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Extraordinary sensitivity of the internal Doppler effect in a superfluid ^{4}He-^{3}He admixture

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Recently, a nontrivial T (temperature) behavior was found for the Doppler shift of the fourth and first sounds in superfluid ^{4}He with internal motion: a plateau in the phonon region and a sharp peak in the beginning of the roton region of the Doppler parameters $\Gamma_{4,1}(T)=(\Delta u_{4,1}/v_s)_{v=0}$. The situation is similar to the case of second sound investigated long ago for $\Gamma_2(T)=(\Delta u_2/v_n)_{v=0}$, but the signs and values of plateaus and peaks indicated some kinds of Doppler anomalies: the "outstripping effect" (OEF), in addition to the "back-entrainment effect" (BEF) described by Khalatnikov (Δu_i is the Doppler shift of ith sound; v_s, v_n, v are the velocities of superfluid and normal components and of the liquid as a whole, respectively). The Doppler anomalies mean the breaking of some "natural" suppositions: that Δu_i is intermediate between v_n and v_s, and that the sign of $(\Delta u_i - v)$ is determined by the velocity of the "dominant" component (at low T this is the superfluid component for first and fourth sounds, $v_d = v_s$, and the normal one for second sound, $v_d = v_n$). The direction of $(\Delta u_i - v)$ can be opposite to the direction of $(v_d - v)$ (BEF) and the center of spreading sound can move faster than the flowing dominant component when the other component is stationary: $\Delta u_i > v_d$ (OEF). The Doppler anomalies as well as the very existence of the nonkinematic (internal) Doppler shift $\Delta u_i - v \neq 0$, and its nontrivial T behavior are special manifestations of the superfluidity. Here we investigate the Doppler phenomenon in the ^{4}He-^{3}He mixture. We find strong sensitivity of the T behavior of the Doppler shift and of the Doppler anomalies to the ^{3}He admixture. At low T this is associated with a general peculiarity of the ^{4}He-^{3}He mixture: the nonanalyticity of its characteristics, i.e., the inequivalence of $T \to 0$, $X \to 0$ to $X \to 0$, $T \to 0$ (X is the concentration of ^{3}He). We find some "key derivatives:" $\partial\rho/\partial w^2$, $\partial\sigma/\partial w^2$, crucial for the T behavior of Γ_i whose role changes at $X \neq 0$ (ρ and $\rho\sigma$ are the mass and entropy densities, respectively). The detailed explanation of the strong sensitivity (including finite T, X) is found by means of an analysis of peculiarities of the quasiparticles. We find (i) a jump of the low-T plateau of the Doppler parameters are D_i (modified version of Γ_i) for all the sounds at infinitesimal X (δX); the T range of the new plateau increases with X; the jump is greater than the variation of the plateau level with X for all $X < 0.06 = X_{max}$; (ii) a sharp decrease of the peak of $D_i(T)$ with increasing X up to its disappearance already at $X \ll X_{max}$; (iii) strong amplification at $X \neq 0$ of some Doppler anomalies.

I. INTRODUCTION

A. Internal Doppler effect and Doppler anomalies

Internal macroscopic motion which is the main peculiarity of superfluids, and several types of sound, imply some unusual manifestations of the Doppler effect. The relative motion between the normal and superfluid components creates in the rest frame of the liquid an anisotropy of the state of a special type. First of all, the purely kinematic coincidence of the velocities of the center of spreading sound, Δu_i, and of the fluid as a whole, v, is *broken*, $\Delta u_i - v \neq 0$: although both components take part in the sound oscillation (even in the case of fourth sound), the participation of the components is not on an equal footing. [In the case of fourth sound the velocity of the normal component does not oscillate ($\mathbf{v}_n = 0$, $\mathbf{v}_n' = 0$) but there exists the possibility of conversion between the components, so that ρ_n oscillates, $\rho_n' \neq 0$.] Thus the different "weights" of the components in the structure of sounds imply a deviation of the velocity of the center of spreading sound Δu_i from the center-of-mass (c.m.) velocity, $\mathbf{v} = \mathbf{j}/\rho = (\rho_s \mathbf{v}_s + \rho_n \mathbf{v}_n)/\rho$.

In the first approximation ($w/u \ll 1$) the nonkinematic part of the Doppler shift, $\Delta u_i - v$, is proportional to the velocity of the relative motion, $\mathbf{w} = \mathbf{v}_n - \mathbf{v}_s$. In the case $\mathbf{u} \| \mathbf{v} \| \mathbf{w}$ (where $\mathbf{u}$ is the sound velocity) we get for the ith sound

$$u_i = u_i^0 \pm \Delta u_i , \quad \Delta u_i - v = f_i(w) \approx \gamma_i w ,$$

or

$$\Delta u_i \approx \left[\frac{\rho_s}{\rho} - \gamma_i \right] v_s + \left[\frac{\rho_n}{\rho} + \gamma_i \right] v_n . \tag{1}$$

At $v = 0$ one can represent

$$\Delta u_2 = \Gamma_2 v_n , \quad \Delta u_{1,4} = \Gamma_{1,4} v_s , \quad \left[\Gamma_2 = \gamma_2 \Big/ \frac{\rho_s}{\rho} , \quad \Gamma_{1,4} = -\gamma_{1,4} \Big/ \frac{\rho_n}{\rho} \right] . \tag{2}$$

The Doppler parameters Γ_i have a simple and universal (qualitatively) T (temperature) behavior: a plateau in the phonon region, $T < \tilde{T}$, and a sharp peak at the beginning of the roton region, $T \sim \tilde{T}$ [$\tilde{T} \approx 0.57$ K is defined by $\rho_{ph}(\tilde{T}) = \rho_r(\tilde{T})$; ρ_{ph} and ρ_r are the phonon and roton contributions to the normal density ρ_n, respectively].

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However, the existence of the internal Doppler shift $(\Delta u_i - v) \neq 0$ is not the only peculiarity of the phenomenon. It seems natural to suppose that the combined oscillation of the two moving components has an *intermediate* rest frame, i.e., $v_n < \Delta u_i < v_s$ (if $v_n < v_s$), and that the sign of $(\Delta u_i - v)$ is determined by the velocity of the dominant component. [At low T, i.e., far enough from T_λ, where $\rho_n/\rho_s \equiv \beta \ll 1$, the normal and superfluid components are dominant in the structure of second($v_n'/v_s' = \beta^{-1}$) and first ($j_s'/j_n' = \beta^{-1}$) (or fourth) sounds, respectively.] This means that the velocity of the center of spreading sound would be intermediate between the c.m. velocity $\mathbf{v}$ and the velocity of the dominant component, $\mathbf{v}_n$ or $\mathbf{v}_s$:

$$0 < \gamma_2 < \gamma_2^c \quad \left[\gamma_2^c = \frac{\rho_s}{\rho} \right] ,$$

$$-\gamma_{1,4}^c < \gamma_{1,4} < 0 \quad \left[\gamma_{1,4}^c = \frac{\rho_n}{\rho} \right] ,$$

or

$$0 < \Gamma_2 < 1 \ \ (\text{i.e., for } v_n > v,\ \ v < \Delta u_2 < v_n) ,$$

$$0 < \Gamma_{1,4} < 1 \ \ (\text{i.e., for } v_s > v,\ \ v < \Delta u_{1,4} < v_s) . \tag{3}$$

The case $\gamma = 0$ $(\Gamma = 0)$ corresponds to the participation of both components on equal footing, $\gamma = \gamma^c$ $(\Gamma = 1)$—to the total disappearance of the influence of the "auxiliary" (i.e., nondominant) component. But these restrictions turn out to be broken [very strongly; see Eq. (6)].

There emerge two effects—the anomalies of the internal Doppler shift.

(i) The direction of the internal Doppler shift $(\Delta u_i - v)$ can be opposite to the direction of motion of the dominant component (in the c.m. frame):

$$\gamma_2 < 0, \quad \gamma_{1,4} > 0, \quad \text{or } \Gamma_i < 0 . \tag{4}$$

This "*back-entrainment*" *effect*[3] (BEF) was described in Ref. 1 for second sound, $\gamma_2 < 0$, i.e., for $v_n > v$, $\Delta u_2 < v$, and in Refs. 2 and 3 for fourth and first sounds, $\gamma_{1,4} > 0$, i.e., for $v_s > v$, $\Delta u_{1,4} < v$.

(ii) In the cases of first and fourth sounds the Doppler shift can exceed the velocity of the dominant component v_s:

$$\gamma_{1,4} < -\frac{\rho_n}{\rho} = -\gamma_{1,4}^c \quad \text{or } \Gamma_{1,4} > 1 . \tag{5}$$

Equation (5) means that the center of the spreading sound moves faster than the flowing superfluid part of the liquid when the normal component is stationary. This "*outstripping*" *effect* (OEF) was described in Refs. 2 and 3. A similar effect is absent for second sound, i.e., the condition $\gamma_2 > \rho_s/\rho = \gamma_2^c$ or $\Gamma_2 > 1$ is not fulfilled: $\max \gamma_2 \simeq 0.9 \gamma_2^c$.

The scale of the Doppler anomalies, i.e., of the breaking of the "natural" conditions [Eq. (3)] is shown by the following list of plateaus and peaks of $\Delta u_{4,1} = \Gamma_{4,1} v_s$, $\Delta u_2 = \Gamma_2 v_n$:

$$\begin{aligned}
&\Delta u_4 \approx 3.79 v_s \ \ (> v_s,\ \text{OEF}) ,\\
&\min \Delta u_4 \approx -25.6 v_s \ \ (< 0,\ \text{BEF}) ,\\
&\Delta u_1 \approx -35.5 v_s \ \ (< 0,\ \text{BEF}) ,\\
&\max \Delta u_1 \approx 43.7 v_s \ \ (> v_s,\ \text{OEF}) ,\\
&\Delta u_2 \approx \tfrac{2}{3} v_s \ \ (< v_n,\ \text{OEF is absent}) ,\\
&\min \Delta u_2 \approx -1.18 v_n \ \ (< 0,\ \text{BEF}) .
\end{aligned} \tag{6}$$

B. Subject of the paper

In this paper we investigate the influence of a ^{3}He admixture on the internal Doppler effect. First of all we analyze in detail the physical origin of the Doppler anomalies and of the peculiarities of the T behavior of the Doppler shift in pure ^{4}He. We explain the common features of the behavior of the Doppler parameters Γ_i (or D_i, see below) for different sounds (a sharp peak in the beginning of the roton region and a "plateau" in the phonon region) and the differences [e.g., the opposite sign of peaks and low-T constants of Γ_1 (D_1) and Γ_4 (D_4) which appear in spite of the similarity of the structure of first and fourth sounds at low T]. Then we compare the role of the thermal and ^{3}He quasiparticles in the Doppler phenomenon. It appears that, unlike the case of pure ^{4}He, the important aspects of the problem at $X \neq 0$ require an exact approach. Here we investigate the problem for all the sounds using exact considerations. In the former investigations of the Doppler effect in mixtures the anomalies and the peculiarities of T behavior either were not discussed at all (Ref. 4) or were considered only for fourth sound in the first approximation ($X \ll 1$) (Ref. 5).

The results are the following. (i) There are jumps of low-T asymptotic values of the Doppler parameters D_i:

$$\Delta D_i = D_i^{(1)} - D_i^{(2)} = D_i(T \to 0, \delta X) - D_i(T \to 0, X = 0) ,$$

and of the second sound velocity u_2. (ii) The values of the jumps, $|\Delta D_i|$, are much greater than the variation of $D_i(T \to 0)$ with $X < 0.06 = X_{\max}$. Thus at small X we get an approximately constant step of $D_i(T)$:

$$D_i(T \ll T(X)) - D_i(T \gg T(X)) \approx \Delta D_i \quad (T < \tilde{T}) .$$

Here $T(X)$ describes the T range of the new level of the plateau. $T(X)$ quickly increases with X so that already at small X $(\ll X_{\max})$ it spans the whole phonon region, $T < \tilde{T}$. The other consequence of increase of X is a sharp decrease of the peaks of D_i at $T \sim \tilde{T}$ up to its disappearance (already at $X \ll 0.06$).

$T(X)$ separates two types of T behavior of thermodynamic or hydrodynamic quantities Q, i.e., the regions of predominance of thermal and ^{3}He quasiparticles: $T \gg T(X)$ (almost pure ^{4}He) and $T \ll T(X)$ ("pronounced mixture"). Actually there exist two characteristic temperatures, "thermodynamic" $[T_1(X)]$ and "hydrodynamic" $[T_2(X)]$; those are defined by $\sigma_{\rm ph}(T_1) = \sigma_3(T_1)$ and $\rho_{\rm ph}(T_2) = \rho_{n3}(T_2)$ [$\sigma_{\rm ph}$, σ_3, $\rho_{\rm ph}$, and ρ_{n3} are the contributions of the phonons and ^{3}He quasiparticles to the entropy (per unit mass) σ and ρ_n, respectively]:

$$T_1 \approx \frac{\hbar c}{k_B}\left[\frac{45\rho c}{2\pi^2 m_3}\right]^{1/3} X^{1/3} \approx 7.4 X^{1/3}\ (\mathrm{K}) ,$$
$$T_2 \approx \frac{\hbar c}{k_B}\left[\frac{45\rho c}{2\pi^2 \hbar}\right]^{1/4} \frac{m_3^*}{m_3} X^{1/4} \approx 12.0 X^{1/4}\ (\mathrm{K}) \quad (7)$$

(c is the sound velocity of phonons, $m_3^* = 2.46 m_3$ is the effective mass of the ^{3}He atom, and ρ is the density of the mixture). At low X we get $T_2 \gg T_1 \gg T_d$, where $T_d \sim X^{2/3}$ is the Fermi degeneracy temperature of ^{3}He. Both T_1 and T_2 appear for $Q = D_i$. There appears an unexpected amplification of the Doppler anomalies: We find two "key" derivatives, $\partial\rho/\partial(w^2/2) = \rho^2 \partial(\rho_n/\rho)/\partial P$ and $\partial\sigma/\partial(w^2/2) = \partial(\rho_n/\rho)/\partial T$, which determine the peculiarities of the T behavior of all D_i ($w = v_n - v_s$ is the relative velocity of the components). We prove that they are independent of X. This points out the thermal-excitation origin of the Doppler anomalies. Nevertheless, it appears that some anomalies substantially increase in the case of predominance of ^{3}He quasiparticles.

C. The main peculiarities of the internal Doppler effect in a ^{4}He-^{3}He mixture

Unlike the case of pure ^{4}He where all the peculiarities of the phenomenon can be described in the simplest approximation[1-3] $\rho_n/\rho \ll 1$, the important questions in the case of mixtures require exact considerations. (i) There is exact cancellation at small T of all the terms of type $O((\rho_n/\rho)^k) = O(X^k)$ in the expression for the OEF parameter of fourth sound:

$$(\Delta u_4 - v_s)/v_s = -\left[\gamma_4 + \frac{\rho_n}{\rho}\right] = O\left[\frac{\rho_{n4}}{\rho}\right] = O(T^4) . \quad (8)$$

This result has a clear physical meaning: the natural "reference point" of the Doppler shift Δu_4 is not v but v_s [cf. Eq. (1)], and the natural scale is not ρ_n/ρ but ρ_{n4}/ρ. In fact, in the case of fourth sound the oscillations of normal velocity are excluded ($v_n' = 0$), so that the participation of the normal component which creates $\Delta u_4 - v_s \neq 0$ is connected only with intertransformations of the normal and superfluid components ($\rho_n' = -\rho_s'$). But this is possible only for ρ_{n4}, the thermal-excitation part of ρ_n. (ii) There is exact separation in the fifth-order equation for first and second sounds of the special solution u_5 [additional to $(\pm u_{1,2}^0 + \gamma_{1,2} w)$]—the velocity of the "concentration waves." We prove its coincidence with the velocity of the normal component [$u_5 = (\rho_s/\rho) w$ at $v = 0$]. (iii) The simplest approximation for γ_2,

$$\gamma_2(T \to 0, X \neq 0) = 1 + O(\rho_n/\rho) ,$$

does not solve the question of the existence of the OEF, $\gamma_2 > \rho_s/\rho$ [unlike the case $\gamma_2(T \to 0, X = 0) = \frac{2}{3}$]. The exact results

$$\gamma_2(T \to 0, X \neq 0) = \frac{\rho_s}{\rho} - \frac{\rho_n}{2\rho} - \gamma_1(T \to 0, X \neq 0) ,$$
$$\gamma_1(T \to 0, X \neq 0) = O\left[\frac{\rho_n}{\rho}\right] > 0 \quad (9)$$

prove the absence of the OEF, $\gamma_2 < \rho_s/\rho$.

According to Eqs. (8) and (9) the following parameters are convenient for consideration of the influence of $X \neq 0$ at small T:

$$\gamma_4 + \frac{\rho_n}{\rho} = O\left[\frac{\rho_{n4}}{\rho}\right], \quad \gamma_1 = O\left[\frac{\rho_n}{\rho}\right] ,$$
$$\gamma_2 - \frac{\rho_s}{\rho} = O\left[\frac{\rho_n}{\rho}\right] . \quad (10)$$

Equations (10) reflect the low-T scale and the "natural reference points" of the Doppler shift at $X \neq 0$: v_s, v, and v_n for fourth, first, and second sound, respectively. In fact, we get

$$\Delta u_4 - v_s = -\left[\gamma_4 + \frac{\rho_n}{\rho}\right] v_s \quad (v_n = 0) ,$$
$$\Delta u_1 - v = \gamma_1 w , \quad (11)$$
$$\Delta u_2 - v_n = \left[\gamma_2 - \frac{\rho_s}{\rho}\right] v_n \quad (v_s = 0) .$$

In terms of the parameters D_i,

$$D_1 = \gamma_1 \Big/ \frac{\rho_n}{\rho} = -\Gamma_1 ,$$
$$D_2 = \gamma_2 \Big/ \frac{\rho_s}{\rho} = \Gamma_2 \approx \gamma_2 \quad (\text{at } T \ll T_\lambda) , \quad (12)$$
$$D_4 = \left[\gamma_4 + \frac{\rho_n}{\rho}\right] \Big/ \frac{\rho_{n4}}{\rho} = -(\Gamma_4 - 1)\frac{\rho_n}{\rho_{n4}} ,$$

the influence of the admixture (^{3}He) acquires a simple and universal (qualitatively) character: jumps of the plateau (which are substantially larger than the variation of D_i with $X < 0.6$) and disappearance of the peaks. For the plateau we obtain

$$D_4 \approx -a \approx -2.79 \quad (T \gg T_1) ,$$
$$-5a \approx -13.95 \quad (T \ll T_1) \quad (\mathrm{OEF}) ,$$
$$D_1 = (3a+1)(a+1) \approx 35.5 \quad (T \gg T_2) ,$$
$$2\theta \frac{m_3}{m_3^*} + \theta^2 \left[\frac{m_3}{m_3^*}\right]^2 \approx 0.66 \quad (T \ll T_1) \quad (\mathrm{BEF}) , \quad (13)$$
$$D_2 \approx \tfrac{2}{3} \quad (T \gg T_2) ,$$
$$1 - \frac{\rho_n}{\rho_s}\left\{\frac{1}{2} + \left[2\theta\frac{m_3}{m_3^*} + \theta^2\left[\frac{m_3}{m_3^*}\right]^2\right]\right\}$$
$$\approx \left[1 - 1.16\frac{\rho_n}{\rho_s}\right]$$
$$\approx (1 - 2.85X) \quad (T \ll T_2)$$

(the condition of the OEF, $D_2>1$, is not fulfilled for the mixture either). Here $a=(\rho/c)(dc/d\rho)$, $\theta=-(1/\rho)(\partial\rho/\partial X)\approx 0.71$, $\rho_n/\rho_s\approx(\rho_{n3}/\rho)=m_3^*/m_3X$; $T\gg T_d$; T_1 and T_2 are defined in Eq. (7).

Both jumps of the plateau, for D_4 and D_1, $-a\rightarrow -5a$, $35.5\rightarrow 0.66$ [Eq. (13)], correspond to a substantial amplification of the anomalies,

$$\max\left[-\left[\gamma_4+\frac{\rho_n}{\rho}\right]\equiv -D_4\frac{\rho_{n4}}{\rho}\right]$$

(OEF parameter) by a factor 15, $\max(\gamma_1\equiv D_1\rho_n/\rho)$ (BEF parameter) by 10^3 at $X=0.06$:

$$\max\left[-\left[\gamma_4+\frac{\rho_n}{\rho}\right]\right]=0.6\times 10^{-5}\quad (X=0)\ ,$$
$$0.9\times 10^{-4}\quad (X\gg 10^{-3})\ ,$$
$$\max\gamma_1=0.8\times 10^{-4}\quad (X=0)\ , \tag{14}$$
$$0.66\frac{\rho_n}{\rho}\approx 1.62X\quad (X\gg 10^{-4})\ ,$$

i.e., up to 0.1 (at $X=0.06$).

In conclusion we give one more example of the nonanalyticity (in addition to D_i)—the second-sound velocity,

$$u_2^0(X,T)^2\approx\frac{\rho_s}{\rho_n}\left[\left[\sigma-X\frac{\partial\sigma}{\partial X}\right]^2\Big/\frac{\partial\sigma}{\partial T}+X^2\partial\left[\frac{Z}{\rho}\right]\Big/\partial X\right] \tag{15}$$

[$Z/\rho=\mu_4-\mu_3$; $m_i\mu_i$ $(i=4,3)$ are the chemical potentials of ^{4}He and ^{3}He, respectively]. We obtain

$$u_2\approx\frac{c}{\sqrt{3}}\quad (T\gg T_1)\ ,$$
$$u_2\approx\left[\frac{5}{3}\frac{kT}{m_3}\right]^{1/2}\quad (T_1\gg T\gg T_d)\ , \tag{16}$$
$$u_2=\frac{V_F}{\sqrt{3}}\quad (T\ll 0.05XT_d)\ ;$$

see Eq. (7); $V_F=(3\pi^2\rho X/m_3)^{1/3}/m_3^*$ is the Fermi velocity and

$$T_d=\frac{\hbar^2(\rho X/m_3)^{2/3}(3\pi^2)^{2/3}}{2k_Bm_3^*}\approx 1.42X^{2/3}\ (\mathrm{K})\ . \tag{17}$$

It is interesting to note that in all cases the results coincide with the velocity of collisional elastic sound of an ideal gas with the spectrum of the corresponding quasiparticles. This is noteworthy since the second sound is the oscillation of the relative velocity of the components but not of the normal component only. Neither the elasticity of phonons, $\partial P_{\rm ph}/\partial\rho_{\rm ph}$ $(\neq c^2/\sqrt{3})$, nor the Doppler shift, $\Delta u_2\neq v_n$, corresponds to the elastic sound in a gas of particles with the spectrum $\epsilon=cp$.

D. Two-fluid hydrodynamics of pure ^{4}He and ^{4}He-^{3}He mixtures

For the numerical calculation of the internal Doppler shift in ^{4}He we use the well-known two-fluid description[6] of the thermodynamics and hydrodynamics of pure ^{4}He. This theory has been verified experimentally as long as the temperature is not close to the transition temperature T_λ and the relative internal motion is not large enough to create turbulence ($w\leq 60$ cm sec^{-1}). The maximum error in the thermodynamic functions as given by the theory is[7] $\sim 3\%$ for $T\leq 1.2$ K and the description is expected to be exact at $T\rightarrow 0$ K.

