$ if $x>0$ and $\theta(x)=0$ if $x<0$. The advanced Green's functions are given by complex conjugate of (17). When the time dependence is included, we have[13]

$$\langle E|g_L|E'\rangle = \langle E|\exp\left[-\frac{i}{2}\Phi\tau_3\right]\bar{g}_L\exp\left[\frac{i}{2}\Phi\tau_3\right]|E'\rangle$$
$$=\frac{1}{2}\delta(E-E')[\bar{g}_L(E+eV)_{11}(1+\tau_3)+\bar{g}_L(E-eV)_{22}(1-\tau_3)]$$
$$+\delta(E-E'+2eV)\bar{g}_L(E+eV)_{12}\tau_+e^{-i\Phi_0}+\delta(E-E'-2eV)\bar{g}_L(E-eV)_{21}\tau_-e^{i\Phi_0}$$
$$=\sum_{k=-1,0,1}\delta(E-E'+2eVk)\bar{g}_L(\varepsilon,\varepsilon+2uk),u=\frac{eV}{\Delta} . \quad (18)$$

The g_B is the quasiparticle Green's function in the barrier with localized state, and because of the lack of homogeneity in the plane of junction it depends on ρ,ρ' (rather than on the difference between ρ,ρ'). We relate g_B to the unperturbed Green's function g_{0B} which describes the system without resonance level[16,22] and corresponds to the one-dimensional problem. Using the Fourier transformation on energy variable, the differential equation (1) in the insulator can be transformed into the integral equation for $g_B(E,r,r')$

$$g_B(E,\mathbf{r},\mathbf{r}')=g_{0B}(E,\mathbf{r},\mathbf{r}')+\int d\mathbf{r}_1 V_{\rm res}(\mathbf{r}_1)g_{0B}(E,\mathbf{r},\mathbf{r}_1)\tau_3 g_B(E,\mathbf{r}_1,\mathbf{r}') , \quad (19)$$

where the short-range potential $V_{\rm res}$ is defined in the range $|\mathbf{r}-\mathbf{r}_0|\le a \ll d$. Here r_0 is the coordinate of the localized state in the insulating layer. Green's function g_{0B} has the zero derivatives at the interfaces and can easily be found. After Fourier transformation on in-plane coordinates of the junction we get

$$g_{0,B}(Ex,x')=-\frac{2m\cosh[(R-x')\chi]\cosh[(L-x)\chi]}{\chi\sinh(2\chi d)}\tau_3 , \quad \text{if } x'>x . \quad (20)$$

Here

$$\chi=\left[\frac{2m}{\hbar^2}(V_B-\mu)+p_\parallel^2\right]^{1/2} . \quad (21)$$

The case $x>x'$ follows from Eq. (20) by simply replacing x by x' and vice versa.

For short-range resonance level potential, the solution of Eq. (19) in momentum representation is given

$$g^r(\mathbf{p}_\parallel\mathbf{p}_\parallel' Ex,x')=\delta_{\mathbf{p}_\parallel,\mathbf{p}_\parallel'}g_{0B}^r(\mathbf{p}_\parallel Ex,x') +g_{0B}^r(\mathbf{p}_\parallel Ex,x_0)\tau_3 L_E^r(x_0) \times g_{0B}^r(\mathbf{p}_\parallel' Ex_0,x') , \quad (22)$$

where

$$L_E^r(x_0)=\left[\int d\mathbf{r}\, V_{\rm res}(\mathbf{r})\right] \times\left[1-\int d\mathbf{r}\, V_{\rm res}(\mathbf{r})g_{0B}^r(E\mathbf{r}_0,\mathbf{r})\tau_3\right]^{-1} . \quad (23)$$

After inserting the explicit form (20) for g_{0B} into the last equation, the latter is reduced to the resonant propagator of a quasiparticle

$$[L_E^r(x_0)]^{-1}=\frac{m}{2\pi}\left[[\alpha-\alpha_0-l(x_0)]+\frac{E\alpha}{2(V_B-\mu)}\tau_3\right] ,$$
$$\alpha_0=\left[\frac{2m}{\hbar^2}(V_B-E_0)\right]^{1/2} , \quad \alpha=\left[\frac{2m}{\hbar^2}(V_B-\mu)\right]^{1/2} . \quad (24)$$

Here E_0 denotes the resonant energy value of the localized state. The function $l(x)$ was obtained in Ref. 23 and takes the form

$$l(x_0)=\frac{1}{2}e^{-2d\alpha}[\bar{f}(x_0)+\bar{f}(-x_0)]/d ,$$
$$\bar{f}(x_0)=\frac{\exp(2x_0\alpha)}{d-x_0}\varphi[p_F^2\alpha(d-x_0)]d ; \quad (25)$$

$\varphi(x)$ is a complicated function which includes a combination of Kummer's functions. In the limiting case $x \to \infty$, the function $\varphi(x)$ turns to unity, and for the small arguments $\varphi(x)=2x/3$. In the following we use approximate formula $l(x_0)=\exp(-2d\alpha)\cosh(2\alpha x_0)/d$, which is correct for short-range potential, and $\alpha < d$. Near resonance we approximate Eq. (24) as follows:

$$(L_E^r)^{-1}=\frac{m\Delta}{4\pi\alpha}[\lambda_1+(\varepsilon+i0)\tau_3] \ . \tag{26}$$

