Analytical results for giant Shapiro steps in Josephson arrays

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We study two-dimensional Josephson arrays driven by a combined dc plus ac current, and with an applied transverse magnetic field of f flux quanta per plaquette. We present *ansatz* solutions for sufficiently large frequencies, which are a generalization of the traveling wave solutions found by Marino and Halsey for the case of dc current driving. For $f=\frac{1}{2}$ and $\frac{1}{3}$, we compute the widths of the first few Shapiro steps for both integer and fractional winding numbers. These expressions consist of products of Bessel functions of (i_{ac}/ω_{ac}), where i_{ac} and ω_{ac} are the amplitude and the frequency of the driving ac current, respectively, times a frequency-dependent factor for fractional steps. In the limit of large frequencies, we find that the fractional steps are suppressed, whereas the maximum integer step widths saturate to a frequency-independent value. We show that the suppression of the fractional steps is due to decrease of the vertical (i.e., perpendicular to the direction of flow of the injected current) supercurrent relative to the normal current, whereas the persistence of the integer steps is due to the existence of zero-frequency (though spatially varying) terms in the expansion for the gauge-invariant phase differences, for which the normal current vanishes. These results are in reasonable agreement with the numerical computations carried out by other groups.

I. INTRODUCTION

When a resistively shunted Josephson junction is driven by a combined dc and ac current $I(t)=I_{dc}+I_{ac}\sin(\Omega_{ac}t)$, the current-voltage characteristic exhibits plateaus in which the time-averaged voltage is equal to an integer multiple of $\hbar\Omega_{ac}/2e$ for a finite interval of dc current. These plateaus are called *Shapiro steps.*[1] An analogous effect is observed when the current is applied to an $N\times N$ square array of Josephson junctions with transverse magnetic flux per plaquette $\Phi=f\Phi_0$, where $\Phi_0=\hbar c/2e$ is the quantum flux, and $f=p/q$ is the frustration, p and q being relatively prime integers. The total voltage across the array is locked at values given by

$$V_N=\frac{n}{q}\frac{N\hbar\Omega_{ac}}{2e}\ . \tag{1}$$

In this case, the steps are called *fractional giant Shapiro steps.*[2] The accepted explanation of this effect is that the $q\times q$ periodic vortex superlattice moves coherently in response to the external ac field.[2,3]

If the parallel shunt resistance is R, and the critical current per junction is i_0, we can then define a dimensionless time $\tau=(2ei_0R/\hbar)t$. Measured in these time units, the external ac frequency is $\omega_{ac}\equiv\hbar\Omega_{ac}/2ei_0R$. The Josephson frequency is defined by $\Omega_J\equiv 2eV_N/N\hbar$, with its normalized version given by $\omega_J\equiv V_N/i_0RN$. In terms of them, the above relation can simply be expressed as $\omega_J=\nu\omega_{ac}$, where $\nu\equiv n/q$ is the winding number.

These steps display a variety of characteristics. It has been observed that there exists a qualitative difference between the cases where ν is an integer *(integer steps),* and when it is a fraction *(fractional steps).*[2–5] This difference becomes manifest when going from low to high frequencies. At low frequencies, both fractional and integer steps behave qualitatively as in the single-junction case. In the high-frequency limit, on the other hand, fractional steps are suppressed, whereas the maximum integer step widths saturate to a frequency-independent value.[4–6]

This phenomenon has been widely studied by several authors.[2–7] Analytical treatments have been provided by Halsey,[7] Lee and Halsey,[6] and by Rzchowski, Sohn, and Tinkham.[4] Nevertheless, a theoretical derivation based on first principles is still lacking. The missing link in the understanding of this problem has been the knowledge of the solutions for dc plus ac current driving. Progress in this direction has already been made. The existence of a family of traveling-wave solutions for dc current-driven arrays has been reported by Marino and Halsey[8] in the limit of high Josephson frequencies. In this paper we present a generalization of these traveling-wave states to the case where the array is driven by an additional ac current. For $f=\frac{1}{2}$, these are also solutions to the model equations used by Rzchowski, Sohn, and Tinkham.[4] In addition to the modes corresponding to the Josephson frequency ω_J and its harmonics, these solutions contain terms oscillating with frequencies given by linear combinations of ω_J and ω_{ac}. If one then computes the current flowing across the entire array, one finds that only the terms with frequencies given by $(k_1q\omega_J+m\omega_{ac})$ survive, where k_1 and m are integers. This is due to the fact that these terms are exactly on phase everywhere on the array, whereas the other terms have phases such that their sums vanish. Shapiro steps result when the linear combination of frequencies is zero. We then derive expressions for the step widths as a function of i_{ac} and ω_{ac} for $f=\frac{1}{2}$ and $f=\frac{1}{3}$, by computing the ensuing dc supercurrent corresponding to these modes across the array for the first few steps. These expressions

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consist of products of q Bessel functions of (i_{ac}/ω_{ac}), and other factors that depend solely on the frequency and an arbitrary constant phase ψ_0.

The main idea ensuing from this analysis is that the difference between the behaviors of the fractional and integer steps at low and high frequencies is determined by the relative size of the vertical (perpendicular to the direction of flow of the injected current) supercurrent relative to the normal current for the different frequency modes contributing to a given step width. At low frequencies the vertical supercurrent is dominant for all modes, and hence both integer and fractional steps behave in the same manner. When the frequency is increased, all the modes with nonvanishing normal currents decrease. The suppression of the fractional steps at high frequencies appears as a consequence of their dependence on these modes, whereas the persistence of the integer steps is due to the existence of zero-frequency modes (and therefore with vanishing normal currents) whose amplitudes are determined by the vertical supercurrent. Consequently, integer steps behave in the same fashion in both the low- and high-frequency regimes. This is the same mechanism that yields the Shapiro steps for a single Josephson junction.

