

J_1 - J_2 quantum Heisenberg antiferromagnet on the triangular lattice: A group-symmetry analysis of order by disorder

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On the triangular lattice, for J_2/J_1 between $\frac{1}{8}$ and 1, the classical Heisenberg model with first- and second-neighbor interactions presents four-sublattice ordered ground states. Spin-wave calculations of Chubukov and Jolicoeur and Korshunov suggest that quantum fluctuations select amongst these states a collinear two-sublattice order. From theoretical requirements, we develop the full symmetry analysis of the low-lying levels of the spin- $\frac{1}{2}$ Hamiltonian in the hypotheses of either a four- or a two-sublattice order. We show for the exact spectra of periodic samples ($N = 12, 16$, and 28) how quantum fluctuations select the collinear order from the four-sublattice order.

I. INTRODUCTION

Symmetry breaking and the selection of a particular macroscopic state amongst many degenerate ones result in part from infinitesimal external causes. In the case of planar Néel order, the plane of antiferromagnetic ordering, for example, is chosen by the environment, whereas the possibility of antiferromagnetic symmetry breaking and the nature of the antiferromagnetic order are intrinsic and deeply rooted in the spectral properties of the low-lying levels of the Hamiltonian on a given lattice.¹⁻³ Two features have to be considered in this respect: “the ground state” and the first excitations of the system. In the past, interest has mainly been focused on the “first excitations”: the so-called antiferromagnetic magnons. The interest in the ground state has been limited to the measurement of the order parameter modulus. The approach of this problem through exact diagonalizations on small samples has led us to focus on the nature of this ground state: the eigenstates of the Heisenberg Hamiltonian on a finite lattice of N sites are eigenvectors of total spin S and, in all presently studied cases, the absolute ground-state is $S = 0$ or $S = \frac{1}{2}$ (depending on the number of sites in the sample). If we consider the even site samples, the $S = 0$ absolute ground state is spherically symmetric: it does not break the rotational symmetry of the Hamiltonian and as such is insufficient to describe a Néel antiferromagnetic state. As underlined by Anderson in 1952,¹ the Néel symmetry-breaking state arises from a linear combination of a macroscopic number of levels $\{\tilde{E}\}$ with different S values which in the thermodynamic limit collapse to the absolute ground state faster than the

softest magnons.

This set of levels $\{\tilde{E}\}$ —called QDJS for quasi degenerate joint states in Refs. 2 and 3—has specific symmetry and dynamical properties which embody the characteristics of the symmetry-breaking phase. Let us recall that for a finite solid, the low-lying levels are eigenstates of the total momentum and they indeed collapse to the ground state in the thermodynamic limit faster than the softest phonons. This is a wave packet of these eigenstates that localizes the center of mass. Here, in an ordered antiferromagnet, the multiplicity $\{\tilde{E}\}$ is associated with the dynamics of the order parameter. In other words, the knowledge of the symmetry and dynamical properties of this set of eigenstates yields the nature of the ordered phase. For the case of a $2d$ Néel phase, this set of levels $\{\tilde{E}\}$ collapses to the ground states as N^{-1} , that is, faster than other quantities, in particular, faster than the softest magnons which converge to the ground state as $N^{-1/2}$.⁴ Understanding the symmetry and dynamical properties of these low-lying levels of the Heisenberg Hamiltonian on the triangular lattice leads to a consistent picture of an ordered ground state with three-sublattice Néel order; this reconciles spin-wave theories and exact diagonalizations approaches.^{2,3}

More subtle symmetry breakings still exist when two or more different kinds of order are classically degenerate. In the pure classical case, Villain *et al.*⁵ have shown that thermal fluctuations could select a specific order. The selected order has softer excitation modes and therefore, for a given low energy, a larger density of excitations and a larger entropy: Villain *et al.*⁵ called this mechanism “order by disorder.” This concept has been

rather fruitful for the studies of classical and quantum antiferromagnets.^{6,7}

The existence of competing interactions is indeed the main cause of classical ground-state degeneracy. As a generic example, one can consider the so-called J_1 - J_2 model on a triangular lattice with two competing antiferromagnetic interactions. This Hamiltonian reads

$$\mathcal{H} = 2J_1 \sum_{\langle i,j \rangle} \mathbf{s}_i \cdot \mathbf{s}_j + 2J_2 \sum_{\langle\langle i,k \rangle\rangle} \mathbf{s}_i \cdot \mathbf{s}_k, \quad (1)$$