Here the dimensionless resonant energy variable λ_1 is introduced. We also set $2m=1$ everywhere but in the α_0, α [the formula (26) contains m which come out from the α_0 and α]. By this choice the current obtained preserves proper units. Below we will neglect the usual tunneling that compares with the resonance one. In this approximation the g Green's functions in Eq. (14) take the forms

$$\begin{aligned}
g_{BL}(\mathbf{p}_\parallel\mathbf{p}'_\parallel E)&=\delta_{\mathbf{p}_\parallel,\mathbf{p}'_\parallel}g_{0BL}(\mathbf{p}_\parallel E)-\beta_R(\mathbf{p}_\parallel)\beta_R(\mathbf{p}'_\parallel)\tau_3 L_E \ ,\\
g_{BR}(\mathbf{p}_\parallel\mathbf{p}'_\parallel E)&=\delta_{\mathbf{p}_\parallel,\mathbf{p}'_\parallel}g_{0BR}(\mathbf{p}_\parallel E)-\beta_L(\mathbf{p}_\parallel)\beta_L(\mathbf{p}'_\parallel)\tau_3 L_E \ ,\\
g_{BLR}(\mathbf{p}_\parallel\mathbf{p}'_\parallel E)&=-\beta_R(\mathbf{p}_\parallel)\beta_L(\mathbf{p}'_\parallel)\tau_3 L_E \ ,\\
g_{BR}(\mathbf{p}_\parallel\mathbf{p}'_\parallel E)&=-\beta_L(\mathbf{p}_\parallel)\beta_R(\mathbf{p}'_\parallel)\tau_3 L_E \ ,
\end{aligned} \tag{27}$$

where the notations are introduced,

$$\begin{aligned}
&g_B(LL)\equiv g_{BL} \ ; \quad g_B(RR)\equiv g_{BR} \ ;\\
&g_B(RL)\equiv g_{BRL} \ ; \quad g_B(LR)\equiv g_{BLR} \ ;
\end{aligned} \tag{28}$$

and

$$\begin{aligned}
&\beta_L(\mathbf{p}_\parallel)=\frac{\cosh[\chi(x_0-L)]}{\chi\sinh(2\chi d)} \ ;\\
&g_{0BR}(\mathbf{p}_\parallel E)=g_{0BL}(\mathbf{p}_\parallel E)=a\tau_3 \ ;\\
&a=-\frac{k(\mathbf{p}_\parallel)}{\chi}\coth(2\chi d) \ .
\end{aligned} \tag{29}$$

The function β_R can be obtained from β_L by simply replacing L by R.

Due to the factorized form of the resonant terms the operator solution of the basic equation (14) for retarded or advanced exact Green's function $S^{a,r}$ can be written as

$$\begin{aligned}
&S_L^r=(g_{1L}^r)^{-1}+q_L^r W^r q_L^r \ ,\\
&S^r(LR)=-q_L^r W^r q_R^r \ ,\\
&S^r(RL)=-q_R^r W^r q_L^r \ ;
\end{aligned} \tag{30}$$

where the matrix elements of coefficients are

$$\begin{aligned}
&\langle \mathbf{p}_\parallel E|q_{L,R}^r|\mathbf{p}'_\parallel E'\rangle=\langle \mathbf{p}_\parallel E|(g_{1L,R}^r)^{-1}|E'\mathbf{p}'_\parallel\rangle\beta_{R,L}(\mathbf{p}_\parallel) \ ,\\
&\langle \mathbf{p}_\parallel E|(g_{1R}^r)^{-1}|E'\mathbf{p}'_\parallel\rangle=-\frac{\delta_{\mathbf{p}_\parallel,\mathbf{p}'_\parallel}\delta(E-E')k(\mathbf{p}_\parallel)}{1+a^2}\tau_3[\overline{g}_R^r(E)-g_{0BR}(\mathbf{p}_\parallel)]\tau_3 \ .
\end{aligned} \tag{31}$$

Here we have omitted the E argument in g_{0BR} because this function does not depend on E [see Eq. (29)]. The matrix elements of $(g_{1L}^r)^{-1}$ consist of the same objects as the last line of Eq. (31) (R should be replaced by L) but rule 18 must be applied to them. This is because of the time dependence of left-hand superconductor Green's functions.

The matrix function W, which is the solution of Eq. (14), represents the complete resolvent Green's function of a quasiparticle which tunnels due to the resonant state in the insulator

$$W^r=[(\hat{L}^r\tau_3)^{-1}-\gamma_0^r-\gamma^r]^{-1} \ , \tag{32}$$

where $\hat{L}^r$ is the operator. In the energy representation it is reduced to L_E; γ_0^r, γ^r are related to the retarded Green's functions at the right and left SI interfaces, respectively,

$$\gamma^r=\langle\beta_R^2(g_{1L}^r)^{-1}\rangle \ ; \tag{33}$$

and angular brackets stand for momentum integration

$$\langle f\rangle=\frac{1}{(2\pi)^2}\int d\mathbf{p}_\parallel f(\mathbf{p}_\parallel) \ . \tag{34}$$