Our solutions for the gauge-invariant phase differences contain the arbitrary phase ψ_0. On a given step, this phase varies over an interval that we hypothesize to be frequency dependent. At high frequencies we shall assume that this interval attains a constant size, which in principle can only be determined by fitting the numerical results to the theoretical predictions. Our ignorance about the details of the nature of the solutions at low frequencies does not allow us to provide expressions for the step widths in this regime. Our study, thus, does not address the problem of trying to find the true variation of this interval with the frequency.

In Sec. II we review the solutions for the dc case and present their generalization to the case of dc plus ac current driving. In Sec. III we use these solutions as the starting point to compute the widths of the first few Shapiro steps for $f=\frac{1}{2}$ and $\frac{1}{3}$. Finally, in Sec. IV we provide results from numerical computations.

II. SOLUTIONS FOR dc PLUS ac CURRENT DRIVING

We consider a square array of $N\times N$ overdamped resistively shunted Josephson junctions in a uniform transverse magnetic field with f flux quanta piercing each plaquette, parallel shunt resistance R, and critical current per junction i_0. We define the gauge-invariant phase differences by $\theta_{ij}\equiv\theta_i-\theta_j-A_{ij}$, where θ_i is the superconducting phase on the ith site on the array, A_{ij} is the line integral of the magnetic vector potential, $A_{ij}=(2\pi/\Phi_0)\int_i^j \mathbf{A}\cdot d\mathbf{x}$, such that $\sum_P A_{ij}=2\pi f$, where the sum is around a plaquette, and j denotes a site that is nearest neighbor with i. Then the current flowing from the ith to the jth site is

$$\tilde{I}_{ij}=\frac{d}{d\tau}(\theta_{ij})+\sin(\theta_{ij}) , \tag{2}$$

where $\tilde{I}_{ij}\equiv I_{ij}/i_0$. The first term is the normal current, while the second one represents the supercurrent. The equations of motion simply express the fact that the total current arriving at each site on the array should equal the current externally injected there

$$\sum_j \tilde{I}_{ij}=i_{i;\text{ext}} , \tag{3}$$

where the external current $i_{i;\text{ext}}$ vanishes everywhere on the array, save for at the boundaries. Henceforth, we shall take the convention that when computing the gauge-invariant phase differences on horizontal bonds j is to be taken to the right of i, and above it on vertical bonds.

For the case in which the system is driven by a uniform dc current injected parallel to one of the axes of the array (which we take to be the horizontal axis with the current flowing from right to left), and with periodic boundary conditions along the vertical direction, Marino and Halsey[8] reported the existence of a family of traveling-wave solutions. These solutions are characterized by a parameter δ that measures the phase shift of the phase oscillations along the horizontal direction. Along the vertical direction the phase shift is simply equal to $2\pi f$, which is consistent with the condition of transverse periodic boundary conditions with period q. These solutions possess a combined spatiotemporal translational symmetry, in the sense that a translational of the solution by one lattice spacing along the horizontal direction is equivalent to the translation of the solution by a time $\tau=\delta/\omega_J$, and a translation of the solution by one lattice spacing along the vertical direction is equivalent to a translation of the solution by a time $\tau=2\pi f/\omega_J$.

Let us define

$$\psi\equiv\omega_J t+2\pi f n_Y+\delta n_X+\psi_0 , \tag{4}$$

where n_X and n_Y are integers and ψ_0 is a constant phase. Then the gauge-invariant phase differences on horizontal and vertical bonds for these solutions are given by

$$\theta_H(\psi)=\psi+f_H(\psi) , \tag{5}$$

$$\theta_V(\psi)=f_V(\psi) , \tag{6}$$

where f_H and f_V are periodic functions with period 2π and zero average. The authors of Ref. 8 worked out the analytical form of these functions in the limit of high voltages, by retaining only the first harmonic in their Fourier expansion.

The generalization of these solutions to the case of combined dc and ac current driving is straightforward. The main modification is that now, in addition to the mode with Josephson frequency ω_J, a mode with frequency ω_{ac} should also be present, due to the external ac current. The presence of this second time scale ruins the spatiotemporal translational symmetry in the cases in which ω_J and ω_{ac} are incommensurate. This entails no problem, as we shall see. The beating of these two modes due to the horizontal supercurrent requires the additional presence of terms with frequencies given by linear combinations of ω_J and ω_{ac} in the expansion for the gauge-invariant phase differences. These solutions must further satisfy several conditions. First, they should reduce to their dc counterparts when i_{ac} and ω_{ac} are set equal to zero; secondly, in order to (indirectly) enforce the boundary conditions, the dc and ac currents (at ω_{ac} and its harmonics) flowing on each horizontal bond should be independent of position, and equal to the values given by the currents injected at the boundaries, whereas on vertical bonds they should vanish, allowing for the possible existence of zero-frequency modes (that occur only on integer Shapiro steps), which should not be regarded strictly as dc currents; and finally the linear term should remain unchanged, because it is still true that the slope should yield the average voltage per junction. Our ansatz for the gauge-invariant phase differences on horizontal bonds then takes the form

$$\theta_H(n_X,n_Y,\tau)=\psi+\sum_{n,m}\alpha^H_{n,m}\cos(n\psi+m\omega_{ac}\tau+\xi^H_{n,m}) , \tag{7}$$

where n (≥ 0) and m are integers, while on vertical bonds we have

$$\theta_V(n_X,n_Y,\tau)=\sum_{n\neq 0,m}\alpha^V_{n,m}\cos(n\psi+m\omega_{ac}\tau+\xi^V_{n,m}) , \tag{8}$$

where ψ is given by Eq. (4), and $\xi^{H,V}_{n,m}$ are constant phases. The phase differences are taken according to our convention. The different terms in this expansion are labeled by the two integers n and m. We shall refer to this component of the phases and currents as the (n,m) mode. Notice that the modes with $n=0$ have no spatial dependence, in agreement with our assumption. This form of solution is good enough to describe the system even at low frequencies.