The hydrodynamic description of the liquid assumes that it contains two components, the normal viscous component that contains all the entropy, and a superfluid component that has no viscosity and has nonrotational flow. The normal component is regarded as an ideal gas of excitations—phonons ($\epsilon=cp$, $c\simeq 2.4\times 10^4$ cm/sec) and rotons [$\epsilon(p)=\Delta+(p-p_0)^2/2\mu$, $\Delta/k_B\simeq 8.65$ K, $p_0/\hbar\simeq 1.91$ Å^{-1}, $\mu\simeq 0.16m_{^4\mathrm{He}}$]. The ideal-gas approximation includes the neglect of the derivatives $(\partial\Delta/\partial T)_\rho$, $(\partial\mu/\partial T)_\rho$, and $(\partial p_0/\partial T)_\rho$, which is correct for $T\leq 1.2$ K.[7,8] Other approximations used in the thermodynamic description are[7,8,3]

$$r=\frac{\rho}{\Delta}\frac{\partial\Delta}{\partial\rho}\simeq -0.59,\quad f_2\equiv 1+3a=1+3\frac{\rho}{c}\frac{\partial c}{\partial\rho}\simeq 9.37\ ,$$
$$\frac{\rho}{p_0}\frac{\partial p_0}{\partial\rho}\simeq\frac{1}{3},\quad \frac{\rho}{\mu}\frac{\partial\mu}{\partial\rho}\simeq -1.2\ . \tag{18}$$

The thermodynamics of the ^{4}He-^{3}He mixture and of the ^{3}He component is taken from Ref. 9. The thermodynamic functions are computed numerically using a series representation which is easy to integrate and calculate. The series have been fitted to experimental results in the range $T\leq 0.25$ K and $\tilde{X}\leq 8\%$, where $\tilde{X}$ is the molar concentration of ^{3}He in the mixture [$X=(M_3/M)\tilde{X}$, $M=\tilde{X}M_3+(1-\tilde{X})M_4$].

We checked the validity of the description by comparison with experimental measurements of the specific heat

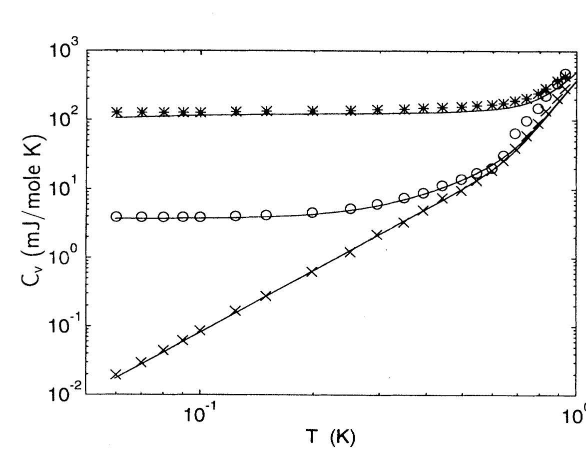

FIG. 1. Verification of the thermodynamics. Comparison of our calculation of the specific heat of ^{3}He-^{4}He mixture (solid lines) and experiment (Ref. 10) (data points and dashed lines).

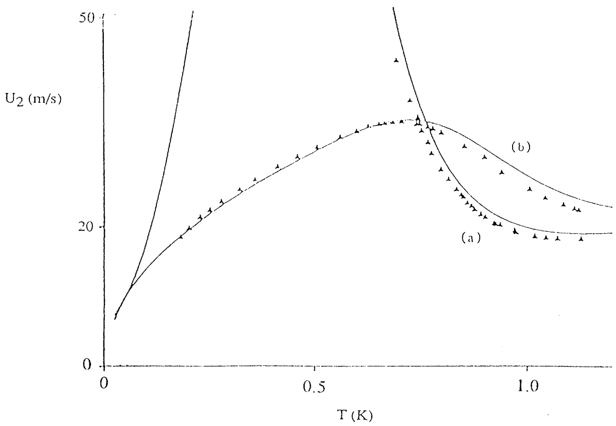

FIG. 2. Velocity of second sound (u_2) for ^{3}He-^{4}He mixtures. Solid lines, results of our calculation; symbols, the experimental results (Ref. 11). Molar concentrations are (a) $10^{-6}\%$ and (b) 0.173%.

of the mixture,[10] as shown in Fig. 1. The excellent agreement with the experiment means that the thermodynamic description we use is good for $T \leq 1.2$ K. Further verification of the thermodynamic description is shown in Figs. 2 and 3 where we compare the calculated velocity of second and fourth sound at different temperatures with experimental results.[11,12] For second sound (Fig. 2), the agreement is good but for fourth sound (Fig. 3) it is only fair. In this second case we see that the experiment and calculation tend to agree at low temperatures ($T \leq 1.0$ K) and diverge as the temperature approaches the phase transition. We see that below ~ 1 K our description gives a maximum relative error of $\sim 10\%$.

The ^{3}He component contributes a practically constant amount to the normal density:

$$\rho_n = \rho_{n4}(T) + \frac{\tilde{X} M_3^*}{V_m} = \rho_{n4} + X\rho \frac{M_3^*}{M_3} , \tag{19}$$

where $\rho_{n4}(T)$ is the contribution to the normal density of

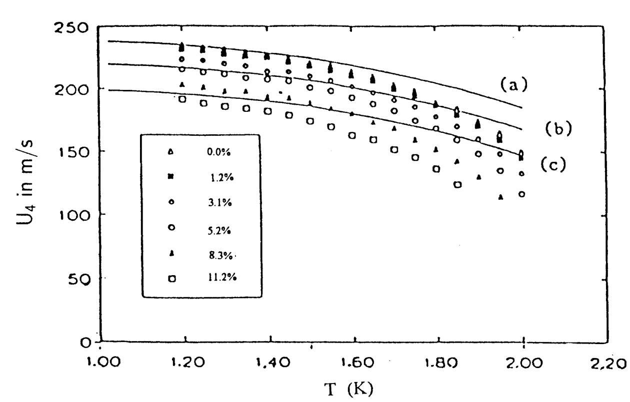

FIG. 3. Velocity of fourth sound (u_4) for ^{3}He-^{4}He mixtures: Solid lines, results of our calculation; symbols, the experimental results (Ref. 12). The molar concentrations are (a) 0% (b) 5.2% and (c) 11.2%. Our calculations are accurate only for $T < 1.3$ K.

the ^{4}He excitations, $M_3^* = 2.46 M_3$ is the effective molar mass of ^{3}He in the mixture, and V_m is the molar volume of the mixture. The ^{3}He component also changes the total density of the mixture:

$$\rho = \frac{\tilde{X} M_3 + (1-\tilde{X}) M_4}{V_m} , \tag{20}$$

where M_4 is the molar mass of ^{4}He. The molar mass of ^{3}He is roughly $\frac{3}{4}$ that of ^{4}He and its molar volume is larger so that the molar volume of the mixture is

$$V_m = V_4 (1 + \tilde{\alpha} \tilde{X}) , \tag{21}$$

where V_4 is the molar volume of ^{4}He and $\tilde{\alpha} = 0.286$ (see Ref. 9). For the exact numerical calculations we used MACSYMA.[13]

II. DOPPLER SHIFT IN PURE ^{4}He: ORIGIN OF THE ANOMALIES AND OF THE PECULIARITIES OF THE TEMPERATURE BEHAVIOR

A. Doppler parameters and initial sound velocities

Let us consider the expressions for the sound velocities in the absence of internal motion ($w = 0$), u_i^0 (with corrections $\propto \rho_n/\rho$), and for the Doppler parameters γ_i, which were obtained in Refs. 2 and 3 on the basis of the solution of the two-fluid hydrodynamic equations for oscillating perturbations.

For fourth sound we get (in the first approximation in ρ_n/ρ)

$$\gamma_4 \approx \gamma_4^{(1)} = \left[-\frac{\rho_n}{\rho} + \frac{\partial \rho_n/\partial P}{\partial \rho/\partial P} \right] - \frac{\sigma \partial \rho_n/\partial T}{\rho \partial \sigma/\partial T} - \frac{\alpha (1/\rho)(\partial \rho_n/\partial T)}{(\partial \rho/\partial P)(\partial \sigma/\partial T)} , \tag{22}$$

$$\alpha = \frac{1}{\rho}\frac{\partial \rho}{\partial T} ,$$

$$(u_4^0)^2 \approx \bar{u}_4^2 + (u_4^{(1)})^2 = \frac{\partial P}{\partial \rho}\left[1 - \frac{\rho_n}{\rho} + \frac{2\alpha\sigma}{\partial \sigma/\partial T} + \frac{\alpha^2}{(\partial \rho/\partial P)(\partial \sigma/\partial T)} \right] + \frac{\sigma^2}{\partial \sigma/\partial T} , \tag{23}$$

$$(u_4^{(1)})^2/\bar{u}_4^2 = O\left[\frac{\rho_n}{\rho}\right], \quad \bar{u}_4^2 = \frac{\partial P}{\partial \rho}$$

[see Eqs. (32) and (44) in Ref. 3]. Here and below all the partial derivatives are taken with all other independent variables held constant.

For first and second sounds let us begin from the exact expressions

$$\gamma_{1,2} = -\frac{R(u_{1,2}^0)^2 + S}{2[2A(u_{1,2}^0)^2 + B]} = \mp \frac{R(u_{1,2}^0)^2 + S}{2\sqrt{B^2 - 4AC}} , \tag{24}$$

$$(u_{1,2}^0)^2 = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A} , \tag{25}$$

$$A=A_0+A', \quad A_0=\rho\frac{\partial\rho}{\partial P}\frac{\partial\sigma}{\partial T},$$

$$A'=-\rho\frac{\partial\rho}{\partial T}\frac{\partial\sigma}{\partial P}\equiv-\alpha^2\rho\ ,$$

$$B=-\rho\left[\frac{\partial\sigma}{\partial T}+\frac{\rho_s\sigma^2}{\rho_n}\frac{\partial\rho}{\partial P}\right], \quad C=\frac{\rho_s\rho\sigma^2}{\rho_n} \tag{26}$$

[see Eqs. (52)–(56) in Ref. 3; the correct sign before B in Eq. (55) and in the expression for B (56b) is plus].

$$R=-\frac{\partial\rho}{\partial P}M+R', \quad S=M+S'\ , \tag{27a}$$

$$M=(\rho+3\rho_s)\frac{\partial\sigma}{\partial T}-2\sigma\frac{\rho^2}{\rho_n}\frac{\partial(\rho_n/\rho)}{\partial T}\ ,$$

$$R'=(\rho+3\rho_s)\frac{1}{\rho^2}\left[\frac{\partial\rho}{\partial T}\right]^2-2\sigma\frac{\rho^2}{\rho_n}\frac{\partial(\rho_n/\rho)}{\partial P}\frac{\partial\rho}{\partial T}\ , \tag{27b}$$

$$S'=4\sigma\frac{\rho_s}{\rho}\frac{\partial\rho}{\partial T}\ .$$

In the exact expressions for A, R, and S we pick out the terms A', R', and S' which contain the small factor $\alpha\sim(1/T)(\rho_n/\rho)$. [The definitions (27b) are somewhat simpler than in Ref. 3.] The terms M, R', and S' can be represented as power series in ρ_n/ρ with finite number of terms:

$$M=M_0+M_1\ ,$$

$$M_0=4\rho\frac{\partial\sigma}{\partial T}-2\sigma\frac{\rho}{\rho_n}\frac{\partial\rho_n}{\partial T}, \quad M_1=-3\rho_n\frac{\partial\sigma}{\partial T}\ , \tag{28a}$$

$$R'=R'_0+R'_1\ ,$$

$$R'_0=4\rho\alpha^2-2\frac{\rho}{\rho_n}\frac{\partial\rho_n}{\partial p}\alpha\sigma, \quad R'_1=-3\rho_n\alpha^2-2\alpha\rho_n\sigma\frac{\partial\rho}{\partial P}\ , \tag{28b}$$

$$S'=S'_0+S'_1, \quad S'_0=4\alpha\rho\sigma, \quad S'_1=-4\alpha\rho_n\sigma \tag{28c}$$

[see Eqs. (57) in Ref. 3].

Substituting in Eqs. (24) and (25) the expansions of the coefficients (26)–(28) we obtain

$$\gamma_{1,2}=\bar\gamma_{1,2}+\gamma'_{1,2}, \quad \gamma'_{1,2}=O(\alpha)\ ,$$

$$\bar\gamma_{1,2}=\frac{M[-(\partial\rho/\partial P)\bar u_{1,2}^2+1]}{2\sqrt{B^2-4A_0C}}\ ,$$

$$u_{1,2}^{0^2}=\bar u_{1,2}^2+(u_{1,2}^{(1)})^2, \quad (u_{1,2}^{(1)})^2/\bar u_{1,2}^2=O\left[\frac{\rho_n}{\rho}\right],$$

$$\bar u_{1,2}^2=\frac{-B\pm\sqrt{B^2-4A_0C}}{2A_0}\ ,$$

$$(u_{1,2}^{(1)})^2\approx-\frac{A'}{A_0}\left[\bar u_{1,2}^2\pm\frac{C}{\sqrt{B^2-4A_0C}}\right], \tag{29}$$

i.e.,

$$\bar u_1^2=\frac{\partial P}{\partial\rho}, \quad \bar u_2^2=\frac{\rho_s\sigma^2}{\rho_n\partial\sigma/\partial T}\ , \tag{30}$$

$$(u_1^{(1)})^2\approx\frac{\alpha^2}{\partial\sigma/\partial T}\frac{\bar u_1^4}{1-\bar u_2^2/\bar u_1^2}\ ,$$

$$(u_2^{(1)})^2\approx-\frac{\alpha^2}{\partial\sigma/\partial T}\frac{\bar u_2^4}{1-\bar u_2^2/\bar u_1^2}\ , \tag{31}$$

$$\bar\gamma_1=\gamma_1^{(0)}=0, \quad \bar\gamma_2=\frac{M}{2\rho\partial\sigma/\partial T}=\gamma_2^{(0)}+\Delta\gamma_2\ , \tag{32a}$$

$$\gamma_2^{(0)}=\frac{M_0}{2\rho\partial\sigma/\partial T}=2-\frac{\sigma}{\rho_n}\frac{\partial\rho_n/\partial T}{\partial\sigma/\partial T}\ , \tag{32b}$$

$$\Delta\gamma_2=\frac{M_1}{2\rho\partial\sigma/\partial T}=-\frac{3}{2}\frac{\rho_n}{\rho}\ . \tag{32c}$$

The expression for γ_1 ($=\gamma'_1$) to first order in ρ_n/ρ can be represented in a form which contains $\gamma_2^{(0)}$:

$$\gamma_1=\gamma'_1=\frac{M(\partial\rho/\partial P)(u_1^{(1)})^2-R'(u_1^0)^2-S'}{2\sqrt{B^2-4AC}}$$

$$\approx\gamma_1^{(1)}=\frac{M_0\partial\rho/\partial P(u_1^{(1)})^2-R'_0\bar u_1^2-S'_0}{2\sqrt{B^2-4A_0C}}$$

$$=\gamma_2^{(0)}\frac{\alpha^2\bar u_1^2}{(\partial\sigma/\partial T)(1-\bar u_2^2/\bar u_1^2)^2}+\frac{\alpha\sigma\{(\rho/\rho_n)[-\rho/\rho_n+(\partial\rho_n/\partial P)/(\partial\rho/\partial P)]-2\}-2\alpha^2\bar u_1^2}{(\partial\sigma/\partial T)(1-\bar u_2^2/\bar u_1^2)}\ . \tag{33}$$

Since we get

$$\gamma'_2=\frac{-M(\partial\rho/\partial P)(u_2^{(1)})^2+R'(u_1^0)^2+S'}{2\sqrt{B^2-4A_0C}}=O(\alpha)$$

[as long as $(u_2^{(1)})^2$, R', and S' contain α] γ_2 can be written in the form

$$\gamma_2=\left[2-\frac{\sigma}{\rho_n}\frac{\partial\rho_n/\partial T}{\partial\sigma/\partial T}\right]-\frac{3}{2}\frac{\rho_n}{\rho}+O(\alpha)\ . \tag{34}$$

Note that the result in Ref. 1,

$$\gamma_2 = 2\frac{\rho_s}{\rho} - \frac{\sigma}{\rho_n}\frac{\partial \rho_n/\partial T}{\partial \sigma/\partial T} + O(\alpha) ,$$

does not contain the exact correction of the first order in ρ_n/ρ on the supposition $\alpha = 0$.

B. Temperature behavior of Doppler parameters

The temperature behavior of D_i which corresponds to $\gamma_2^{(0)}$ and $\gamma_{4,1}^{(1)}$ [see Eqs. (32b), (22), and (33)], is plotted in Figs. 4–6. Here it is compared with the results of the exact numerical calculation on the basis of the determinants (6) and (51) in Ref. 3. The functions D_i have the following common features: an almost constant behavior (plateau) in the phonon region, a drastic change in the beginning of the roton region (sharp peak of $|D_i|$), and almost constant behavior (not large but different from the kinematic value 0) at high temperature.

The main differences between the functions D_i are the following: The low-T plateau is negative for D_4 (OEF, $\gamma_4/(\rho_n/\rho) < -1$) and positive for D_1 (BEF) and D_2; the peak is positive (maximum) for γ_4 (BEF) and negative (minimum) for γ_1 (OEF, $\gamma_1/(\rho_n/\rho) < -1$) and γ_2 (BEF).

Using the expressions in the phonon region,[3]

$$\frac{\partial \rho_n}{\partial P} \approx -\frac{5a}{c^2}\frac{\rho_n}{\rho}, \quad \frac{\partial \rho_n}{\partial T} \approx \frac{4\rho_n}{T}, \quad \frac{\partial \rho}{\partial P} \approx \frac{1}{c^2} ,$$

$$\alpha = \frac{1}{\rho}\frac{\partial \rho}{\partial T} = \rho\frac{\partial \sigma}{\partial P} = \frac{f_2 \rho_n}{T\rho}, \quad \sigma \approx \frac{c^2 \rho_n}{T}, \quad \frac{\partial \sigma}{\partial T} \approx 3\frac{c^2 \rho_n}{T^2} , \tag{35}$$

$$f_2 = 3a + 1 \approx 9.37, \quad a = \frac{\rho}{c}\frac{dc}{d\rho}$$

we find

$$\gamma_4(T < 0.4\ \mathrm{K}) \Big/ \frac{\rho_n}{\rho} \approx -(a+1) \approx -3.79 \quad \text{(OEF)} ,$$

$$\gamma_1(T < 0.4\ \mathrm{K}) \Big/ \frac{\rho_n}{\rho} \approx (3a+1)(a+1) \approx 35.5 \quad \text{(BEF)} ,$$

$$\gamma_2(T < 0.4\ \mathrm{K}) \approx \tfrac{2}{3} . \tag{36}$$

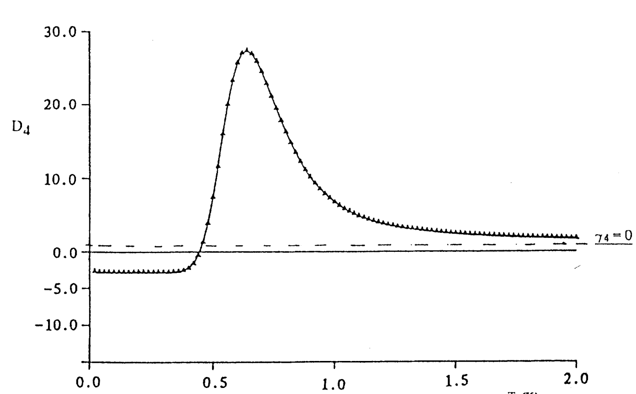

FIG. 4. Doppler parameter $D_4(T) = (\gamma_4 + \rho_n/\rho)/(\rho_{n4}/\rho)$ for pure ^{4}He. Solid line corresponds to the first-order expression [Eq. (22)]; triangles, an exact solution of the initial determinant [see Eq. (6) in Ref. 3].

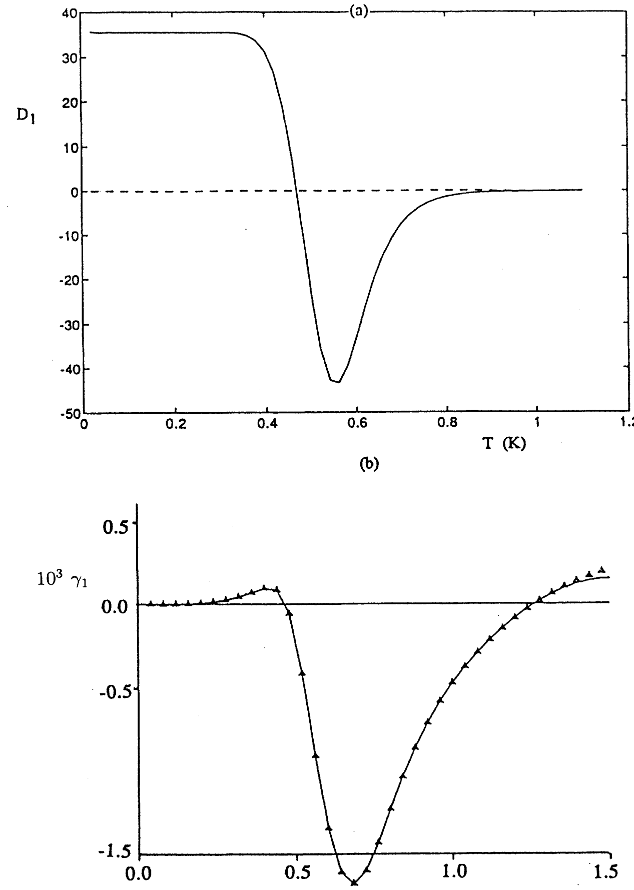

FIG. 5. (a) Doppler parameter $D_1(T) = \gamma_1/(\rho_n/\rho)$ for pure ^{4}He. (b) Doppler coefficient $\gamma_1(T)$ for pure ^{4}He. Solid line corresponds to the first-order expression [Eq. (33)]; triangles, the exact calculation of the initial determinant [see Eq. (51) in Ref. 3].

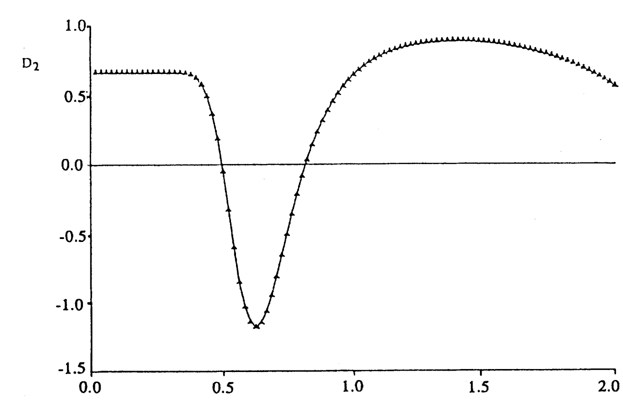

FIG. 6. Doppler parameter $D_2(T) = \gamma_2/(\rho_s/\rho)$ for pure ^{4}He. Solid line corresponds to the simplest expression [Eq. (32b)]; triangles, the exact calculation of the initial determinant [see Eq. (51) in Ref. 3].

See Eqs. (36) and (71) in Ref. 3. Note the corrections to Eqs. (68) and (71) in Ref. 3: the correct expressions are

$$K_1(T)=\underline{1+}\frac{\partial\rho}{\partial T}\left\{\frac{1}{\rho^2}\frac{\partial p}{\partial T}\frac{\partial p}{\partial\rho}[\cdots]/[\cdots]^2 - \left[\frac{\partial p}{\partial\rho}[\cdots]+\underline{3}\frac{\sigma}{\rho}\right]\right\}\Big/[\cdots]\ ,$$

$$K_1(T)=1+(3a+1)(a+\underline{1})\ ;$$

corrected terms are underlined. We see that the low-T value of $|\gamma_4|$ is smaller by one order of magnitude than that of γ_1. We discuss below the detailed explanation of all the differences and the similarities in the behavior of γ_i. Now we only note that γ_4 is the result of the "competition" (subtraction) of two large terms, the first and third in the expression (22) (see Fig. 7) whereas γ_1 contains only the first term

$$\left[-\frac{\rho_n}{\rho}+\frac{\partial\rho_n/\partial P}{\partial\rho/\partial p}\right],$$

with negative factor $\propto\alpha<0$ [see Eqs. (33) and (22)]. We discuss below the physical origin of these terms and their very different behavior due to ^{3}He admixture, $X\neq 0$. Further, we get

$$\max[D_4]\approx 26.6\quad (T\approx 0.63\ \mathrm{K})\quad (\mathrm{BEF})\ ,$$
$$\max[(-D_1)]\approx 43.7\quad (T\approx 0.56\ \mathrm{K})\quad (\mathrm{OEF})\ , \qquad (37)$$
$$\max[(-D_2)]\approx 1.18\quad (T\approx 0.61\ \mathrm{K})\quad (\mathrm{BEF})\ .$$

The absolute value of the maximum OEF parameter for first sound turns out to be much more (by two orders of magnitude) than that of fourth sound, $\max(-\gamma_4-\rho_n/\rho)\approx 0.6\times 10^{-5}$ ($T\approx 0.4$ K) and $\max(-\gamma_1-\rho_n/\rho)\approx 1.7\times 10^{-3}$ ($T\approx 0.67$ K).

