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Topological solitons and geometrical frustration

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We study classical Heisenberg spins coupled by an isotropic or an anisotropic spin-spin interaction on an infinite elastic cylinder. In the continuum limit, the Hamiltonian of the system is given by a nonlinear σ model. We investigate the cylindrically symmetric solutions of the sine-Gordon equation (the Euler-Lagrange equation for this Hamiltonian). The periodic solution as well as the anisotropic one-soliton solution do not satisfy the self-dual equations of Bogomol'nyi [Sov. J. Nucl. Phys. **24**, 449 (1976)] which are a necessary condition to reach the minimum energy configuration in each homotopy class. This generates geometrical frustration and produces a geometric effect: a shrinking of the cylinder coupled with nontrivial spin distributions.

I. INTRODUCTION

The role of nonlinear excitations in the study of low-dimensional, artificially structured materials is becoming increasingly important due to their observable effects on the physical properties of a realizable condensed matter system. The underlying physics becomes much richer when the interplay between the geometry and topology is also considered. Here we illustrate this point by exploring classical Heisenberg-coupled spins on an infinite elastic cylinder. In the continuum limit, classical one- and two-dimensional static Heisenberg (ferromagnetic and antiferromagnetic) spins are described by a nonlinear σ model.[1–5] Under homogeneous boundary conditions, the spin distributions fall into different homotopy classes.[1,2,6] The latter are associated with the minimum energy configuration for each spin distribution.

In the first part of this article, we apply this model to isotropic spin-spin coupling on a rigid cylinder (i.e., a cylinder with constant radius ρ_0). In this case, a solution of the sine-Gordon equation (the Euler-Lagrange equation for this Hamiltonian) is given by a nontrivial spin distribution (one-soliton) which satisfies the self-dual equations[6] (Sec. II). These equations represent a necessary condition to attain the absolute minimum of the energy in each homotopy class (Sec. III). Therefore, the one-soliton solution realizes the minimum of the magnetic energy in the first homotopy class, and no perturbation (perturbation of the spin distribution or deformation of the elastic cylinder) could lower the energy of the system (Sec. IV).

Next, we study the nonlinear σ model on a cylindrically symmetric support, and show that in this case, Bogomol'nyi's decomposition can still be applied (Sec. V). The self-dual equations are first order differential equations and all solutions of these equations satisfy the sine-Gordon equation, which is a second order differential equation, but not vice versa. We explore an ansatz for the two-soliton solution (Sec. VI) as well as the exact multisoliton solutions of the sine-Gordon equation (Sec. VII), and in fact we find that, due to the soliton-soliton interaction, these spin distributions do not satisfy the self-dual equations, and therefore do not reach the minimum energy (per soliton) in each homotopy class.[1] This is due to a geometrical frustration, which stems from a misfit between the width of the soliton and the radius of the cylinder, which is the characteristic length of the system. Thus, a variation in the geometry of the cylinder, compensating for the misfit, can lead to a lowering of the energy. Indeed we find that the increase of the elastic energy for a periodic shrinking of the cylinder is more than compensated by the lowering of the magnetic energy associated with the lattice soliton (Sec. VIII). Here we note the analogy with the Peierls instability in low-dimensional, interacting electron-phonon systems.

In a different approach, we used the multisoliton solution as an exact solution describing magnetic solitons on a cylinder of finite length (Sec. IX). This enables us to look for approximate solutions of the nonlinear σ model on a sphere, when compactifying the finite cylinder into a sphere. In the last sections, because the misfit between the magnetic length and the characteristic length of the geometry of the support plays an important role in the frustration, we introduce anisotropy in the spin-spin coupling and thus renormalize the characteristic magnetic length (Sec. X). We also deal with a moving soliton in view of the relativistic contraction of the length (Sec. XI). Finally, we suggest some materials such as magnetically coated cylindrical thin films on which our model and the magnetoelastic effects predicted here could be tested experimentally (Sec. XII).

II. THE RIGID CYLINDER

The nonlinear σ model is the continuum limit of the Heisenberg Hamiltonian for classical ferromagnets or antiferromagnets.[1–5] On an infinite rigid cylindrical support, for isotropic spin-spin coupling, it is given by

$$H_{\text{isotropic}} = J \iint_{\text{cylinder}} (\nabla \vec{n})^2 \, dS, \tag{1}$$

with $|\vec{n}|^2 = 1$ and J is the spin-spin coupling constant. The order parameter of the classical Heisenberg model covers the sphere S^2. In order to incorporate this constraint in Eq. (1), we will work with the two fields (θ, Φ) where $\vec{n} = (\cos\theta,\ \sin\theta\ \cos\Phi,\ \sin\theta\ \sin\Phi)$. Here θ is the colatitude and Φ is the azimuthal angle. Then, in cylindrical coordinates (ρ,x,φ) we write Eq. (1) as

$$H_{\text{isotropic}} = J \iint_{\text{cylinder}} \left[(\partial_x\theta)^2 + \sin^2\theta\, (\partial_x\Phi)^2 + \frac{(\partial_\varphi\theta)^2}{\rho_0^2} + \frac{\sin^2\theta}{\rho_0^2} (\partial_\varphi\Phi)^2 \right] \rho \, dx \, d\varphi. \tag{2}$$

We will work with homogeneous boundary conditions, because we desire $H_{\text{isotropic}}$ to be finite. By homogeneous boundary conditions, we mean $\lim_{x\to\infty}\theta \equiv 0[\pi]$ and $\lim_{x\to\infty}\frac{d\theta}{dx} = 0$. The way $\frac{d\theta}{dx}$ goes to zero should ensure the convergence of the integral in Eq. (2). With these boundary conditions, we can identify all points at infinity and compactify the infinite cylinder into a sphere (the ends of the cylinder become the two poles of the sphere). The mapping of this sphere to S^2, the order parameter manifold, gives us a homotopy group isomorphic to **Z**, the group of the relative integers, because the order parameter manifold S^2 is a simply connected manifold. Thus the spin distributions on the infinite cylinder can be classified in different classes of topologically nontrivial spin distributions[1,7,8] [i.e., $\theta = \theta(x,\varphi)$]. Inside each class, the spin distributions are topologically equivalent: they belong to the same homotopy class.