The gauge-invariant phase differences are not independent. The sum of their oscillating parts around a plaquette has to vanish. We impose this condition to each frequency mode and obtain

$$\alpha^V_{n,m}=\beta\alpha^H_{n,m} , \tag{9}$$

$$\xi^V_{n,m}=\xi^H_{n,m}+n(\pi f-\delta/2) , \tag{10}$$

where $\beta=\sin n\pi f/\sin(n\delta/2)$ if $n\neq\dot{q}$ ($n=\dot{q}$ is a shorthand for $n=q$ mod0) and 0 otherwise. Thus the modes with $n=\dot{q}$ are absent on vertical bonds. This result is the same one that was obtained in the dc case. The reader is referred to Ref. 8 for the details of the derivation.

On the Shapiro steps we will assume that $\delta=2\pi f$, due to our requirement of ac translational invariance [see the discussion preceding Eq. (7)]. This implies $\beta=1$ for $n\neq\dot{q}$, in which case α and ξ are the same on both horizontal and vertical bonds, which considerably simplifies matters. Consequently, we shall hereafter drop the superscripts H and V in our expressions.

We shall perform a mode expansion of the horizontal supercurrent $\sin\theta_H$ in the following manner:

$$\sin\left[\psi+\sum_{n,m}\alpha_{n,m}\cos(n\psi+m\omega_{ac}\tau+\xi_{n,m})\right] = \sum_{n,m}S_{n,m}\cos(n\psi+m\omega_{ac}\tau+\Xi_{n,m}) . \tag{11}$$

The different components of the supercurrent can be computed in a straightforward manner by performing a Fourier-Bessel expansion of the left-hand side of Eq. (11). A generic term in this expansion is of the form

$$S_{n,m}\cos(n\psi+m\omega_{ac}\tau+\Xi_{n,m})=\sum_{\{(n_j,m_j)\}}\text{Im}\left\{\left[\prod J_{k_j}(\alpha_{n_j,m_j})i^{k_j}\right]\exp\left[i\psi+i\sum_j k_j(n_j\psi+m_j\omega_{ac}\tau+\xi_{n_j,m_j})\right]\right\} . \tag{12}$$

The sum is over the set of sets of pairs $\{\{(n_j,m_j)\}\}$ such that there exists a set of coefficients $\{k_j\}$ for which the following relationship (understood as a vector identity) holds:

$$(n,m)=(1,0)_L+\sum_j k_j(n_j,m_j) , \tag{13}$$

where $(1,0)_L$ denotes the contribution due to the linear term, which is absent on vertical bonds. These expressions are in general quite complicated. We shall assume that they can approximately be computed starting from the lowest-order (in n and m) modes, since the magnitude of the different Bessel functions decays quickly with increasing order. This approximation should be good enough at high frequencies, but we do not expect it to remain accurate for lower frequencies. The feedback of the higher-order modes on the lower ones should become more important as one approaches the critical current. On vertical bonds the (n,m) component of the supercurrent is (neglecting higher-order corrections) equal to $2J_1(\alpha_{n,m})$. It can be shown for simple cases that other contributions vanish.

We now turn to the equations of current conservation [Eq. (3)]. Once again, we have to distinguish between the cases $n\neq\dot{q}$ and $n=\dot{q}$. In the former case, the equation for current conservation for our ansatz solution is

$$-2\omega_{n,m}\alpha_{n,m}\sin(n\psi+m\omega_{ac}\tau+\xi_{n,m}) +2J_1(\alpha_{n,m})\cos(n\psi+m\omega_{ac}\tau+\xi_{n,m}) +S_{n,m}\cos(n\psi+m\omega_{ac}\tau+\Xi_{n,m})=0 , \tag{14}$$

where $\omega_{n,m}=n\omega_J+m\omega_{ac}$. There is an overall factor of $2\sin\pi f$ that goes away. The first term in the above equation represents the combined effect of both the horizontal and vertical normal currents, each of them contributing the same amount; the second one is due to the vertical supercurrent, and the last one comes from the horizontal supercurrent, which has to be computed for each mode. For $n=\dot{q}$ the current is trivially conserved at each site. Furthermore, the current corresponding to the (0,1) mode should equal the external ac current:

$$-\omega_{ac}\alpha_{0,1}\sin(\omega_{ac}\tau+\xi_{0,1})+S_{0,1}\cos(\omega_{ac}\tau+\Xi_{0,1}) = -i_{ac}\sin(\omega_{ac}\tau) \; . \tag{15}$$

For convenience, and without loss of generality, we have introduced a minus sign at the right-hand side of this equation. If we neglect the vertical supercurrent and replace the factor of 2 multiplying the first term in Eq. (14) by 1, then these two last equations describe an overdamped single junction.

In general, we expect $\alpha_{0,1} \sim i_{ac}/\omega_{ac}$, and the asymptotic behavior of the different amplitudes at high and low frequencies to be of the form

$$\alpha_{n,m} \sim h(\omega_J) \prod_{\sum n_i k_i = m} J_{n_i}^{k_i}(\alpha_{0,1}) \tag{16}$$

for $n \neq \dot{q}$, where $h(x) \sim c$ (c = const ≤ 1), as $x \to 0$, and $h(x) \sim x^{-n}$ as $x \to \infty$. This will be made clear below. Consequently, it is safe to assume (save for $\alpha_{0,1}$) that $J_1(\alpha_{n,m}) \approx \alpha_{n,m}/2$. Equation (14) can then be solved in terms of the expression for the horizontal supercurrent:

$$\alpha_{n,m} = \frac{S_{n,m}}{\sqrt{4\omega_{n,m}^2+1}} \; , \tag{17}$$

$$\xi_{nm} = \pi + \Xi_{n,m} - \arctan(2\omega_{n,m}) \; . \tag{18}$$

For $\omega_{m,m} \gg 1$ the effect of the vertical supercurrent can be neglected and $\alpha_{n,m} \sim S_{n,m}/(2\omega_{n,m})$, whereas for $\omega_{n,m} \ll 1$ the vertical supercurrent dominates and $\alpha_{n,m} \sim S_{n,m}$. Equation (16) can then be proven in the following manner. Since $(0,m)=(1,0)_L + m(0,1)$ then to leading order $S_{0,m} = J_m(i_{ac}/\omega_{ac})$. From Eq. (17) it follows that $\alpha_{0,m}$ has the asymptotic behavior given by Eq. (16). The proof for $\alpha_{n,m}$ can be made by induction.