where J_1 and $J_2 = \alpha J_1$ are positive and the first and second sums run on the first and second neighbors, respectively. The classical study of this model has been developed by Jolicoeur *et al.*⁸ They have shown that for small α ($\alpha < \frac{1}{8}$) the ground state corresponds to a three-sublattice Néel order with magnetizations at 120° from each other, whereas for $\frac{1}{8} < \alpha < 1$, there is a degeneracy between a two-sublattice Néel and a four-sublattice Néel order (see Fig. 1). Chubukov and Jolicoeur⁹ and Korshunov¹⁰ have then shown that quantum fluctuations (evaluated in a spin-wave approach) could, like thermal ones, lift this degeneracy of the classical ground states and lead to a selection of the collinear state (see Fig. 1). As usual for spin- $\frac{1}{2}$ systems, the validity of the spin-wave theory has to be checked. The first study of the exact spectrum of Eq. (1) done by Jolicoeur *et al.*⁸ was not incompatible with this conclusion, but was insufficient to yield it immediately. Deutscher and Everts¹¹ found good agreement between spin-wave results for the collinear state and exact diagonalizations but their sample geometries were too restricted to fully accommodate the four-sublattice order. We show in this paper that a study of the complete dynamical “ground-state multiplicity” leads to this conclusion.

In order to understand the origin of this thermodynamical multiplicity we first study exactly solvable models which display either four-sublattice order or collinear order (Sec. II). Then, on exact spectra of small samples, we show how quantum fluctuations of increasing wavelength select the collinear order (Sec. III).

II. EXACT SOLVABLE QUANTUM MODELS OF ORDERED SYSTEMS

These model Hamiltonians are obtained by retaining the Fourier components of the Heisenberg Hamiltonian which are compatible either with the two- or the four-sublattice order. In Fourier components, the Heisenberg Hamiltonian [Eq. (1)] reads

$${}^4E(S, S_A, S_B, S_C, S_D) = \frac{8}{N}(J_1 + J_2)[S(S+1) - S_A(S_A+1) - S_B(S_B+1) - S_C(S_C+1) - S_D(S_D+1)], \quad (4)$$

where the quantum numbers S_A, S_B, S_C, S_D run from 0 to $N/8$ and the total spin results from a coupling of the four spins S_A, S_B, S_C, S_D .

The low-lying levels of Eq. (4) are obtained for $S_A =$

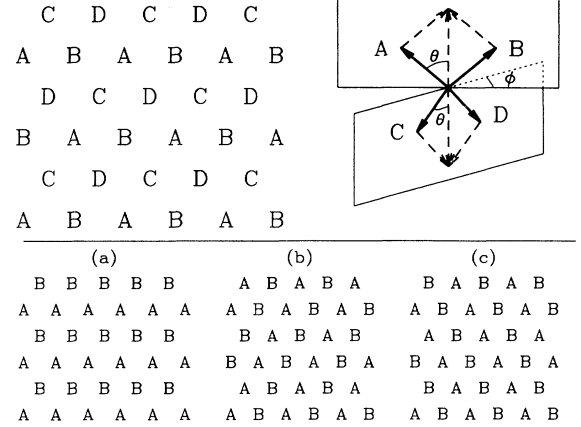


FIG. 1. Top: four-sublattice classical ground state. Spins in the sublattices A and B , as well as spins in C and D , make an angle 2θ . The plane of the spins of A and B makes an angle ϕ with the plane of the spins of C and D . Bottom: the collinear solutions with the three possible arrangements (in this case, classical spins in sublattices A and B are antiparallel).

$$\mathcal{H} = 6J_1 \sum_{\mathbf{k}} \mathbf{S}_{\mathbf{k}} \cdot \mathbf{S}_{-\mathbf{k}} \left[\gamma_{\mathbf{k}} + \frac{\alpha}{3} [\cos \mathbf{k} \cdot (2\mathbf{u}_1 + \mathbf{u}_2) + \cos \mathbf{k} \cdot (\mathbf{u}_1 + 2\mathbf{u}_2) + \cos \mathbf{k} \cdot (\mathbf{u}_2 - \mathbf{u}_1)] \right], \quad (2)$$

where $\mathbf{S}_{\mathbf{k}} = \frac{1}{\sqrt{N}} \sum_i \mathbf{s}_i \exp i\mathbf{k} \cdot \mathbf{r}_i$ and $\gamma_{\mathbf{k}} = 1/3 \sum_{\mu} \cos \mathbf{k} \cdot \mathbf{u}_{\mu}$ ($\mathbf{u}_{\mu}$ are three vectors at 120° from each other, connecting a given site to first neighbors). In Eq. (2), the $\mathbf{k}$ components associated with the $\mathbf{k}$ vectors which keep the sublattices invariant provide the essential features of the dynamics of the order parameter. We successively study the case with four-sublattice order and the case with two-sublattice collinear order.