In this paper, we restrict ourselves only to solutions with cylindrical symmetry, which will be sufficient for our purposes. This means that θ and Φ will satisfy the following conditions:

$$\Phi = \varphi, \qquad \frac{\partial\theta}{\partial\varphi} = 0. \tag{3}$$

The Hamiltonian (1) then becomes

$$H_{\text{isotropic}} = 2\pi\rho_0 J \int_{-\infty}^{+\infty} \left[\left(\frac{d\theta}{dx}\right)^2 + \frac{\sin^2\theta}{\rho_0^2} \right] dx. \tag{4}$$

After variation of the Hamiltonian, the Euler-Lagrange equation $\delta H = 0$ leads to

$$\frac{d^2\theta(x)}{dx^2} = \frac{1}{2\rho_0^2}\sin 2\theta. \tag{5}$$

This second order differential equation is the sine-Gordon equation whose solutions are solitons. This equation appears in a wide variety of physical systems, including charge-density-wave materials, splay waves in membranes, magnetic flux in Josephson lines, torsion-coupled pendula, propagation of crystal dislocations, Bloch wall motion in magnetic crystals, two-dimensional models of elementary particles, etc. A solution for a single spin twist, i.e., a solution for a distribution belonging to the first homotopy class, is given by

$$\theta = 2\arctan\exp\frac{x}{\rho_0}. \tag{6}$$

It is represented schematically in Fig. 1. The energy associated with this distribution is given by

$$H^1_{\text{isotropic}} = 8\pi J.$$

The solution (6) depends explicitly on the radius of the cylinder: the soliton has a fixed width ρ_0. But $H^1_{\text{isotropic}}$, which is the minimum energy level in the first homotopy class, is independent of ρ_0. It is equal to the value of the first energy level in the case of a two-dimensional (2D) plane,[1] where, because of the invariance of the Hamiltonian under homothety, any soliton could be shrunk into a metastable point, the result being a degeneracy of the energy levels. A homothety in the 2D plane is a length scaling transformation f, such that $f : x \to \lambda x,\ y \to \lambda y$. In our case, when studying the nonlinear σ model on a cylindrical geometry, and thus introducing a characteristic length that is the radius ρ_0, we lift this degeneracy. But when working on an infinite rigid cylinder, the change of variable $x \to u = \frac{x}{\rho_0}$ in the Hamiltonian eliminates any dependence on ρ_0:

$$H_{\text{isotropic}} = 2\pi J \int_{-\infty}^{+\infty} \left[\left(\frac{d\theta}{du}\right)^2 + \sin\theta^2 \right] du. \tag{7}$$

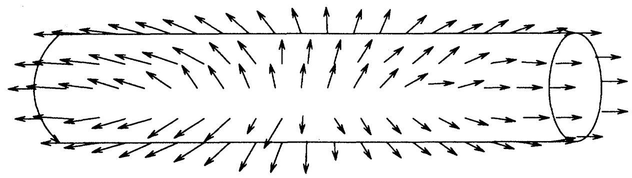

FIG. 1. Cylindrically symmetric $0 \to \pi$ twist soliton on an infinite cylinder.

III. THE SELF-DUAL EQUATIONS FOR THE RIGID CYLINDER

The choice of cylindrically symmetric solutions (3) is justified by the fact that we are able to reach the topological minimum energy in the first homotopy class, $H^1_{\text{isotropic}}$ for these solutions. All other, more general, distributions belonging to the same homotopy class have higher, or equal, energies.

The solutions corresponding to the absolute minimum energy in each homotopy class satisfy the self-duality equations:

$$\rho_0\,\partial_x\theta = \pm\,\sin\theta\,\partial_\varphi\Phi, \quad \partial_\varphi\theta = \mp\,\rho_0\,\sin\theta\,\partial_x\Phi. \tag{8}$$

These first order differential equations are obtained in the general case $[\theta{=}\theta(x,\varphi)$ and $\Phi{=}\Phi(x,\varphi)]$ by using the technique employed by Belavin and Polyakov[1] and Bogomol'nyi.[6] Using the obvious expressions

$$\left(\partial_x\theta \pm \frac{\sin\theta}{\rho_0}\partial_\varphi\Phi\right)^2 + \left(\frac{\partial_\varphi\theta}{\rho_0} \mp \sin\theta\,\partial_x\Phi\right)^2 \geq 0,$$

we can obtain the following inequality:

$$\mathcal{H}_{\text{isotropic}} \geq 2\frac{\sin\theta}{\rho_0}\;|\partial_x\theta\partial_\varphi\Phi - \partial_\varphi\theta\partial_x\Phi|.$$

Therefore, when integrating over the whole support, we get

$$\begin{aligned} H_{\text{isotropic}} &= J\iint_{\text{cylinder}} \mathcal{H}_{\text{isotropic}}\;\rho_0 dx\,d\varphi \\ &\geq 2J\iint \sin\theta\;\,d\theta\,d\Phi = 8\pi J|Q|. \end{aligned} \tag{9}$$

The right-hand side of the inequality (9) is just the winding number of the solution, Q, and does not depend on the geometry of the support. The equality holds when Eqs. (8) are satisfied. Thus, Bogomol'nyi's decomposition allows us to show the existence of equidistant energy levels and generates first order differential equations, the self-dual equations, which have to be satisfied in order to reach the absolute minimum energy in each homotopy class.

Equations (8) can be written as Laplace equations in U and Φ:

$$\partial^2_{\varphi\varphi}U + \partial^2_{uu}U = 0, \quad \partial^2_{\varphi\varphi}\Phi + \partial^2_{uu}\Phi = 0, \tag{10}$$

where $U = \ln\left(\tan\frac{\theta}{2}\right)$ and $u = \frac{x}{\rho_0}$. In the case of a cylindrically symmetric solution, Eqs. (8) reduce to

$$\rho_0\,\partial_x\theta = \pm\,\sin\theta, \tag{11}$$

and Eqs. (10) lead to $\partial^2_{uu}U = 0$. The solution (6) satisfies both the self-dual and the Euler-Lagrange equations. The derivative of Eq. (8), with respect to the variable x, leads to the sine-Gordon equation (5).

If we look for more general solutions of Eqs. (8) and Eqs. (10), we find that

$$\Phi = ax + n\varphi, \quad U = \frac{n}{\rho_0}x + a\rho_0\varphi.$$

Because of the 2π periodicity of Φ and because θ has to be single valued, a should be zero and n is an integer. The cylindrically symmetric solution corresponds to $n{=}1$. When using the above solution for Φ in Eq. (9), we see that the spin distribution will belong to the homotopy class of winding number n, and the width of the longitudinal soliton is $\frac{\rho_0}{n}$. Then, if we are looking for a solution of the self-dual equations for the rigid cylinder belonging to the first homotopy class, we must take $n{=}1$. Therefore, the solution (6) is the only solution corresponding to the first excited level.

IV. ONE-SOLITON DISTRIBUTION ON AN ELASTIC CYLINDER

In this section, we will study the spin-elastic coupling for a nontrivial spin distribution on the cylindrical support, i.e., the effect of a variable cylindrical geometry on the excitation spectrum. Henceforth we do not impose the constraint $\rho = \rho_0$. The cylinder radius ρ becomes a function of $x : \rho(x)$. The geometry of the support of the spin distribution is dependent on the magnetic distribution and vice versa. We are interested in the possible modification of the spin distribution due to a modification of the geometry, which could lead to a lowering of the energy.