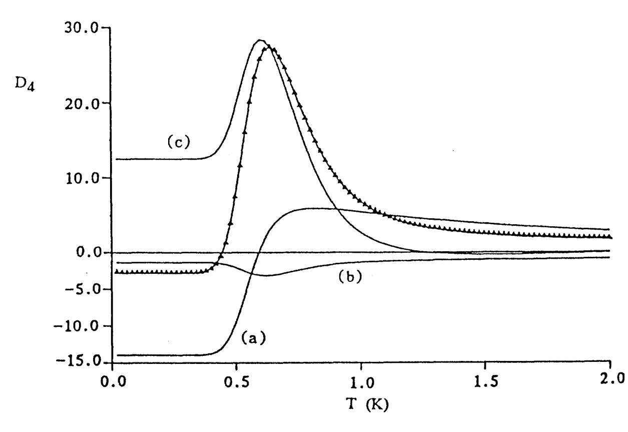

FIG. 7. Doppler parameter $D_4(T)=(\gamma_4+\rho_n/\rho)/(\rho_{n4}/\rho)$ for pure ^{4}He. Contribution of density $F(T)$ and entropy $G(T)$ key factors: (a) $(\gamma_4^{(\rho)}+\rho_n/\rho)/(\rho_{n4}/\rho)=1+F$; (b) $\gamma_4^{(\sigma)}/(\rho_{n4}/\rho)=-[\sigma/T(\partial\sigma/\partial T)]G$; (c) $\gamma_4^{(\alpha)}/(\rho_{n4}/\rho)=-[\alpha\bar{u}_1^2/T(\partial\sigma/\partial T)]G$ [Eqs. (38) and (46)].

C. Doppler parameters and structures of sounds

The expressions for γ_i (22), (32b), and (33) are determined on the one hand by the terms in the coefficients in the hydrodynamic equations which are proportional to the relative velocity w, and on the other hand by the structure of the sounds at $w=0$. The expressions for γ_i can be represented in the form

$$\gamma_4^{(1)}=\frac{\partial\rho/\partial(v_s^2/2)}{\rho\partial\rho/\partial P}-\frac{\sigma\partial\sigma/\partial(v_s^2/2)}{\partial\sigma/\partial T}-\alpha\frac{\partial\sigma/\partial(v_s^2/2)}{(\partial\rho/\partial P)(\partial\sigma/\partial T)}$$
$$\equiv\gamma_4^{(\rho)}+\gamma_4^{(\sigma)}+\gamma_4^{(\alpha)}\ , \qquad (38)$$

$$\gamma_2^{(0)}=2-\frac{\rho}{\rho_n}\frac{\sigma\partial\sigma/\partial(w^2/2)}{\partial\sigma/\partial T}=2+\gamma_2^{(\sigma)}=2+\frac{\rho}{\rho_n}\gamma_4^{(\sigma)}\ , \qquad (39)$$

$$\gamma_1^{(1)}=\left[\frac{\rho}{\rho_n}\gamma_4^{(\sigma)}+2\frac{\bar{u}_2^2}{\bar{u}_1^2}\right]\frac{\alpha^2\bar{u}_1^2}{(\partial\sigma/\partial T)(1-\bar{u}_2^2/\bar{u}_1^2)^2}+\left[\frac{\rho}{\rho_n}\gamma_4^{(\rho)}-2\right]\frac{\alpha\sigma}{(\partial\sigma/\partial T)(1-\bar{u}_2^2/\bar{u}_1^2)}\ . \qquad (40)$$

The derivatives in the numerators of the terms of γ_i,

$$\rho^2 f=\left[\frac{\partial\rho}{\partial(w^2/2)}\right]_v=\left[\frac{\partial\rho}{\partial(v_s^2/2)}\right]_{v_n}=\rho^2\frac{\partial(\rho_n/\rho)}{\partial P}\ ,$$

$$g=\left[\frac{\partial\sigma}{\partial(w^2/2)}\right]_v=\left[\frac{\partial\sigma}{\partial(v_s^2/2)}\right]_{v_n}=\frac{\partial(\rho_n/\rho)}{\partial T}\ ,$$

$$\alpha=\frac{1}{\rho}\frac{\partial\rho}{\partial T}=\rho\frac{\partial\sigma}{\partial P}\ , \qquad (41)$$

describe the dependence of the densities of mass and entropy (per unit mass) on the relative velocity and the "coupling" between the oscillations

$$P'=u^2\rho'=u\rho v',\quad T'=u^2\frac{\rho_n}{\rho_s\sigma^2}\sigma'=\frac{u\rho_n}{\rho\sigma}w'\ , \qquad (42)$$

respectively. The latter are independent at $\alpha=0$ in the case of free sounds (i.e., without the condition $v_n=v_n'=0$). The derivatives in the denominators of γ_i, $\partial\rho/\partial P$ and $\partial\sigma/\partial T$, characterize the relation between the oscillating quantities ρ' and P', σ' and T'. The origin of the derivatives (41) in γ_i is the following. The densities and the currents in the hydrodynamic equations contain terms proportional to w^2 and Aw, respectively (A is some thermodynamic quantity). These terms give in the equations for the sound amplitudes linear (in w') terms ($\propto ww', w'A$) with the factors (41) (as well as with the quantities $\partial\rho/\partial P$, $\partial\sigma/\partial T$, and ρ_n/ρ). Correspondingly, the structure of the sounds acquires at $w\neq 0$ without fail the terms proportional to w'. However, the sound velocity changes in the first order in w' only in the case of the presence of w oscillation (w') already in the initial structure of the sound. In fact, in the case $\alpha=0$ when the oscillations (42) correspond to independent sounds, the first sound has no Doppler correction, $\bar{\gamma}_1=0$, whereas the second sound has it, $\bar{\gamma}_2\neq 0$ [Eq. (32a)]. The cause of this becomes evident if we exclude the variables T' and P' from the equations for T', P', w', and v' which correspond to the determinant (51) in Ref. 3:

$$uw\left[-\frac{\partial\rho}{\partial(\omega^2/2)}+\frac{2\rho_n\rho_s}{\rho}\frac{\partial\rho}{\partial P}\right]w'+\rho\left[1-u^2\frac{\partial\rho}{\partial P}\right]v'=0\ ,$$

$$\left[-\frac{\rho_n}{\rho}\left[\frac{\bar{u}_2^2}{u}-u\right]-w\left[\frac{\rho_n\bar{u}_2^2}{\rho u^2}\frac{\partial(\rho_s\sigma)/\partial T}{\rho\partial\sigma/\partial T}-\frac{(\rho_s+\rho)\sigma}{\rho\partial\sigma/\partial T}\frac{\partial\sigma}{\partial(w^2/2)}+\frac{\rho_n}{\rho}+\frac{2\rho_n\rho_s}{\rho^2}\right]\right]w'$$
$$-w\left[\frac{u^2}{\rho}\frac{\partial\rho}{\partial(w^2/2)}-\frac{\rho_n}{\rho}-\frac{\sigma\partial(\rho_s\sigma)/\partial p}{\partial\sigma/\partial T}\right]v'=0\ . \tag{43}$$

In the linear approximation we obtain

$$u_1^2=\frac{\partial P}{\partial\rho},\quad u_2^2=\bar{u}_2^2+\bar{\gamma}_2 w\ ,$$
$$\bar{\gamma}_2=2-\frac{\sigma\partial\rho_n/\partial T}{\rho_n\partial\sigma/\partial T}\approx 2-g\frac{\rho\sigma}{\rho_n\partial\sigma/\partial T}\ . \tag{44}$$

Although the independent oscillations of v' and w' (42) are mixed in Eqs. (43), the linear (in w) correction to the first-sound velocity is absent. Further, both the terms f and g introduce changes in the structure of sounds which are linear in w, but only g leads to the first-order Doppler correction.

This consideration explains the existence of α in γ_1: only because of the inequality $\alpha\neq 0$ does the structure of first sound contain at $w=0$ the w' oscillations (with the factor α). In the structure of fourth sound the oscillations v' and w' are mixed independently of α: because of the condition $v_n'=0$ the independent mode is proportional to $v_s'=v'-(\rho_n/\rho)w'$. The small coefficient of w' implies a small factor in γ_4, $\gamma_4\approx\gamma_4^{(1)}$. The term $\gamma_4^{(\sigma)}$ is proportional to the similar term in $\gamma_2^{(0)}$ [Eqs. (38) and (39)], but in γ_4 this term is supplemented by the direct contribution of the ρ' oscillation which contains w', $\gamma_4^{(\rho)}$ [owing to the breaking of the relations (42) in the case of fourth sound], in addition to the indirect contribution which is proportional to α, $\gamma_4^{(\alpha)}$.

We show below that $\gamma_4^{(\rho)}$ ensures in the phonon region a negative constant of γ_4 and (with the factor $\alpha<0$) a positive constant of γ_1, whereas $\gamma_2^{(\sigma)}$ ensures a deep minimum for γ_2 and [with the factor $\alpha^2\rho/\rho_n=O(\rho_n/\rho)>0$] for γ_1, and simultaneously a sharp maximum for γ_4: here $\gamma_4^{(\alpha)}\approx -f_2\gamma_2^{(\sigma)}\rho_n/\rho$, so that

$$\gamma_4^{(\sigma)}+\gamma_4^{(\alpha)}\approx -(f_2-1)\gamma_2^{(\sigma)}\frac{\rho_n}{\rho}$$

[$f_2-1=8.27$ see Eq. (35)].

D. Origin of T behavior of D_i: Key derivatives, properties of phonons and rotons

The common peculiarities of the Doppler parameters, D_i or Γ_i, for pure ^{4}He, the plateau in the phonon region ($T<\tilde{T}$) and the sharp peak after it ($T\sim\tilde{T}$), are determined by the density (F) and entropy (G) "key factors"

$$F=f\Big/\frac{\rho_{n4}}{\rho^2}\frac{\partial\rho}{\partial P},\quad G=g\Big/\frac{\rho_{n4}}{\rho T}\ . \tag{45}$$

[Note that f and g in Eq. (41) do not change at $X\neq 0$: $\partial(\rho_n/\rho)/\partial P=\partial(\rho_{n4}/\rho)/\partial P$, $\partial(\rho_n/\rho)/\partial T=\partial(\rho_{n4}/\rho)/\partial T$. The dimensionless form F,G of the "key derivatives" f,g corresponds to the simplest T dependence.] In fact, the results of Eqs. (32b), (22), and (33) can be represented in the following form (here $\rho_n=\rho_{n4}$):

$$D_2\approx\gamma_2,\quad \gamma_2-\frac{\rho_s}{\rho}=\left[1-\frac{\sigma}{T\partial\sigma/\partial T}G\right]+\Delta\gamma_2\ ,$$
$$\Delta\gamma_2=-\frac{\rho_n}{2\rho}-\gamma_1+O(\alpha)\ ,$$
$$D_4=F+1-\frac{\sigma+\alpha\bar{u}_1^2}{T\partial\sigma/\partial T}G\ , \tag{46}$$
$$D_1=\alpha T\frac{\rho}{\rho_n}\left[1-\frac{\bar{u}_2^2}{\bar{u}_1^2}\right]^{-1}\left[\frac{\sigma}{T\partial\sigma/\partial T}F-2\frac{\sigma+\alpha\bar{u}_1^2}{T\partial\sigma/\partial T}\right]$$
$$+\frac{\rho}{\rho_n}\frac{\alpha^2\bar{u}_1^2}{\partial\sigma/\partial T}\left[1-\frac{\bar{u}_2^2}{\bar{u}_1^2}\right]^{-2}\left[2-\frac{\sigma}{T\partial\sigma/\partial T}G\right]\ .$$

All the terms in the expressions for D_i including F and G, are almost constant at $T<\tilde{T}$, but the "key" factors introduce a substantial change at $T\sim\tilde{T}$: F corresponds to a sharp increase (large step) and G to a positive peak. In $D_2\approx\gamma_2$ the only factor G (with negative coefficient) causes a lowering of the plateau (so that $\gamma_2-\rho_s/\rho<0$, absence of the OEF) and a negative peak (BEF).

In D_4 a "competition" occurs between two factors: G (with positive coefficient $-\alpha\bar{u}_1^2\gg\sigma$) substantially diminishes the modulus of the negative plateau which is caused by $(F+1)$ (OEF) and gives a positive peak (BEF). In D_1 a similar competition occurs, but F and G appear with opposite signs, so that G lowers the positive plateau (BEF) and gives a negative peak (OEF).

The behavior mentioned above of the key factors and of their coefficients is the consequence of the peculiarities of the phonons and rotons. In fact, we get

$$F=\frac{-(5a+1)\rho_{\rm ph}+[(\Delta/T)|r|-\frac{1}{3}]\rho_r}{\rho_{\rm ph}+\rho_r}\,,$$
$$G=\frac{4\rho_{\rm ph}+(\Delta/T)\rho_r}{\rho_{\rm ph}+\rho_r}\,, \tag{47}$$

$$\frac{\sigma}{T\partial\sigma/\partial T}=\frac{\rho_{\rm ph}+\lambda(T/\Delta)\rho_r}{3\rho_{\rm ph}+\lambda\rho_r}\,,$$
$$\frac{\alpha\bar{u}_1^2}{T\partial\sigma/\partial T}=\frac{-f_2\rho_{\rm ph}+\lambda|r|\rho_r}{3\rho_{\rm ph}+\lambda\rho_r}\,, \tag{48}$$

$$\frac{\alpha T\rho}{\rho_n}=\frac{-f_2\rho_{\rm ph}+\lambda|r|\rho_r}{\rho_{\rm ph}+\rho_r}\,,$$
$$\frac{\bar{u}_2^2}{\bar{u}_1^2}=\frac{[\rho_{\rm ph}+\lambda(T/\Delta)\rho_r]^2}{(\rho_{\rm ph}+\rho_r)(3\rho_{\rm ph}+\lambda\rho_r)}\,, \tag{49}$$

$$f_2\equiv 3a+1\approx 9.37,\quad r=\frac{\rho}{\Delta}\frac{\partial\Delta}{\partial\rho}=-0.59\,,$$
$$\lambda=\frac{3(k_B\Delta)^2}{(cp_0)^2}\approx 0.18 \tag{50}$$

(see Ref. 3; Δ and p_0 are the energy and momentum of the roton minimum, respectively).

A negative sign of $(F+1)$ (the source of the OEF) corresponds to the negative "compressibility" of the phonons $[(F+1)\propto\partial\rho_n/\partial P<0]$: the equilibrium phonon concentration *decreases* by compression $(a>0)$.

The behavior of the fractions (47)–(49) at $T\sim\tilde{T}$ [large step of F and sharp peak of G which is the basis of the peaks of D_1 (OEF), D_2 (BEF), and D_4 (BEF); slow change of other fractions] is determined by two peculiarities of the phonon and roton contribution. (i) The fast exponential increase of roton concentration: the index $\Delta/\tilde{T}\approx 15.17$. (ii) The unusual smallness of the roton contribution to entropy: $\lambda\tilde{T}/\Delta\approx 10^{-2}$. The latter is explained by the very large "effective mass" of the rotons, $\tilde{m}_r\equiv\rho_r/n_r\approx(p_0^2/3k_B\tilde{T})$ in comparison with the phonon case, $\tilde{m}_{\rm ph}\equiv\rho_{\rm ph}/n_{\rm ph}\approx k_B\tilde{T}/c^2$:

$$\frac{\tilde{m}_r}{\tilde{m}_{\rm ph}}\approx\frac{c^2p_0^2}{3k_B^2\tilde{T}}=\frac{\Delta^2}{\lambda\tilde{T}^2}=1.28\times 10^3\,.$$

Some additional considerations relative to the role of the phonon and roton peculiarities as well as to the pressure dependence of D_i are given in Appendix A.

III. FOURTH SOUND IN A ^{4}He-^{3}He MIXTURE

A. Exact equation for the fourth-sound velocity

In the case of a completely locked normal component the basic two-fluid hydrodynamic equations are the following:[6]

$$\frac{\partial\rho}{\partial t}+{\rm div}\,\rho_s\mathbf{v}_s=0\,,$$
$$\frac{\partial(\rho\sigma)}{\partial t}=0\,,$$
$$\frac{\partial\mathbf{v}_s}{\partial t}+(\mathbf{v}_s\nabla)\mathbf{v}_s=-\nabla\left[\mu-X\frac{Z}{\rho}\right]\,, \tag{51}$$
$$\frac{\partial(\rho X)}{\partial t}=0\,,$$

$$\mu=X\mu_3+(1-X)\mu_4,\quad\frac{Z}{\rho}=\mu_3-\mu_4\,,$$
$$d\mu=\frac{1}{\rho}dP-\sigma dT+\frac{Z}{\rho}dX-\frac{\rho_n}{\rho}\mathbf{v}_s d\mathbf{v}_s\,. \tag{52}$$

As in Refs. 2 and 3 for pure ^{4}He, we consider the case of fourth sound in a capillary, where we do not have to worry about the effect of porosity which modulates the velocity of fourth sound in porous media.[15]

Regarding all the variables in Eq. (51) as functions of P, T, X, and v_s and substituting the latter variables with the corrections $(P',\dots)\propto\exp[i\omega(x-ut)]$ which correspond to the sound propagating along the x axis, $Ox\|\mathbf{v}_s$, we obtain a system of linear equations for the amplitudes P', T', X', and v_s' $(v_n=v_n'=0)$ and correspondingly the equation for the fourth-sound velocity u:

$$\begin{vmatrix}
-u\frac{\partial\rho}{\partial P}+v_s\frac{\partial\rho_s}{\partial P} & -u\frac{\partial\rho}{\partial T}+v_s\frac{\partial\rho_s}{\partial T} & -uv_s\rho^2\frac{\partial(\rho_n/\rho)}{\underline{\partial P}}+\rho_s & -u\frac{\partial\rho}{\partial X}+v_s\frac{\partial\rho_s}{\partial X}\\
\frac{\partial(\rho\sigma)}{\partial P} & \frac{\partial(\rho\sigma)}{\partial T} & \left[\rho\frac{\partial(\rho_n/\rho)}{\underline{\partial T}}+\sigma\rho^2\frac{\partial(\rho_n/\rho)}{\underline{\partial P}}\right]v_s & \frac{\partial(\rho\sigma)}{\partial X}\\
\frac{1}{\rho}-X\frac{\partial(1/\rho)}{\partial X} & -\left[\sigma-X\frac{\partial\sigma}{\partial X}\right] & -u+v_s\left[1-\left[\frac{\rho_n}{\rho}-X\frac{\partial(\rho_n/\rho)}{\underline{\partial X}}\right]\right] & -X\frac{\partial(X/\rho)}{\partial X}\\
X\frac{\partial\rho}{\partial P} & X\frac{\partial\rho}{\partial T} & Xv_s\rho^2\frac{\partial(\rho_n/\rho)}{\underline{\partial P}} & \frac{\partial(\rho X)}{\partial X}
\end{vmatrix}=0\,. \tag{53}$$

The peculiarity of the underlined factors will be discussed below [see Eq. (B18c)].

We used here the Maxwell relations based on Eq. (52)

$$\frac{\partial\rho}{\partial(v_s^2/2)}=\rho^2\frac{\partial(\rho_n/\rho)}{\partial P},\quad \frac{\partial\sigma}{\partial(v_s^2/2)}=\frac{\partial(\rho_n/\rho)}{\partial T},$$
$$\frac{\partial\mu}{\partial(v_s^2/2)}=-\frac{\rho_n}{\rho},\quad \frac{\partial(Z/\rho)}{\partial(v_s^2/2)}=-\frac{\partial(\rho_n/\rho)}{\partial X}. \tag{54}$$

Supposing $v_s \ll u$ we find, as in the case of pure ^4He [cf. Eqs. (10)–(12) in Ref. 3],

$$Du^2-\rho_s\tilde{B}_1+uv_s\left\{-\left[\frac{\rho_s}{\rho}+1\right]D+\tilde{A}+\tilde{B}+\tilde{C}\right\}=0, \tag{55a}$$

$$u=u_0+\Delta u,$$

$$u_0^2=\rho_s\frac{\tilde{B}_1}{D},\quad \Delta u=\left[1-\frac{\rho_n}{2\rho}-\frac{\tilde{A}+\tilde{B}+\tilde{C}}{2D}\right]v_s, \tag{55b}$$

or

$$\Delta u=\left[\frac{\rho_s}{\rho}-\gamma_4\right]v_s,\quad \gamma_4=-\frac{\rho_n}{2\rho}+\frac{\tilde{A}+\tilde{B}+\tilde{C}}{2D}, \tag{55c}$$

where

$$\rho D=\rho\left[\frac{\partial\rho}{\partial P}\frac{\partial(\rho\sigma)}{\partial T}-\frac{\partial\rho}{\partial T}\frac{\partial(\rho\sigma)}{\partial P}\right] \tag{56}$$

is the coefficient of u^2 in the determinant (53) (note that it does not depend on X), $-\rho\rho_s\tilde{B}_1$ is the term of det (53) which is independent of u (and v_s),

$$\tilde{B}_1=\left[1+\frac{X}{\rho}\frac{\partial\rho}{\partial X}\right]\left[\left[\sigma-X\frac{\partial\sigma}{\partial X}\right]\frac{\partial(\rho\sigma)}{\partial P}+\left[\frac{1}{\rho}-X\frac{\partial(1/\rho)}{\partial X}\right]\frac{\partial(\rho\sigma)}{\partial T}\right]$$
$$-\frac{X}{\rho}\frac{\partial\rho}{\partial T}\left[X\frac{\partial(Z/\rho)}{\partial X}\frac{\partial(\rho\sigma)}{\partial P}+\frac{\partial(\rho\sigma)}{\partial X}\left[\frac{1}{\rho}-X\frac{\partial(1/\rho)}{\partial X}\right]\right]+\frac{X}{\rho}\frac{\partial\rho}{\partial P}\left[X\frac{\partial(Z/\rho)}{\partial X}\frac{\partial(\rho\sigma)}{\partial T}-\frac{\partial(\rho\sigma)}{\partial X}\left[\sigma-X\frac{\partial\sigma}{\partial X}\right]\right], \tag{57}$$

and

$$\rho\left\{-\left[\frac{\rho_s}{\rho}+1\right]D+\tilde{A}+\tilde{B}+\tilde{C}\right\}\equiv\rho E, \tag{58a}$$

where

$$\tilde{B}=\tilde{B}_1B_2,\quad B_2=\rho^2\frac{\partial(\rho_n/\rho)}{\partial P} \tag{58b}$$

[cf. Eq. (12) in Ref. 3] is the coefficient of uv_s.

The origins of all the terms, $\tilde{A}$, $\tilde{B}$, $\tilde{C}$, and D, their first-order approximation, and the dependence on T and X, as well as the proof of the exact cancellation of the finite corrections $\propto O(X^k)$ at $T\to 0$, are discussed in Appendix B.