The four vectors which keep the four-sublattice order invariant are $\mathbf{k} = \mathbf{0}$ and the three middles of the Brillouin-zone boundaries (called in the following $\mathbf{k}_I$, $\mathbf{k}_H$, and $\mathbf{k}_G$). In this study, we will exclusively consider finite samples with $N = 4p$ sites and with periodic conditions: these samples do not frustrate the four- (nor the two-) sublattice order and they effectively present the above-mentioned $\mathbf{k}$ vectors in their Brillouin zone. It is straightforward to write the contribution of these Fourier components to $\mathcal{H}$ in the form

$${}^4\mathcal{H}_0 = \frac{8}{N}(J_1 + J_2)(\mathbf{S}^2 - \mathbf{S}_A^2 - \mathbf{S}_B^2 - \mathbf{S}_C^2 - \mathbf{S}_D^2), \quad (3)$$

where $\mathbf{S}$ is the total spin operator and the $\mathbf{S}_{\alpha}$ are the total spin operator of each sublattice. ${}^4\mathcal{H}_0, \mathbf{S}^2, S_z, \mathbf{S}_A^2, \mathbf{S}_B^2, \mathbf{S}_C^2$, and $\mathbf{S}_D^2$ form a set of commuting observables. The eigenstates of ${}^4\mathcal{H}_0$ have the following energies:

$$S_B = S_C = S_D = N/8:$$

$${}^4E_0(S) = -\frac{J_1 + J_2}{2}(N + 8) + \frac{8}{N}(J_1 + J_2)S(S + 1). \quad (5)$$

These states, which have maximal sublattice magnetizations $S_A^2 = S_B^2 = S_C^2 = S_D^2 = (N/8 + 1)N/8$, are the *rotationally invariant projections of the bare Néel states with four sublattices*. This is the single physical origin of all properties of $\{^4\tilde{E}\}$. These levels have an energy collapsing to the absolute ground state as N^{-1} justifying the name of tower of states or “ground-state multiplicity” given to $\{^4\tilde{E}\}$. In this exactly solvable model there are no quantum fluctuations to renormalize the sublattice magnetization; quantum fluctuations will be introduced by the discarded part of the Hamiltonian [Eq. (2)].

As we will now show, this multiplicity $\{^4\tilde{E}\}$ can be entirely and uniquely described by its symmetry proper-

ties under spin rotation and transformation of the space group of the lattice. Let us begin by the SU(2) properties induced by the fact that these states represent the coupling of four identical spins. The degeneracy of each S level is $(2S + 1)N_S$ where the factor $(2S + 1)$ comes from the magnetic degeneracy and N_S is the number of different couplings of four spins, each of length $N/8$, giving a total spin S . This number is readily evaluated by using the decomposition of the product of four spins $n/8$ representations of SU(2) ($\mathcal{D}^{N/8}$):

$$\{^4\tilde{E}\} = \mathcal{D}^{N/8} \otimes \mathcal{D}^{N/8} \otimes \mathcal{D}^{N/8} \otimes \mathcal{D}^{N/8} \quad (6)$$

in spin S irreducible representations ($\mathcal{D}^S$). One obtains

$$\begin{aligned} N_S &= \frac{1}{2} \left(-3S^2 + S(N + 1) + 2 + \frac{N}{2} \right) \quad \text{for } S \leq \frac{N}{4}, \\ &= \frac{1}{2} \left(\frac{N}{2} - S + 1 \right) \left(\frac{N}{2} - S + 2 \right) \quad \text{for } S \geq \frac{N}{4} + 1. \end{aligned} \quad (7)$$

Note that this degeneracy depends both on S and N and not only on the total spin S as is the case for two- or three-sublattice problems. In fact, in the latter two cases, which describe Néel order on a square or triangular lattice, the objects to be considered stem from the coupling of two or three angular momenta: they have perfect counterparts in the orbital three-dimensional world which are rigid rotators and tops with well-known quantum numbers, depending only on S . More generally, a Néel order with p sublattices on a finite sample of N spins gives rise to a “ground-state multiplicity” of the order of N^p .