We consider solutions with cylindrical symmetry $\left(\Phi = \varphi \text{ and } \frac{\partial\theta}{\partial\varphi} = 0\right)$, i.e., we are dealing with a quasi-one-dimensional case. Therefore, we may take into account the results of Cieplak and Turski[9] concerning the renormalization of the spin-spin coupling resulting from the addition of a spin-elastic term to the Hamiltonian. As a consequence, we add to the nonlinear σ model Hamiltonian only the elastic term

$$\mathcal{H}_{\text{el}} = \frac{1}{4}\chi_1\left(\frac{d\rho}{dx}\right)^2 + \frac{\chi_2}{\rho_0^2}(\rho-\rho_0)^2, \tag{12}$$

where χ_1 and χ_2 are elastic constants of the cylinder for deformations along the axial and radial directions, respectively. This Hamiltonian density physically models the cost in energy of elastic deformations of the support and imposes additional boundary conditions: $\lim_{|x|\to\infty}\rho(x) = \rho_0$ and $\lim_{|x|\to\infty}\frac{d\rho}{dx}(x) = 0$ in order to have $\lim_{|x|\to\infty}\mathcal{H}_{\text{el}} = 0$ (then, the cost in energy of elastic deformations is finite). The new Hamiltonian reads

$$\begin{aligned} H_{\text{isotropic+el}} = 2\pi\int\Bigg\{ & J\left[\rho\left(\frac{d\theta}{dx}\right)^2 + \frac{\sin^2\theta}{\rho}\right] \\ & + \frac{1}{4}\chi_1\left(\frac{d\rho}{dx}\right)^2\rho + \frac{\chi_2}{\rho_0^2}(\rho-\rho_0)^2\rho\Bigg\}dx, \end{aligned}$$

where the coupling constant J does not depend on the geometrical deformations. Now we vary the radius of the cylinder in the presence of a quasi-one-dimensional magnetic soliton. The variation with respect to the variables θ and ρ leads to the following differential equations (Euler-Lagrange equations):

$$\rho'\theta' + \rho\theta'' = \frac{\sin 2\theta}{2\rho}, \tag{13}$$

$$\chi_1\left[\rho''\rho + \frac{(\rho')^2}{2}\right] - 2\frac{\chi_2}{\rho_0^2}(\rho-\rho_0)(2\rho-\rho_0) = J\left[(\theta')^2 - \frac{\sin^2\theta}{\rho^2}\right], \tag{14}$$

where ($'$) denotes $\frac{d}{dx}$. Note that after direct integration of Eq. (13), we get the following expression:

$$\rho^2(\theta')^2 = \sin^2\theta + C. \tag{15}$$

If the integration constant C is equal to zero, Eq. (15) is nothing but the self-duality equation (11). Then the right-hand side (rhs) of Eq. (14) vanishes. This happens, in particular, when we apply homogeneous boundary conditions. But in general, this constant is different from zero. This case is discussed in the following sections.

A trivial solution of these highly nonlinear differential equations, compatible with the boundary conditions, is given by $\rho=\rho_0$ and $\theta = 2\arctan\exp\left(\frac{x}{\rho_0}\right)$. Obviously, the system does not exploit the "new degree of freedom" $\rho = \rho(x)$. Any other solution of these equations different from $\rho=\rho_0$ would lead to a higher energy because an elastic deformation would give a positive contribution to the Hamiltonian. This happens because the solution (6) satisfies the self-dual equation (11). Therefore, it corresponds to a metastable state in the first homotopy class, for which any perturbation would drive it to a higher energy level.

An important feature of the Bogomol'nyi's inequality (9) is that the Hamiltonian, i.e., a metric-dependent term, and a topological term appear on different sides of the inequality (9). The nonlinear σ model is "autotuned," in the sense that, in the one-soliton case, the width of the spin distribution is equal to the characteristic length of the geometric support: ρ_0. This characteristic length is introduced in the Hamiltonian via both the gradient and the element of surface.

V. HEISENBERG SPINS ON A 2D SURFACE

We will show now that on an arbitrary smooth surface S, we will still be able to use the above method based on the existence of a topological invariant. The gradient is to be replaced by the covariant derivative, $\mathbf{D}$, which reflects the metric. The differential element of surface becomes

$$d\vec{S} = \sqrt{|g|}\ d\Omega,$$

where g is the metric tensor of the support. We must now extract the second order Euler-Lagrange equations from the following Hamiltonian represented by the generalized nonlinear σ model:

$$H_{\text{isotropic}} = J\iint_S (\mathbf{D}\vec{n})^2\ \sqrt{|g|}\ d\Omega.$$

Bogomol'nyi's technique is still valid and leads to first order differential equations.

When dealing with a cylindrically symmetric support, the exact Hamiltonian is given by

$$H_{\text{isotropic}} = J\iint\left[\frac{(\partial_x\theta)^2}{1+(\partial_x\rho)^2} + \frac{\sin^2\theta}{1+(\partial_x\rho)^2}(\partial_x\Phi)^2 + \frac{(\partial_\varphi\theta)^2}{\rho^2} + \frac{\sin^2\theta}{\rho^2}(\partial_\varphi\Phi)^2\right] \times\rho\sqrt{1+(\partial_x\rho)^2}\ dx\,d\varphi. \tag{16}$$

Starting from the obvious inequalities

$$\left(\frac{\partial_x\theta}{\sqrt{1+\partial_x\rho^2}} \pm \frac{\sin\theta}{\rho}\partial_\varphi\Phi\right)^2 + \left(\frac{\partial_\varphi\theta}{\rho} \mp \frac{\sin\theta}{\sqrt{1+\partial_x\rho^2}}\partial_x\Phi\right)^2 \geq 0,$$

we prove the existence of stable energy levels by means of the inequality $H_{\text{isotropic}} \geq 8\pi J|Q|$. This gives rise to the same energy spectrum as in the rigid cylinder case, and also to the following self-dual equations:

$$\frac{1}{\sqrt{1+(\partial_x\rho)^2}}\left(\frac{\partial\theta}{\partial x}\right) = \pm\frac{\sin\theta}{\rho}\left(\frac{\partial\Phi}{\partial\varphi}\right), \quad \left(\frac{\partial\theta}{\partial\varphi}\right) = \mp\frac{\rho\sin\theta}{\sqrt{1+(\partial_x\rho)^2}}\left(\frac{\partial\Phi}{\partial x}\right). \tag{17}$$

If dealing with a more general 2D surface S, the nonlinear σ model Hamiltonian for static Heisenberg spins, represented by the local order parameter $\vec{n}$, is given by[10]

$$H_{\text{isotropic}} = \iint_S \mathcal{H}_{\text{isotropic}}\ \sqrt{|g|}d\Omega = J\iint_S \sqrt{|g|}\ g^{ij}h_{\alpha\beta}\ \partial_i n^\alpha \partial_j n^\beta d\Omega,$$

where $h_{\alpha\beta}$ is the metric tensor of the differential manifold which represents the order parameter $\vec{n}$, and g^{ij} is the dual of g_{ij}, the metric tensor of the support whose determinant is $\sqrt{|g|}$. As the metric tensors and their duals are real, symmetric, positive definite matrices, we can define the tensors $(\sqrt{h})_{\alpha\beta}$ and $(\sqrt{g})_{ij}$ in such a way that $(\sqrt{h})^2 = h$ and $(\sqrt{g})^2 = g$.