B. Direct calculations at $T\to 0$

Now we prove the result at $T\to 0(\gamma_4+\rho_n/\rho)=O(\rho_{n4}/\rho)$ by means of the direct calculation of the determinant (53). We find

$$\frac{\partial\rho}{\partial P}\approx\frac{1}{c^2}\left[1-\frac{TX}{T_0}+O\left[\frac{\rho_{n4}}{\rho}\right]\right]\equiv\frac{1}{\bar{c}^2}\left[1+O\left[\frac{\rho_{n4}}{\rho}\right]\right];$$

cf. Eqs. (23) and (24) in Ref. 3; we get here

$$\phi=\phi(^4\mathrm{He})-\frac{kT\rho X}{m_4};$$

see Eq. (B7). Further we obtain

$$\frac{\partial\rho_s}{\partial P}\approx\frac{1}{\bar{c}^2}\frac{\rho_s}{\rho}\left[1+O\left[\frac{\rho_{n4}}{\rho}\right]\right],$$

$$\frac{\partial\rho}{\partial T}\approx\frac{\rho}{T_0}\left[X+\frac{T_0}{T}O\left[\frac{\rho_{n4}}{\rho}\right]\right],$$

$$\frac{\partial\rho_s}{\partial T}\approx-\frac{\rho_s}{T_0}\left[X+\frac{T_0}{T}O\left[\frac{\rho_{n4}}{\rho}\right]\right],$$

$$\frac{\partial\rho}{\partial X}\approx-\left[(1+\tilde{\alpha})\frac{m_4}{m_3}-1\right]\rho\equiv-\theta\rho\approx-0.71\rho,$$

$$\frac{\partial\rho_s}{\partial X}\approx-\theta\rho_s-\frac{\rho_n}{X}$$

[see Eqs. (B8), (B14), and (B16)].

Thus we obtain for the determinant (53)

$$\begin{vmatrix} -\frac{u}{\bar{c}^2}+\frac{v_s}{\bar{c}^2}\frac{\rho_s}{\rho} & u\frac{\rho X}{T_0}-v_s\frac{\rho_s X}{T_0} & \rho_s & u\theta\rho - v_s\left[\theta\rho_s+\frac{\rho_n}{X}\right] \\ \frac{\partial(\rho\sigma)}{\partial P} & \frac{\partial(\rho\sigma)}{\partial T} & 0 & \frac{\partial(\rho\sigma)}{\partial X} \\ \frac{1}{\rho}-X\frac{\partial(1/\rho)}{\partial X} & -\left[\sigma-X\frac{\partial\sigma}{\partial X}\right] & -u+v_s & -X\frac{\partial(Z/\rho)}{\partial X} \\ \frac{X}{\bar{c}^2} & -\frac{X^2\rho}{T_0} & 0 & \rho(1-X\theta) \end{vmatrix}$$

$$=(u^2-2uv_s)\left[\frac{\partial(\rho\sigma)}{\partial P}\frac{\rho X}{T_0}+\frac{\partial(\rho\sigma)}{\partial T}\frac{1}{\bar{c}^2}\right]+O(v_s^2)+\text{terms independent of }(u,v_s)\left[+O\left[\frac{\rho_{n4}}{\rho}\right]\right]=0\,.$$

Substituting $u=u_0+(\rho_s/\rho-\gamma_4)v_s$ we find the exact result for $\gamma_4^{X\neq 0}(T\to 0)$:

$$\left[\frac{\rho_s}{\rho}-\gamma_4^{X\neq 0}(T\to 0)\right]=1+O\left[\frac{\rho_{n4}}{\rho}\right],\quad\text{i.e.,}\quad \gamma_4^{X\neq 0}(T\to 0)+\frac{\rho_n}{\rho}=O\left[\frac{\rho_{n4}}{\rho}\right]. \tag{59}$$

Another proof is given in Appendix B [see Eqs. (B18)].

C. Temperature behavior of $\gamma_4(T)$ in a ^{4}He-^{3}He mixture

In order to represent the result (59) in the approximate expression γ_4 [Eq. (B6)] exactly it is sufficient to add one correction of the second order from the first term of $\tilde{B}_1$ in Eq. (B2a) which gives the following change in Eq. (B6):

$$\frac{\partial\rho_n}{\partial P}\Big/\frac{\partial\rho}{\partial P}\to\left[\frac{\partial\rho_n}{\partial P}\Big/\frac{\partial\rho}{\partial P}\right]\left[1+\frac{X}{\rho}\frac{\partial\rho_n}{\partial X}\right].$$

Thus we get

$$\Delta u_4=\left[\frac{\rho_s}{\rho}-\gamma_4\right]v_s\,,$$

$$\gamma_4\approx\left[-\frac{\rho_n}{\rho}+\frac{\partial\rho_n/\partial P}{\partial\rho/\partial P}\left[1+\frac{X}{\rho}\frac{\partial\rho}{\partial X}\right]-\frac{X}{\rho}\frac{\partial\rho_n}{\partial X}\right]-\frac{(\sigma-X\partial\sigma/\partial X)\frac{1}{2}[\partial\rho_n/\partial T+\rho\partial(\rho_n/\rho)/\partial T]}{\rho\partial\sigma/\partial T}-\frac{(\alpha/\rho)\frac{1}{2}[\partial\rho_n/\partial T+\rho\partial(\rho_n/\rho)/\partial T]}{(\partial\rho/\partial P)(\partial\sigma/\partial T)}$$

$$=\frac{\partial P}{\partial\rho}\rho f+\left[-\frac{\rho_n}{\rho}+h\right]-\left[\left[\sigma-X\frac{\partial\sigma}{\partial X}\right]\Big/\frac{\partial\sigma}{\partial T}\right]g-\left[\frac{\partial P}{\partial\rho}\alpha\Big/\frac{\partial\sigma}{\partial T}\right]g+O\left[\frac{\rho_{n4}}{\rho}X\right]$$

$$=\gamma_4^{(\rho)}+\gamma_4^{(Z/\rho)}+\gamma_4^{(\sigma)}+\gamma_4^{(\alpha)}+O\left[\frac{\rho_{n4}}{\rho}X\right]=-\frac{\rho_n}{\rho}+O\left[\frac{\rho_{n4}}{\rho}\right]. \tag{60a}$$

see Eq. (B18c), cf. Eq. (38);

$$\gamma_4^{(Z/\rho)}\equiv X\left[\partial\left[\frac{Z}{\rho}\right]\Big/\partial\frac{v_s^2}{2}\right]_{v_n}=-X\partial\left[\frac{\rho_n}{\rho}\right]\Big/\partial X=-\frac{\rho_n}{\rho}+h=-\frac{\rho_n}{\rho}+\frac{\rho_{n4}}{\rho}+O\left[\frac{\rho_{n4}}{\rho}X\right];$$

$$\gamma_4^{(\sigma)}=-\left[\left[\sigma-X\frac{\partial\sigma}{\partial X}\right]\Big/\frac{\partial\sigma}{\partial T}\right]g\,.$$

This result coincides with that in Eqs. (6) and (7) in Ref. 5. [Here we redefine γ_4: $\Delta u_4=\gamma_4 v_s\to\Delta u_4=(\rho_s/\rho-\gamma_4)v_s$.] Substituting Eqs. (B8) and (B11)–(B15) in Eq. (60a) we find at low T (including the beginning of the roton region)

$$\gamma_4 \approx \left[-\frac{\rho_n}{\rho}+\frac{\partial\rho_{n4}/\partial P}{\partial\rho/\partial P}\right]-\frac{[\rho_{ph}/\rho+(T/T_0)X][4\rho_{ph}/\rho+(\Delta/T-\frac{1}{2})\rho_r/\rho-\rho_{n3}XT/2\rho T_0]}{3[\rho_{ph}/\rho+(T/2T_0)X]}$$
$$+\frac{[f_2\rho_{ph}/\rho+(T/T_0)X][4\rho_{ph}/\rho+(\Delta/T-\frac{1}{2})\rho_r/\rho-\rho_{n3}XT/2\rho T_0]}{3[\rho_{ph}/\rho+(T/2T_0)X]}+O\left[\frac{\rho_{n4}}{\rho}X\right]$$
$$=-\frac{\rho_n}{\rho}+\left[\gamma_4^{(\rho)}+\frac{\rho_{n4}}{\rho}\right]+K\gamma_4^{(\sigma)}+O\left[\frac{\rho_{n4}}{\rho}X\right], \qquad (60b)$$

where

$$K=(1+\gamma_4^{(\alpha)}/\gamma_4^{(\sigma)})$$
$$=\left[1+c^2\alpha\Big/\left[\sigma-X\frac{\partial\sigma}{\partial X}\right]\right]$$
$$=\left[1-\left[f_2\frac{\rho_{ph}}{\rho}+\frac{TX}{T_0}\right]\Big/\left[\frac{\rho_{ph}}{\rho}+\frac{TX}{T_0}\right]\right],$$

$$f_2=3a+1,\quad \gamma_4^{(\rho),X\neq0}=\rho f\Big/\frac{\partial\rho}{\partial P}=\gamma_4^{(\rho),X=0},$$

$$\gamma_4^{(\rho)}+\frac{\rho_{n4}}{\rho}=-5a\frac{\rho_{ph}}{\rho}-\left[\frac{\Delta}{T}r-\frac{2}{3}\right]\frac{\rho_r}{\rho},$$

$$\gamma_4^{(\sigma),X\neq0}=\left[-4\frac{\rho_{ph}}{\rho}\left[\frac{\rho_{ph}}{\rho}+\frac{TX}{T_0}\right]\Big/3\left[\frac{\rho_{ph}}{\rho}+\frac{TX}{2T_0}\right]\right]$$
$$+O(X^2).$$

We see that, although the scale of γ_4 turns out to be of the order of ρ_n/ρ,

$$\gamma_4(T\to0)=\frac{\rho_n(T=0)}{\rho}=\frac{m_3^*}{m_3}X,$$

so that unlike the case of pure ^{4}He $\gamma_4(T=0)\neq0$, the deviation from $\gamma_4^c=\rho_n/\rho$ becomes substantially smaller, of the order of ρ_{n4}/ρ.

The physical reason for that is clear: at low T the atoms of the admixture are stopped and correspondingly do not take part in the sound oscillations. This means that the Doppler coefficient in the expression $\Delta u_4=(\rho_s/\rho-\gamma_4)v_s$ at $T\to0$ goes to 1, i.e., $\gamma_4(T\to0)\to-\rho_n/\rho$. The deviation from this special kinematic result at $T\neq0$ is caused by the ^{4}He excitations only, which do take part in the oscillations owing to the possibility of the transformation between ρ_{n4} and ρ_s: $v_n'=0$ but $\rho_n'=\rho_{n4}'\neq0$. This was pointed out already in Refs. 2 and 3.

However, now we obtain an unexpected result: for all that the role of the admixture is substantial. In fact at $X\gg5\times10^{-4}$ the contribution of the admixture not only suppresses the scale of $(\gamma_4+\rho_n/\rho)$ (from ρ_n/ρ up to ρ_{n4}/ρ), but also leads to the special suppression of the contribution $\gamma_4^{(\sigma)}+\gamma_4^{(\alpha)}$ [practically to the cancellation of these terms, cf. Eq. (38)]. The factor $K=\{1+c^2\alpha/[\sigma-X(\partial\sigma/\partial\lambda)]\}$ in the relation $\gamma_4^{(\sigma)}+\gamma_4^{(\alpha)}=\gamma_4^{(\sigma)}K$ jumps by the transition $\rho_{n4}\gg\rho_{n3}\to\rho_{n4}\ll\rho_{n3}$ from $-3a$ up to $O(\rho_{n4}/\rho_n)$ [the jump of $\gamma_4^{(\sigma)}$ is not so big: $\frac{4}{3}\rho_{ph}/\rho\to O(X^2)$]. The "competition" between the first and third terms of $(\gamma_4+\rho_n/\rho)$ (Fig. 7) which determines the coefficient $(\gamma_4+\rho_n/\rho)$ at $X=0$ is replaced by the predominance of the first term, $\gamma_4+\rho_n/\rho\approx\gamma_4^{(\rho)}+\rho_{n4}/\rho$. This implies a substantial increase of the constant $|[\gamma_4(T\to0)+\rho_n/\rho]/(\rho_{n4}/\rho)|$ *at arbitrarily small X*, i.e., a jump, $(-a)\to(-5a)$,

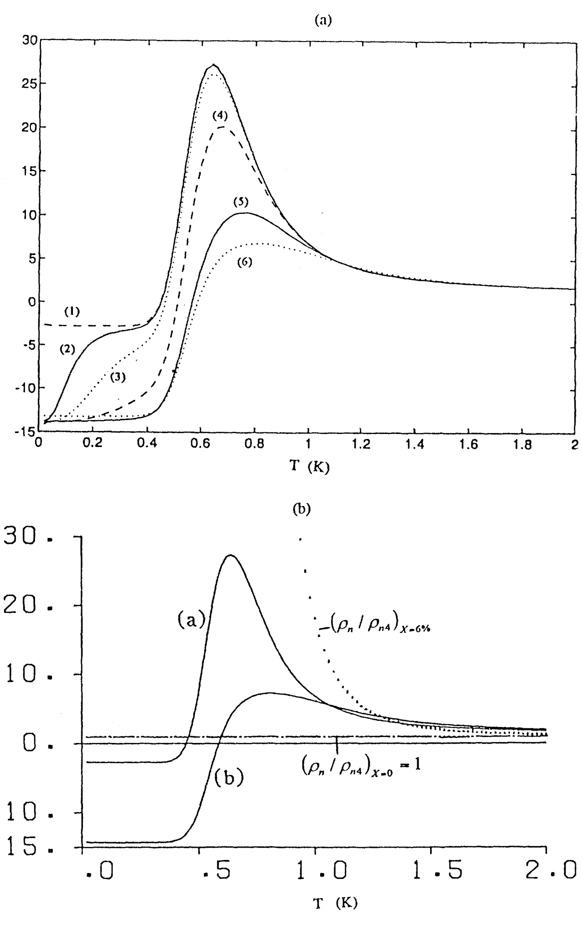

FIG. 8. (a) Doppler parameter D_4 for ^{3}He-^{4}He mixtures. Suppression of the term with the entropy factor G with increasing concentration x: (1) 0% (2) 10^{-3}% (3) 10^{-2}% (4) 10^{-1}% (5) 1% and (6) 6%. (b) Temperature regions of Doppler anomalies (OEF and BEF) for pure ^{4}He (curve a) and 6% ^{3}He (curve b). OEF when $D_4<0$, BEF when $D_4>\rho_n/\rho_{n4}$.

$$\left[\gamma_4^{X\neq 0}(T\to 0)+\frac{\rho_n}{\rho}\right]\Big/\frac{\rho_{n4}}{\rho}\approx\left[\gamma_4^{(\rho)}(T\to 0)+\frac{\rho_n}{\rho}\right]\Big/\frac{\rho_{n4}}{\rho}=-5a\approx -13.95 \tag{61}$$

instead of

$$\left[\gamma_4^{X=0}(T\to 0)+\frac{\rho_n}{\rho}\right]\Big/\frac{\rho_{n4}}{\rho}=-a\approx -2.79 \ .$$

The cancellation of $(\gamma_4^{(\sigma)}+\gamma_4^{(\alpha)})$ implies also the disappearance at $X \gg 10^{-3}$ of the sharp peak of γ_4 which is caused in the case of pure ^{4}He by the term $\gamma_4^{(\alpha)}$ (Fig. 8) as well as a broadening of the temperature region of the OEF, $(\gamma_4+\rho_n/\rho)<0$, which leads to an additional increase of the maximum of the OEF parameter:

$$\max\left[-\left[\gamma_4^{X=0.6}+\frac{\rho_n}{\rho}\right]\right]\approx 0.89\times 10^{-4} \quad (T\approx 0.52 \text{ K}) \tag{62}$$

instead of

$$\max\left[-\left[\gamma_4^{X=0}+\frac{\rho_n}{\rho}\right]\right]\approx 0.67\times 10^{-5} \quad (T\approx 0.4 \text{ K}) \ ;$$

see Fig. 9. Thus we obtain for the mixture with $X \gg 10^{-3}$ a behavior of $\gamma_4(T)/(\rho_{n4}/\rho)$ which differs substantially from the case of pure ^{4}He throughout the most interesting region, $T<\tilde{T}$, $T\sim\tilde{T}$ and is practically independent of X.

Thus concerning the role of the ^{3}He admixture in ^{4}He we find the following results.

(1) A jump of the low-T limit of the Doppler coefficient, $D_4=(\gamma_4+\rho_n/\rho)/(\rho_{n4}/\rho)$, which appears at an arbitrarily small concentration, $X\neq 0$, and is practically independent of X.

(2) A drastic change of the whole picture of the temperature dependence of the Doppler coefficient, $(\gamma_4+\rho_n/\rho)/(\rho_{n4}/\rho)$, throughout the interesting region, $0<T\sim\tilde{T}$, already at small concentrations $10^{-3} \ll X \ll 0.06 = X_{\max}$, and the practical invariability of this picture by changes of X by an order of magnitude, up to the maximum possible value of X. Note that at too small values of X we get for the low-T behavior of $(\gamma_4+\rho_n/\rho)/(\rho_{n4}/\rho)$ a variation between the two levels: the high-T ("pure") value, $-a$, and the low-T ("mixture") one, $-5a$ (see Figs. 8 and 10).

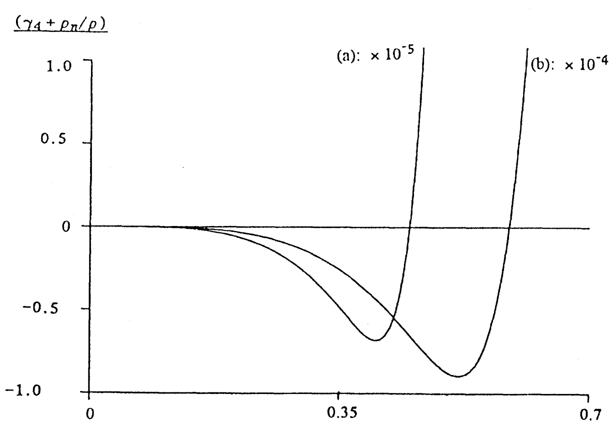

FIG. 9. "Outstripping parameter" of fourth sound: $(\gamma_4+\rho_n/\rho)=D_4\rho_{n4}/\rho<0$, for pure ^{4}He (a) and ^{3}He-^{4}He mixture ($x=6\%$) (b).

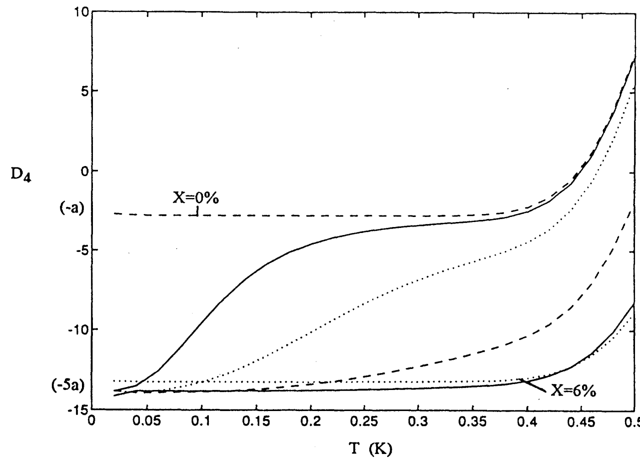

FIG. 10. Two levels of the Doppler parameter D_4: $(-a)$ $[T>T(X)]$ and $(-5a)$ $[T<T(X)]$ [see Eq. (13)].

IV. FIRST AND SECOND SOUNDS IN A ^{4}He-^{3}He MIXTURE

A. Basic equations

Now we have to use the complete set of two-fluid hydrodynamic equations:[6]

$$\frac{\partial\rho}{\partial t}+\operatorname{div}\mathbf{j}=0, \quad \mathbf{j}=\rho_n\mathbf{v}_n+\rho_s\mathbf{v}_s \ ,$$

$$\frac{\partial j_i}{\partial t}+\nabla_k(P\delta_{ik}+\rho_n v_{ni}v_{nk}+\rho_s v_{si}v_{sk})=0 \ ,$$

$$\frac{\partial(\rho\sigma)}{\partial t}+\operatorname{div}(\rho\sigma\mathbf{v}_n)=0 \ , \tag{63}$$

$$\frac{\partial\mathbf{v}_s}{\partial t}+(\mathbf{v}_s\nabla)\mathbf{v}_s=-\nabla\left[\mu-X\frac{Z}{\rho}\right] \ ,$$

$$\frac{\partial(\rho X)}{\partial t}+\operatorname{div}(\rho X\mathbf{v}_n)=0 \ ,$$

$$\mu=X\mu_3+(1-X)\mu_4, \quad \frac{Z}{\rho}=\mu_3-\mu_4 \ . \tag{64}$$

For simplicity we consider the sound oscillations propagating along the x axis, $Ox\|\mathbf{w}=\mathbf{v}_n-\mathbf{v}_s$ and choose the reference frame so that $\mathbf{v}\equiv\mathbf{j}/\rho\|Ox$. Taking into account also the formula

$$d\mu=\frac{1}{\rho}dP-\sigma dT+\frac{Z}{\rho}dX-\frac{\rho_n}{\rho}w\,dw \tag{65}$$

and substituting all the variables A_i with the oscillating corrections

$$A_i'\equiv(P',T',w',v',X')\propto\exp[ik(x-ut)] \ ,$$

we obtain a set of linear equations for A_i' which gives the equation for the sound velocities of the normal modes:

$$\begin{vmatrix} -U\frac{\partial\rho}{\partial P} & -U\frac{\partial\rho}{\partial T} & -U\rho^2\frac{\partial(\rho_n/\rho)}{\partial P}w & \rho & -U\frac{\partial\rho}{\partial X} \\ 1 & 0 & \frac{2\rho_n\rho_s}{\rho}w & -U\rho & 0 \\ \left[-U\rho\frac{\partial\rho}{\partial P}+w\frac{\partial(\rho_s\sigma)}{\partial P}\right] & \left[-U\rho\frac{\partial\sigma}{\partial T}+w\frac{\partial(\rho_s\sigma)}{\partial T}\right] & \left[-U\rho\frac{\partial(\rho_n/\rho)}{\partial T}w+\rho_s\sigma\right] & 0 & \left[-U\rho\frac{\partial\sigma}{\partial X}+w\frac{\partial(\rho_s\sigma)}{\partial X}\right] \\ \left[Uw\frac{\partial(\rho_n/\rho)}{\partial P}\frac{1}{\rho}-X\frac{\partial(1/\rho)}{\partial X}\right] & \left[Uw\frac{\partial(\rho_n/\rho)}{\partial T}-\left[\sigma-X\frac{\partial\sigma}{\partial X}\right]\right] & \left[U\frac{\rho_n}{\rho}-w\left[\frac{\rho_n}{\rho}-X\frac{\partial(\rho_n/\rho)}{\partial X}\right]\right] & -\left[U+\frac{\rho_n}{\rho}w\right] & \left[Uw\frac{\partial(\rho_n/\rho)}{\partial X}-X\frac{\partial(Z/\rho)}{\partial X}\right] \\ wX\frac{\partial\rho_s}{\partial P} & wX\frac{\partial\rho_s}{\partial T} & \rho_s X & 0 & -\rho_U+\frac{\partial(\rho_s X)}{\partial X}w \end{vmatrix}=0\ . \tag{66}$$

As in the case of pure ^{4}He, the sound velocity u appears (owing to the Galilean invariance) only in the combination $\mathbf{U}=\mathbf{u}-\mathbf{v}$ ($\mathbf{U}$ is the sound velocity in the c.m. reference frame). In the derivation of Eq. (66) we used the Maxwell relations which follow from the expression for the chemical potential (65):

$$\begin{aligned}
\left[\frac{\partial\rho}{\partial w^2/2}\right]_{P,T,X} &= \rho^2\left[\frac{\partial(\rho_n/\rho)}{\partial P}\right]_{T,X,w}\ ,\\
\left[\frac{\partial\sigma}{\partial w^2/2}\right]_{P,T,X} &= \left[\frac{\partial(\rho_n/\rho)}{\partial T}\right]_{P,X,w}\ ,\\
\left[\frac{\partial(Z/\rho)}{\partial P}\right]_{T,X,w} &= \left[\frac{\partial(1/\rho)}{\partial X}\right]_{P,T,w}\ ,\\
\left[\frac{\partial(Z/\rho)}{\partial T}\right]_{P,X,w} &= -\left[\frac{\partial\sigma}{\partial X}\right]_{P,T,w}\ ,\\
\left[\frac{\partial(Z/\rho)}{\partial w^2/2}\right]_{P,T,X} &= -\left[\frac{\partial(\rho_n/\rho)}{\partial X}\right]_{P,T,w}\ .
\end{aligned} \tag{67}$$

A detailed analysis of Eq. (66) and of its exact solution is given in Appendix C.