The propagators W^r and W^a are of central importance for calculating the current through the junction because they directly define the nonequilibrium Keldysh Green's function S^K. Indeed with the help of Eq. (10) we obtain

$$\begin{aligned}
-S^K=&S_L^r(g_L^K+g_{BL}^K)S_L^a-S_L^r g_{BLR}^K S^a(RL)\\
&-S^r(LR)g_{BRL}^K S_L^a+S^r(LR)(g_R^K+g_{BR}^K)S^a(RL) \ ,
\end{aligned} \tag{35}$$

where the functions g^K are connected to distribution function f which in the energy representation is $\tanh(E/2T)$ (T being the temperature) by relation

$$g^K=g^a f-f g^r$$

and the rule of finding the matrix elements of g_L^K is the same as for g_L [see Eqs. (16) and (18)]. The relation between S^K, W^r, and W^a follows from Eq. (30).

III. CURRENT IN JOSEPHSON JUNCTION

We rewrite Eq. (12), which gives the expression for the current through the Josephson junction using the dimensionless energy variables ε,ε' and the chosen units ($2m\to 1$). Inserting the formulas (35) and (30) in Eq. (12), after some algebra we get

$$J(t)=-4e\Delta\,\mathrm{Re}\left\{\int\frac{d\varepsilon}{2\pi}\int d\varepsilon'\,e^{i\Delta(\varepsilon-\varepsilon')t}\langle\varepsilon'|\mathrm{Sp}\left[\frac{1+\tau_3}{2}(B_R W^a+D^r W^r F W^a)\right]|\varepsilon\rangle\right\}, \tag{36}$$

where

$$B_R=\gamma_0^r f-f\gamma_0^a\;;\quad D^r=\gamma_0^r-(\hat{L}^r\tau_3)^{-1}\;;\quad F=B_R+B_L\;;\quad B_L=\gamma^r f-f\gamma^a\,. \tag{37}$$

The matrix elements of these quantities follow directly from the definition of the layer Green's functions g_i [Eqs. (17) and (33)], from the time transformation rule for g_i and the distribution function f [see Eq. (18)]

$$\langle\varepsilon|\gamma^r|\varepsilon'\rangle=\frac{\bar{f}(-x_0)}{\bar{f}(x_0)}\langle\varepsilon|\exp\left[\frac{-i\Phi\tau_3}{2}\right]\gamma_0^r\exp\left[\frac{i\Phi\tau_3}{2}\right]|\varepsilon'\rangle\,, \tag{38}$$

$$\langle\varepsilon|\gamma_0^r|\varepsilon'\rangle=-A\,\delta(\varepsilon-\varepsilon')[i(\varepsilon_1^r(\varepsilon)-\eta^r(\varepsilon)\tau_1)+\tau_3 p_F/\alpha]\bar{f}(x_0)\,, \tag{39}$$

$$\langle\varepsilon|B_R|\varepsilon'\rangle=-2iA\,\delta(\varepsilon-\varepsilon')\bar{f}(x_0)f(\varepsilon)N(\varepsilon)(1-\tau_1\varepsilon^{-1})\;; \tag{40}$$

$$\bar{D}=\frac{1}{2}[(1+\tau_3)D^r-D^a(1+\tau_3)]\,, \tag{41}$$

$$\langle\varepsilon|\bar{D}|\varepsilon'\rangle=-iA\,\delta(\varepsilon-\varepsilon')\bar{f}(x_0)[(1+\tau_3-\tau_1\varepsilon^{-1})N(\varepsilon)-\tau_2K(\varepsilon)]\,, \tag{42}$$

$$\begin{aligned}\frac{\langle\varepsilon|F|\varepsilon'\rangle}{-2iA}=\delta(\varepsilon-\varepsilon')&\left\{\bar{f}(-x_0)\left[f(\varepsilon+u)N(\varepsilon+u)\frac{1+\tau_3}{2}+f(\varepsilon-u)N(\varepsilon-u)\frac{1-\tau_3}{2}\right]\right.\\&\left.+\bar{f}(x_0)f(\varepsilon)N(\varepsilon)(1-\tau_1\varepsilon^{-1})\right\}-\bar{f}(-x_0)\left[\delta(\varepsilon-\varepsilon'+2u)\tau_+e^{-i\Phi}\frac{f(\varepsilon+u)N(\varepsilon+u)}{\varepsilon+u}\right.\\&\left.+\delta(\varepsilon-\varepsilon'-2u)\tau_-e^{i\Phi}\frac{f(\varepsilon-u)N(\varepsilon-u)}{\varepsilon-u}\right]\\&=\sum_{n=-1,0,1}\delta(\varepsilon-\varepsilon'+2un)\hat{F}(\varepsilon,\varepsilon+2un)\,.\end{aligned} \tag{43}$$

Here $A=(\alpha/4\pi p_F)e^{-2d\alpha}/d$ contains the parameters of the Josephson junction; the function $\bar{D}$ will be used below. The two functions $N(\varepsilon)$ and $K(\varepsilon)$ select the quasiparticle states outside and inside the superconducting gap, respectively,

$$N(\varepsilon)=\frac{|\varepsilon|}{\sqrt{\varepsilon^2-1}}\theta(|\varepsilon|-1)\;;\quad K(\varepsilon)=\frac{\theta(1-|\varepsilon|)}{\sqrt{1-\varepsilon^2}}\,. \tag{44}$$