In the present calculation we will neglect the supercurrent in Eq. (15). This is a good approximation for $\omega_{ac} \gg 1$. Thence, $\alpha_{0,1} = i_{ac}/\omega_{ac}$ and $\xi_{0,1}=0$. Using this, $(1,0)=(1,0)_L+0(0,1)$, and $J_0(\alpha_{n,m}) \approx 1$ for all the other modes, we find

$$\alpha_{1,0} = \frac{J_0(i_{ac}/\omega_{ac})}{\sqrt{4\omega_J^2+1}} \; , \tag{19}$$

$$\xi_{1,0} = \frac{\pi}{2} - \arctan(2\omega_J) \; . \tag{20}$$

This solution also holds in the dc case ($i_{ac}=0$), and represents a generalization of the solutions presented in Ref. 8. Unlike them, this solution remains regular as $\omega_J \to 0$, owing to the vertical supercurrent. Similarly, $(1,\pm 1)=(1,0)_L \pm (0,1)$. Then,

$$\alpha_{1,\pm 1} = \frac{J_1(i_{ac}/\omega_{ac})}{\sqrt{4(\omega_J \pm \omega_{ac})^2+1}} \; , \tag{21}$$

$$\xi_{1,\pm 1} = \pi - \arctan[2(\omega_J \pm \omega_{ac})] \; . \tag{22}$$

Other modes can be computed in a similar fashion.

III. THE SHAPIRO STEPS

It is clear that both the normal current and the supercurrent have the same harmonic dependence as the gauge-invariant phase differences. In particular, this implies that when computing the total voltage and supercurrent across the array only the modes with $n = k_1 q$ survive. It can immediately be checked that all the other terms cancel out. This can be interpreted in terms of the vortex configuration by saying that a vortex moves q times during a period of $2\pi/\omega_J$. This is key in order to understand the phenomenon of the Shapiro steps.

Looking back at Eq. (11), in the presence of an external ac current, an additional dc supercurrent will appear across the array for frequencies satisfying $k_1 q \omega_J + m\omega_{ac} = 0$, with k_1 and m relative primes. The Shapiro steps ensue. We see that the proposed rigid motion of the $q \times q$ vortex superlattice in response to the external ac field has a very natural explanation within our theory. For $k_1 > 1$ we have subharmonic steps. We will not consider this possibility here, because these steps are in general too small to be observed.

The first step in our calculation is to identify the modes that yield the largest contribution to the dc supercurrent corresponding to the different Shapiro steps. For integer steps ($\nu = n$) the choice is unambiguous: it is the set of zero-frequency modes $(k', -k'n)$, where $1 \leq k' < q$. These modes have the virtue that their associated normal currents vanish, and thus they are determined by the vertical supercurrent. The Shapiro step widths in this case depend only on (i_{ac}/ω_{ac}). The case of fractional steps ($\nu = n/q$) can be analyzed in a similar manner. The most important contributions are due to the modes (n', n''), with $n' \leq n$ and $n'' < q$. The amplitudes of these modes decrease with the frequency because they are mainly determined by the normal current.

The distinction between the low- and high-frequency behaviors just amounts to saying that at low frequencies both fractional and integer steps are in a supercurrent-dominated regime, whereas at high frequencies only the integer steps are, thanks to the zero-frequency modes. Saying that the steps display single-junction behavior is just another way of rephrasing this fact.

All of our expressions depend on the arbitrary phase ψ_0. In particular, the dc supercurrent on a given step depends on this phase, and thus the width of the step will depend on the range of variation of it. The interval of variation of this phase should depend on the dynamic stability of these solutions and also on the nature of the solution for the (0,1) mode at low frequencies [recall that we neglected the supercurrent term in Eq. (15)]. We conjecture that the size of this interval varies with the frequency, growing from zero to an interval of constant size at high frequencies, and that this is the mechanism underlying the growth and saturation of the steps. This assumption seems to be good enough to reproduce all the observed qualitative features of the steps. According to this, our results for a given frequency should differ at most by a constant factor (for all values of the i_{ac}) from the results obtained from simulations or experiments. We shall not attempt to resolve this issue here; rather, we shall assume that the range of this phase is an adjustable parameter that is to be determined by fitting the observed data to the theoretical prediction. Nevertheless, a rough estimate can be made by using a generalization of an ansatz used by Halsey.[7] Our results have a factor of the form $\cos(q\psi_0+\phi_{q,\nu})$. We shall assume that at high frequencies $(q\psi_0+\phi_{q,\nu})$ is centered at $\pi/2$ for q even, and at 0 for q odd. Furthermore, for both q even and odd, we shall assume that this phase varies over the interval $[-\pi/2,\pi/2]$ for fractional steps, and over $[-\pi/2q,\pi/2q]$ for integer steps. We shall thus restrict ourselves to making estimates of values of the step widths only for large enough frequencies.

We now turn to specific examples. We shall only consider the cases $f=\frac{1}{2}$ and $\frac{1}{3}$, because the number of modes that have to be included in a given calculation increases quickly with q.