B. Jumps of second-sound velocity

In the pure ^{4}He case the ratio $(u_2^0)^2/c^2$ does not contain any small factor at low T: in the phonon region we get $u_2^{0^2}/c^2\approx\frac{1}{3}$ [Eq. (A3)], which formally corresponds to the elastic (collision) sound in a system of particles with the linear dispersion law $\epsilon=cp$. The situation drastically changes in the case of a ^{3}He-^{4}He mixture in the region of the admixture dominance. The point is that at low T the small factor of the ratio $[(\sigma-X\partial\sigma/\partial X)/(\partial\sigma/\partial T)]\sim T$ is not compensated here by the ratio $(\sigma-X\partial\sigma/\partial X)/(\rho_n/\rho_s)$ unlike in the pure ^{4}He case in the phonon region where $\sigma/(\rho_n/\rho_s)=c^2/T$. We get [see Eqs. (C16), (B10), and (B13)]

$$\bar{u}_2^{\,2}=\bar{\bar{u}}_2^{\,2}+\frac{\rho_s}{\rho_n}X^2\frac{\partial(Z/\rho)}{\partial X}\ , \tag{68}$$

$$\begin{aligned}\bar{\bar{u}}_2^{\,2}&=\frac{\rho_s(\sigma_4+kX/m_3)^2}{(\rho_{n4}+X\rho m_3^*/m_3)(\partial\sigma_4/\partial T+\frac{3}{2}kX/Tm_3)}\\ &\approx\frac{2}{3}\frac{\rho_s}{\rho}\frac{kT}{m_3^*}\approx\frac{2}{3}\frac{kT}{m_3^*}\ .\end{aligned} \tag{69}$$

The last term in the expression (68) is of the same order as the first one. In fact, using the formulas for an ideal solution (see Ref. 6)

$$\begin{aligned}
&Z=\rho(\mu_3-\mu_4)\ ,\\
&\mu_3\approx\mu_{30}+\frac{kT}{m_3}\ln X\ ,\\
&\mu_4\approx\mu_{40}+\frac{kT}{m_4}\ln(1-X)\ ,
\end{aligned} \tag{70}$$

we get

$$\frac{\rho_s}{\rho_n}X^2\frac{\partial(Z/\rho)}{\partial X}\approx\frac{m_3}{m_3^*}X\left[\frac{kT}{m_3X}-\frac{kT}{m_4(1-X)}\right]\approx\frac{kT}{m_3^*}. \tag{71}$$

Thus we obtain

$$u_2^{0^2}\approx\frac{5}{3}\frac{kT}{m_3^*}. \tag{72a}$$

Let $\overline{T}$ reflect the difference of the real ^{3}He-^{4}He mixture from an ideal solution. We get

$$(u_2^0)^2\approx\frac{k}{m_3^*}(\tfrac{2}{3}T+\overline{T}). \tag{72b}$$

However, at low T we get practically $\overline{T}\approx T$; see Fig. 11.

We see that the second-sound velocity in the region of admixture dominance coincides approximately with the velocity of elastic (collision) sound in the ideal Boltzmann gas ($P=(\rho/m)kT$, $PV^\gamma=$const, $\gamma=c_P/c_v=\frac{5}{3}$):

$$u^2=\left[\frac{\partial P}{\partial\rho}\right]_S=\gamma\frac{P}{\rho}=\frac{5}{3}\frac{kT}{m}. \tag{73}$$

The result (72) is correct beyond the region of the quantum (Fermi) degeneracy, $T\gg T_d$ [Eq. (17)]. Inside the latter region ($T\ll T_d$) we obtain

$$\overline{\overline{u}}_2^2\approx\frac{25}{9}\left[\frac{\pi}{3}\right]^{2/3}\frac{k^2T^2}{\hbar^2X(\rho X/m_3)^{2/3}}=\frac{25\pi^2}{18X}\left[\frac{T}{T_d}\right]^2\frac{kT_d}{m_3^*}. \tag{74}$$

Here we used the expression for the ideal Fermi gas

$$\sigma_3\approx\left[\frac{\pi}{3}\right]^{2/3}\frac{m_3^*}{\rho X\hbar^2}\left[\frac{\rho X}{m_3}\right]^{1/3}k^2T=k\frac{\pi^2}{2m_3}\frac{T}{T_d}.$$

At $T/T_d\ll 12X/25\pi^2\approx0.05X$, this term turns out to be small in comparison with the last term of the expression for $\overline{u}_2^2$ [Eq. (68)]:

$$\frac{\rho_s}{\rho_n}X^2\frac{\partial(Z/\rho)}{\partial X}\approx\frac{m_3}{m_3^*}X\frac{d(\mu_3-\mu_4)}{dX}$$
$$\approx\frac{m_3}{m_3^*}\frac{2}{3}\mu_3=\frac{2}{3}\frac{p_F^2}{2m_3^{*2}}=\frac{1}{3}\left[\frac{p_F}{m_3^*}\right]^2$$
$$=\frac{v_F^2}{3}=\frac{2}{3}\frac{kT_d}{m_3} \tag{75}$$

(p_F and v_F are the momentum and the velocity on the Fermi surface, respectively). We took into account here that

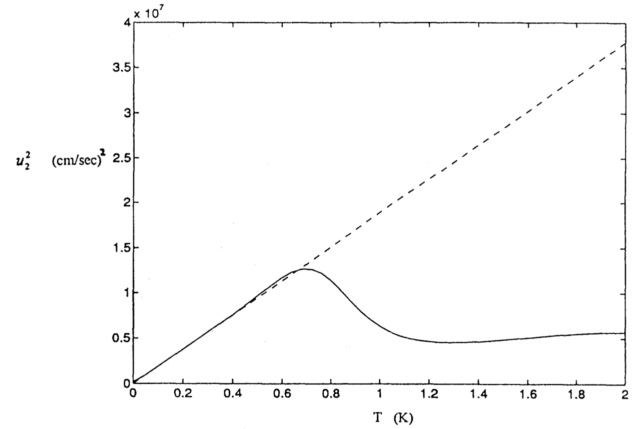

FIG. 11. Second-sound velocity in the ideal-solution approximation $u_2^0=\sqrt{\frac{5}{3}kT/m_3}$ (dashed line) and as a result of numerical calculation (solid line).

$$\mu_3=\frac{p_F^2}{2m_3^*m_3}=(3\pi^2\rho X/m_3)^{2/3}/2m_3^*m_3$$

($\mu_3=\epsilon_F/m_3$ is the chemical potential per unit mass).

The result in the quantum region $(u_2^0)^2\approx(v_F/\sqrt{3})^2$ corresponds to the elastic (collision) sound in the ideal Fermi gas.

We see that at low T [$T\ll T_1$, see Eq. (7)] already the arbitrarily small concentration of the ^{3}He admixture, $X\to0$, leads to a finite (even large) change of the second-sound velocity, i.e., we get a jump. In particular, we find

$$\frac{c}{\sqrt{3}}\to\left[\frac{5}{3}\frac{kT}{m_3^*}\right]^{1/2}\quad(T\gg T_d),$$
$$\frac{v_F}{\sqrt{3}}=(3\pi^2\rho X/m_3)^{1/3}/m_3^*\sqrt{3}\quad(T\ll0.5XT_d). \tag{76}$$

C. Doppler parameters in terms of thermodynamic derivatives

The exact results [Eqs. (C10), (C12), and (C13)] allow one to find the Doppler parameters γ_i.

First of all, using the relation

$$(u_1^0)^2-(u_2^0)^2=\sqrt{B^2-4AC}/A \tag{77}$$

[see Eq. (C9)] we obtain the exact formula

$$\gamma_2=-\frac{R}{2A}-\gamma_1, \tag{78a}$$

i.e.,

$$\gamma_2=\left\{1+\frac{\rho}{\rho_n}h-\frac{(\sigma-X\partial\sigma/\partial X)}{[(\partial\rho/\partial P)(\partial\sigma/\partial T)-\alpha^2]}\frac{\rho}{\rho_n}\left[\frac{\partial\rho}{\partial P}g-\frac{\partial\rho}{\partial T}f\right]\right.$$
$$\left.-\frac{3\rho_n}{2\rho}-\frac{X\partial\rho/\partial X}{[(\partial\rho/\partial P)(\partial\sigma/\partial T)-\alpha^2]}\frac{\rho}{\rho_n}\left[\frac{\partial\sigma}{\partial P}g-\frac{\partial\sigma}{\partial T}f\right]\right\}-\gamma_1, \tag{78b}$$

or

$$\gamma_2=\left\{\left[2-\frac{\rho}{\rho_n}X\frac{\partial(\rho_n/\rho)}{\partial X}-\frac{\sigma-X\partial\sigma/\partial X}{\partial\sigma/\partial T}\frac{\rho}{\rho_n}\frac{\partial(\rho_n/\rho)}{\partial T}\right]-\frac{3}{2}\frac{\rho_n}{\rho}+O(\alpha)+O(X)\right\}-\gamma_1 . \tag{79a}$$

Note that Eq. (78) is correct for the case of pure ^{4}He too (the case $X=0$) and Eq. (34) corresponds to Eq. (79a). The term in square brackets in Eq. (79a) corresponds to the simplest (zero) approximation for γ_2 ($\gamma_2^{(0)}$) since $\gamma_1\sim O(\rho_n/\rho)$ (see below); cf. Eq. (32b).

$$\gamma_2^{(0)}=2-\frac{\rho}{\rho_n}X\frac{\partial(\rho_n/\rho)}{\partial X}-\frac{\sigma-X\partial\sigma/\partial X}{\partial\sigma/\partial T}\frac{\rho}{\rho_n}\frac{\partial(\rho_n/\rho)}{\partial T}$$
$$=1+\frac{\rho}{\rho_n}h-\frac{\sigma-X\partial\sigma/\partial X}{\partial\sigma/\partial T}\frac{\rho}{\rho_n}g$$
$$\equiv\frac{M_0}{2\rho\partial\sigma/\partial T}=1+O\left[\frac{\rho_{n4}}{\rho}\right] .$$

We mean here the expansion in ρ_n/ρ, independently of ρ_{n4}/ρ; the latter small parameter is reflected by the factors f, g, and h [Eq. (C1)]. We can also write

$$\gamma_2^{(0)}=2+\gamma_2^{(Z/\rho)}+\gamma_2^{(\sigma)}$$
$$=2+\frac{\rho}{\rho_n}(\gamma_4^{(Z/\rho)}+\gamma_4^{(\sigma)}) \tag{79b}$$

[cf. definitions in Eqs. (38)–(40) and (60b)].

Equation (78b) can also be represented in the similar form

$$\gamma_2=\left\{2-\frac{3\rho_n}{2\rho}+\frac{\rho}{\rho_n}\gamma_4^{(Z/\rho)}+\frac{(\rho/\rho_n)\gamma_4^{(\sigma)}}{1-\alpha^2/(\partial\rho/\partial P)(\partial\sigma/\partial T)}+\frac{\alpha(\sigma-X\partial\sigma/\partial X)(\rho/\rho_n)\gamma_4^{(\rho)}}{(\partial\sigma/\partial T)[1-\alpha^2/(\partial\rho/\partial P)(\partial\sigma/\partial T)]}+\frac{(X/\rho)(\partial\rho/\partial X)}{[1-\alpha^2/(\partial\rho/\partial P)(\partial\sigma/\partial T)]}\frac{\rho}{\rho_n}(\gamma_4^{(\rho)}+\gamma_4^{(\alpha)})\right\}-\gamma_1 . \tag{79c}$$

In order to calculate γ_1 in the first approximation, $\gamma_1^{(1)}$ (hence, $\Delta\gamma_2^{(1)}$), we use the representations $(u_{1,2}^0)^2=\bar{u}_{1,2}^2+(u_{1,2}^{(1)})^2$ and $A=A_0+A_1$ [see Eqs. (C14)–(C17)] as well as $R=-(\partial\rho/\partial P)M+R'$ and $S=M+S'$ [Eq. (C13)] where R, S, and M are the quantities of the first order in ρ_n/ρ, R', and S'—of the second order. Besides that we must use the simplest approximations for M, R', and S': M_0, R'_0, and S'_0.

The exact expression for γ_1 is the following:

$$\gamma_1=-\frac{R(u_1^0)^2+S}{2\sqrt{B^2-4AC}}=-\frac{M[1-(u_1^0)^2/\bar{u}_1^2]+R'(u^0)^2+S'}{2A(u_1^{0^2}-u^{0^2})}$$
$$=\frac{M(u_1^{(1)})^2/\bar{u}_1^2-R'(u_1^0)^2-S'}{2A_0(1+A'/A_0)\{\bar{u}_1^2(1-\bar{u}_2^2/\bar{u}_1^2)+[(u_1^{(1)})^2-(u_2^{(1)})^2]\}} , \tag{80a}$$

where

$$M=\frac{\partial\sigma}{\partial T}\left[3\rho_s-\rho+\frac{2\rho^2}{\rho_n}h\right]-2\frac{\rho^2}{\rho_n}\left[\sigma-X\frac{\partial\sigma}{\partial X}\right]g ,$$

$$R'=\alpha^2\left[3\rho_s-\rho+\frac{2\rho^2}{\rho_n}h\right]-2\frac{\rho^2}{\rho_n}\left[\sigma-X\frac{\partial\sigma}{\partial X}\right]\frac{\partial\rho}{\partial T}f+\frac{2\rho^2}{\rho_n}X\frac{\partial\rho}{\partial X}\left[\frac{\partial\sigma}{\partial P}g-\frac{\partial\sigma}{\partial T}f\right] , \tag{80b}$$

$$S'=4\frac{\rho_s}{\rho}\left[\left[\sigma-X\frac{\partial\sigma}{\partial X}\right]\frac{\partial\rho}{\partial T}+\frac{\partial\sigma}{\partial T}X\frac{\partial\rho}{\partial X}\right]$$

[see Eqs. (C13) and (C1); A_0, A', $\bar{u}_{1,2}$, and $u_{1,2}^{(1)}$ are given in Eqs. (C12), (C14), (C16), and (C17)]. Thus we get

$$\gamma_1=\frac{1}{[1-\alpha^2/(\partial\rho/\partial P)(\partial\sigma/\partial T)]\{(1-\bar{u}_2^2/\bar{u}_1^2)+[(u_1^{(1)})^2-(u_2^{(1)})^2]/\bar{u}_1^2\}}$$
$$\times\left\{\left[1-\frac{3\rho_n}{2\rho}+\frac{\rho}{\rho_n}h-\frac{\rho}{\rho_n}\frac{\sigma-X\partial\sigma/\partial X}{\partial\sigma/\partial T}g\right]\frac{(u_1^{(1)})^2}{\bar{u}_1^2}\right.$$
$$-\left[\frac{\alpha^2}{(\partial\rho/\partial P)(\partial\sigma/\partial T)}\left[1-\frac{3\rho_n}{2\rho}+\frac{\rho}{\rho_n}h\right]-\frac{\rho^2}{\rho_n}\frac{\alpha(\sigma-X\partial\sigma/\partial X)}{(\partial\rho/\partial P)(\partial\sigma/\partial T)}f\right.$$
$$\left.\left.+\frac{\rho}{\rho_n}\frac{X\partial\rho/\partial X}{(\partial\rho/\partial P)(\partial\sigma/\partial T)}\left[\frac{\partial\sigma}{\partial P}g-\frac{\partial\sigma}{\partial T}f\right]\right]\left[1+\frac{(u_1^{(1)})^2}{\bar{u}_1^2}\right]-2\frac{\rho_s}{\rho\partial\sigma/\partial T}\left[\alpha\left[\sigma-X\frac{\partial\sigma}{\partial X}\right]+\frac{\partial\sigma}{\partial T}\frac{X}{\rho}\frac{\partial\rho}{\partial X}\right]\right\}. \tag{80c}$$

Equations (80c) and (80b) give the exact expressions for γ_1 and γ_2.

In the simplest approximation we find

$$\gamma_1^{(1)}=\frac{M_0}{2A_0\bar{u}_1^2(1-\bar{u}_2^2/\bar{u}_1^2)}\frac{(u^{(1)})^2}{\bar{u}_1^2}-\frac{R_0'\bar{u}_1^2+S_0'}{2A_0\bar{u}_1^2(1-\bar{u}_2^2/\bar{u}_1^2)}, \tag{81a}$$

where

$$M_0=2\rho\frac{\partial\sigma}{\partial T}\left[1+\frac{\rho}{\rho_n}h-\frac{\rho}{\rho_n}\frac{(\sigma-X\partial\sigma/\partial X)}{\partial\sigma/\partial T}g\right]=2\rho\frac{\partial\sigma}{\partial T}\gamma_2^{(0)},$$
$$R_0'=2\alpha^2\rho\left[1+\frac{\rho}{\rho_n}h\right]-2\frac{\rho^2}{\rho_n}\left[\sigma-X\frac{\partial\sigma}{\partial X}\right]\frac{\partial\rho}{\partial T}f+\frac{2\rho^2}{\rho_n}X\frac{\partial\rho}{\partial X}\left[\frac{\partial\sigma}{\partial P}g-\frac{\partial\sigma}{\partial T}f\right], \tag{81b}$$
$$S_0'=4\left[\left[\sigma-X\frac{\partial\sigma}{\partial X}\right]\frac{\partial\rho}{\partial T}+\frac{\partial\sigma}{\partial T}X\frac{\partial\rho}{\partial X}\right].$$

Thus we obtain

$$\gamma_1^{(1)}=\gamma_2^{(0)}\frac{u_1^{(1)^2}}{\bar{u}_1^2(1-\bar{u}_2^2/\bar{u}_1^2)}+\frac{1}{(\partial\sigma/\partial T)(1-\bar{u}_2^2/\bar{u}_1^2)}$$
$$\times\left\{\alpha\left[\sigma-X\frac{\partial\sigma}{\partial X}\right]\left[\frac{\rho^2}{\rho_n\partial\rho/\partial P}f-2\right]-\alpha^2\bar{u}_1^2\left[1+\frac{\rho}{\rho_n}h\right]+\frac{\rho}{\rho_n}X\frac{\partial\rho}{\partial X}\bar{u}_1^2\left[\frac{\partial\sigma}{\partial T}f-\frac{\partial\sigma}{\partial P}g\right]-2\frac{X}{\rho}\frac{\partial\rho}{\partial X}\frac{\partial\sigma}{\partial T}\right\}. \tag{81c}$$

The representation

$$\gamma_1^{(1)}=\frac{\gamma_2^{(0)}}{1-\bar{u}_2^2/\bar{u}_1^2}\left[\frac{\alpha^2[\bar{u}_1^2-(\rho_s/\rho_n)X^2\partial(Z/\rho)/\partial\lambda]+2\alpha(X/\rho)(\partial\rho/\partial X)(\rho_s/\rho_n)(\sigma-X\partial\sigma/\partial X)}{(\partial\sigma/\partial T)(1-\bar{u}_2^2/\bar{u}_1^2)}\right.$$
$$\left.+\frac{(\rho_s/\rho_n)(X^2/\rho^2)(\partial\rho/\partial X)^2}{1-\bar{u}_2^2/\bar{u}_1^2}\right]$$
$$+\left\{\left[\alpha\left[\sigma-X\frac{\partial\sigma}{\partial X}\right]+\frac{\partial\sigma}{\partial T}\frac{X}{\rho}\frac{\partial\rho}{\partial X}\right]\left[\frac{\rho}{\rho_n}\left[-\frac{\rho}{\rho_n}+\frac{\partial\rho_n/\partial\rho}{\partial\rho/\partial P}\right]-2\right]\right.$$
$$\left.-\alpha^2\bar{u}_1^2\left[1+\frac{\rho_n}{\rho}\left[\frac{\rho_n}{\rho}-X\frac{\partial(\rho_n/\rho)}{\partial X}\right]\right]-\frac{\alpha\bar{u}_1^2}{\rho_n}\frac{\partial\rho_n}{\partial T}\frac{X}{\rho}\frac{\partial\rho}{\partial X}\right\}\Big/\frac{\partial\sigma}{\partial T}\left[1-\frac{\bar{u}_2^2}{\bar{u}_1^2}\right] \tag{82}$$

is more convenient by comparison with the case of pure ^{4}He [Eq. (33)].

D. Low-T expressions for Doppler parameters

Let us consider the behavior of γ_2 and γ_1 at $T\to 0$ (i.e., $f,g,h\to 0$). We get

$$\gamma_2^{(0)}=1, \tag{83}$$

$$\gamma_2=\frac{\rho_s}{\rho}-\frac{\rho_n}{2\rho}-\gamma_1, \tag{84}$$

$$\gamma_1 = \frac{(\rho_s/\rho - \rho_n/2\rho)\{(u_1^{(1)})^2/\bar{u}_1^2 - [\alpha^2/(\partial\rho/\partial P)(\partial\sigma/\partial T)][1+(u_1^{(1)})^2/\bar{u}_1^2]\} - 2(\rho_s/\rho)\{\alpha[\sigma - X(\partial\sigma/\partial X)]/(\partial\sigma/\partial T) + (X/\rho)(\partial\rho/\partial X)\}}{[1-\alpha^2/(\partial\rho/\partial P)(\partial\sigma/\partial T)]\{(1-\bar{u}_2^2/\bar{u}_1^2)+[(u_1^{(1)})^2-(u_2^{(1)})^2]/\bar{u}_1^2\}} \tag{85}$$

[see Eqs. (78b) and (80c)].

The result (83) shows the jump of $\gamma_2(T\to 0)$, $\frac{2}{3}\to 1$, already at an infinitesimal X. But it corresponds formally to the OEF: $\gamma_2 > \rho_s/\rho$. Actually the accuracy of the calculation (a zero-order approximation) is not sufficient for such a conclusion: $\gamma_2 - \rho_s/\rho = O(\rho_n/\rho)$ is a quantity of the first order in ρ_n/ρ. We need the first-order correction, $\Delta\gamma_2^{(1)}$. According to Eq. (84)

$$\gamma_2^{(0)} + \Delta\gamma_2^{(1)} = \frac{\rho_s}{\rho} - \frac{\rho_n}{2\rho} - \gamma_1^{(1)} . \tag{86}$$

Neglecting the terms with the factors $(\partial\sigma/\partial T)^{-1} \propto T$ we get at $T\to 0$ [see Eq. (85)]

$$\gamma_1^{(1)} = \left[\frac{(u_1^{(1)})^2}{\bar{u}_1^2} - 2\frac{X}{\rho}\frac{\partial\rho}{\partial X} \right] ,$$

where

$$\frac{(u_1^{(1)})^2}{\bar{u}_1^2} = \frac{\rho_s}{\rho_n}\frac{X^2}{\rho^2}\left[\frac{\partial\rho}{\partial X}\right]^2 \tag{87}$$

[see Eqs. (C18)], i.e.,

$$\gamma_1^{(1)} \approx 2\theta X + \theta^2 X \frac{m_3}{m_3^*} . \tag{88}$$

Thus we find

$$\gamma_1^{(1)}(T\to 0) \approx 2\theta X > 0 \quad \text{(BEF)} , \tag{89a}$$

$$\gamma_2^{(1)}(T\to 0) = \frac{\rho_s}{\rho} - \frac{\rho_n}{2\rho} - 2\theta X = \frac{\rho_s}{\rho} - \frac{\rho_n}{\rho}\left[\frac{1}{2} + 2\theta\frac{m_3}{m_3^*}\right]$$
$$< \frac{\rho_s}{\rho} \tag{89b}$$

(absence of OEF).