The junctions of sufficiently large area which were used in the experiments[1,2] on αSi are in the self-averaging region. For such junctions we need to average the current over the coordinates and the energies of the localized states. This averaging results in two additional integrations of the formula (36), one over x_0 and the other over λ_1. It is convenient to rescale the resonant level $\lambda_1\to\lambda=\Gamma\lambda_1$ so that the important dependence from characteristic resonant parameter Γ is explicitly revealed,

$$\Gamma=(\Delta\alpha^{-2})mdp_F\exp(2d\alpha)=\frac{\Delta d}{V_B-\mu}\left[\frac{\mu}{V_B-\mu}\right]^{1/2}\alpha\exp(2d\alpha)\,. \tag{45}$$

We include the energy independent term $\tau_3 p_F/\alpha$ in the variable of integration $\lambda\to\lambda+p_F/\alpha$, and finally get for the full propagator (32) the expression

$$\begin{aligned}\langle\varepsilon|(\hat{W}^r)^{-1}|\varepsilon'\rangle=\delta(\varepsilon-\varepsilon')&\left[\lambda\tau_3+\Gamma\varepsilon+i\bar{f}(-x_0)\begin{bmatrix}\varepsilon_1^r(\varepsilon+u)&0\\0&\varepsilon_1^r(\varepsilon-u)\end{bmatrix}+i\bar{f}(x_0)(\varepsilon_1^r(\varepsilon)-\eta_1^r(\varepsilon)\tau_1)\right]\\&-i\bar{f}(-x_0)[\delta(\varepsilon-\varepsilon'+2u)\tau_+\eta_1^r(\varepsilon+u)+\delta(\varepsilon-\varepsilon'-2u)\tau_-\eta_1^r(\varepsilon-u)]\,.\end{aligned} \tag{46}$$

Here we introduce the notation: $W=A^{-1}\hat{W}$. Now the formula for the total time dependent current averaged over positions and energies of the resonance states can be written as

$$J(t)=-\frac{4\alpha\Delta}{e\pi^2\rho_n}\mathrm{Re}\left[\int_{-d}^{d}dx_0\int d\lambda\int d\varepsilon\int d\varepsilon'\,e^{i\Delta(\varepsilon'-\varepsilon)t}\langle\varepsilon'|\mathrm{Sp}\left[i\frac{1+\tau_3}{2}(\hat{B}_R\hat{W}^a+\hat{D}^r\hat{W}^r\hat{F}\hat{W}^a)\right]|\varepsilon\rangle\right], \tag{47}$$

where values B_R, F, D^r, and later $\bar{D}$ with the cap are equal to the corresponding values B_R, F, D^r, and $2\bar{D}$ divided on the coefficient $-2iA$. Here ρ_n is part of the resistivity of the Josephson junction which is determined by tunneling processes due to the localized level

$$\rho_n^{-1}=\frac{e^2\pi n(E_0)n(x_0)}{md}\left[\frac{V_B-\mu}{\mu}\right]^{1/2}\exp(-2d\alpha)=\frac{e^2\pi n(E_0)n(x_0)\Delta}{md\Gamma(V_B-\mu)}\ , \tag{48}$$

where $n(E_0)$ and $n(x_0)$ are, respectively, the density of energy states of the localized level and the density of the resonance scatters. The part of the resistivity which is determined by direct tunneling processes ρ_{dir}, if we consider the usual NIN normal junction, has an additional, in comparison with Eq. (48), factor 2 in the exponent. Also, ρ_{dir} includes the density of states of the normal-metal electrodes which replaces the density of the localized states in ρ_n. Therefore, if the thickness of the insulator is sufficient to suppress the rather large density of states of the normal metal, then the resonance tunneling contribution becomes dominant as was observed in experiment.[1,2]

The calculation of the current density (47) can be simplified if one notes that the energy difference $\varepsilon-\varepsilon'$ of the resolvent Green's function $\langle\varepsilon|W^{r,a}|\varepsilon'\rangle$ is an integer multiple of 2 eu, the same as in Ref. 13, i.e., $\langle\varepsilon|W^{r,a}|\varepsilon'\rangle$ is given by the sum

$$\langle\varepsilon|\hat{W}|\varepsilon'\rangle=\sum_n\delta(\varepsilon-\varepsilon'+2un)\hat{W}(\varepsilon,\varepsilon+2un)\ , \tag{49}$$

$$\hat{W}(\varepsilon+2um,\varepsilon+2un)=\hat{W}(\varepsilon|m,n)\ . \tag{50}$$

Here n and m are any positive or negative integers.

Equation (50) introduces matrix notations (the analogous definitions are true for the other functions F, B_R, and D). Below we will restrict our calculations to the zero harmonic contribution to the current. From Eq. (47) we obtain

$$J_0=-\frac{2\alpha\Delta}{e\pi^2\rho_n}\int_{-d}^{d}dx_0\int d\lambda\int d\varepsilon\,\mathrm{Sp}[(iP_+(\hat{B}_R\hat{W}^a-\hat{W}^r\hat{B}_R)+\hat{\hat{D}}\hat{W}^r\hat{F}\hat{W}^a)(\varepsilon,\varepsilon)]\ , \tag{51}$$

where $P_+=(1+\tau_3)/2$ and $(f)(\varepsilon,\varepsilon)$ means $f(\varepsilon,\varepsilon)$ as in Eq. (50).