Now we can define the two tensors $\overline{T}^k_\gamma$

$$\overline{T}^k_\gamma = (\sqrt{h})_{\gamma\alpha}(\sqrt{g})^{ki}\partial_i n^\alpha \pm\epsilon^k_j\epsilon^\beta_\gamma(\sqrt{h})_{\beta\delta}(\sqrt{g})^{jl}\partial_l n^\delta,$$

where ϵ is the antisymmetric tensor. We then write $\sum_{k,\gamma}(\overline{T}^k_\gamma)^2 \geq 0$.

By developing these inequalities, we get the following expression:

$$g^{ij}h_{\alpha\beta}\ \partial_i n^\alpha \partial_j n^\beta \geq 2\sqrt{|h|}\frac{1}{\sqrt{|g|}}\ |\partial_i n^\alpha \partial_j n^\beta - \partial_j n^\alpha \partial_i n^\beta|.$$

Multiplying by $\sqrt{|g|}$ and integrating over the whole surface S, the rhs of this inequality becomes the Pontryagin index Q (or the winding number[5]). We can thus write, for any smooth 2D surface, the following inequality:

$$H_{\text{isotropic}} \geq 8\pi J|Q|. \tag{18}$$

Therefore, on any smooth 2D surface, the nonlinear σ model leads to the same energy spectrum (18), independent of the metric. We can obtain the self-dual equations, if we require the tensors $\overline{T}$ to be identically zero. These equations represent a necessary condition to attain the absolute minimum of the energy in each homotopy class. They are first order differential equations and all solutions of these equations satisfy the Euler-Lagrange equations, which are second order differential equations. We can associate to each static soliton, which is a solution of the self-dual equations, a self-energy $8\pi J$ (or mass energy). Thus, the value of the energy associated with a topological metastable solution is independent of the metric. This discussion (and Sec. IX) are germane to membranes, vesicles, and bubbles of magnetic materials.

VI. THE TWO-SOLITON ANSATZ

In the following we will approximate the $0 \to 2\pi$ soliton on the rigid cylinder by an ansatz[11] describing two cylindrically symmetric static solitons separated by a distance $2d$,

$$\theta = 2\arcsin\left[\frac{e^{\frac{x-d}{\rho_0}} - e^{\frac{-x-d}{\rho_0}}}{\sqrt{1+(e^{\frac{x-d}{\rho_0}} + e^{\frac{-x-d}{\rho_0}})^2}}\right]. \tag{19}$$

Note that the sine-Gordon equation does not have an exact static two-soliton solution.[12] However, an exact moving two-soliton solution exists.[11] The above function corresponds to a distribution belonging to the second homotopy class. Up to a term proportional to $e^{-\frac{2d}{\rho_0}}$, it satisfies both the sine-Gordon equation (5) and the self-dual equation (11). Therefore, when the two twists are well separated (compared to ρ_0, the characteristic length in our problem), this function can be considered as a good analytical approximation of the exact solution.[12] We calculate up to the first order the magnetic energy associated with this spin distribution, and find the following expression:

$$H^{(2)}_{\text{isotropic}} = 16\pi J + 64\pi J e^{-\frac{2d}{\rho_0}} + \cdots . \tag{20}$$

The first term of the rhs of Eq. (20) is the energy associated with a soliton belonging to the second homotopy class (the magnetic distribution covers the sphere S^2 twice, therefore Q=2). In addition to the minimum energy, there is a positive term which describes the exponential repulsive interaction between the two solitons.[13,14] This additional energy appears because the function in Eq. (19) does not satisfy exactly the self-dual equation (11).

VII. THE PERIODIC SOLITON SOLUTION

Next, we turn to an exact solution of the second order differential equation: the periodic solution of the sine-Gordon equation, more specifically the soliton lattice.[13,15] Note that the Bogomol'nyi argument can be applied between any two points on the cylinder where the variation of $\vec{n}$, the order parameter, covers the sphere S^2 an integer number of times. Therefore, when dealing with a rigid cylinder, the first order self-dual equations (11) are still valid for this interval. Any function which satisfies them would satisfy the sine-Gordon equation (5) and gives the minimum energy for this interval. The periodic solution of the sine-Gordon equation (5) can be obtained directly using a Poisson sum.[16] It is given by the following expression:

$$\theta = \arccos\left[\text{sn}\left(\frac{x}{k\rho_0}, k\right)\right]. \tag{21}$$

The period of this solution is $4d = 4\rho_0 k K(k)$, where k is the modulus of the Jacobian elliptic function sn (sine amplitude), and $K(k)$ is the complete elliptic integral of the first kind. In the limit $k \to 1$, as $\lim_{k\to 1} K(k) \to \infty$, the half period $2d$ tends to infinity. At the boundaries, we recover the homogeneous conditions discussed in Sec. II. Therefore, we get the single twist soliton (Eq. 6). Note that the soliton lattice (and repulsive solitonic interaction) also arise in the context of doped bond-order-wave and charge-density-wave materials (e.g. conducting polymers[13]), fluctuations in Josephson junctions[17] and cholesteric liquid crystals,[18] vortex lines in superconducting barriers,[19] etc.