A. Integer steps: $\nu=n$

1. $f=\frac{1}{2}$

A dc supercurrent occurs at the mode

$$(2,-2n)=(1,0)_L+(1,-n)-n(0,1)\ .$$

In this case $\alpha_{1,-n}=J_n(i_{\rm ac}/\omega_{\rm ac})$, and $\xi_{1,-n}=(n+1)\pi/2$. We thus obtain the expression

$$\begin{aligned} i_{1/2,n}&=J_1(\alpha_{1,-n})J_n(\alpha_{0,1})\sin[2\psi_0+(2n+1)\pi/2]\\ &=\tfrac{1}{2}[J_n(i_{\rm ac}/\omega_{\rm ac})]^2\sin[2\psi_0+(2n+1)\pi/2]\ , \end{aligned} \tag{23}$$

where $i_{1/2,n}$ denotes the ensuing dc supercurrent. This is not yet the expression for the step width. The final answer depends on the range of variation of ψ_0. Using our ansatz for this variation at high frequencies we find

$$\Delta i_{1/2,n}=\frac{\sqrt{2}}{2}[J_n(i_{\rm ac}/\omega_{\rm ac})]^2\ . \tag{24}$$

2. $f=\frac{1}{3}$

Here we have two contributions, namely,

$$(3,-3n)=(1,0)_L+(2,-2n)-n(0,1)\ ,$$

and

$$(3,-3n)=(1,0)_L+2(1,-n)-n(0,1)\ .$$

Now, $\alpha_{2,-2n}=[J_n(i_{\rm ac}/\omega_{\rm ac})]^2/2$ and $\xi_{2,-2n}=(n+3/2)\pi$. The total dc supercurrent corresponding to this step is

$$\begin{aligned} i_{1/3,n}&=J_n(i_{\rm ac}/\omega_{\rm ac})J_2(\alpha_{1,-n})\sin(3\psi_0+3n\pi/2)\\ &\quad+J_1(\alpha_{2,-2n})J_n(i_{\rm ac}/\omega_{\rm ac})\sin(3\psi_0+3n\pi/2)\\ &=\tfrac{3}{8}[J_n(i_{\rm ac}/\omega_{ac})]^3\sin(3\psi_0+3n\pi/2)\ . \end{aligned} \tag{25}$$

At high frequencies this becomes

$$\Delta i_{1/3,n}=\tfrac{3}{16}[J_n(i_{\rm ac}/\omega_{\rm ac})]^3\ . \tag{26}$$

In conclusion, we find that the integer step widths are independent of the frequency. It should be clear that in general $i_{p/q,n}\sim[J_n(i_{\rm ac}/\omega_{\rm ac})]^q$. For $q=1$, this reduces to the single-junction result, or equivalently to the result for the unfrustrated case $f=0$.

B. Fractional steps: $\nu=n/q$

We need to compute the $(q,-n)$ component of the horizontal supercurrent. As mentioned above, the most important contributions come from the modes (n',n'') such that $n'<q$ and $n''\le n$. The number of modes that are relevant for the determination of the steps grows quickly with n though, so we will only work out explicitly a few simple cases.

1. $f=\frac{1}{2},\nu=\frac{1}{2}$ or $\frac{3}{2}$

a. $\nu=\frac{1}{2}$. We can write either

$$(2,-1)=(1,0)_L+(1,-1)+0(0,1)$$

or

$$(2,-1)=(1,0)_L+(1,0)-(0,1)\ .$$

Omitting some details we get

$$\begin{aligned} i_{1/2,1/2}&=-J_0(\alpha_{0,1})J_1(\alpha_{1,-1})\cos(2\psi_0+\arctan\omega_{\rm ac})\\ &\quad-J_1(\alpha_{1,0})J_1(\alpha_{0,1})\cos(2\psi_0-\arctan\omega_{\rm ac})\ . \end{aligned} \tag{27}$$

Using the expressions for $\alpha_{1,0}$, $\alpha_{0,1}$, and $\alpha_{1,-1}$, this turns into

$$i_{1/2,1/2}=-\frac{J_0(i_{\rm ac}/\omega_{\rm ac})J_1(i_{\rm ac}/\omega_{\rm ac})}{(\omega_{\rm ac}^2+1)}\cos(2\psi_0) \tag{28}$$

and at high frequencies

$$\Delta i_{1/2,1/2}=\frac{2J_0(i_{\rm ac}/\omega_{\rm ac})J_1(i_{\rm ac}/\omega_{\rm ac})}{(\omega_{\rm ac}^2+1)}\ . \tag{29}$$

We find that the step width decays like $1/\omega_{\rm ac}^2$ at high frequencies, in disagreement with what has been assumed by other authors.[4,5] At low frequencies, this result reduces to a frequency-independent expression, which is characteristic of single-junction behavior. It is also possible to compute higher-order corrections [in Bessel functions of $(i_{\rm ac}/\omega_{\rm ac})$] by considering the combinations of modes

$$\begin{aligned} (2,-1)&=(1,0)_L+(1,1)-2(0,1)\\ &=(1,0)_L+(1,-2)+(0,1)\ . \end{aligned}$$

It is not hard to see that these corrections vary as

$$J_1(i_{\rm ac}/\omega_{\rm ac})J_2(i_{\rm ac}/\omega_{\rm ac})/(9\omega_{\rm ac}^2+1)\ .$$

This is negligible compared to the expression given in Eq. (29) for most cases of interest.

b. $\nu=\frac{3}{2}$. In this case we have

$$(2,-3)=(1,0)_L+(1,-1)-2(0,1)$$

and

$$(2,-3)=(1,0)_L+(1,-2)+(1,-1)+0(0,1)\ .$$

We will neglect other contributions. The calculation is identical to the previous one and yields

$$i_{1/2,3/2}=\frac{J_1(i_{ac}/\omega_{ac})J_2(i_{ac}/\omega_{ac})}{(\omega_{ac}^2+1)}\cos(2\psi_0)\ . \quad (30)$$

In the limit of high frequencies

$$\Delta i_{1/2,3/2}=\frac{2J_1(i_{ac}/\omega_{ac})J_2(i_{ac}/\omega_{ac})}{(\omega_{ac}^2+1)}\ . \quad (31)$$

The next-order correction varies like $J_0(i_{ac}/\omega_{ac})J_3(i_{ac}/\omega_{ac})$, which can be neglected. We see that this step width has the same dependence on ω_{ac} as for $\nu=1/2$.