The current density J_0 can be presented in a form which explicitly consists of the differences of distribution functions shifted on nu values of argument and $f(\varepsilon)$. To do this we rewrite with the help of Eqs. (38) and (40) the first term of Eq. (51) as the function of these differences. Performing the trace in Nambu space we get

$$\mathrm{Sp}[P_+(\hat{B}_R\hat{W}^a-\hat{W}^r\hat{B}_R)(\varepsilon,\varepsilon)]=\bar{f}(x_0)N(\varepsilon)f(\varepsilon)\Big\{[\hat{W}^r((\hat{W}^r)^{-1}-(\hat{W}^a)^{-1})\hat{W}^a]_{11}-\frac{1}{2\varepsilon}[\hat{W}^r((\hat{W}^r)^{-1}-(\hat{W}^a)^{-1})\hat{W}^a]_{21}$$
$$+(1\Longleftrightarrow 2)+(\hat{W}^a+\hat{W}^r)_{21}-(1\Longleftrightarrow 2)\Big\}(\varepsilon,\varepsilon)\ ; \tag{52}$$

here the subscripts stand for matrix notations in the Nambu space. Combining Eq. (52) with the matrix elements of F Eq. (43) and D Eq. (42) we find

$$J_0=J_1+J_2\ ,$$

where

$$J_1=-\frac{2\alpha\Delta}{e\pi^2\rho_n}\int_{-d}^{d}dx_0\int d\lambda\int d\varepsilon N(\varepsilon)\Big\{\frac{if(\varepsilon)}{2\varepsilon}((\hat{W}^a+\hat{W}^r)_{12}-(1\Longleftrightarrow 2))(\varepsilon,\varepsilon)+$$
$$+2(\hat{W}^r\tilde{F}\hat{W}^a)_{11}(\varepsilon,\varepsilon)-\varepsilon^{-1}((\hat{W}^r\tilde{F}\hat{W}^a)_{21}+(\hat{W}^r\tilde{F}\hat{W}^a)_{12})(\varepsilon,\varepsilon)\Big\}\bar{f}(x_0)\ , \tag{53}$$
$$J_2=-\frac{i2\alpha\Delta}{e\pi^2\rho_n}\int_{-d}^{d}dx_0\int d\lambda\int d\varepsilon K(\varepsilon)[((\hat{W}^r\tilde{F}\hat{W}^a)_{21}-(\hat{W}^r\tilde{F}\hat{W}^a)_{12})(\varepsilon,\varepsilon)]\bar{f}(x_0)\ .$$

The matrix elements of $\tilde{F}$ follow from the corresponding matrix elements $\langle\varepsilon_1|\hat{F}|\varepsilon_2\rangle$ if we replace the distribution functions in expression (43) by the differences of these distribution functions with $f(\varepsilon)$. Equations (53) are convenient for numerical analyses. To solve the problem numerically, we truncate the basis $\{n,m\}$ [see Eq. (50)] in the energy space and consider a matrix of size $(2N+1)\times(2N+1)$ with the N value related to the number of Andreev reflections. The diagonal element $(\varepsilon,\varepsilon)$ will be chosen in such a way that it corresponds to the matrix indices $N+1,N+1$. It is the central site of the matrix.

IV. ZERO-BIAS JOSEPHSON CURRENT. NORMAL METAL JUNCTIONS. NUMERICAL RESULTS

In this section we will apply the theory which was developed above to analyze the current through the barrier in the tunnel junctions. In the two limits investigated here we achieve analytical results. The first is the symmetric NIN junction when the electrodes are normal metals (not superconductors). In this case the matrices $(W^{a,r})^{-1}$ and $W^{a,r}$ of the resolvent operators are diagonal in the energy as well as in Nambu space

$$(\hat{W}^r)^{-1}(E,E)=\lambda\tau_3+\Gamma_T\frac{E}{T}+2i\cosh(2\alpha x_0)\ . \tag{54}$$

The other functions which describe the current [see Eqs. (40)–(43)] are reduced to diagonal form as well

$$\hat{F}(E,E)=\begin{bmatrix} f(E+eV)\bar{f}(-x_0)+f(E)\bar{f}(x_0) & 0 \\ 0 & f(E-eV)\bar{f}(-x_0)+f(E)\bar{f}(x_0)\end{bmatrix},$$

$$\hat{\bar{D}}(E,E)=P_+\bar{f}(x_0)\ ;\ \ \hat{\bar{B}}_R=\bar{f}(x_0)f(E)\ .$$

Here $\Gamma_T=\Gamma(T/\Delta)$, and we have returned to the dimensional notation for the energy variable E. The second term in Eq. (51) takes the form

$$\mathrm{Sp}(\hat{\bar{D}}\hat{W}^r\hat{F}\hat{W}^a)(E,E)=\frac{\bar{f}(x_0)[\bar{f}(-x_0)f(E+eV)+\bar{f}(x_0)f(E)]}{\left[\lambda+\Gamma_T\frac{E}{T}+2i\cosh(2\alpha x_0)\right]\left[\lambda+\Gamma_T\frac{E}{T}-2i\cosh(2\alpha x_0)\right]}\ . \tag{55}$$