The exact magnetic energy per soliton (or the energy density per half period, $2d$; as $\theta(\pm d) \equiv 0[\pi]$) now reads

$$H_{\text{isotropic}} = \frac{8\pi J}{k}\left[E(k) - \frac{k'^2 K(k)}{2}\right], \tag{22}$$

where k' is the complementary modulus ($k'^2 = 1 - k^2$) and $E(k)$ is the complete elliptic integral of the second kind. In the dilute limit, i.e., $k \to 1$ [then $E(k) \to 1$], we expand the exact solution (22) and find that the energy per soliton is given by

$$\begin{aligned} H_{\text{isotropic}} &= 8\pi J + 32\pi J \exp\left(-\frac{2d}{\rho_0}\right) + \cdots \\ &= 8\pi J + 2\pi J k'^2 + \cdots . \end{aligned} \tag{23}$$

We get the result of the two-soliton ansatz (Eq. 20). In addition to the self-energy of a soliton (which corresponds to a topological consideration), there is a repulsive exponential interaction energy between two solitons. The periodic solution (21) does not reach the minimum energy per soliton $H^1_{\mathrm{isotropic}}$. This is because the periodic solution (Eq. 21) *does not* satisfy the self-duality equations for the rigid cylinder (11), even though it is an exact solution of the sine-Gordon equation. The periodic soliton solution rather satisfies the relation

$$\rho_0{}^2\,(\partial_x\theta)^2 = \sin^2\theta + \frac{k'^2}{k^2}. \tag{24}$$

Equation (24) represents the modified "equipartition" relation between "kinetic" energy $(\partial_x\theta)^2$ and "pseudopotential" energy $\sin^2\theta$. For a multisoliton solution, in addition to the "pseudopotential" energy, there is an exponential repulsive interaction energy density: $(\frac{k'}{k})^2$. In the limit of a single twist soliton ($d \to \infty$ and $k' = 0$) the term $(\frac{k'}{k})^2$ vanishes. We recover the energy $H^1_{\mathrm{isotropic}}$ of the single twist soliton: the interaction term goes to zero and "equipartition" holds. As the self-dual equations are not satisfied, this implies that we can minimize the magnetic energy by an elastic deformation of the cylinder. Indeed, Eq. (14) does not allow $\rho = \rho_0$ as a solution anymore, since if the self-duality is not satisfied, the rhs of Eq. (14) does not vanish anymore. This is due to the soliton-soliton interaction: the width of the lattice soliton, $k\rho_0$, is no longer equal to ρ_0, the radius of the cylinder. Therefore, under the influence of the neighboring solitons, the spin distribution is "squeezed"— there is a geometrical frustration. In the case of a single twist soliton, the width of the soliton (6) appears naturally to be ρ_0 (the characteristic length of the problem). In order to recover the original correspondence between the magnetic length and the characteristic length of the support, i.e., in order to remove the geometrical frustration, we will next introduce a deformation of the geometrical support.

VIII. THE PERIODICALLY PINCHED CYLINDER

If we relax the constraint $\rho = \rho_0$ and allow ρ to be x dependent, i.e., if we allow elastic deformations of the cylinder, the Hamiltonian of the nonlinear σ model becomes Eq. (16). The modified self-dual equations for cylindrically symmetric support are then given by Eq. (17).

We do not solve exactly the highly nonlinear Eq. (17), but rather show that with an appropriate ansatz for $\rho = \rho(x)$ we can minimize $H_{\mathrm{isotropic+el}}$. From the observation that the magnetic width is smaller than ρ_0, we can expect a pinching of the support in order to locally reduce the radius of the cylinder. This will eliminate the geometrical frustration. We take the following cylindrically symmetric, periodic ansatz for ρ and for the spin distribution $\vec{n}$:

$$\rho = \rho_0 - \epsilon\rho_0 \mathrm{cn}^2\left(\frac{x}{k\rho_0}, k\right)$$

and

$$(\theta, \Phi) = \left\{\arccos\left[\mathrm{sn}\left(\frac{x}{k\rho_0}, k\right)\right], \varphi\right\},$$

where cn is the Jacobian elliptic function, cosine amplitude. The periodic magnetic soliton solution and the accompanying periodic pinch on the elastic cylinder are schematically depicted in Fig. 2. Note that the pinch appears exactly at those positions where spins are perpendicular to the cylinder ($\theta = \frac{\pi}{2}$). Assuming that the quantities $\frac{Jk'^2}{\chi_1\rho_0{}^2}$ and $\frac{Jk'^2}{\chi_2\rho_0^2}$ are small compared to 1, we expand the Hamiltonian density up to the third order in ϵ:

$$\rho\sqrt{1+(\partial_x\rho)^2}\mathcal{H}_{\mathrm{isotropic+el}} = J\left[2\rho_0\mathrm{cn}^2\left(\frac{x}{k\rho_0}, k\right) + \frac{k'^2}{k^2\rho_0}\right] + \epsilon\left[-\frac{Jk'^2}{k^2\rho_0} + \epsilon\rho_0\left(\frac{\chi_1}{k^2} + \chi_2\right)\right]\mathrm{cn}^2\left(\frac{x}{k\rho_0}, k\right) - \epsilon^2\mathcal{H}_\alpha - O(\epsilon^3),$$

where $\mathcal{H}_\alpha$ is a positive function. If we integrate over half a period ($2d$), we find

$$H_{\mathrm{isotropic+el}} = \frac{8\pi J}{k}\left[E(k) - \frac{k'^2K(k)}{2}\right] - \epsilon\left[\frac{Jk'^2}{\rho_0} - \epsilon\rho_0(\chi_1 + \chi_2 k^2)\right]\frac{8\pi J}{k}E(k) - \epsilon^2 H_\alpha - O(\epsilon^3).$$

In order to minimize $H_{\mathrm{isotropic}}$, we choose $\epsilon = \frac{Jk'^2}{2[\chi_1+\chi_2k^2]\rho_0^2}$. Then the total energy for the elastic case is smaller than the energy associated with the rigid case. The ansatz for $\theta(x)$ does not correspond to the exact spin distribution on the periodically deformed cylinder [defined by $\rho(x)$] which would reach the minimum energy for that given geometry. This particular distribution corresponds to a higher energy of the magnetic soliton. Nevertheless, even with this ansatz, we can lower the total energy, i.e., $H_{\mathrm{elastic}} < H_{\mathrm{rigid}}$. Therefore, using the

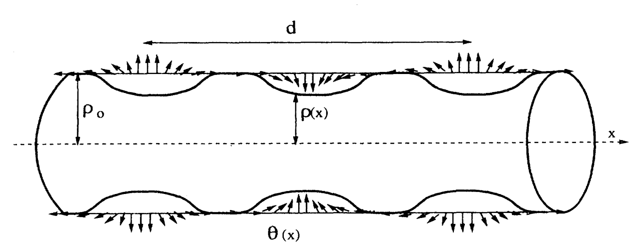

FIG. 2. Magnetic soliton lattice solution and periodic pinch on an elastic cylinder.