The determination of the space symmetries of these eigenstates allows a complete specification of $\{^4\tilde{E}\}$. The four-sublattice order is invariant in a two-fold rotation ($\mathcal{R}_\pi$): thus the eigenstates of $\{^4\tilde{E}\}$ belong to the trivial representation of C_2 . As it arises from the coupling of four identical spins, this subset of levels forms a representation space of S_4 , the permutation group of four elements. The eigenstates of $\{^4\tilde{E}\}$ could thus be labeled by the irreducible representations of S_4 (see Table I).

TABLE I. Character table of the permutation group S_4 . The first line indicates classes of permutations. The second line gives an element of the space symmetry class corresponding to the class of permutation. These space symmetries are t , the one-step translation ($A \rightarrow B$), $\mathcal{R}_{2\pi/3}$ (resp. $\mathcal{R}'_{2\pi/3}$) the threefold rotation around a site of the D (respectively, B) sublattice, and σ , the axial symmetry keeping invariant C and D . N_{el} is the number of elements in each class.

S_4	I	$(A, B)(C, D)$	(A, B, C)	(A, B)	(A, B, C, D)
$\mathcal{G}$	I	t	$\mathcal{R}_{2\pi/3}$	σ	$\mathcal{R}'_{2\pi/3}\sigma$
N_{el}	1	3	8	6	6
Γ_1	1	1	1	1	1
Γ_2	1	1	1	-1	-1
Γ_3	2	2	-1	0	0
Γ_4	3	-1	0	1	-1
Γ_5	3	-1	0	-1	1

Indeed, the complete analysis of all the eigenstates of Eq. (1) is usually done through the more general point of view of the space group of the lattice. But it is straightforward to show that in the four-sublattice subset of solutions, each element of the space group maps onto a permutation of S_4 : one step translations map onto products of transpositions as $(A, B)(C, D)$, threefold rotations onto circular permutations of three sublattices (A, B, C) and so on. The complete mapping of the space symmetries of the four-sublattice order onto the permutations of S_4 is given in Table I together with the character table of S_4 . Each irreducible representation of S_4 can thus be characterized in terms of its space symmetry properties. As noted above they are all invariant in $\mathcal{R}_\pi$. Analysis of the properties under translation shows that Γ_1 , Γ_2 , and Γ_3 correspond to the wave vector $\mathbf{0}$, whereas Γ_4 and Γ_5 have a wave vector $\mathbf{k}_H$, $\mathbf{k}_I$, or $\mathbf{k}_J$. Γ_1 and Γ_2 are invariant under C_3 , whereas Γ_3 is associated with the two complex representations of this same group. Finally, Γ_1 and Γ_4 are even under axial symmetry whereas Γ_2 and Γ_5 are odd. The number of replicas of Γ_i that should appear for each S is then computed in the S, M_S subspace with the help of the trace of the permutations of S_4 :

$$n_{\Gamma_i}^{(S)} = \frac{1}{24} \sum_k \text{Tr}(R_k|_S) \chi_i(k) N_{\text{el}}(k), \quad (8)$$

where R_k is an element of the class k of S_4 , $N_{\text{el}}(k)$ is the number of elements in this class, and $\chi_i(k)$ is the character of the class k in the irreducible representation Γ_i (see Table I). The values of the traces for a given total spin S are then found as

$$\begin{aligned} \text{Tr} \left((A, B, C) \Big|_S \right) &= \text{Tr} \left((A, B, C) \Big|_{M_S=S} \right) \\ &\quad - \text{Tr} \left((A, B, C) \Big|_{M_S=S+1} \right). \end{aligned} \quad (9)$$