We now consider the case $d>1/\alpha$. Inserting Eq. (55) into Eq. (51) and performing the integration over x_0 and λ, we obtain the expected formula for the current of the normal junctions in which the tunneling occurs due to the localized state,

$$J_n=\frac{1}{e\rho_n}\int dE[f(E+eV)-f(E)]\ . \tag{56}$$

Let us now consider the second example: zero bias symmetric Josephson junction. This is the equilibrium case, and therefore we use the thermodynamic (Matsubara) representation for Green's functions and the current. From Eq. (12) we get

$$\begin{aligned}J_s&=2eT\sum_{\mathbf{p}_\parallel\omega}\langle\mathbf{p}_\parallel|\mathrm{Sp}[P_+(S_L^Tg_L^T-g_L^TS_L^T)_\omega]|\mathbf{p}_\parallel\rangle\\&=2eT\sum_{\omega}\mathrm{Sp}[((P_+\gamma^T-\gamma^TP_+)W^T)_\omega]\ .\end{aligned} \tag{57}$$

Here $\omega=\pi T(2k+1)$, where k is a positive or negative integer, and the superscript T stands for identification of the Matsubara functions S, g_L, W, and γ [see Eq. (33)]. These functions can be obtained from their retarded form by replacing $E\to i\omega-\delta$. As we did before for the I-V characteristic, Josephson current, J_s, (57) needs to average over the coordinates and the energies of the localized state. In Eq. (57) the functions with cap [see Eq. (47)] are given

$$\begin{aligned}&P_+\hat{\gamma}^T-\hat{\gamma}^TP_+=\tau_2\exp(i\Phi\tau_3)\bar{f}(-x_0)\ ;\\&\hat{W}^T=\Big[\lambda\tau_3+i\Gamma_T\frac{\omega}{T}+2i\varepsilon_1(i\omega)\cosh(2\alpha x_0)\\&\qquad -i\eta(i\omega)[\bar{f}(x_0)+\bar{f}(-x_0)\exp(i\Phi\tau_3)]\Big]^{-1},\end{aligned} \tag{58}$$

where $\varepsilon_1(i\omega)=\omega(\omega^2+\Delta^2)^{-0.5}$ and $\eta(i\omega)=\varepsilon_1(i\omega)\Delta/i\omega$. With the help of Eq. (58) we get

$$J_S=\frac{4\alpha\sin\Phi}{\pi e\rho_n}T\sum_{\omega}\int_{-d}^{d}dx_0\int d\lambda\,\varphi(x_0,\lambda,\omega)\eta^2(i\omega)\ ,$$

$$\varphi(x_0,\lambda,\omega)=\left[\lambda^2+\left[\Gamma_T\frac{\omega}{T}+2\varepsilon_1(i\omega)\cosh(2\alpha x_0)\right]^2-2\eta^2(i\omega)\left[\cosh(4\alpha x_0)+\cos\Phi\right]\right]^{-1}.$$

After averaging over the energies of the localized states, we arrive at the following expression for the Josephson current:

$$J_S=\frac{4\alpha\sin\Phi}{\pi e\rho_n}T\sum_{\omega}\int_{-d}^{d}dx_0\frac{(\omega^2+\Delta^2)^{-1}}{\left[\left[\Gamma_T\frac{\omega}{T}+2\varepsilon_1(i\omega)\cosh(2\alpha x_0)\right]^2-2\eta^2(i\omega)[\cosh(4\alpha x_0)+\cos\Phi]\right]^{1/2}}\ .$$

If the dimensionless parameter Γ is small, then the last formula can be simplified:

$$J_s=\frac{\pi\Delta^2\sin\Phi}{e\rho_n}T\sum_{\omega}\frac{1}{\omega^2+\Delta^2}s\left[\frac{\Delta^2\sin^2\frac{\Phi}{2}}{\omega^2+\Delta^2}\right], \tag{59}$$

where

$$s(y)=\frac{1}{\pi}\int_{-2\alpha d}^{2\alpha d}\frac{dz}{\sqrt{\cosh^2z-y}}\ ,\quad y<1\ . \tag{60}$$

If the effective radius of localized state $1/\alpha$ is less compared with the thickness of the weak link $2d$, the function $s(y)$ weakly depends on α and its argument y. For T not too close to zero temperature, $s(y)\approx 1$. For $\Gamma\ll 1$ and for the temperature in the vicinity of the critical temperature, J_s coincides with the result which was obtained in Ref. 24 for metal contact. Near T_c Eqs. (59) and (60) also represent the Josephson current which has been recently calculated in Ref. 25, though deviation appears at low temperatures.

There are other systems where the resonance tunneling can be handled directly on the basis of the suggested theory. The simplest example is the three layer NIS junctions where one of the electrodes is normal metal. The I-V characteristic can be derived from Eq. (53). In this case the resolvent operator is diagonal in the energy variables.