E. Comparison with the case of pure ^{4}He

Similarly to Eqs. (38)–(40), (60a) and (79b) we can represent the results (81c) and (80c), respectively,

$$\gamma_1^{(1)} = \frac{1}{1-\bar{u}_2^2/\bar{u}_1^2}\left\{ \left[2 + \frac{\rho}{\rho_n}(\gamma_4^{(Z/\rho)} + \gamma_4^{(\sigma)})\right]\frac{(u^{(1)})^2}{\bar{u}_1^2} + \frac{\alpha(\sigma - X\partial\sigma/\partial X)}{\partial\sigma/\partial T}\left[\frac{\rho}{\rho_n}\gamma_4^{(\rho)} - 2\right] \right.$$
$$\left. - \frac{\alpha^2\bar{u}_1^2}{\partial\sigma/\partial T}\left[2 + \frac{\rho}{\rho_n}\gamma_4^{(Z/\rho)}\right] + \frac{X}{\rho}\frac{\partial\rho}{\partial X}\left[\frac{\rho}{\rho_n}(\gamma_4^{(\rho)} + \gamma_4^{(\alpha)}) - 2\right]\right\} , \tag{90}$$

$$\gamma_1 = \left[1 - \frac{\alpha^2}{(\partial\rho/\partial P)(\partial\sigma/\partial T)}\right]^{-1}\left[\left[1 - \frac{\bar{u}_2^2}{\bar{u}_1^2}\right] + \frac{(u^{(1)})^2 - (u_2^{(1)})^2}{\bar{u}_1^2}\right]^{-1}$$
$$\times \left\{ \left[2 - \frac{3\rho_n}{2\rho} + \frac{\rho}{\rho_n}(\gamma_4^{(Z/\rho)} + \gamma_4^{(\sigma)})\right]\frac{(u^{(1)})^2}{\bar{u}_1^2} + \frac{\alpha(\sigma - X\partial\sigma/\partial X)}{\partial\sigma/\partial T}\left[\frac{\rho}{\rho_n}\gamma_4^{(\rho)}\left[1 + \frac{(u^{(1)})^2}{\bar{u}_1^2}\right] - 2\frac{\rho_s}{\rho}\right] \right.$$
$$\left. - \frac{\alpha^2\bar{u}_1^2}{\partial\sigma/\partial T}\left[2 - \frac{3\rho_n}{2\rho} + \frac{\rho}{\rho_n}\gamma_4^{(Z/\rho)}\right]\left[1 + \frac{(u^{(1)})^2}{\bar{u}_1^2}\right] + \frac{X}{\rho}\frac{\partial\rho}{\partial X}\left[\frac{\rho}{\rho_n}(\gamma_4^{(\rho)} + \gamma_4^{(\alpha)})\left[1 + \frac{(u^{(1)})^2}{\bar{u}_1^2}\right] - 2\frac{\rho_s}{\rho}\right]\right\} . \tag{91}$$

Comparing the expressions for γ_2 and γ_1 [Eqs. (80c), (81c), (90), and (91)] with the case $X=0$ [Eqs. (39) and (40)] we see some natural complications: the addition of terms with factors $\gamma_4^{(Z/\rho)}$ and $X\partial\rho/\partial X$, the replacement $\sigma \to \sigma - \partial\sigma/\partial X$, including the definition of $\gamma_2^{(\sigma)}$. However, the main changes are introduced by the terms of the former type: with the factors $(\rho/\rho_n)\gamma_4^{(\sigma)}$ and $(\rho/\rho_n)\gamma_4^{(\rho)}$. As we showed above just these terms caused the nonmonotonic character of γ_2 and $\gamma_1/(\rho_n/\rho)$ at $X=0$: the sharp negative peaks of γ_2 (BEF) and γ_1 (OEF) in the beginning of the roton region. However, at $X\neq 0$ there appears the small factor ρ_{n4}/ρ_n which suppresses these terms $[\gamma_4^{(\sigma)}, \gamma_4^{(\rho)} \propto O(\rho_{n4}/\rho)]$ unlike the terms $\propto O(\rho_n/\rho)$. This means that at $X \gg 10^{-5}$ (10^{-5} is the estimate of ρ_n/ρ in the beginning of the roton region) the nonmonotonicity is suppressed (Fig. 12). Let us note that the situation in the cases of γ_1 and γ_2 somewhat differs from the case of γ_4, where instead of the factor ρ_{n4}/ρ_n there happened a cancellation of some terms ($\gamma_4^{(\sigma)}$ and $\gamma_4^{(\alpha)}$), in addition to the transformation of the scale of the

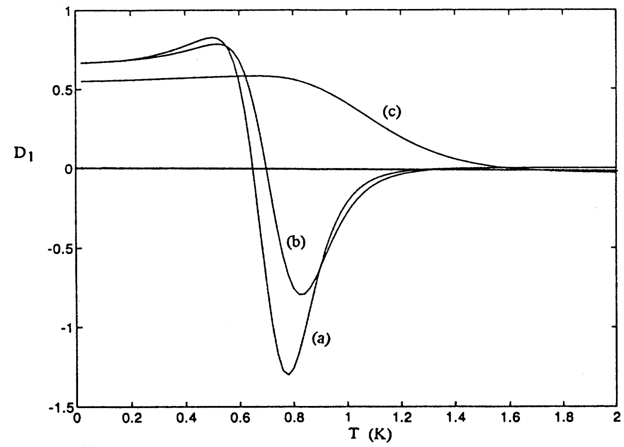

FIG. 12. Doppler parameter D_1 for ^{3}He-^{4}He mixture: (a) $X=0.05\%$, (b) $X=0.1\%$, and (c) $X=6\%$.

contributions of $\gamma_4^{(\rho)}$, $\gamma_4^{(\sigma)}$, and $\gamma_4^{(\alpha)}$ to $(\gamma_4+\rho_n/\rho)$, $\rho_n/\rho \to \rho_{n4}/\rho$.

It is important to note that, in spite of the suppression of the nonmonotonicity of γ_i in the mixture that is natural since the thermal excitations are here a small fraction of the normal component, the BEF parameter, $\gamma_1>0$, turns out to be much more than in the case of pure ^{4}He: up to three orders of magnitude. The point is that the scale of $\gamma_1>0$ is ρ_n/ρ but not ρ_{n4}/ρ, e.g., at $T\to 0$

$$\gamma_1 \approx \left[2\theta\frac{m_3}{m_3^*} + \left[\theta\frac{m_3}{m_3^*}\right]^2 \right] \frac{\rho_n}{\rho} \approx 0.66\frac{\rho_n}{\rho}$$

[see Eq. (88)].

At $T\to 0$ we obtain the jumps of the low-T limit of γ_2 and $\gamma_1/(\rho_n/\rho)$ at arbitrarily small X.

$$\gamma_2(T\to 0):$$

$$\tfrac{2}{3}\to 1 \left\{ \text{more exactly } \frac{2}{3} \to \left[\frac{\rho_s}{\rho} - \frac{\rho_n}{\rho}\left[\frac{1}{2}+2\theta\frac{m_3}{m_3^*}\right]\right]\right\}$$

$$\gamma_1(T\to 0)/\rho_n/\rho:$$

$$(3a+1)(a+1) \approx 35.5 \to \left[2\theta\frac{m_3}{m_3^*} + \left[\theta\frac{m_3}{m_3^*}\right]^2 \right] \approx 0.66 \qquad (92)$$

[see Eq. (36)]. At quite small $X \gg 10^{-5}$ these constants embrace the whole phonon region. At too small X the "jumps" appear in the curves $\gamma_2(T)$ and $\gamma_1(T)/(\rho_n/\rho)$ by the transition from the "high-T" part of the phonon region ($\rho_{n4} \gg \rho_{n3}$) to the "low-T" part ($\rho_{n4} \ll \rho_{n3}$); see Fig. 13. The result $\gamma_1(T\to 0)/(\rho_n/\rho) \approx 2\theta m_3/m_3^* > 0$ means (i) the BEF with large parameter $\propto \rho_n/\rho$ for first sound; (ii) the scale ρ_n/ρ for γ_1 and $\gamma_1+\rho_n/\rho$, unlike the case of fourth sound where $\gamma_4+\rho_n/\rho = O(\rho_{n4}/\rho)$; (iii) the absence of the OEF for second sound (in spite of the jump of γ_2^0, $\frac{2}{3}\to 1$):

$$\gamma_2(T\to 0) = \frac{\rho_s}{\rho} - \frac{\rho_n}{2\rho} - \gamma_1(T\to 0) < \frac{\rho_s}{\rho} .$$

F. What happens to Doppler parameters $D_{4,1,2}$ at $X\neq 0$

The results at $X\neq 0$ can be represented as follows [cf. Eqs. (12)–(14) and (45)–(49)]:

$$D_2 \approx \gamma_2 ,$$

$$\gamma_2 - \frac{\rho_s}{\rho} = \left[\frac{\rho_{n4}}{\rho_n} - \frac{\sigma - X\partial\sigma/\partial X}{T\partial\sigma/\partial T}\frac{\rho_{n4}}{\rho_n} G \right] + \Delta\gamma_2 ,$$

$$\Delta\gamma_2 = -\frac{\rho_n}{2\rho} - \gamma_1 + O\left[\frac{\rho_{n4}}{\rho}X\right] , \qquad (93)$$

$$D_4 = \left[\gamma_4 + \frac{\rho_n}{\rho} \right] \Big/ \frac{\rho_{n4}}{\rho}$$

$$\approx F + 1 - \frac{(\sigma - X\partial\sigma/\partial X) + \alpha\bar{u}_1^2}{T\partial\sigma/\partial T} G , \qquad (94)$$

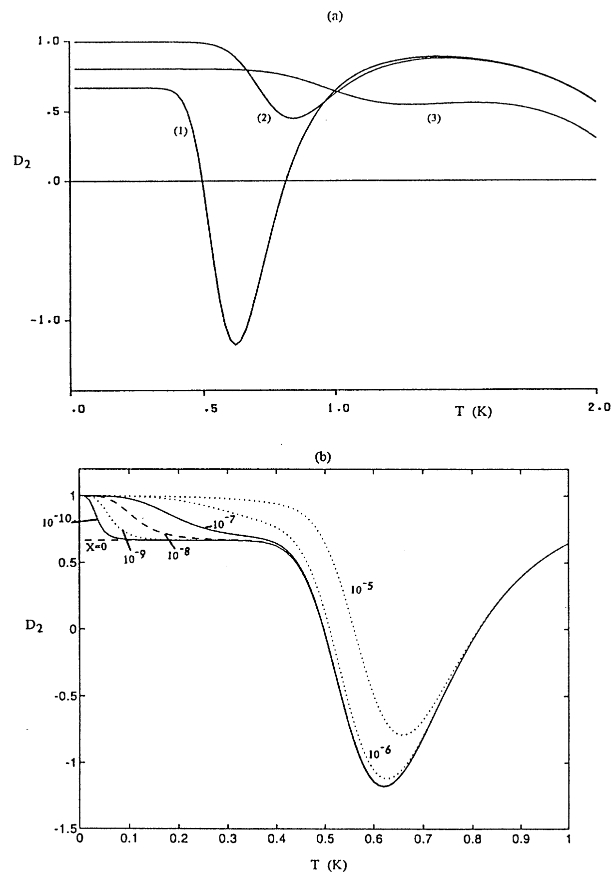

FIG. 13. (a) Doppler parameter D_2 for ^{3}He-^{4}He mixture: disappearance of BEF and of the minimum. (1) $X=0.0\%$, (2) $X=0.1\%$, and (3) $X=6\%$. (b) Two levels of the Doppler parameter D_2: $\frac{2}{3}$ [$T>T(X)$] and $1-O(X)$ [$T<T(X)$] [see Eqs. (7) and (13)]. Absence of OEF, $D_2<1$.

$$\frac{\sigma - X\partial\sigma/\partial X}{T\partial\sigma/\partial T} = \frac{\rho_{\rm ph}+\lambda(T/\Delta)\rho_r+(T/T_0)X\rho}{3\rho_{\rm ph}+\lambda\rho_r+\frac{3}{2}(T/T_0)X\rho} ,$$

$$T_0 \equiv \frac{m_3 c^2}{k_b} \approx 20.8 \text{ K} , \qquad (95)$$

$$\frac{\alpha \bar{u}_1^2}{T\partial\sigma/\partial T} = \frac{-f_2\rho_{\rm ph}+\lambda|r|\rho_r-(T/T_0)X\rho}{3\rho_{\rm ph}+\lambda\rho_r+\frac{3}{2}(T/T_0)X\rho} ,$$

$$D_1 = \gamma_1 \Big/ \frac{\rho_n}{\rho}$$

$$\approx \left[-2\frac{X}{\rho_n}\frac{\partial\rho}{\partial X} + \frac{X^2}{\rho_n^2}\left[\frac{\partial\rho}{\partial X}\right]^2 \right]$$

$$+ \left[\text{terms with factors } \frac{\rho_{n4}}{\rho_n}F, \frac{\rho_{n4}}{\rho_n}G \right]$$

$$\approx 0.66 + O\left[\frac{\rho_{n4}}{\rho_n}\right] . \qquad (96)$$

This allows us to explain briefly the universal changes of Doppler parameters D_i at $X \neq 0$ (Figs. 8, 12, and 13). In the expressions for $(\gamma_2^{(0)}-\rho_s/\rho)$ and $(\gamma_4^{(1)}+\rho_n/\rho)$, at $X\neq 0$ [Eqs. (93) and (94)] all the terms acquire the small factor ρ_{n4}/ρ whereas the term $(-\rho_n/2\rho)$ in $\Delta\gamma_2$ becomes large $(\sim X)$ and γ_1 acquires a term $\sim X$ [Eqs. (93) and (96)]. The other changes of D_i are the following: $\sigma \to \sigma - X\partial\sigma/\partial X$ and the quantities σ, $T\partial\sigma/\partial T$, and α depend substantially on X. This is important for D_4. A cancellation occurs of the coefficient of G [Eqs. (94) and (95)] so that this term leaves the competition of the case of pure ^{4}He where it diminishes the contribution of the source of the OEF: the term $(F+1)\propto \partial\rho_{n4}/\partial P<0$. So we get a jump of the plateau of D_4, $(-a)\to(-5a)$ (amplification of the OEF), together with a quick diminution by increasing X of the BEF peak at $T\sim\tilde{T}$ which is caused by G (up to its disappearance at $X \ll 0.06$, Fig. 8). The factor ρ_{n4}/ρ_n in $(\gamma_2-\rho_s/\rho)$ and D_1 means a suppression of the terms containing F and G in comparison with $\Delta\gamma_2$ [Eq. (93)] and the first term (≈ 0.66) of D_1 [Eq. (96)]. This implies a quick diminution by increasing X of the BEF peak of γ_2 and the OEF peak of D_1 at $T\sim\tilde{T}$ (up to their disappearance at $X \ll 0.06$). The term $0.66\rho_n/\rho$ in γ_1 (plateau) means a substantial amplification of the BEF (and the absence of the OEF for second sound: $\gamma_2-\rho_s/\rho\approx\Delta\gamma_2\approx-1.16\rho_n/\rho<0$), Figs. 14 and 15.

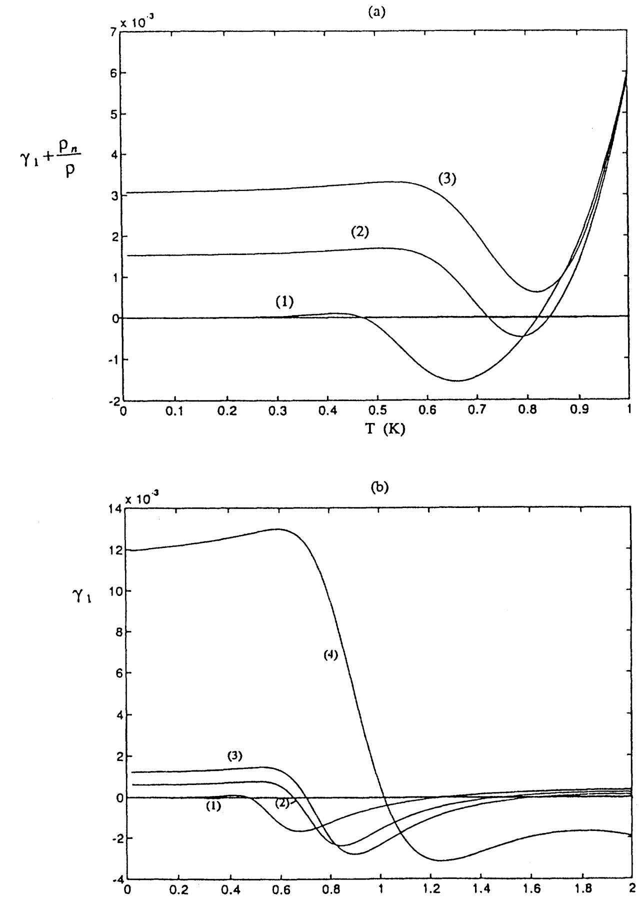

FIG. 14. (a) OEF parameter of first sound for ^{4}He-^{3}He mixtures, $\gamma_1+\rho_n/\rho<0$; disappearance of OEF with increasing ^{3}He concentration. (1) $X=0.0\%$, (2) $X=0.05\%$, and (3) $X=0.1\%$. (b) BEF parameter of first sound $\gamma_1>0$ for ^{4}He-^{3}He mixtures. (1) $X=0.0\%$, (2) $X=0.1\%$, (3) $X=1\%$, and (4) $X=6\%$.

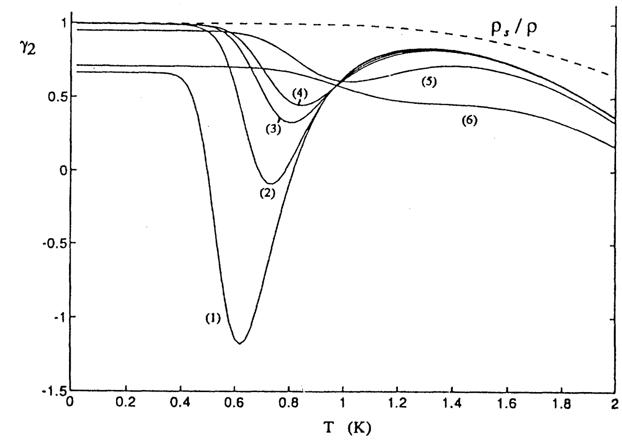

FIG. 15. Doppler parameter γ_2 for pure ^{4}He-^{3}He mixtures: absence of OEF, $\gamma_2\leq\rho_s/\rho$. (1) $X=0.0\%$, (2) $X=0.01\%$, (3) $X=0.05\%$, (4) $X=0.1\%$, (5) $X=1\%$, and (6) $X=6\%$.

Thus we see that the influence of the admixture of ^{3}He on the Doppler anomalies is strong and nontrivial: jumps of plateaus and decrease of peaks imply a suppression of one type of anomalies but an amplification of another type.

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APPENDIX A: PURE ^{4}He: DOPPLER COEFFICIENTS AND THERMAL EXCITATIONS

1. Phonon and roton peculiarities

It is interesting to note that the negative values of $\gamma_4^{(p)}/(\rho_n/\rho)=-(5a+1)<0$ in the phonon region and of

$$\gamma_4^{(\alpha)}\Big/\frac{\rho_n}{\rho}\approx-|r|\frac{\Delta}{T}\left[1-\frac{f_2}{\lambda|r|}\frac{\rho_{\rm ph}}{\rho_r}\right]$$

in the roton region correspond to some anomalies: negative "compressibility," $\partial\rho_n/\partial P<0$, and positive thermal expansion coefficient, $\alpha=1/\rho\partial\rho/\partial T>0$, respectively [see Eqs. (38), (41), (45), and (47)–(50)]. The anomalies originate from the special properties of the phonons and rotons, $\partial\rho_n/\partial P$ is negative since the increasing pressure P causes an increase of the sound velocity,

$$\frac{dc}{dP}\approx\frac{dc}{d\rho}\frac{\partial\rho}{\partial P}\approx\frac{a}{\rho c}>0\ ,$$

which implies a decrease of the phonon number: the latter is not conserved, it is determined by the temperature and the spectrum. The coefficient α becomes anomalous—positive in the roton region because of the decrease of the roton minimum by increasing density $[r=(\rho/\Delta)(d\Delta/d\rho)<0]$. The latter leads to the anomalous (negative) contribution of the rotons to the pressure, $P=P_0+P_{\rm ph}+P_r$, $P_{\rm ph}=\pi^2(kT)^4/90c^3\hbar^3$, $P_r\approx kTn_rs$, $s=1+(\Delta/T)r<0$, and this provides the positive term in the expression for α

$$\left[\frac{\partial\rho}{\partial T}\right]_P=-\frac{(\partial\phi/\partial T)_{\rho,P}}{(\partial\phi/\partial\rho)_{T,P}}\ ,\qquad \phi=P-f_1-f_2\frac{\pi^2(kT)^4}{90c^3\hbar^3}-kTn_rs \tag{A1}$$

[see Eqs. (22) and (24) in Ref. 3].

Let us comment on the negative "compressibility." This is not the true compressibility of the "normal subsystem," $(\partial\rho_n/\partial P)_T\neq(\partial\rho_n/\partial P_n)_T$. Negative compressibility implies the instability of the system because of an amplification of density perturbations. Here we get only the "acceleration" of the sound excitation in the direction of superfluid motion. The true compressibility of the phonons $(\partial\rho_{\rm ph}/\partial P_{\rm ph})_T$ is positive. In fact, $P_{\rm ph}=(c^2/4)f_2(\rho)\rho_{\rm ph}$, $f_2=1+3a$, $a=(\rho/c)(dc/d\rho)$; $\rho_{\rm ph}=\rho_{\rm ph}(\rho,T)$, $P_{\rm ph}=P_{\rm ph}(\rho,T)$ [see Eqs. (17) and (21) in Ref. 3], i.e.,

$$\left[\frac{\partial P_{\rm ph}}{\partial\rho_{\rm ph}}\right]_T=\frac{(\partial P_{\rm ph}/\partial\rho)_T}{(\partial\rho_{\rm ph}/\partial\rho)_T}=\frac{c^2}{4}\left[f_2+\frac{\rho}{3}\frac{df_2}{d\rho}\right]=\frac{c^2}{4}\left[1+3a+\frac{\rho}{a}\frac{da}{d\rho}\right]\ . \tag{A2}$$

It is interesting to compare this result with the interpretation of second sound. At low T we can regard second sound as an oscillation of T, or σ [see Eq. (42)], or $\rho_n\approx\rho_{\rm ph}$ $(\rho_{\rm ph}=(\rho T/c^2)\sigma_{\rm ph}$, ρ here being approximately constant). Thus it is a collisional sound in the gas of excitations (phonons). This interpretation is supported by the value of the second-sound velocity at low T:

$$u_2^2(T\to0)=\frac{\rho_s\sigma^2}{\rho_n\partial\sigma/\partial T}\approx\frac{c^2}{3}\ . \tag{A3}$$

The result $u_2=c/\sqrt{3}$ is exact for the collisional (i.e., local-equilibrium) sound in a gas of free particles with the dispersion law $\epsilon_p=cp$ (like photons c does not depend on ρ)—it corresponds to the elasticity of the gas. Using the formulas for the gas

$$F_g=VT\int\frac{d^3p}{(2\pi)^3}\ln(1-e^{-cp/T})=-V\frac{\pi^2}{90}\frac{(kT)^4}{(\hbar c)^3}\ ,$$

$$\rho_g=\int\frac{d^3p}{(2\pi)^3}m(p)(e^{cp/T}-1)=\frac{\pi^2}{30}\frac{(kT)^4}{(\hbar c)^3c^2}\ ,$$

$$m(p)=\frac{p}{d\epsilon_p/dp}=\frac{p}{c}\ ,$$

$$P_g=\frac{\pi^2}{90}\frac{(kT)^4}{(\hbar c)^3}=\tfrac{1}{3}\rho_gc^2\ ,$$

we obtain

$$u_g^2=\left[\frac{\partial P_g}{\partial\rho_g}\right]_\sigma=\frac{dP_g}{d\rho_g}=\frac{c^2}{3}\ . \tag{A4}$$

Returning to the phonon case we see that the simple connection between the sound velocity and elasticity is absent here: the relation between $P_{\rm ph}$ and $\rho_{\rm ph}$ differs from that in the photon case, cf. Eqs. (A2), (A3), and (A4). The difference is explained by (i) the special inertial properties of the phonons, $\rho_{\rm ph}=\frac{4}{3}\rho_g$, which are connected with the broken Galilean symmetry;[14] correspondingly, $c^2/3\to c^2/4$; (ii) the dependence of the spectrum (velocity c) on ρ, $f_2=1+3a$, $a=(\rho/c)(dc/d\rho)\neq0$. Both causes are connected with the fact that phonons are quasiparticles (in a medium) but not particles (in the vacuum).