In each M_S subspace of $\{^4\tilde{E}\}$, it is straightforward to find the trace of the elements of S_4 :

$$\begin{aligned} \text{Tr}(I_d|_{M_S}) &= \sum_{t,v,x,y=-N/8}^{N/8} \delta_{t+v+x+y, M_S}, \\ \text{Tr}((A, B)(C, D)|_{M_S}) &= \sum_{t,v=-N/8}^{N/8} \delta_{2t+2v, M_S}, \\ \text{Tr}((A, B, C)|_{M_S}) &= \sum_{t,v=-N/8}^{N/8} \delta_{3t+v, M_S}, \\ \text{Tr}((A, B)|_{M_S}) &= \sum_{t,v,x=-N/8}^{N/8} \delta_{2t+v+x, M_S}, \\ \text{Tr}(A, B, C, D)|_{M_S} &= \sum_{t=-N/8}^{N/8} \delta_{4t, M_S}, \end{aligned} \quad (10)$$

where t, v, x, y are the z components of the total spin of each sublattice (constrained to vary between $N/8$ and $-N/8$) and $\delta_{i,j}$ denotes the Kronecker symbol. Using Eqs. (8), (9), and (10) one readily obtains the number of occurrences of each Γ_i for each S subset of $\{^4\tilde{E}\}$ (Table II). We have thus obtained the complete determination (all quantum numbers, and all the degeneracies) of the family of low-lying levels describing the ground-state multiplicity of the four-sublattice Néel solutions.

Let us now consider the collinear solutions (Fig. 1). They are particular solutions of the four-sublattice case and we will rapidly go through the same scheme of analysis, indicating mainly the new points. The two vectors which keep the two sublattices invariant are $\mathbf{0}$ and the middle of one side of the Brillouin zone [the vectors $\mathbf{k}_I$, $\mathbf{k}_H$, and $\mathbf{k}_G$ correspond, respectively, to the collinear solutions (a), (b), and (c) in Fig. 1]. Extracting a specific set of two wave vectors from Eq. (2), we find the following contribution to the total Hamiltonian:

$$^2\mathcal{H}_0 = \frac{8}{N}(J_1 + J_2) \left[S^2 - \frac{1}{2}(S_\alpha^2 + S_\beta^2) \right]. \quad (11)$$

The corresponding low-energy spectrum for $S_\alpha = S_\beta = N/4$ is

$$^2E_0(S) = -\frac{J_1 + J_2}{2}(N + 8) + \frac{8}{N}(J_1 + J_2)S(S + 1) \quad (12)$$

and is degenerate with the four-sublattice low-energy spectrum [see Eq. (5)]. Here, the two sublattices have maximal spins $S_\alpha = S_\beta = N/4$. These new solutions arise from the symmetric coupling of the spins of two sublattices of the four-sublattice order: $\mathbf{S}_\alpha = \mathbf{S}_\mathbf{A} + \mathbf{S}_\mathbf{B}$ or $\mathbf{S}_\alpha = \mathbf{S}_\mathbf{A} + \mathbf{S}_\mathbf{C}$ or $\mathbf{S}_\alpha = \mathbf{S}_\mathbf{A} + \mathbf{S}_\mathbf{D}$ with the counterparts for $\mathbf{S}_\beta$. As there are three ways to do this coupling, the collinear solutions have a Z_3 degeneracy. The representation space is thus the sum of three products $\mathcal{D}^{N/4} \otimes \mathcal{D}^{N/4}$. It is not a direct sum since $\mathcal{D}^{N/4}(A, B) \otimes \mathcal{D}^{N/4}(C, D)$ and $\mathcal{D}^{N/4}(A, C) \otimes \mathcal{D}^{N/4}(B, D)$ have in common the same (symmetric) irreducible representation with a total spin $N/2$. On a N sample, the representation space of the ground state of the collinear solution is

$$\{^2\tilde{E}\} = 3\mathcal{D}^{S=0} \oplus 3\mathcal{D}^{S=1} \oplus \dots \oplus 3\mathcal{D}^{S=N/2-1} \oplus \mathcal{D}^{S=N/2}. \quad (13)$$

The degeneracy is thus $3(2S + 1)$ for all S values except for $S = N/2$ where it is $(2S + 1)$.

As for the four-sublattice order, the space-group analysis is done as for the two-sublattice order, but the number of occurrences of each irreducible representations Γ_i is now different since the space $\{^2\tilde{E}\}$ is smaller than $\{^4\tilde{E}\}$. The calculation could be done along the same lines as for the four-sublattice order. The problem, however, is much simpler because for each S value there are only three replicas of $\mathcal{D}^S$ arising from the Z_3 group [Eq. (13) and Fig. (1)]. This allows direct computation of the permutation traces in each S subset of $\{^2\tilde{E}\}$. Using the coupling rules of two angular momenta (and in particular the fact that the S eigenstate resulting from the coupling of two integer spins changes sign as $(-1)^S$ with the interchange of the two parent spins) one obtains (for $S \neq N/2$):

TABLE II. Number of occurrences $n_{\Gamma_i}(S)$ of each irreducible representation Γ_i with respect to the total spin S . For $N = 28$, n_{Γ_1} , and n_{Γ_2} as well as n_{Γ_4} and n_{Γ_5} have been added because this sample does not present any axial symmetry.