2. $f=\frac{1}{3}$, $\nu=\frac{1}{3}$ or $\frac{2}{3}$

a. $\nu=\frac{1}{3}$. This calculation involves higher-order modes than the ones hitherto used. The number of terms, therefore, considerably increases. In fact, we now have the possible combinations

$$(3,-1)=(1,0)_L+(1,-1)+(1,0)+0(0,1)\ ,$$

$$(3,-1)=(1,0)_L+2(1,0)-(0,1)\ ,$$

$$(3,-1)=(1,0)_L+(2,0)-(0,1)\ ,$$

and

$$(3,-1)=(1,0)_L+(2,-1)+0(0,1)\ .$$

There are two contributions to the $(2,-1)$ mode, which will be considered separately. These are

$$(2,-1)=(1,0)_L+(1,-1)+0(0,1)$$
$$=(1,0)_L+(1,0)-(0,1)\ .$$

Putting everything together we obtain

$$i_{1/3,1/3}=\frac{J_0^2(i_{ac}/\omega_{ac})J_1(i_{ac}/\omega_{ac})}{4\sqrt{4\omega_{ac}^2/9+1}}\times\left[\frac{\cos[3\psi_0-2\arctan(2\omega_{ac}/3)]}{2\sqrt{4\omega_{ac}^2/9+1}}+\frac{\cos[3\psi_0+\arctan(2\omega_{ac}/3)+\arctan(4\omega_{ac}/3)]}{\sqrt{16\omega_{ac}^2/9+1}}\right.$$
$$+\frac{\cos[3\psi_0+\arctan(4\omega_{ac}/3)-\arctan(2\omega_{ac}/3)]}{\sqrt{16\omega_{ac}^2/9+1}}$$
$$\left.+\frac{\cos[3\psi_0-\arctan(4\omega_{ac}/3)-\arctan(2\omega_{ac}/3)]}{\sqrt{16\omega_{ac}^2/9+1}}+\frac{\cos(3\psi_0)}{\sqrt{4\omega_{ac}^2/9+1}}\right]\ . \quad (32)$$

After some algebraic manipulations this becomes

$$i_{1/3,1/3}=\frac{J_0^2(i_{ac}/\omega_{ac})J_1(i_{ac}/\omega_{ac})}{4(4\omega_{ac}^2/9+1)}\left[\left[\frac{3-8\omega_{ac}^2/9}{(16\omega_{ac}^2/9+1)}+\frac{2\omega_{ac}^2/9+\frac{3}{2}}{(4\omega_{ac}^2/9+1)}\right]\cos(3\psi_0)\right.$$
$$\left.+\left[\frac{2\omega_{ac}/3}{(4\omega_{ac}^2/9+1)}-\frac{2\omega_{ac}/3}{(16\omega_{ac}^2/9+1)}\right]\sin(3\psi_0)\right]\ . \quad (33)$$

In the limit of high frequencies this is

$$i_{1/3,1/3}=\frac{81}{128\omega_{ac}^3}J_0^2(i_{ac}/\omega_{ac})J_1(i_{ac}/\omega_{ac})\sin(3\psi_0)\ , \quad (34)$$

and with our ansatz for ψ_0

$$\Delta i_{1/3,1/3}=\frac{81}{128\omega_{ac}^3}J_0^2(i_{ac}/\omega_{ac})J_1(i_{ac}/\omega_{ac})\ , \quad (35)$$

whereas in the low-frequency limit we find

$$i_{1/3,1/3}=\tfrac{9}{8}J_0^2(i_{ac}/\omega_{ac})J_1(i_{ac}/\omega_{ac})\cos(3\psi_0)\ . \quad (36)$$

b. $\nu=\frac{2}{3}$. The number of modes to be included in the calculation keeps growing, as promised:

$$(3,-2)=(1,0)_L+2(1,-1)+0(0,1)\ ,$$

$$(3,-2)=(1,0)_L+2(1,0)-2(0,1)\ ,$$

$$(3,-2)=(1,0)_L+(2,-1)-(0,1)\ ,$$

$$(3,-2)=(1,0)_L+(2,0)-2(0,1)\ ,$$

$$(3,-2)=(1,0)_L+(2,-2)+0(0,1)\ ,$$

$$(3,-2)=(1,0)_L+(1,-1)+(1,0)-(0,1)\ ,$$

and

$$(3,-2)=(1,0)_L+(1,-2)+(1,0)+0(0,1)\ .$$

In order to carry out the calculation we need $(1,-2)=(1,0)_L-2(0,1)$ and

$$(2,-2)=(1,0)_L+(1,-1)-(0,1)$$
$$=(1,0)_L+(1,0)-2(0,1)$$
$$=(1,0)_L+(1,-2)+0(0,1)\ .$$

The calculation is analogous to the one done in the previous section. In the high-frequency limit we obtain

$$i_{1/3,2/3}=\frac{81}{128\omega_{ac}^{3}}J_0(i_{ac}/\omega_{ac})\times[J_1^2(i_{ac}/\omega_{ac})+J_0(i_{ac}/\omega_{ac})J_2(i_{ac}/\omega_{ac})/8]\times\cos(3\psi_0)\ , \tag{37}$$

and after the usual assumption for ψ_0

$$\Delta i_{1/3,2/3}=\frac{81}{128\omega_{ac}^{3}}J_0(i_{ac}/\omega_{ac})\times[J_1^2(i_{ac}/\omega_{ac})+J_0(i_{ac}/\omega_{ac})J_2(i_{ac}/\omega_{ac})/8]\ . \tag{38}$$

We have included higher-order corrections in Bessel functions that this time yield a non-negligible contribution. Once again, the step width decays as $1/\omega_{ac}^3$. The low-frequency limit yields

$$i_{1/3,2/3}=\frac{9J_0(i_{ac}/\omega_{ac})}{8}\times[J_1^2(i_{ac}/\omega_{ac})+J_0(i_{ac}/\omega_{ac})J_2(i_{ac}/\omega_{ac})]\times\sin(3\psi_0)\ . \tag{39}$$

We get the expected single-junction behavior.