Now we will address the general problem of the subgap current in Josephson junctions under applied bias voltage. For the numerical calculations, we consider a simplified limit when the localized state has the coordinate $x_0=0$, i.e., it is situated in the middle of the weak link. We also put the constant phase Φ_0 equal to zero because the current does not depend upon it. We write all terms in the expression for current (53) in the matrix form using the basis $\{n,m\}$ [see Eq. (50)]. For example, we have

$$(\hat{W}^r\hat{F}\hat{W}^a)_{21} \rightarrow \hat{W}^r_{2\alpha}(\varepsilon|N+1,m)\hat{F}_{\alpha\beta}(\varepsilon|m,n)\hat{W}^r_{\beta 1}(\varepsilon|n,N+1)\,, \quad (61)$$

where the Greek subscripts indicate Nambu space. From Eq. (43) follows

$$\begin{aligned}
\hat{F}_{11}(\varepsilon|m,n)&=\delta_{m,n}[f(\varepsilon+u+2mu)N(\varepsilon+u+2mu)\\
&\quad+f(\varepsilon+2mu)N(\varepsilon+2mu)]\,,\\
\hat{F}_{12}(\varepsilon|m,n)&=-\delta_{m,n}\frac{f(\varepsilon+2mu)N(\varepsilon+2mu)}{\varepsilon+2mu}\\
&\quad-\delta_{m+1,n}\frac{f(\varepsilon+u+2mu)N(\varepsilon+u+2mu)}{\varepsilon+u+2mu}\,,\\
\hat{F}_{21}(\varepsilon|m,n)&=-\delta_{m,n}\frac{f(\varepsilon+2mu)N(\varepsilon+2mu)}{\varepsilon+2mu}\\
&\quad-\delta_{m-1,n}\frac{f(\varepsilon-u+2mu)N(\varepsilon-u+2mu)}{\varepsilon-u+2mu}\,,\\
\hat{F}_{22}(\varepsilon|m,n)&=\delta_{m,n}[f(\varepsilon-u+2mu)N(\varepsilon-u+2mu)\\
&\quad+f(\varepsilon+2mu)N(\varepsilon+2mu)]\,.
\end{aligned} \quad (62)$$

In a similar way, with the help of Eq. (46), we get the matrix elements of the resolvent $(\hat{W})^{-1}_{\alpha\beta}$.

We have considered the case of the subgap voltage $eV<\Delta$ and low temperatures $T<\Delta$ (here $T=0.1\Delta$) because the most pronounced signature of multiple Andreev reflections belongs to these regions. The subgap current is proportional to ρ_n^{-1}. Due to resolvent Green's functions, it also has the complicated, implicit dependence from resonant parameter Γ. Therefore, we may represent the results of our numerical calculations [Fig. 1] as a plot for the product $J_q=aJ_0\rho_n$ versus voltage (in the units of superconducting gap) at the different values of resonant parameter Γ. Here $a=e^2\pi/4\alpha d$ is the function of the barrier thickness d which yields the preexponent to the exponential d dependence of the resistivity.

We can expect a weaker dependence of a or even for it

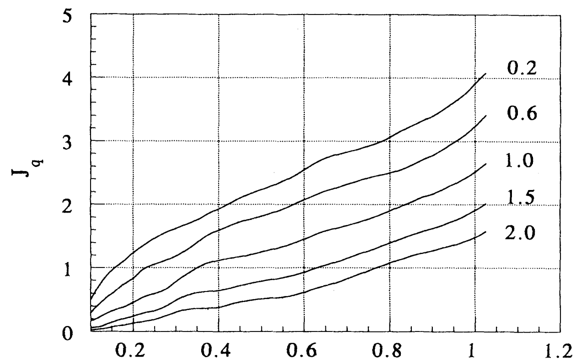

FIG. 1. Normalized tunneling current $J_q=a\rho_nJ_0$ versus voltage for Josephson junction with the resonant scatters in the barrier. The different curves correspond to the resonant parameter Γ between 0.2 and 2. The temperature $T=0.1\Delta$; the coefficient $a=e^2\pi/4\alpha d$. If the only variable parameter of the tunnel structure is the thickness of the weak link, the plot $J=J_q\Gamma^{-1}$ displays the full current density dependence on Γ.

to be a constant if an averaging over the positions of resonant levels were performed, as it had been for the Josephson zero-biased junction [see Eq. (59)]. In cases when the only variable parameter of the tunnel structure is the thickness of the weak link, we have $\rho_n\sim\Gamma$, and the plot $J=aJ_0\rho_n\Gamma^{-1}=J_q\Gamma^{-1}$ displays the full current density dependence on the resonant parameter. This plot can be easily obtained from Fig. 1. The results represented on Fig. 1 clearly show the existence of the subgap current. From Fig. 1 one sees decrease in the current density with the enhancing of Γ. Such behavior is more pronounced in Fig. 2 where J_q is plotted as the function of resonant parameter at the two fixed values of the applied voltage $eV=\Delta$ and $\Delta/2$. The fact that the subgap current becomes less when Γ grows is related to the physical meaning of the resonance parameter Γ. The latter is provided by the ratio of two characteristic times: $\Gamma=\tau_r/\tau_\Delta$ where $\tau_r\sim e^{2d\alpha}$ is the decay time of a localized state into the conduction electron states and $\tau_\Delta\approx\hbar/\Delta$ define the correlation time of the electrons in a Cooper pair. If $\tau_\Delta\gg\tau_r$