$N = 16$									
S		0	1	2	3	4	5	6	7
$n_{\Gamma_1}(S)$		1	0	2	0	2	1	1	0
$n_{\Gamma_2}(S)$		0	0	1	0	0	0	0	0
$n_{\Gamma_3}(S)$		2	0	2	1	2	0	1	0
$n_{\Gamma_4}(S)$		0	2	2	3	2	2	1	1
$n_{\Gamma_5}(S)$		0	2	1	2	1	1	0	0

$N = 28$															
S		0	1	2	3	4	5	6	7	8	9	10	11	12	13
$n_{\Gamma_1}(S) + n_{\Gamma_2}(S)$		2	0	5	1	5	3	4	2	4	1	2	1	1	0
$n_{\Gamma_3}(S)$		3	0	4	2	5	2	5	2	3	1	2	0	1	0
$n_{\Gamma_4}(S) + n_{\Gamma_5}(S)$		0	7	6	11	9	12	9	10	6	6	3	3	1	0

$$\begin{aligned}
\text{Tr}(I_d|_S) &= 3, \\
\text{Tr}((A, B)(C, D)|_S) &= 1 + 2(-1)^S, \\
\text{Tr}((A, B, C)|_S) &= 0, \\
\text{Tr}((A, B)|_S) &= 1, \\
\text{Tr}((A, B, C, D)|_S) &= (-1)^S.
\end{aligned} \tag{14}$$

Therefore, the collinear solution is simply characterized by Γ_1 and Γ_3 for even S and Γ_4 for odd S .

From these equations [(8), (9), (10), and (14)], the symmetries of all states of the tower are fully determined both for the four-sublattice order $\{^4\tilde{E}\}$ and for the collinear order $\{^2\tilde{E}\}$.

Going back now to the original (J_1 - J_2) model, we have to account for quantum fluctuations generated by the discarded part of $\mathcal{H}$. This perturbation does not commute with sublattice total spins and consequently reduces the sublattice magnetization. Nevertheless, it preserves all the symmetries of the Néel state and thus also the ones of the levels of $\{^4\tilde{E}\}$ or $\{^2\tilde{E}\}$. Then the question is: do quantum fluctuations conserve qualitatively the dynamics of these levels or not? If these levels remain the low-lying ones of the exact spectra with overall dynamics qualitatively similar to that of the bare Néel state [Eqs. (5) and (12)], then the quantum model will be ordered at $T = 0$. By qualitatively, we mean that the leading term of the energy of the exact subset $\{E\}$ behaves as $\beta \frac{8}{N} (J_1 + J_2) S(S+1)$, where β is a renormalization factor. This factor is related to the spherical homogeneous susceptibility of the sample,³ even if, in general, the tensor of susceptibilities is not spherical because quantum fluctuations lift the degeneracies of $\{^4\tilde{E}\}$ of the exactly solvable model.

III. EXACT SPECTRA OF SMALL PERIODIC SAMPLES

We have determined the low- (and high-) energy levels of the J_1 - J_2 Hamiltonian in each irreducible representation of $SU(2)$ and of the space group of the triangular lattice for small periodic samples with $N=12, 16$, and 28 . The spectra are displayed in Figs. 2 and 3. We directly see in the upper parts of these figures the set of QDJS (“ground-state multiplicity”) well separated from the set of levels corresponding to the one-magnon excitations. We have verified that the QDJS form a set of levels with the exact properties of the above defined $\{^4\tilde{E}\}$ subset. The action of quantum fluctuations could then be read in the lower parts of the figures. As expected, quantum fluctuations lift the degeneracies which are present in the exactly solvable model and stabilize the eigenstates with the lower S values. Nevertheless, the low-lying energies per site still group around a line of equation $E_\infty + 8\beta S(S+1)(J_1 + J_2)/N^2$ with $\beta = 1.004$ (resp. 1.055) for $N = 16$ (resp. 28). The number and space symmetries of these levels for each S and N value are exactly those required by the above analysis of the four-sublattice Néel order. Moreover, it is already visible on the $N = 16$ sample and quite clear on the $N = 28$ sample that a dichotomy appears in this family (see Fig. 4).