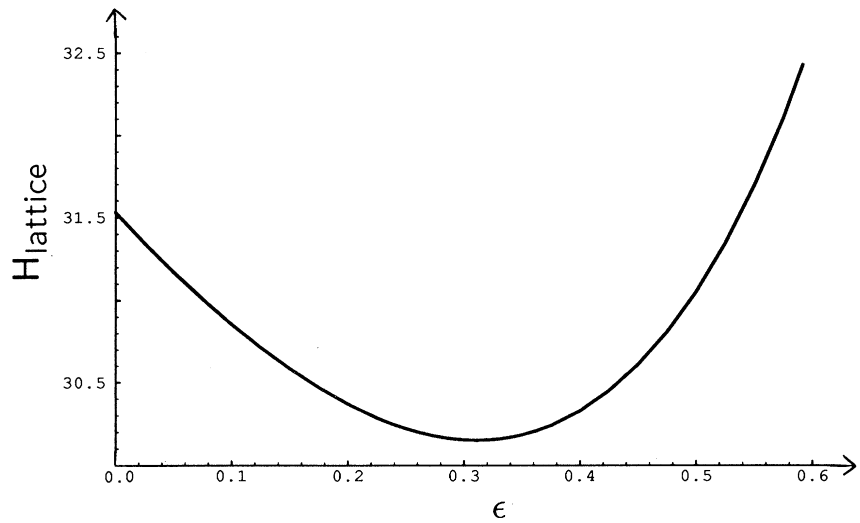

FIG. 3. Energy of the elastic nonlinear σ model as a function of ϵ, the depth of the pinch of the support, for the periodic soliton solution when k=0.5 and J=1.

Ritz theorem, we can deduce that the exact solution of lowest energy in the first homotopy class will require a deformation of the support. This deformation of the support enables a screening of the repulsive soliton-soliton interaction. We plot in Fig. 3 the elastic and magnetic energies as a function of ϵ, the magnitude of the pinch of the support. Note that there is a minimum in energy for a value of $\epsilon \neq 0$. We have verified that this minimum exists for all $k \neq 1$. The periodic deformation of the cylinder is the magnetoelastic analog of the Peierls effect. Here the lowering of magnetic energy is analogous to the gain in electronic energy in Peierls distorted materials.

In the case $k \to 1$ ($k' \to 0$), $\epsilon \to 0$ and we recover the result of Sec. IV: for a single twist soliton, no elastic perturbation of the support is able to lower the magnetic energy. The one-soliton solution, Eq. (6), satisfies the self-duality equation for the rigid cylinder, Eq. (11). However, two solitons on an elastic cylinder will cause a deformation due to the interaction between the two solitons.

IX. THE NONLINEAR σ MODEL ON A SPHERE

The above discussion is also valid when dealing with a cylinder of finite length $2L$. Exact solutions of our problem (i.e., cylindrically symmetric distributions which are extrema of the Hamiltonian on a finite cylinder) are lattice solitons where k satisfies $L = N\rho_0 kK(k)$ (here, N is the number of solitons). In the case N=1, the period of the lattice soliton is twice the length of the support and we recover the single twist soliton (Sec. II), but on a finite cylinder. The interaction term in the energy density (Eq. 23) vanishes when $L \to \infty$.

At $\pm L$, the field $\theta(x)$ is zero (modulo π). Hence, at the boundaries of the finite cylinder, the spins are all oriented along the x axis, and the order parameter covers the sphere S^2 an integer number of times. But its derivative at the boundaries is nonzero, because $\frac{d\theta}{dx} = \frac{1}{k\rho_0}\mathrm{dn}(\frac{x}{k\rho_0}, k)$ is a strictly positive function. Here dn is the Jacobian elliptic function, the delta amplitude. As a consequence, the derivative of the field $\vec{n}$ at $\pm L$ has the norm $\frac{k'}{k\rho_0}$ and its orientation is φ dependent. We do not get the homogeneous boundary conditions of Sec. II. In fact, these conditions are not required anymore because the Hamiltonian density is defined on a finite support. It is important to note that, in the case of a finite support, the change of variables already used in Eq. (7) is no longer valid.

Next, we apply the nonlinear σ model on a sphere of radius unity. We look for spin distributions with a rotational symmetry around the z axis (the axis of the two poles), i.e., $\Phi = \varphi$ and $\theta = \theta(w)$, where w is the colatitude and φ is the azimuth. The Hamiltonian now reads

$$H_{\mathbf{sphere}} = 2\pi J \int_0^\pi \left[\left(\frac{d\theta}{dw} \right)^2 + \frac{\sin^2\theta}{\sin^2 w} \right] \sin w \, dw.$$

The only condition required for this Hamiltonian to converge is that the smooth field $\theta(w)$ goes to 0 when w goes to $0[\pi]$. The Euler-Lagrange equation for this Hamiltonian is

$$\frac{d^2\theta}{dw^2}\sin w + \frac{d\theta}{dw}\cos w = \frac{\sin\theta \ \cos\theta}{\sin w}, \tag{25}$$

and the self-dual equations are now given by the following expression:

$$\frac{d\theta}{dw}\sin w = \pm \ \sin\theta. \tag{26}$$

After integrating once Eq. (25), we get the following expression:

$$\left(\frac{d\theta}{dw}\right)^2 \sin^2 w = \sin^2\theta + C.$$

As we have dropped the homogeneous boundary conditions (because we are working on a finite support), the constant C may be nonzero. If it is zero, then the above expression reduces to the self-dual equation (26).

The simplest nontrivial solution of this first order differential equation [and of Eq. (25)] is the "hedgehog" solution,[5] given by $\theta = \pm w$ for Eq. (26) with the + sign and $\theta = \pi \pm w$ for Eq. (26) with the − sign. It belongs to the first homotopy class (Q=1). Therefore, $H^1_{\text{sphere}} = 8\pi J$.

We now explore a family of approximate solutions of Eq. (26). If L is a fixed real number, and if k satisfies $kK(k)Q = L$, then the solution (21) represents a Q soliton on a rigid cylinder of length $2L$, denoted by $F_Q(x)$. The function $F_Q \circ F_1^{-1}(w)$ represents a distribution of spins over the sphere which belongs to the Qth homotopy class. It can be written as

$$\theta(w) = \arccos\left[\text{sn}\left(\frac{k^*}{k}\text{sn}^{-1}(\cos w, k^*), k\right)\right], \qquad (27)$$

where $QkK(k) = k^*K(k^*) = L$. Note that when Q=1 (i.e., $k = k^*$), we recover the "hedgehog" solution.[5] If $k \neq k^*$, the solution (27) does not satisfy either Eq. (25) or Eq. (26), except in the limit $k \to 1$, i.e., when $L \to \infty$. In this case, the interaction term vanishes. Equation (25) is then satisfied and we recover $H_{\text{sphere}} = 8\pi JQ$. Only in this limit do we have at the poles $\lim_{L\to\infty} \frac{d\theta}{dw} = 0$, in accord with the homogeneous boundary conditions of Sec. II.