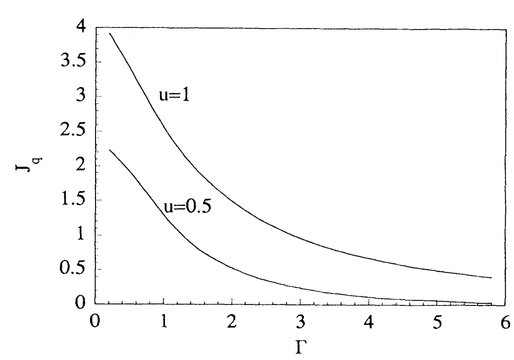

FIG. 2. The plot of $J_q=a\rho_nJ_0$ versus Γ for the particular bias voltages $u=1$ and 0.5 ($u=eV/\Delta$).

then the superconducting correlations in the weak link are not destroyed during Andreev reflections. In this case the perturbative approach (the tunneling Hamiltonian theory) breaks, all orders tunneling have to be considered, and the resonance tunneling processes assume importance. The same phenomenon occurred for Josephson current in the junctions with resonance tunneling via the localized level in the Kondo limit.[17] If the opposite inequality takes place $\tau_\Delta \ll \tau_r$ the Andreev reflections make only a small contribution to the subgap current, so the latter will be strongly suppressed.

In order to clarify the origin of the subgap current, we point out that it can be defined by resonance tunneling while the normal current for $eV > 2\Delta$ affects the direct tunneling region. To demonstrate this, let us introduce, similar to Γ, parameter $\Gamma_{dir} = \tau_{dir}/\tau_\Delta$ where $\tau_{dir} \sim e^{4d\alpha}$ represents the direct tunneling decay time. Since the values Γ, Γ_{dir} do not include the density of states, we expect $\Gamma_{dir} \gg \Gamma$, and thus in spite of $\rho_n > \rho_{dir}$, it is possible to provide a larger current (see Fig. 2) for the resonance tunneling.

We end this section by estimating for Γ in the case of a αSi|SiO$_x$ barrier.[1] For the resonant parameter we use Eq. (45) written in the terms of the Fermi momentum p_F and the coherence length ξ_0 of superconductors,

$$\Gamma \approx \frac{dp_F^2}{\pi \xi_0 \alpha^2} \exp(2d\alpha) .$$

Let us take Al ($\xi_0 = 1.8 \times 10^{-4}$ cm) as the superconducting electrodes with long coherence length. For the thickness of αSi barrier $2d = d_{cr} \approx 51$ Å, which belongs to the crossover between direct and indirect regimes[1] and the values of other parameters, $\alpha^{-1} = 8.25$ Å and $p_F \sim 1$ Å^{-1}, we have $\Gamma \sim 15$. For $2d = d_{cr} - 2.3\alpha^{-1}$ the normal Ohmic current will be dominated by the direct tunneling of electrons. As for the subgap current, we estimate $\Gamma \sim 1$ and $\rho_n \sim \rho_{dir} e^{2.3} \sim 10\rho_{dir}$, while $\Gamma_{dir} \approx 50 \gg \Gamma$. Current density J_0 for voltages below the gap will be defined by the tunneling via the resonant scatter. This is because the large Γ_{dir} drives the direct tunneling contribution to negligibly small values (see Fig. 2), a smallness which could not be overcome by the larger than ρ_n^{-1} value of conductivity ρ_{dir}^{-1}.

V. CONCLUSION

In this paper we have presented a quantitative analysis of the multiple Andreev reflections in Josephson tunnel junctions when the tunneling processes occur due to the localized state in the weak link. This approximation is justified if the effective radius of the localized state is less than the thickness of insulator which separates the superconductors. If $\alpha^{-1} \geq 2d$, then ordinary tunneling through the barrier is relevant and I-V characteristics become the functions of two parameters: direct tunneling probability across an insulating layer and the resonance tunneling via a localized state. It is possible to generalize our formulas for the current; in this case, however, the numerical calculations are more complicated. We have, moreover, assumed noninteracting quasiparticles. This approximation is relevant for the tunneling structures with rather small resonant parameter, i.e., in cases in which decaying time determined by the resonance tunneling is less than the time of superconducting correlations. The influence of Coulomb repulsion on the resonant supercurrent was studied in Ref. 17. A generalization of this theory on a nonequilibrium case considering multiple Andreev reflections in SIS Josephson junctions would be interesting. We shall leave this case for future study.

Starting with the equation for the exact Green's function of the SIS three layers system we made several approximations to achieve a solution for this equation. We have considered that the localized states do not cause self-energy corrections to the quasiparticle energy in the electrodes, an approximation which restricts our analysis to the short-range potential of localized states.

In the above analysis we took a simple time dependence for the phase of the order parameter [Eq. (16)]. This is not a self-consistent description. However, because the voltage drop is mainly in the region of weak link, the approximation is qualitatively correct.

The dependence of the I-V characteristics from resonant tunneling parameter Γ shows that we have a larger subgap current for smaller Γ. Our theory, which is based on BCS approximation, can be applied to the high-T_c superconductors to estimate subgap current density, though a possible symmetry of the order parameter, especially $d_{x^2-y^2}$ symmetry, has yet to be considered.

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