Thus there is not a simple analogy between collisional sounds in the gases of phonons and photons.

2. Pressure dependence of D_i

In conclusion we discuss the pressure (P) dependence of the Doppler shift. By increasing P at fixed T the phonon density quickly diminishes:

$$\frac{\rho_{\rm ph}(P)}{\rho_{\rm ph}^{\rm SPV}}\approx\left[\frac{c(P)}{c}\right]^{-5},\quad\frac{dc}{dP}\approx\frac{a}{\rho c},\quad a=\frac{\rho}{c}\frac{dc}{d\rho}\approx2.79\ , \tag{A5}$$

whereas the roton density sharply increases:

$$\frac{\rho_r(P)}{\rho_r^{\rm SPV}}\approx e^{-[\Delta(P)-\Delta]/T},\quad\frac{d\Delta}{dP}\approx\frac{r\Delta}{\rho c^2},$$

$$r=\frac{\rho}{\Delta}\frac{d\Delta}{d\rho}\approx-0.59\ . \tag{A6}$$

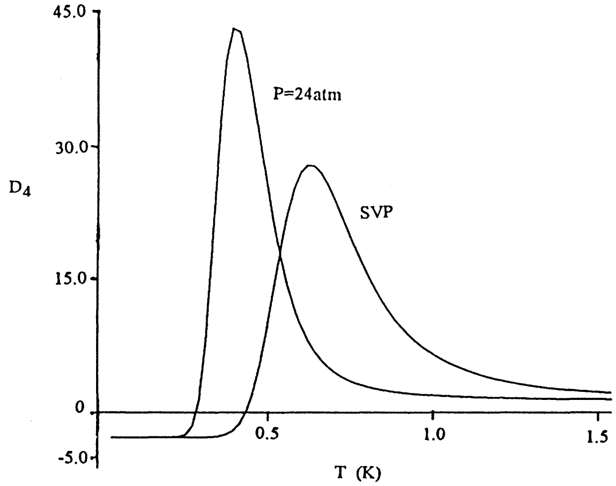

FIG. 16. Comparison of the Doppler parameter $D_4(T)$ at saturated vapor pressure (SVP) and at $P=24$ atm.

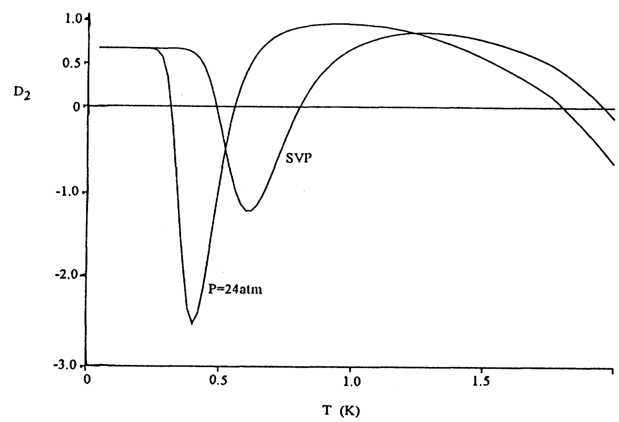

FIG. 18. Comparison of the Doppler parameter $D_2(T)$ at SVP and at $P=24$ atm.

This means that the temperature of the beginning of the roton region, $\tilde{T}(P)$ $[g(\tilde{T})=\rho_r/\rho_{\rm ph}=1]$, becomes lower, together with the position of the peaks of the functions $D_i(T)$ which become in addition sharper and higher: the "competition" between the phonon and roton contributions by increasing T happens earlier and is "more hard." The results of numerical calculations are plotted in Figs. 16–18. In particular, we get at $P=24$ atm

$$\begin{aligned} &\max D_4=43.3 \quad (T=0.39\ \mathrm{K}) , \\ &\min(D_2\approx\gamma_2)=-2.53 \quad (T=0.39\ \mathrm{K}) , \\ &\min D_1=-61 \quad (T=0.4\ \mathrm{K}) , \end{aligned} \tag{A7}$$

[cf. Eq. (37)].

However, $\max\gamma_4(T)$ and $|\min\gamma_1(T)|$ decrease at high P because of the displacement of the extrema to lower T: this leads to a decrease of the factor ρ_n/ρ. Note that the total normal density $\rho_n(T)$ decreases in the phonon region and increases in the roton one; the most substantial increase occurs not far from the beginning of the roton region:

$$\rho_n\approx\rho_r\approx\rho_r(P=0)e^{[\Delta-\Delta(P)]/T} . \tag{A8}$$

This is confirmed by the numerical calculations, e.g., for

$$\frac{\rho_n}{\rho}(P=24\ \mathrm{atm})/\frac{\rho_n}{\rho}(P=0)$$

we get

$$0.128 \ \ (T=0.3\ \mathrm{K}),\ \ 4.936 \ \ (T=0.5\ \mathrm{K}) ,$$

$$8.751 \ \ (T=0.6\ \mathrm{K}),\ \ 7.636 \ \ (T=0.7\ \mathrm{K}) .$$

For an estimate of $\gamma_i(T,P)$ we can use the linear approximation of the basic quantities in their expressions [see Eqs. (A5) and (A6)]

$$\frac{\delta c}{c}\approx a\frac{P}{\rho c^2},\quad \frac{\delta\Delta}{\Delta}\approx r\frac{P}{\rho c^2} , \tag{A9}$$

$$g(P)=\frac{\rho_r(P)}{\rho_{\rm ph}(P)}=\left[1+\frac{\delta c}{c}\right]^5 e^{\delta\Delta/T} , \tag{A10}$$

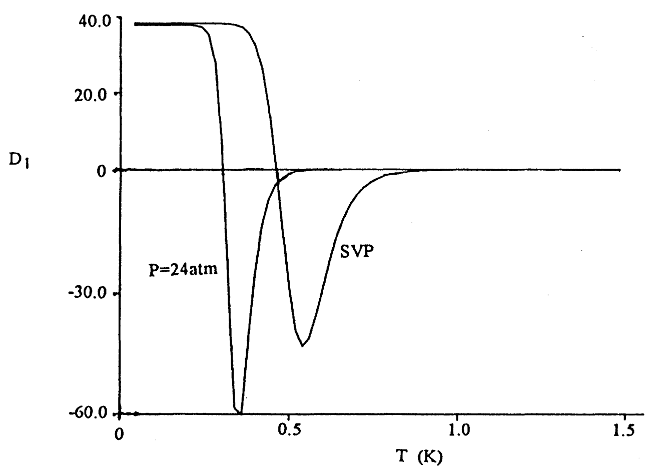

FIG. 17. Comparison of the Doppler parameter $D_1(T)$ at SVP and at $P=24$ atm.

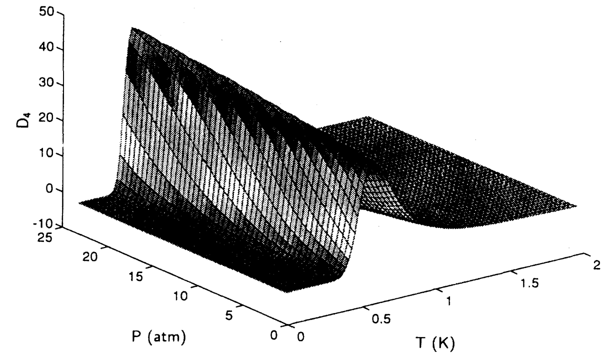

FIG. 19. Doppler parameter D_4 as a function of temperature (T) and pressure (P).

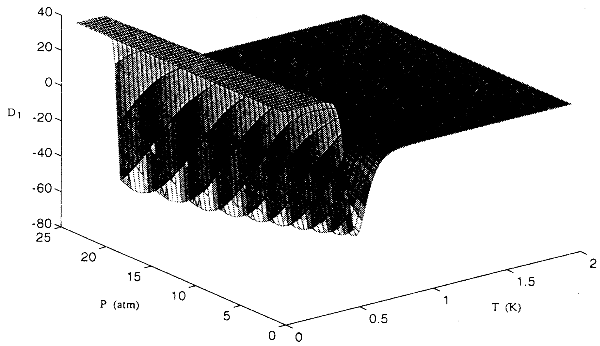

FIG. 20. Doppler parameter D_1 as a function of temperature (T) and pressure (P).

$$\lambda(P)=\frac{3(k_B\Delta)^2}{(c\rho_0)^2}\approx\frac{(1+\delta\Delta/\Delta)^2}{(1+\Delta c/c)^2}\lambda,\quad \lambda=0.18\ ,$$

e.g., for $P=24$ atm we get

$$\frac{P}{\rho c^2}\approx 0.28,\quad \frac{\delta c}{c}\approx 0.78,\quad \frac{\delta\Delta}{\Delta}\approx -0.165\ ,$$

$$\lambda(P)\approx 0.22\lambda\approx 0.04.$$

The dependence of the Doppler shift on P at fixed T is plotted in Figs. 19–21. Near the beginning of the roton region we get graphs of $D_i(P)$ which have some similarity to the temperature curves of these quantities: the graphs contain a sharp extremum (this happens in a very small temperature interval) or a sharp monotonic increase or decrease.

APPENDIX B: ^{4}He-^{3}He MIXTURE: EQUATION FOR FOURTH-SOUND VELOCITY AND ITS SOLUTION AT $T\rightarrow 0$

1. Coefficients in the exact equation

Let us explain the origin of the terms of ρE as well as of ρD and $-\rho\rho_s\tilde{B}_1$ [see Eqs. (55)–(58)]. All these quantities correspond to the determinants which are formed from det (53) by the division of its third column in four parts. The latter are proportional to u, 1, uv_s, and v_s, respectively,

$$(\text{i})=(0,0,-u,0),\quad (\text{ii})=(\rho_s,0,0,0)\ ,$$
$$(\text{iii})=(-uv_s\rho^2 f,0,0,0)\ , \qquad \text{(B1a)}$$
$$(\text{iv})=(0,[\rho g+\sigma\rho^2 f]v_s,(1-h)v_s,\rho^2 f v_s)\ ,$$

where

$$f=\frac{\partial(\rho_n/\rho)}{\partial P},\quad g=\frac{\partial(\rho_n/\rho)}{\partial T},\quad h=\frac{\rho_n}{\rho}-X\frac{\partial(\rho_n/\rho)}{\partial X}\ . \qquad \text{(B1b)}$$

The case (i) gives two contributions which correspond to the division of the first line of the determinant (i) into two parts, $\propto u$ and $\propto v_s$, respectively,

$$(\text{i1})\rightarrow\left[-u\frac{\partial\rho}{\partial P},-u\frac{\partial\rho}{\partial T},0,-u\frac{\partial\rho}{\partial X}\right]\ ,$$
$$(\text{i2})\rightarrow\left[v_s\frac{\partial\rho_s}{\partial P},v_s\frac{\partial\rho_s}{\partial T},0,v_s\frac{\partial\rho_s}{\partial X}\right]\ . \qquad \text{(B1c)}$$

The case (i1) corresponds to the term ρDu^2 and the version (ii) describes the term $-\rho\rho_s\tilde{B}_1$ [see Eqs. (56) and (57)]. The independence of D of X becomes obvious if one subtracts the first line of the determinant (i1) with the factor $X/(-u)$ from its fourth line.

The case

$$(\text{i2})\rightarrow\left[v_s\frac{\partial\rho}{\partial P}v_s\frac{\partial\rho}{\partial T},0,v_s\frac{\partial\rho}{\partial X}\right]$$
$$-\left[v_s\frac{\partial\rho_n}{\partial P},v_s\frac{\partial\rho_n}{\partial T},0,v_s\frac{\partial\rho_n}{\partial X}\right]$$

gives two terms: $uv_s(-\rho D)+uv_s\rho\tilde{A}$, respectively [cf. the case (i1)].

The case (iii) gives the term $uv_s(-\rho^2 f)(-\rho\tilde{B}_1)\equiv uv_s\rho\tilde{B}$ [cf. the case (ii)]. In the remaining version, (iv), we take into account only the first part of the first line [as in the case (i1)] since the terms $\propto v_s^2$ are neglected. Dividing the third column of (iv) into two parts,

$$(\text{iv})=\left[0,0,\frac{\rho_s}{\rho}v_s,0\right]$$
$$+\left[0,[\rho g+\sigma\rho^2 f]v_s,X\frac{\partial(\rho_n/\rho)}{\partial X}v_s,\rho^2 f v_s\right]\ ,$$

we get two terms: $uv_s(-\rho_s D)+uv_s\rho\tilde{C}$, respectively [cf. the case (i1)].

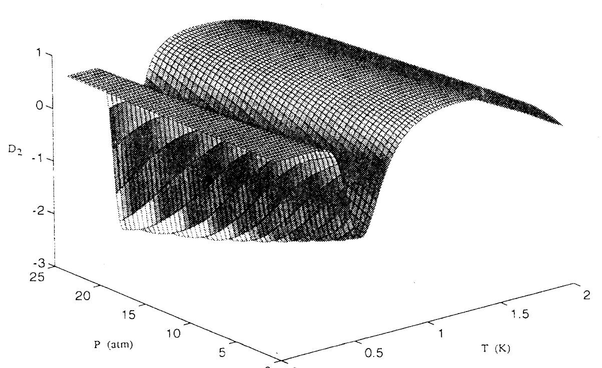

FIG. 21. Doppler parameter D_2 as a function of temperature (T) and pressure (P).

2. First-order approximation

Up to the first order in X we find

$$\tilde{B}_1\approx\left[1+\frac{2X}{\rho}\frac{\partial\rho}{\partial X}\right]\frac{\partial\sigma}{\partial T}+\frac{2}{\rho}\frac{\partial\rho}{\partial T}\left[\sigma-X\frac{\partial\sigma}{\partial X}\right]$$
$$+\frac{\partial\rho}{\partial P}\left[\sigma-X\frac{\partial\sigma}{\partial X}\right]^2 \qquad \text{(B2a)}$$

and

$$(u_4^0)^2 \approx \frac{\partial P}{\partial \rho}\left[1-\frac{\rho_n}{\rho}+\frac{2\alpha(\sigma - X\partial\sigma/\partial X)}{\partial\sigma/\partial T} + \frac{\alpha^2}{(\partial\rho/\partial P)(\partial\sigma/\partial T)} + \frac{(\sigma - X\partial\sigma/\partial X)^2\partial\rho/\partial P}{\partial\sigma/\partial T}+\frac{2X}{\rho}\frac{\partial\rho}{\partial X}\right]. \quad \text{(B2b)}$$

In comparison with the result (23) for pure ^{4}He we see the following changes:

$$\sigma \to \sigma - X\frac{\partial\sigma}{\partial X}, \quad \frac{\partial P}{\partial\rho} \to \frac{\partial P}{\partial \rho}\left[1+\frac{2X}{\rho}\frac{\partial\rho}{\partial X}\right]. \quad \text{(B3)}$$

Omitting the complicated exact expression for $\tilde{A}$ and $\tilde{C}$ let us write the coefficient of uv_s

$$\rho\left\{-\left[\frac{\rho_s}{\rho}+1\right]D+\tilde{A}+\tilde{B}+\tilde{C}\right\} \quad \text{(B4)}$$

in the first order (in X) approximation:

$$\tilde{A}=\rho\left[\frac{\partial\rho_n}{\partial P}\frac{\partial\sigma}{\partial T}-\frac{\partial\rho_n}{\partial T}\frac{\partial\sigma}{\partial P}\right] - \frac{\partial\rho}{\partial P}\left[\frac{\partial\rho_n}{\partial T}\left[\sigma - X\frac{\partial\sigma}{\partial X}\right]-X\frac{\partial\rho_s}{\partial X}\frac{\partial\sigma}{\partial T}\right],$$

$$\tilde{B}=\tilde{B}_1B_2 \approx B_1B_2 \approx \rho^2\frac{\partial(\rho_n/\rho)}{\partial P}\frac{\partial\sigma}{\partial T},$$

$$\tilde{C}=-\rho\frac{\partial(\rho_n/\rho)}{\partial T}\left[\frac{1}{\rho}\frac{\partial\rho}{\partial T}+\left[\sigma - X\frac{\partial\sigma}{\partial X}\right]\frac{\partial\rho}{\partial P}\right] - \frac{\partial\rho}{\partial P}\frac{\partial\sigma}{\partial T}X\frac{\partial\rho_n}{\partial X}+X\frac{\partial\sigma}{\partial X}\frac{\partial\rho}{\partial P}\frac{\partial\rho}{\partial T}+\frac{\partial\rho}{\partial T}\frac{\partial\rho}{\partial P}X\frac{\partial\sigma}{\partial X} + X\frac{\partial\rho}{\partial X}\frac{\partial\rho}{\partial P}\frac{\partial\sigma}{\partial T} \quad \text{(B5)}$$

[cf. Eqs. (12) in Ref. 3].

Substituting Eqs. (B5) into Eq. (55) we obtain

$$\Delta u_4 \approx \Delta u_4^{(1)} = \left\{1-\left[\frac{\partial P}{\partial\rho}\frac{\partial\rho_n}{\partial P}-\frac{\sigma - X\partial\sigma/\partial X}{2\rho}\left[\frac{\partial\rho_n}{\partial T}+\rho\frac{\partial(\rho_n/\rho)}{\partial T}\right]\left[\frac{\partial\sigma}{\partial T}\right]^{-1} - \frac{\alpha}{2\rho}\left[\frac{\partial\rho_n}{\partial T}+\rho\frac{\partial(\rho_n/\rho)}{\partial T}\right]\left[\frac{\partial\rho}{\partial P}\right]^{-1}\left[\frac{\partial\sigma}{\partial T}\right]^{-1}-\frac{X}{\rho}\frac{\partial\rho_n}{\partial X}\right]\right\}v_s$$

$$= \left[\frac{\rho_s}{\rho}-\gamma_4^{(1)}\right]v_s = \left[1-D_4^{(1)}\frac{\rho_{n4}}{\rho}\right]v_s \quad \text{(B6)}$$

[cf. Eq. (1) for $v_n=0$ and Eq. (22)].

3. Thermodynamic and hydrodynamic quantities of the mixture

Comparing $\gamma_4^{(1)}(T)$ in pure ^{4}He and in the ^{4}He-^{3}He mixture one must take into account not only the difference between the expressions for them [Eqs. (22) and (74)] but also the changes of the thermodynamics and hydrodynamics at $X \neq 0$. On the basis of the relation

$$\left[\frac{\partial\rho}{\partial T}\right]_{P,X} = -\frac{(\partial\phi/\partial T)_{\rho,P,X}}{(\partial\phi/\partial\rho)_{T,P,X}},$$

$$\phi \equiv P - f_1 - f_2\frac{\pi^2(kT)^4}{90c^3\hbar^3}-kTn_r s - \frac{kT\rho X}{m_3}, \quad \text{(B7)}$$

$$s = 1+\frac{\Delta}{T}r,$$

we find

$$\alpha = \frac{1}{\rho}\left[\frac{\partial\rho}{\partial T}\right]_{P,X} = \alpha_4+\alpha_3,$$

$$\alpha_4 = -\frac{f_2}{T}\frac{\rho_{\text{ph}}}{\rho}+\frac{\lambda}{T}\left[|r|-\frac{T}{\Delta}\right]\frac{\rho_r}{\rho}, \quad \alpha_3 = -\frac{X}{T_0}, \quad \text{(B8)}$$

$$\lambda = \frac{3(k\Delta)^2}{(cp_0)^2} \approx 0.18, \quad T_0 = \frac{m_3c^2}{k} \approx 20.8 \text{ K}$$

[cf. Eqs. (22)–(25) in Ref. 3]. We took into account that $(\partial P/\partial\rho)_{T,X}=c^2$. The accuracy of this ideal-gas description of the ^{3}He admixture is shown in Fig. 22. Further, we obtain

$$\sigma = \sigma_4+\sigma_3, \quad \sigma_4 = \frac{c^2}{T}\frac{\rho_{\text{ph}}}{\rho}+\lambda\frac{c^2}{\Delta}\frac{\rho_r}{\rho},$$

$$\sigma_3 = \frac{kX}{m_3}\left[\ln\frac{2m_3}{\rho X}\left[\frac{m_3^* kT}{2\pi\hbar^2}\right]^{3/2}+\frac{5}{2}\right], \quad \text{(B9)}$$

$$m_3^*/m_3 = 2.46$$

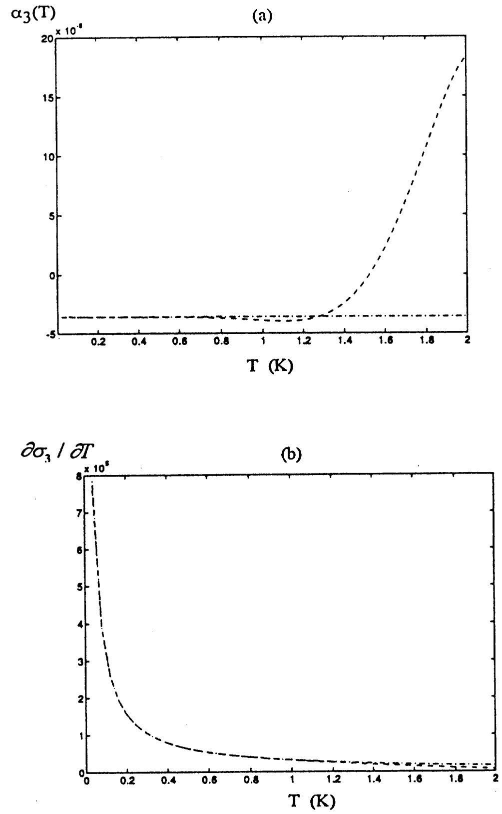

FIG. 22. Accuracy of the ideal-gas description (IGD) of the ^{3}He admixture ($x=0.1\%$). Dash-dot line, IGD; dashed line, numerical calculations; see Ref. 9. (a) $\alpha_3(T)$ and (b) $\partial\sigma_3/\partial T$.