X. SPIN ANISOTROPY

Here we will show that the geometrical effect discussed above (Sec. VIII) can also be generated via a different physical mechanism. Many magnetic materials with anisotropic spin interaction are known as discussed below. We will again consider cylindrically symmetric distributions on a cylinder [Eq. (3)] and we introduce anisotropy in the spin-spin coupling. This anisotropy is expressed by the following perturbative Hamiltonian:

$$H_{\text{anisotropic}} = w_0^2 \sin^2\theta. \qquad (28)$$

The Hamiltonian of our problem is now the easy-axis Hamiltonian.[4] This anisotropy can physically model the difference between the radial and the axial Heisenberg coupling. It forces the alignment of spins with the axis of the cylinder and tends to "squeeze" the soliton in order to reduce the region of unaligned spins. Therefore, we will see that the anisotropy introduces a difference between the width of the magnetic distribution and the characteristic length of the support.

If we consider the rigid cylinder case, the Euler-Lagrange equation is again a sine-Gordon equation:

$$\frac{\partial^2\theta(x)}{\partial x^2} = \frac{1}{2}\left(\frac{1}{\rho_0^2} + w_0^2\right)\sin 2\theta. \qquad (29)$$

The simplest nontrivial solution is the soliton $\theta = 2\arctan e^{\frac{x}{R}}$, where R is defined as $\frac{1}{R^2} = \frac{1}{\rho_0^2} + w_0^2$. Then the new value of the first energy level becomes $H^1_{\text{easy-axis}} = 8\pi J\frac{\rho_0}{R}$.

It is important to notice that this result is explicitly dependent on the geometry of the support. Due to the anisotropy of the spin-spin coupling, the length of the soliton, i.e., the magnetic length, is renormalized and becomes R. Therefore, the characteristic length of the system (the radius of the cylindrical support) is no longer equal to the length of the magnetic distribution. As a consequence, the system cannot reach its minimum in energy anymore; there is a geometrical frustration. We will show that a perturbation of the geometry of an elastic cylinder can again lower the energy of the system.

The Euler-Lagrange equations generated by the complete Hamiltonian (easy-axis plus elastic terms), after elimination of the differential equation for θ, lead to

$$\chi_1\left[\rho''\rho + \frac{(\rho')^2}{2}\right] - 2\frac{\chi_2}{\rho_0^2}(\rho - \rho_0)(2\rho - \rho_0) = 4Jw_0^2\sin^2\theta.$$

This equation differs qualitatively from Eq. (14). Here, if $\theta(x)$ is nontrivial, then $\rho = \rho_0$ is not a solution anymore, even if the spin distribution satisfies the self-duality for the nonlinear σ model on a rigid cylinder (Eq. 11). Thus, the solution which minimizes the energy in any homotopy class, different from the zeroth, requires a deformation of the cylinder. As seen above, the new soliton length R is smaller than ρ_0. Therefore, we expect a pinch of the cylinder to locally restore the correspondence between the magnetic width and the characteristic length of the support. This will lower the energy. We can still write Bogomol'nyi's inequality, because both the elastic Hamiltonian density (12) and the anisotropic perturbative Hamiltonian (28) are positive. Thus, there still exist nontrivial magnetic distributions.

If we consider the quantities $\frac{Jw_0^2}{\chi_1}$ and $\frac{Jw_0^2}{\chi_2}$ as small compared to 1, we can take as ansatz the following pair of functions for the magnetic distribution:

$$\theta = 2\arctan e^{\frac{x}{R}} \quad \text{and} \quad \Phi = \varphi$$

(i.e., a nontrivial spin distribution belonging to the first homotopy class) and for $\rho(x)$ we choose $\rho = \rho_0 - \epsilon R\,\text{sech}^2(\frac{x}{R})$. Then, we find that

$$\mathcal{H}_{\text{easy-axis+el}} = 2J\frac{\rho_0}{R^2}\text{sech}^2\left(\frac{x}{R}\right) + [\rho_0\epsilon^2(\chi_1 + \chi_2) - Jw_0^2\epsilon R]\,\text{sech}^4\left(\frac{x}{R}\right) - \epsilon^2\mathcal{H}_\alpha - O(\epsilon^3),$$

where $\mathcal{H}_\alpha$ is positive. If we integrate over the whole support, we find the following expression:

$$H_{\text{easy-axis+el}} = 8\pi J\frac{\rho_0}{R} + \frac{8\pi}{3}[\rho_0 R\epsilon^2(\chi_1 + \chi_2) - Jw_0^2\epsilon R^2] - \epsilon^2 H_\alpha - O(\epsilon^3).$$

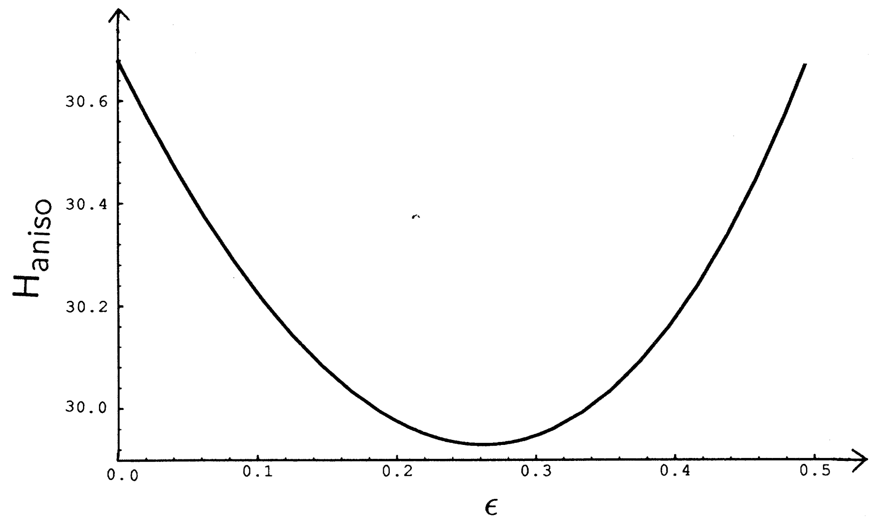

FIG. 4. Energy of the elastic easy-axis model as a function of ϵ, the depth of the pinch of the support, for the one-soliton solution when w_0=0.7 and J=1.

If we now choose $\epsilon = Jw_0^2 R/[2(\chi_1 + \chi_2)\rho_0]$ then we get $H_{\text{easy-axis+el}} < 8\pi\, J(\rho_0/R)$.

We plot in Fig. 4 the energy of the elastic, easy-axis model as a function of ϵ, the depth of the pinch of the support, when there is a nontrivial spin distribution (belonging to the first homotopy class) on an infinite cylinder. Again, note that there is a minimum in energy for a value of $\epsilon \neq 0$.