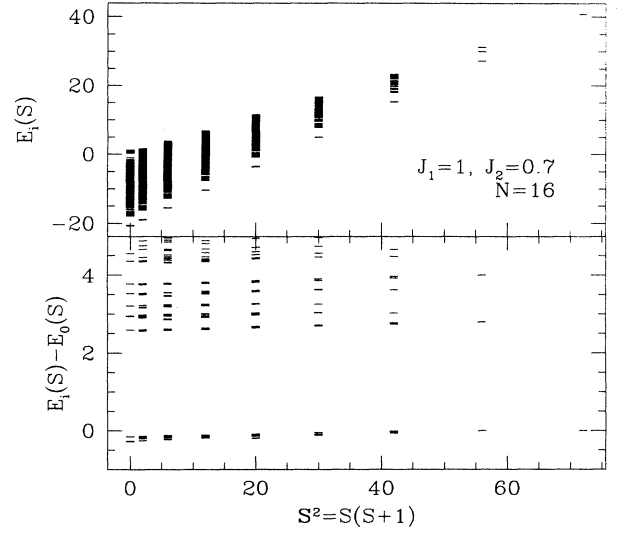


FIG. 2. Top: complete spectrum of the $N = 16$ periodic sample with respect to S^2 for $J_2/J_1 = 0.7$. Bottom: enlargement of the difference between the exact spectrum and the energy of the low-lying levels of the model Hamiltonian [Eq. (5) or Eq. (12)]. The ground-state multiplicity $\{^4\tilde{E}\}$ is well separated from the magnons.

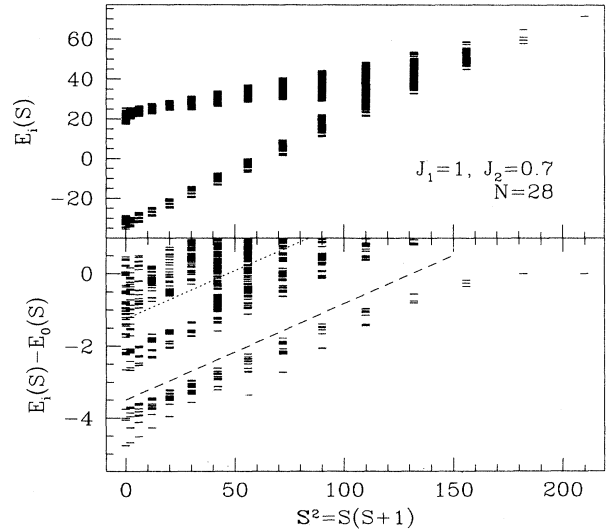


FIG. 3. Partial spectrum of the $N = 28$ periodic sample (same legend as for Fig. 2). Bottom: the tower of states of the four-sublattice order $\{^4\tilde{E}\}$ lays under the dashed line. Above appear the first magnons. Above the dotted line are represented the first excited homogeneous states. In the magnon multiplicity ($\mathbf{k} \neq 0, \mathbf{k}_H, \mathbf{k}_I$, or $\mathbf{k}_J$), for $S \leq 5$, only the five lowest states of each irreducible representation have been computed.