[see Eq. (16) in Refs. 3 and 6],

$$\sigma - X\frac{\partial\sigma}{\partial X} \approx \sigma_4 + \frac{kX}{m_3} . \tag{B10}$$

In the phonon region and in the beginning of the roton one we get

$$\sigma - X\frac{\partial\sigma}{\partial X} = \frac{c^2}{T}\left[\frac{\rho_{\rm ph}}{\rho} + \frac{T}{T_0}X\right] , \tag{B11}$$

$$\frac{\partial\sigma}{\partial T} = \frac{\partial\sigma_4}{\partial T} + \frac{\partial\sigma_3}{\partial T} \approx \frac{3c^2}{T^2}\left[\frac{\rho_{\rm ph}}{\rho} + \frac{T}{2T_0}X\right] . \tag{B12}$$

Finally, we obtain

$$\rho_n = \rho_{n4} + \rho_{n3}, \quad \rho_{n4} = \rho_{\rm ph} + \rho_r, \quad \rho_{n3} = X\rho\frac{m_3^*}{m_3} , \tag{B13}$$

$$\frac{\partial\rho_n}{\partial P} = \frac{\partial\rho_{n4}}{\partial P} + \frac{\partial\rho_{n3}}{\partial P} ,$$

$$\frac{\partial\rho_{n4}}{\partial P} \approx \frac{5a}{c^2}\frac{\rho_{\rm ph}}{\rho} - \frac{1}{c^2}\frac{\rho_r}{\rho}\left[\frac{\Delta}{T}r - \frac{2}{3}\right] ,$$

$$\frac{\partial\rho_{n3}}{\partial P} \approx \frac{1}{c^2}X\frac{m_3^*}{m_3} = \frac{1}{c^2}\frac{\rho_{n3}}{\rho} , \tag{B14}$$

$$\frac{\partial\rho_n}{\partial T} = \frac{\partial\rho_{n4}}{\partial T} + \frac{\partial\rho_{n3}}{\partial T}, \quad \frac{\partial\rho_{n4}}{\partial T} \approx \frac{4\rho_{\rm ph}}{T} + \frac{\rho_r}{T}\left[\frac{\Delta}{T} - \frac{1}{2}\right] ,$$

$$\frac{\partial\rho_{n3}}{\partial T} = \frac{m_3^*}{m_3}X\frac{\partial\rho}{\partial T} = \frac{m_3^*}{m_3}X\rho\sigma_3 = -\rho_{n3}\frac{X}{T_0} = O\left[\frac{X^2}{T_0}\right] ,$$

$$\frac{\partial\rho_n}{\partial T} = \frac{1}{T}\left\{4\rho_{\rm ph} + \rho_r\left[\frac{\Delta}{T} - \frac{1}{2}\right] - \rho_{n3}\frac{XT}{T_0}\right\}$$

[see Eqs. (26) and (27) in Ref. 3 and Eq. (19)].

$$\frac{\partial\rho_n}{\partial X} = \rho\frac{m_3^*}{m_3} + X\frac{m_3^*}{m_3}\frac{\partial\rho}{\partial X} . \tag{B15}$$

$\partial\rho/\partial X$ can be obtained from the definitions

$$\rho = \frac{\tilde{X}M_3 + (1-\tilde{X})M_4}{V_4(1+\tilde{\alpha}\tilde{X})} \equiv \frac{M}{V_M}$$

$$[\tilde{\alpha} = 0.286, \quad \text{see Eq. (20)}] ,$$

$$X = \frac{M_3}{M}\tilde{X} = \frac{m_3}{m}\tilde{X} .$$

In fact, we get

$$\frac{\partial\rho}{\partial X} = \frac{\partial\rho}{\partial\tilde{X}}\frac{d\tilde{X}}{dX} = \left[\frac{M_3 - M_4}{V_m} - \frac{M\tilde{\alpha}V_4}{V_m^2}\right]\frac{d\tilde{X}}{dX}, \quad \frac{dX}{d\tilde{X}} = \frac{M_3M_4}{M_2^2} ,$$

so that [see Eq. (B15)]

$$\frac{\partial\rho}{\partial X} = \frac{M\rho}{M_3M_4}\left[M_3 - M_4 - \frac{\tilde{\alpha}M}{1+\tilde{\alpha}\tilde{X}}\right]$$

$$\approx -\left[(1+\tilde{\alpha})\frac{m_4}{m_3} - 1\right]\rho \equiv -\theta\rho , \tag{B16}$$

$$\theta \approx \left[\frac{m_4}{m_3} + \frac{4}{3}\tilde{\alpha} - 1\right] \approx 0.71 ,$$

$$\frac{\partial\rho_n}{\partial X} = \rho\frac{m_3^*}{m_3}\left\{1 - X\left[(1+\tilde{\alpha})\frac{m_4}{m_3} - 1\right]\right\} \equiv \rho\frac{m_3^*}{m_3}(1-\theta X) . \tag{B17}$$

4. Exact cancellation of the corrections $O(X^k)$ at $T\to 0$

Using the results of calculations in Eqs. (B8)–(B17) we find that for $(\gamma_4 + \rho_n/\rho)$ all the terms of first order in X cancel so that there remain only the terms of first order in ρ_{n4}/ρ. However, this result, $\gamma_4 + \rho_n/\rho = O(\rho_{n4}/\rho)$, must be proved exactly since at $T\to 0$ the quantity ρ_{n4}/ρ becomes smaller than a finite quantity of any order in ρ_n/ρ, $O([\rho_n/\rho]^k)$, $k=1,2,\ldots$.

Let us prove the exact cancellations in $\gamma_4(T=0)$ of the terms of all orders in ρ_n/ρ. The result $\gamma_4 + \rho_n/\rho = O(\rho_{n4}/\rho)$, or $\Delta u_4 = [1 + O(\rho_{n4}/\rho)]v_s$, implies that the coefficient of uv_s in Eq. (55a) is

$$\left\{-2D+O\left[\frac{\rho_{n4}}{\rho}\right]\right\} . \quad \text{(B18a)}$$

In order to prove Eq. (B18a) let us, first of all, take into account that according to the definition

$$\rho_n=\rho_{n3}+\rho_{n4}=\rho\left[\frac{m_3^*}{m_3}X+O\left[\frac{\rho_{n4}}{\rho}\right]\right] \quad \text{(B18b)}$$

we get $f=\partial(\rho_{n4}/\rho)/\partial P$, $g=\partial(\rho_{n4}/\rho)/\partial T$,

$$h=\frac{\rho_{n4}}{\rho}-X\frac{\partial(\rho_{n4}/\rho)}{\partial X}=\frac{\rho_{n4}}{\rho}[1+O(X)] ,$$

i.e.,

$$(f,g,h)=O\left[\frac{\rho_{n4}}{\rho}\right] \quad \text{(B18c)}$$

[see Eq. (B1b)]. Thus we get

$$\tilde{B}\equiv\rho^2 f\tilde{B}_1=O\left[\frac{\rho_{n4}}{\rho}\right] \quad \text{(B18d)}$$

[see Eqs. (58b) and (57)].

Estimates of $\tilde{A}$ and $\tilde{C}$ can be found if we somewhat change the divisions which were considered above in the cases (i2) and (iv) [Eqs. (B1a) and (B1c)]. Dividing the third column of det(iv) into two parts in the following way:

$$(\text{iv})=(0,0,v_s,0)+(0,[\rho g+\sigma\rho^2 f]v_s,-hv_s,\rho^2 fv_s) ,$$

we get two terms: $\rho uv_s(-D)+uv_s O(\rho_{n4}/\rho)$ [cf. the case (i1) which corresponds to ρDu^2 and Eq. (B18c)], i.e.,

$$\tilde{C}=-\frac{\rho_n}{\rho}D+O\left[\frac{\rho_{n4}}{\rho}\right] . \quad \text{(B18e)}$$

Further, we use the equalities

$$\frac{\partial\rho_n}{\partial P}=\rho\frac{\partial(\rho_n/\rho)}{\partial P}+\frac{\rho_n}{\rho}\frac{\partial\rho}{\partial P}=\frac{\rho_n}{\rho}\frac{\partial\rho}{\partial P}+O\left[\frac{\rho_{n4}}{\rho}\right] ,$$

$$\frac{\partial\rho_n}{\partial T}=\frac{\rho_n}{\rho}\frac{\partial\rho}{\partial T}+O\left[\frac{\rho_{n4}}{\rho}\right] ,$$

$$\frac{\partial\rho_n}{\partial X}=\frac{\rho_n}{\rho}\frac{\partial\rho}{\partial X}+\rho\frac{\partial(\rho_n/\rho)}{\partial X}=\frac{\rho_n}{\rho}\frac{\partial\rho}{\partial X}+\frac{\rho_n}{X}+O\left[\frac{\rho_{n4}}{\rho}\right] ,$$

and according to them divide the first line of det(i2) in the following form:

$$(\text{i2})\rightarrow\left[v_s\frac{\rho-\rho_n}{\rho}\frac{\partial\rho}{\partial P},v_s\frac{\rho-\rho_n}{\rho}\frac{\partial\rho}{\partial T},0,v_s\frac{\rho-\rho_n}{\rho}\frac{\partial\rho}{\partial X}\right] + \left[0,0,0,-v_s\frac{\rho_n}{X}\right] .$$

This gives two terms: $-uv_s(\rho-\rho_n)D-uv_s\rho_n D$. The result for the first term follows from the analogy with the case (i1). The second determinant is easily calculated. Thus we get in addition to Eqs. (B18d) and (B18e)

$$\tilde{A}=O\left[\frac{\rho_{n4}}{\rho}\right] , \quad \text{(B18f)}$$

and Eq. (B18a) is proved:

$$\left\{-\left[\frac{\rho_s}{\rho}+1\right]D+\tilde{A}+\tilde{B}+\tilde{C}\right\}=\left\{-2D+O\left[\frac{\rho_{n4}}{\rho}\right]\right\}$$

or [see Eqs. (55c), (B18d), (B18e), and (B18f)]

$$\gamma_4=-\frac{\rho_n}{2\rho}+\frac{\tilde{A}+\tilde{B}+\tilde{C}}{2D}=-\frac{\rho_n}{\rho}+O\left[\frac{\rho_{n4}}{\rho}\right] .$$

APPENDIX C: ⁴He-³He MIXTURE: EQUATION FOR FIRST- AND SECOND-SOUND VELOCITIES AND ITS EXACT SOLUTION

1. Peculiarities of the basic determinant

Before the calculation of the coefficients in the equation for U (66) it is necessary to take into account some general considerations.

The matrix elements of the determinant (66) contain quantities of different orders of magnitude. In the dimensionless form some of them are of the order of unity:

$$\frac{\partial\rho}{\partial P}\Big/\frac{1}{c^2},\quad \frac{\partial\rho}{\partial X}\Big/\rho,\quad \frac{\partial\sigma}{\partial X}\Big/\frac{k}{m_3},\quad \frac{\partial\rho_n}{\partial X}\Big/\rho,\quad \frac{\partial\rho_s}{\partial X}\Big/\rho ,$$

and others are proportional to the small parameter $\rho_n/\rho\sim X$:

$$\frac{\partial\rho}{\partial T}\Big/\frac{\rho}{T},\quad \frac{\partial\rho_s}{\partial T}\Big/\frac{\rho}{T},\quad \frac{\partial\rho_n}{\partial T}\Big/\frac{\rho}{T},\quad \sigma\Big/\frac{k}{m_3} ,$$

$$\frac{\partial\sigma}{\partial T}\Big/\frac{k}{m_3 T},\quad \frac{\partial\sigma}{\partial P}\Big/\frac{k}{m_3\rho c^2} .$$

The derivative $\partial(Z/\rho)/\partial X$ corresponds to a special estimate: it varies inversely as X but is $\propto T$ [see Eq. (71)]. However, it is especially important to pick out the factors which are proportional to $\rho_{n4}/\rho\sim T^4$ [they are underlined in Eq. (66)]:

$$f/(\rho c^2)^{-1},\quad g/T^{-1},\quad h\propto\frac{\rho_{n4}}{\rho} ,$$

$$f\equiv\frac{\partial(\rho_n/\rho)}{\partial P}=\frac{\partial(\rho_{n4}/\rho)}{\partial P},\quad g\equiv\frac{\partial(\rho_n/\rho)}{\partial T}=\frac{\partial(\rho_{n4}/\rho)}{\partial T} ,$$

$$h=\frac{\rho_n}{\rho}-X\frac{\partial(\rho_n/\rho)}{\partial X}=\frac{\rho_{n4}}{\rho}[1+O(X)] ,$$

$$\rho_n=\rho_{n4}+\rho_{n3}=\rho_{n4}+\frac{m_3^*}{m_3}\rho X \quad \text{(C1)}$$

[cf. Eqs. (B1b), (B18c), and (B18b)].

Further, the quintic equation for U (66) can be reduced to a quartic one.

The point is that the set of five solutions of Eq. (66) contains besides the four values of the sound velocities for the first and second sounds in the directions along Ox

$$U_{1,2}=\pm u_{1,2}^0+\gamma_{1,2}w,\quad U_{1,2}\gg w , \quad \text{(C2)}$$

the special fifth solution, $U_5 \sim w$, which describes the velocity of the "concentration wave" that can be set up by injecting a pulse of ^{3}He into the flow. The distortion of the concentration only, X', must move together with the normal component, so that we get

$$U_5 = v_n - v = \frac{\rho_s}{\rho} w \ . \tag{C3}$$

The existence of the exact solution $U_5 = (\rho_s/\rho)w + O(w^2)$ and its separation from the other solutions, $U_1, \ldots, U_4$, can be proved by the following simple consideration. Let us multiply the fifth line of the determinant (66) by $(-\sigma/X)$ and add it to the third line. The result will be proportional to $[U-(\rho_s/\rho)W]$ with the exception of the third term:

$$-\rho\frac{\partial\sigma}{\partial P}\left[U-\frac{\rho_s}{\rho}w\right], \quad -\rho\frac{\partial\sigma}{\partial T}\left[U-\frac{\rho_s}{\rho}w\right],$$
$$-U\rho\frac{\partial(\rho_n/\rho)}{\partial T}w, \ 0, \ \frac{\rho}{X}\left[\sigma - X\frac{\partial\sigma}{\partial X}\right]\left[U-\frac{\rho_s}{\rho}w\right].$$

However, with an accuracy to w^2 we can represent the third term in the form $-[U-(\rho_s/\rho)w]\rho\partial(\rho_n/\rho)/\partial T w$. Thus we get $U_5 = (\rho_s/\rho)w + O(w^2)$. Further, we have now obtained the equality

$$\det(66) = \left[U - \frac{\rho_s}{\rho}w\right]\det_1 + O(w^2) \ , \tag{C4}$$

where $\det_1$ differs from det(66) in the third line. The latter is

$$\left[-\rho\frac{\partial\sigma}{\partial P}, -\rho\frac{\partial\sigma}{\partial T}, -\rho\frac{\partial(\rho_n/\rho)}{\partial T}w, 0, \frac{\rho}{X}\left[\sigma - X\frac{\partial\sigma}{\partial X}\right]\right] . \tag{C5}$$

Equation (C4) can be used for the calculation of $u_1, \ldots, u_4$.

2. Exact equation for sound velocities

In order to find the solutions $U_{i<5} \gg w$, we calculate the coefficients, A, B, C, R, and S in the expression for $\det_1$,

$$\det_1 = AU^4 + BU^2 + C + RU^3 w + SUw \ . \tag{C6}$$

Substituting $U_i = u_i^0 + \gamma_i w$, $u_i^0 \gg w$, in $\det_1 = 0$ we find the equations for $(u_{1,2}^0)^2$,

$$A(u_i^0)^4 + B(u_i^0)^2 + C = 0 \ , \tag{C7}$$

and for γ_i,

$$2\gamma_i[2A(u_i^0)^2 + B] + R(u_i^0)^2 + S = 0 \ , \tag{C8}$$

which give

$$(u_{1,2}^0)^2 = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A} \ , \tag{C9}$$

$$\gamma_{1,2} = \mp \frac{R(u_{1,2}^0)^2 + S}{2\sqrt{B^2 - 4AC}} \tag{C10}$$

[cf. (24) and (25)].

Calculating the determinant

$$\det_1 = \begin{vmatrix} -U\frac{\partial\rho}{\partial P} & -U\frac{\partial\rho}{\partial T} & -Uw\rho^2 f & \rho & -U\frac{\partial\rho}{\partial X} \\ 1 & 0 & \frac{2\rho_n\rho_s}{\rho}w & -U\rho & 0 \\ -\rho\frac{\partial\sigma}{\partial P} & -\rho\frac{\partial\sigma}{\partial T} & -\rho g w & 0 & \frac{\rho}{X}\left[\sigma - X\frac{\partial\sigma}{\partial X}\right] \\ \frac{1}{\rho}\left[1+\frac{X}{\rho}\frac{\partial\rho}{\partial X}\right]+Uwf & -\left[\sigma - X\frac{\partial\sigma}{\partial X}\right]+Uwg & U\frac{\rho_n}{\rho} - w\left[\frac{\rho_{n4}}{\rho} - X\delta\right] & -U - \frac{\rho_n}{\rho}w & -X\frac{\partial(Z/\rho)}{\partial X} + Uw\left[\frac{m_3^*}{m_3}+\delta\right] \\ wX\frac{\partial\rho_s}{\partial P} & wX\frac{\partial\rho_s}{\partial T} & \rho_s X & 0 & -\rho U + w\frac{\partial(\rho_s X)}{\partial X} \end{vmatrix} , \tag{C11}$$

$$\delta = \frac{\partial(\rho_{n4}/\rho)}{\partial X} = (5a+1)\theta\frac{\rho_{n4}}{\rho}$$

[see Eqs. (18) and (B16)], we find the exact expressions for the coefficients A, B, C, R, and S:

$$A = \rho\left[\frac{\partial\rho}{\partial P}\frac{\partial\sigma}{\partial T} - \alpha^2\right],$$

$$B = -\rho\left[\frac{\partial\sigma}{\partial T} + \frac{\rho_s}{\rho_n}\left[\sigma - X\frac{\partial\sigma}{\partial X}\right]^2\frac{\partial\rho}{\partial P} + \frac{\rho_s}{\rho_n}\frac{X^2}{\rho^2}\left[\frac{\partial\rho}{\partial X}\right]^2\frac{\partial\sigma}{\partial T} + \frac{\rho_s}{\rho_n}2\alpha\frac{X}{\rho}\frac{\partial\rho}{\partial X}\left[\sigma - X\frac{\partial\sigma}{\partial X}\right] + \frac{\rho_s}{\rho_n}X^2\frac{\partial(Z/\rho)}{\partial X}\left[\frac{\partial\rho}{\partial P}\frac{\partial\sigma}{\partial T} - \alpha^2\right]\right],$$

$$C = \rho\frac{\rho_s}{\rho_n}\left[\left[\sigma - X\frac{\partial\sigma}{\partial X}\right]^2 + X^2\frac{\partial(Z/\rho)}{\partial X}\frac{\partial\sigma}{\partial T}\right], \tag{C12}$$

$$R = 2\underline{\frac{\rho^2}{\rho_n}\left[\sigma - X\frac{\partial\sigma}{\partial X}\right]\left[\frac{\partial\rho}{\partial P}g}\, - \frac{\partial\rho}{\partial T}f\right] - \underline{\left[\frac{\partial\rho}{\partial P}\frac{\partial\sigma}{\partial T} - \alpha^2\right]}\,\underline{\left[\rho + 3\rho_s - \frac{2\rho^2}{\rho_n}X\frac{\partial(\rho_n/\rho)}{\partial X}\right]} + \frac{2\rho^2}{\rho_n}X\frac{\partial\rho}{\partial X}\left[\frac{\partial\sigma}{\partial P}g - \frac{\partial\sigma}{\partial T}f\right] \equiv -\frac{\partial\rho}{\partial P}M + R', \tag{C13a}$$

$$S = -2\underline{\frac{\rho^2}{\rho_n}\left[\sigma - X\frac{\partial\sigma}{\partial X}\right]}\left[\underline{g} - 2\frac{\rho_n\rho_s}{\rho^3}\frac{\partial\rho}{\partial T}\right] + \underline{\frac{\partial\sigma}{\partial T}}\left[\underline{\rho + 3\rho_s - \frac{2\rho^2}{\rho_n}X\frac{\partial(\rho_n/\rho)}{\partial X}} + 4\frac{\rho_s}{\rho}X\frac{\partial\rho}{\partial X}\right] = M + S'. \tag{C13b}$$

As in the case of pure ^{4}He [Eqs. (26) and (27)] we omit the common factor $\rho\rho_n$ in the coefficients of $\det_1$ [Eq. (C6)]. The underlined terms in Eq. (C13) correspond to $-(\partial\rho/\partial P)M$ [in Eq. (C13a)] and M in Eq. (C13b).

At $X=0$ Eqs. (C12) and (C13) coincide with Eqs. (26) and (27).

3. Sound velocities in the absence of internal motion

In order to calculate $u^0_{1,2}$ [see Eq. (C9)] let us pick out small corrections in the exact expression for A and B [Eq. (C12); cf. Eq. (26)]

$$A = A_0 + A', \quad B = B_0 + B', \quad A_0 = \rho\frac{\partial\rho}{\partial P}\frac{\partial\sigma}{\partial T},$$

$$B_0 = -\rho\frac{\partial\rho}{\partial P}\frac{\partial\sigma}{\partial T}\left[\frac{\partial P}{\partial\rho} + \frac{\rho_s}{\rho_n}\frac{(\sigma - X\partial\sigma/\partial X)^2}{\partial\sigma/\partial T} + \frac{\rho_s}{\rho_n}X^2\frac{\partial(Z/\rho)}{\partial X}\right]. \tag{C14}$$

We find

$$(u^0_{1,2})^2 \equiv \bar{u}^2_{1,2} + (u^{(1)}_{1,2})^2,$$

$$\bar{u}^2_{1,2} = \frac{-B_0 \pm \sqrt{B_0^2 - 4A_0C}}{2A_0},$$

$$(u^{(1)}_{1,2})^2 = (u^0_{1,2})^2 - \bar{u}^2_{1,2} \approx -\frac{A'}{A_0}\left[\bar{u}^2_{1,2} \pm \frac{C}{\sqrt{B_0^2 - 4A_0C}}\right] - \frac{B'}{2A_0}\left[1 \pm \frac{B_0}{\sqrt{B_0^2 - 4A_0C}}\right]. \tag{C15}$$

Substituting A_0, A', B_0, and B' [see Eqs. (C12) and (C14)] we get

$$\bar{u}_1^2 = \frac{\partial P}{\partial\rho}, \quad \bar{u}_2^2 = \frac{\rho_s}{\rho_n}\left[\frac{(\sigma - X\partial\sigma/\partial X)^2}{\partial\sigma/\partial T} + X^2\frac{\partial(Z/\rho)}{\partial X}\right], \tag{C16}$$

$$(u^{(1)}_{1,2})^2 \approx \frac{\alpha^2}{(\partial\rho/\partial P)(\partial\sigma/\partial T)}\left[\bar{u}^2_{1,2} \pm \frac{\bar{u}_2^2}{1 - \bar{u}_2^2/\bar{u}_1^2}\right] + \frac{\bar{u}_1^2}{2}\left[1 \pm \frac{1 + \bar{u}_2^2/\bar{u}_1^2}{1 - \bar{u}_2^2/\bar{u}_1^2}\right] \times \left\{\frac{\rho_s}{\rho_n}\frac{X^2}{\rho^2}\left[\frac{\partial\rho}{\partial X}\right]^2 + \frac{(2\rho_s/\rho_n)(\sigma - X\partial\sigma/\partial X)\alpha(X/\rho)(\partial\rho/\partial X) - (\rho_s/\rho_n)\alpha^2X^2\partial(Z/\rho)/\partial X}{\partial\sigma/\partial T}\right\}. \tag{C17}$$

Here we took advantage of the equations

$$B_0 = -A_0(\bar{u}_1^2 + \bar{u}_2^2), \quad \sqrt{B_0^2 - 4A_0C} = A_0(\bar{u}_1^2 - \bar{u}_2^2).$$

Thus we obtain

$$(u^0_1)^2 \approx \bar{u}_1^2\left[1 + \frac{\alpha^2\bar{u}_1^2}{(\partial\sigma/\partial T)(1 - \bar{u}_2^2/\bar{u}_1^2)} + \frac{k}{1 - \bar{u}_2^2/\bar{u}_1^2}\right],$$

$$(u^0_2)^2 \approx \bar{u}_2^2\left[1 - \frac{\alpha^2\bar{u}_2^2}{\partial\sigma/\partial T(1 - \bar{u}_2^2/\bar{u}_1^2)} - \frac{k}{1 - \bar{u}_2^2/\bar{u}_1^2}\right],$$

where

$$k = \frac{\rho_s}{\rho_n}\frac{X^2}{\rho^2}\left[\frac{\partial\rho}{\partial X}\right]^2 + \alpha\frac{\rho_s}{\rho_n}\left[2\left[\sigma - X\frac{\partial\sigma}{\partial X}\right]\frac{X}{\rho}\frac{\partial\rho}{\partial X} - \alpha X^2\frac{\partial(Z/\rho)}{\partial X}\right]\left[\frac{\partial\sigma}{\partial T}\right]^{-1} \tag{C18}$$

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