We see that at high frequencies the step widths for $f=\frac{1}{3}$ decrease like $1/\omega_{ac}^3$. The terms that vary like $1/\omega_{ac}^2$ cancel out. Compare this result with the ones obtained for $f=\frac{1}{2}$. In those cases we found that the step width varied like $1/\omega_{ac}^2$. The terms varying like $1/\omega_{ac}$ also did cancel out. This result seems to be quite general, and there is a way of understanding it. The fractional steps in the present case correspond to the subharmonic steps for a single junction. We can say that the vertical super-

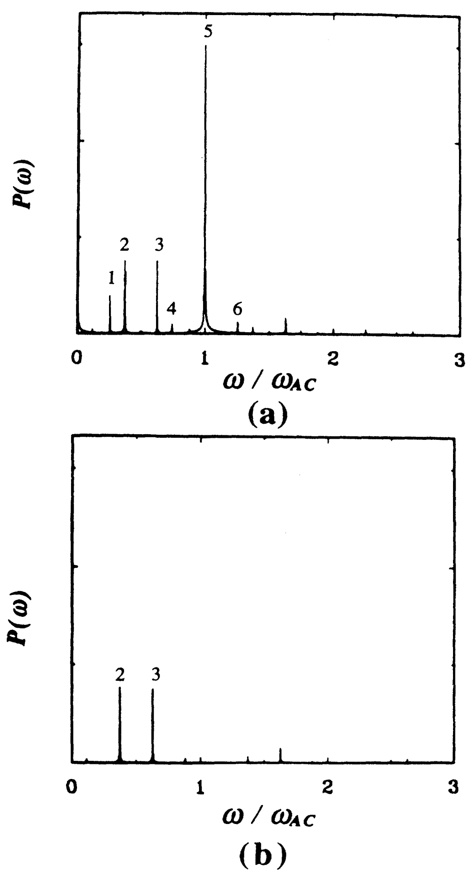

FIG. 1. (a) Power spectrum $P(\omega)$ of horizontal phase oscillations for $f=\frac{1}{2}$, away from any step. The peaks 1–6 correspond to the frequencies $(2\omega_J-\omega_{ac})$, $(\omega_{ac}-\omega_J)$, ω_J, $(2\omega_{ac}-2\omega_J)$, ω_{ac}, and $2\omega_J$, respectively. (b) Power spectrum of vertical phase oscillations. Notice the absence of peaks 1, 4, 5, and 6, which is in accord with the selection rule for the existence of modes on vertical bonds [see the remark following Eq. (10)].

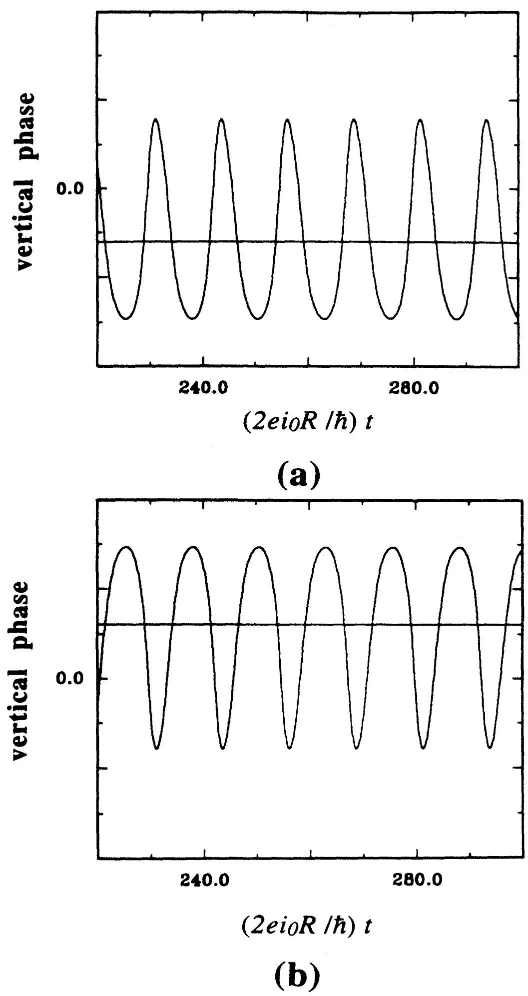

FIG. 2. Gauge-invariant phase differences on two vertical bonds lying on the same column for $f=\frac{1}{2}$ on the first integer step. They are out of phase by $\delta=\pi$, as assumed in our work. The average value of the phase oscillations is nonvanishing and is depicted by a solid line in the figures. This represents the zero-frequency mode, which changes phase by π when translated by one lattice spacing on the array.

current plays the role of an effective additional degree of freedom, analogous to the role played by the capacitive term for a single junction, in which case subharmonic steps do appear. At high frequencies the vertical supercurrent becomes negligible, and then the equation for current conservation effectively becomes the equation for an overdamped single junction, for which the subharmonic steps are absent.

Another observation that can be made at this point is that in all the cases that have been studied the expression of the step widths for $f=p/q$ and $\nu=n/q$ involve the products of q Bessel functions of (i_{ac}/ω_{ac}) such that the sum or the difference of their orders is equal to n. This is in accord with Eq. (16). The prefactor has a different dependence on the frequency due to cancellations occurring among the different modes.

IV. NUMERICAL RESULTS

We use the same method of numerical integration used in Ref. 8. We briefly review it here, for convenience. The equations for current conservation can be written as a matrix equation

$$\underline{M}_{ij}\frac{d\theta_j}{dt}=F(\{\theta_{i'}-\theta i\}) , \tag{40}$$

where i' denotes a nearest neighbor to i. The matrix $\underline{M}$ is then inverted yielding a set of coupled first-order differential equations which we integrate using the fourth-order Runge-Kutta method.[9] The current was uniformly injected at the left boundary of an $N\times N$ array. Furthermore, we used periodic (with period q) boundary conditions in the direction perpendicular to that of the injected current. We normally used arrays of size $N=3q-5q$, in order to avoid effects due to the boundary conditions. We used staircase configurations as initial states in all of our simulations. We restricted our observations to the cases $f=\frac{1}{2}$ and $\frac{1}{3}$, for the reasons already mentioned.