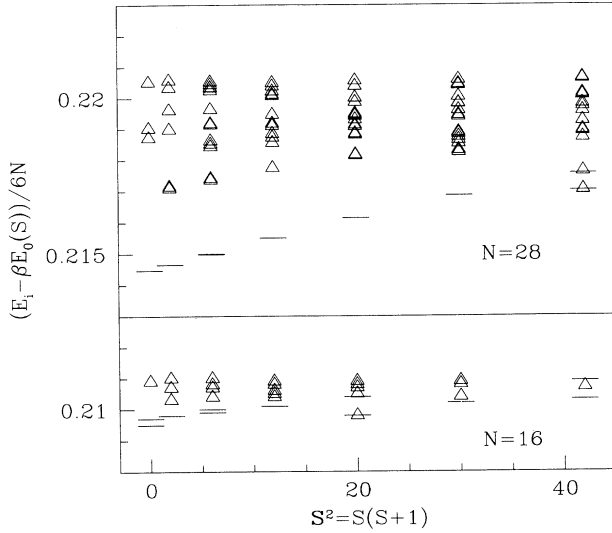


FIG. 4. Enlargement of the $N = 16$ and $N = 28$ QDJS. A global contribution $\beta E_0(S)$ is subtracted from the exact spectrum. This contribution describes the overall dynamics of the order parameter in this finite sample, β measures the renormalization of this dynamics by quantum fluctuations (see text and Ref. 3). The bars represent eigenstates which belong both to $\{^2\tilde{E}\}$ and $\{^4\tilde{E}\}$. The triangles indicate states which belong to $\{^4\tilde{E}\}$ but not to $\{^2\tilde{E}\}$. We see that, with increasing sizes, the tower of states of the collinear order separates from the four-sublattice order. For $N = 28$, the two states of $\{^2\tilde{E}\}$ with even S are quasidegenerate and cannot be distinguished at the scale of the figure.

The lowest levels of this tower of states appear to be Γ_1, Γ_3 , or Γ_4 representations depending on the parity of the total spin. They precisely build the family $\{^2\tilde{E}\}$ of isotropic projections of the collinear solutions given above [Eq. (14)].

We see in Fig. 4 that the difference between the energy

per bond of the collinear states and that of the other states of the four-sublattice order roughly increases by a factor of 4 from the $N = 16$ to the $N = 28$ sample. This strongly suggests that the four-sublattice order will disappear in the thermodynamic limit and only the collinear order will persist. This result supports the conclusion of the spin-wave expansion⁸⁻¹⁰ concerning the selection of the collinear state in the J_1 - J_2 model for $\frac{1}{8} < \alpha < 1$ for spins $\frac{1}{2}$.

IV. CONCLUDING REMARKS

It appears from the two situations that we have studied (triangular Heisenberg model^{2,3} and this model) that the symmetry and dynamical analysis of the low-lying levels of a Hamiltonian likely to exhibit ordered solutions give a rather straightforward answer to the kind of order to be expected.¹² The method is rapid, powerful, and unbiased: it does not require any *a priori* symmetry-breaking choice: if a specific order is selected, one should see it directly on the exact spectra. Moreover, as it is essentially exact, there are no questions relative to the convergence of the expansion as in the spin-wave approach. On the other hand, as the sizes amenable to computation are limited, there is, in the exact approach, a cutoff of the long-wavelength fluctuations. Results so obtained should thus be examined in the light of a finite-size scaling analysis. The present work nevertheless shows that it is not necessary to invoke quantum fluctuations with very long wavelengths to select the collinear order.

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¹ P. W. Anderson, Phys. Rev. **86**, 694 (1952).

² B. Bernu, C. Lhuillier, and L. Pierre, Phys. Rev. Lett. **69**, 2590 (1992).

³ B. Bernu, P. Lecheminant, C. Lhuillier, and L. Pierre, Phys. Rev. B **50**, 10048 (1994).

⁴ The transition from an ordered ground state to a quantum critical situation would be signed by an $N^{-1/2}$ collapse of this family; P. Azaria, B. Delamotte, and D. Mouhanna, Phys. Rev. Lett. **70**, 2483 (1993).

⁵ J. Villain, R. Bidaux, J. P. Carton, and R. Conte, J. Phys. (Paris) **41**, 1263 (1980).

⁶ E. F. Shender, Sov. Phys. JETP **56**, 178 (1982).

⁷ C.L. Henley, Phys. Rev. Lett. **62**, 2056 (1989).

⁸ T. Jolicoeur, E. Dagotto, E. Gagliano, and S. Bacci, Phys. Rev. B **42**, 4800 (1990).

⁹ A. Chubukov and T. Jolicoeur, Phys. Rev. B **46**, 11137 (1992).

¹⁰ S.E. Korshunov, Phys. Rev. B **47**, 6165 (1993).

¹¹ R. Deutscher and H. U. Everts, Z. Phys. B **93**, 77 (1993).

¹² On the contrary, no low-lying states are found on the Kagomé lattice. In particular, the states corresponding to the $\sqrt{3} \times \sqrt{3}$ order are not the states of lower energy; B. Bernu, P. Lecheminant, C. Lhuillier, and L. Pierre, Phys. Scr. **49**, 192 (1993); P. Lecheminant, B. Bernu, C. Lhuillier, and L. Pierre (unpublished).