We have shown that in the rigid cylinder case, we cannot reach the minimum of the total Hamiltonian (easy-axis plus elastic term). There exists a triplet of nontrivial functions (ρ, θ, Φ) which is a solution of our problem. Thus we have proved that due to the anisotropy, a nontrivial magnetic distribution has an influence on the geometry of the cylinder.

This discussion is also valid when the spin distribution is the soliton lattice $\theta(x) = \arccos[\text{sn}(\frac{x}{kR}, k)]$, which is also an exact solution of Eq. (29). We can choose for $\rho(x)$ the following ansatz:

$$\rho(x) = \rho_0 - \epsilon R\, \text{cn}^2\left(\frac{x}{kR}, k\right).$$

If now we choose $\epsilon = Jk'^2/2[\chi_1 + \chi_2 k^2]\rho_0^2$, we are able to lower the total (easy-axis plus elastic) energy of the cylindrically symmetric support. Note that in both cases, the pinch appears exactly at those positions where spins are perpendicular to the cylinder ($\theta \equiv \frac{\pi}{2}[\pi]$).

XI. THE MOVING SOLITON ON A RIGID CYLINDER

In this section, we consider a dynamic solution on an infinite rigid cylinder, with an isotropic coupling. The calculation for the easy-axis case is the same, except ρ_0 is replaced with R in Eqs. (30) and (31) below. The continuum limit of the Lagrangian for antiferromagnetic Heisenberg spins is given by[4]

$$H_{\text{isotropic}} = J \iint_{\text{cylinder}} \left[\frac{1}{c^2}\left(\frac{\partial \vec{n}}{\partial t}\right)^2 + (\nabla \vec{n})^2\right] dS,$$

where c is the characteristic velocity of the spin-support system. With our notation and when looking for cylindrically symmetric solution [$\Phi = \varphi$, $\theta = \theta(x,t)$], this is written as

$$H_{\text{isotropic}} = 2\pi\rho_0 J \int_{-\infty}^{+\infty} \left[\frac{1}{c^2}\left(\frac{\partial\theta}{\partial t}\right)^2 + \left(\frac{\partial\theta}{\partial x}\right)^2 + \frac{\sin^2\theta}{\rho_0^2}\right] dx$$

and leads to the following Euler-Lagrange equation:

$$\frac{\partial^2\theta}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2\theta}{\partial t^2} = \frac{1}{2\rho_0^2}\sin 2\theta.$$

This is the geometrical form of the sine-Gordon equation which is Lorentz invariant. The solutions of this problem are the analytical functions given in the previous sections, with a Lorentzian boost applied, i.e. $x - vt/\sqrt{1-\left(\frac{v}{c}\right)^2} = \gamma(x - vt)$ is substituted for x:

$$\theta(x,t) = \arccos\left[\text{sn}\left(\frac{x - vt}{k\rho_0\sqrt{1-\left(\frac{v}{c}\right)^2}}, k\right)\right], \tag{30}$$

or, when k=1,

$$\theta(x,t) = 2\arctan\exp\left(\frac{x - vt}{\rho_0\sqrt{1-\left(\frac{v}{c}\right)^2}}\right). \tag{31}$$

In addition to the above, an exact moving two-soliton solution is also known.[11] However, this solution cannot be obtained by Lorentzian boost and we do not consider it here. The energy per soliton associated with solutions

(30) and (31) is $H_{\text{isotropic}} = \gamma 8\pi \frac{J}{k}[E(k) - k'^2 K(k)/2]$, i.e., γ times the static energy. In addition, we see that the width of a soliton has become $k\rho_0\sqrt{1-\left(\frac{v}{c}\right)^2}$, in agreement with the Lorentzian contraction of the length. When k=1, the solution (31) does not lead to geometrical frustration, even if the width of the soliton is smaller than the radius of the rigid cylinder. Indeed, this distribution reaches the topological minimum energy in the first homotopy class, which is γ times the energy in rest frame. The moving soliton sees a cylinder of radius $\frac{\rho_0}{\gamma}$ and as a consequence, in this case, the nonlinear σ model is "autotuned". We remark that when $v \to c$, this magnetic length tends to zero and the functions (30) and (31) become nonanalytic. Also, when working with an elastic cylinder and when v is equal to the velocity of the elastic waves, we reach the shock wave regime: there is a coupling between the moving magnetic solitons and the elastic waves.[20]

XII. CONCLUSION

We have considered a classical Heisenberg model on an infinite cylinder and found that both a periodic topological spin soliton and an anisotropy of the spin-spin coupling produce a pinch of the cylinder. This is a consequence of the violation of the self-duality equations, which on the contrary are satisfied in the isotropic single soliton case. For the periodic case, violation of self-duality results from interaction between the solitons and generates geometric frustration. Here we note the analogy with the Peierls instability in low-dimensional, interacting electron-phonon systems, although the origin of the periodic distortion is quite different. In particular, the spin-spin coupling constant J in our problem does not depend on the geometrical deformation.

The sine-Gordon equation also supports an exact dynamical solution, namely, the breather.[21] In this case one would expect temporal (not spatial) pinching of the cylinder in the region of the breather. For periodic boundary conditions, the cylinder can be compactified into a torus. We have also considered the nonlinear σ model on a torus but do not discuss the details here. In this paper we confined ourselves to the cylindrically symmetric case. Removing this constraint would lead to much richer magnetoelastic effects including spin domain walls.

The geometrical effect predicted here may be observable with the use of ultrasonic techniques[22] in cylindrically wrapped thin films of magnetic materials, specifically layered 2D Heisenberg magnets. First measure the normal modes of an undeformed, nonmagnetic, elastic cylinder. Then apply a thin coating of a magnetic material on the cylinder. The change in normal modes of the cylinder caused by magnetoelastic pinching can be measured by resonant ultrasound and compared to calculated values.[23] Similarly, attenuation and phase shift associated with a solitonic deformation can also be measured. This is accomplished by measuring the change in travel time of an ultrasonic wave passing through the cylinder using interferometry.[22]

Some specific examples of layered 2D Heisenberg magnetic materials are $(C_nH_{2n+1}NH_3)_2MX_4$ and $[NH_3(CH_2)NH_3]MX_4$ for $n \leq 16$, where M = Cr, Mn, Fe, Cu, Cd and X = Cl, Br.[24] Other examples, including easy-axis anisotropy, are K_2CuF_4, Ca_2MnO_4, Rb_2FeF_4, etc.,[24] magnetic Langmuir-Blodgett films of manganese stearate $Mn(C_{18}H_{35}O_2)_2$,[25] and possibly, recently synthesized carbon nanotubules[26,27] with appropriate magnetic coatings. Interestingly, periodic shrinking (a stable, finite amplitude, peristaltic state) of tubular fluid membranes was recently observed,[28] although for a different physical reason.[29]

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