Figure 1 shows the power spectra of the oscillating pieces corresponding to the gauge-invariant phase differences on horizontal and vertical bonds for $f=\frac{1}{2}$, away from any step. The slope of the linear term on horizontal bonds agrees with the value of the Josephson fre-

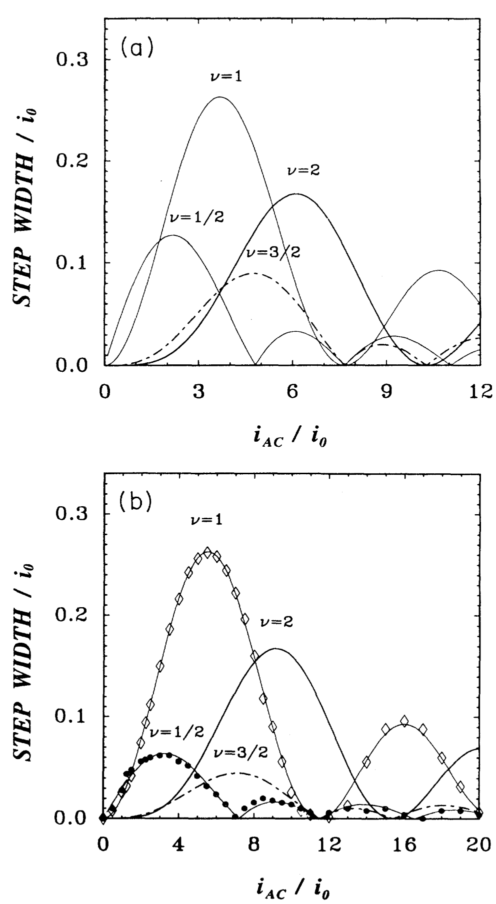

FIG. 3. Step widths for $f=\frac{1}{2}$ and $\nu=\frac{1}{2}$, 1, $\frac{3}{2}$, and 2 for (a) $\omega_{ac}=2.0$ and (b) $\omega_{ac}=3.0$. Fractional steps are suppressed relative to the integer steps. In (b) we compare our predictions to the numerical results obtained by Octavio *et al.* (Ref. 5) for $\nu=\frac{1}{2}$ and 1. These are represented by the symbols. We performed a least-squares fit to determine the range of variation of ψ_0. The value of χ^2 in both cases was of the order of 10^{-4}.

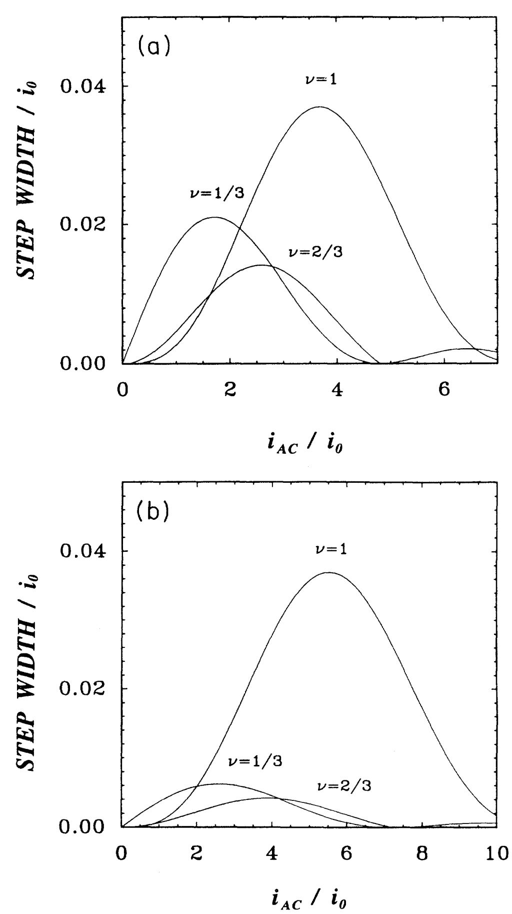

FIG. 4. Step widths for $f=\frac{1}{3}$, $\nu=\frac{1}{3}$, $\frac{2}{3}$, and 1, and $\omega_{ac}=$ (a) 2.0; (b) 3.0. The steps are considerably smaller than those for $f=\frac{1}{2}$.

quency observed in the power spectrum. Peaks 1, 4, 5, and 6 are absent on vertical bonds; in particular, notice the absence of the (0,1) mode. This is in accord with our assumptions. This same behavior has been observed for $f=\frac{1}{3}$. Peak 1 in Fig. 1(a) corresponds to the $(2,-1)$ mode. On the first fractional step the frequency corresponding to this mode is zero, and the corresponding supercurrent yields the additional dc current on the step; the same is true for peak 4 on the first integer step. Also, in the latter case, the mode corresponding to peak 2 becomes the zero-frequency mode. On the different steps we get the same pictures for the power spectra, with the difference that in these cases the motion is periodic.

Figure 2 displays the behavior of the gauge-invariant phase differences on horizontal and vertical bonds for $f=\frac{1}{2}$ and $\nu=1$. The presence of a zero-frequency (spatially varying dc component of the phase) component is clear. This is also in agreement with our results.

In Fig. 3 we show the different step widths for $f=\frac{1}{2}$ as a function of i_{ac} for $\omega_{ac}=2$ and 3. At higher frequencies the fractional steps become suppressed. In (b) we compare our predictions to the numerical results obtained by Octavio *et al.*[5] The agreement is quite good. The half-lengths of the intervals of variation of ψ_0 determined by fitting these data to our expressions are 1.214 and 0.888 for $\nu=\frac{1}{2}$ and 1, respectively, which differ from the values assumed in our rough estimates of $\pi/2$ and $\pi/4$.

In Fig. 4 we show the variation of the step widths for $f=\frac{1}{3}$ with i_{ac} for $\omega_{ac}=2.0$ and 3.0. The qualitative behavior of the step widths is the same that is observed for $f=\frac{1}{2}$.

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