Interactions of collective excitations with vortices in superfluid systems

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We investigate the interactions of collective excitations with vortices in superfluid systms, including ^{4}He and superconductors. The dynamical equations are obtained by the aid of the many-body wave function and the density-density correlation function. The scattering cross section of collective excitations with a vortex is calculated in the Born approximation (valid at long wavelengths), and is expressed in terms of the Feynman model spectrum $\omega(q)$ of the collective excitations by the simple and general formula $\sigma=(\pi/2)\{q/[\omega'(q)]^2\}\cot^2(\theta/2)$, where q is the wave number of the excitation, and θ is the scattering angle. At short wavelengths, the classical equations of motion are derived.

I. INTRODUCTION

The motion of vortices is thought to be the main source of dissipation in superfluid systems. At finite temperatures, the existence of collective excitations and their interactions with the vortex lines create the phenomenon known as mutual friction.[1–4] The phenomenological parameters of mutual friction can be derived from the two-dimensional (2D) scattering cross section of the collective excitations.[5,6]

Different methods have been used in the literature for the calculation of the scattering cross sections. One method is based on the classical dynamics of phonons[7] or rotons[1,4,5] treated as particles. In the presence of a vortex, the Hamiltonian for such a particle is given, for instance, by $H=\epsilon_0(\mathbf{p})+\mathbf{V}(r)\cdot\mathbf{p}$ where $\epsilon_0(\mathbf{p})$ is the energy spectrum of the collective excitations, and the second term represents the Doppler shift of energy due to the velocity field $\mathbf{V}(r)$ associated with the vortex. Effects of the vortex core and density modulations have also been considered. Another method is to treat such a particle quantum mechanically by introducing a quantum wave function for it, using the operator version of the above Hamiltonian to describe the dynamics. This method has been applied to the roton case, with the cross section calculated in the Born approximation.[9] There is then the more conventional method based on the hydrodynamic wave equations. Born approximation[8,5] and phase-shift analysis[10] have been used for the calculation of phonon-scattering cross sections.

We observed that the phenomenon can be considered in more general terms. Besides ^{4}He, there are other systems, such as superconductors, quantum Hall systems,[11] quantum spin systems, that support vortices and collective excitations with different microscopic structure and ground-state properties. The purpose of this paper is to establish the relation between the different methods, and to present a calculation of the scattering cross section that does not refer to the details of the microscopic structure of the systems. For long-wavelength scattering, the cross section may be calculated using the Born approximation, and we found it to be given by the simple and general expression:

$$\sigma(q,\theta)=\frac{\pi}{2}\frac{1}{[\omega'(q)]^2}\cos^2\frac{\theta}{2}\ , \tag{1}$$

where q is the momentum, ω is the Feynman excitation spectrum related to the static structure factor $\mathcal{S}(q)$ by $q^2/2\mathcal{S}(q)$, and θ is the scattering angle of the excitations. We also show that a classical interpretation of the scattering is possible in the opposite limit, i.e., at short wavelengths.

In Sec. II, we will give the basic ingredients of our method. In Sec. III, we will calculate the cross section in the Born approximation, and present the classical approach. In Sec. IV, we will apply the results we have derived on some systems. In Sec. V, we will present our conclusions.

II. DYNAMICAL EQUATIONS

In this section we will lay out the general formulation of the problem, and obtain the dynamical equations for the elementary excitations in the presence of a vortex. The core of the idea is to obtain a Hamiltonian for the elementary excitations, and use the Hamilton's equations to describe the motion of the excitations.

A. General formulation

The elementary excitations are represented by small variations in the local density and phase, and are described by the following many-body wave function[12,13]

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$$\Psi=\exp\left[\sum_{i=1}^{N}\alpha(\mathbf{x}_i)+iS(\mathbf{x}_i)\right]\Psi_0\ , \tag{2}$$

where Ψ_0 is the ground-state wave function, S and α are arbitrary real functions with $|\alpha(\mathbf{x})|\ll 1$, and they do not vary rapidly over the characteristic length scales of the system, such as the interparticle spacing in ^{4}He and the coherence length ξ_0 in superconductors.

We set $\hbar=m=1$ in the rest of the discussion unless otherwise stated, and follow the lines in[14] to obtain the energy in this state as

$$E-E_0=\frac{1}{2}\int\rho(\mathbf{x})\{|\nabla S(\mathbf{x})|^2+|\nabla\alpha(\mathbf{x})|^2\}d\mathbf{x}\ , \tag{3}$$

where the integration is over the spatial dimensions of the system, and $\rho(\mathbf{x})$ is the density

$$\rho(\mathbf{x})=\langle\Psi|\hat{\rho}(\mathbf{x})|\Psi\rangle$$
$$=\langle\Psi_0|\hat{\rho}(\mathbf{x})\exp\left[2\int\hat{\rho}(\mathbf{x}')\alpha(\mathbf{x}')d\mathbf{x}'\right]|\Psi_0\rangle\ , \tag{4}$$

with $\hat{\rho}(\mathbf{x})=\sum_i\delta(\mathbf{x}-\mathbf{x}_i)$.

We want to express Eq. (3) in terms of the canonically conjugate variables $\delta\rho=\rho-\rho_0$ and S,[15] and regard it as the Hamiltonian of the collective excitations. Equation (3) is exact, but we have to make an approximation in order to make progress. Thus we will linearize the density in α using the assumption that the fluctuations are small, yielding

$$\rho(\mathbf{x})=\rho_0+2\int\langle\Psi_0|\hat{\rho}(\mathbf{x})\hat{\rho}(\mathbf{x}')|\Psi_0\rangle\alpha(\mathbf{x}')d\mathbf{x} \tag{5}$$

$$=\rho_0+2\rho_0\int g(\mathbf{x}-\mathbf{x}')\alpha(\mathbf{x}')d\mathbf{x}'\ , \tag{6}$$

where ρ_0 is the density of the system in the ground state, and the function $g(\mathbf{x}-\mathbf{x}')$ is the density-density correlation.[12] We can invert this relation and solve for $\alpha(\mathbf{x})$

$$\alpha(\mathbf{x})=\frac{1}{2\rho_0}\int\delta\rho(\mathbf{x}')g^{-1}(\mathbf{x}-\mathbf{x}')d\mathbf{x}'\ , \tag{7}$$

with the definition of the inverse function by

$$\int g^{-1}(\mathbf{x}-\mathbf{x}')g(\mathbf{x}')d\mathbf{x}'=\delta(\mathbf{x})\ , \tag{8}$$

Relation (7) enables us to write the energy function Eq. (3) in terms of the variables $\delta\rho(\mathbf{x})$, $S(\mathbf{x})$ and the correlation function $g(\mathbf{x})$. The properties of the ground state and the interactions between particles are included in $g(\mathbf{x})$. The normal modes of these equations will be the excitation spectrum of the fluctuations.

B. Collective excitations in the absence of a vortex

After substituting Eq. (7), the Hamiltonian of the system without any vortex in it can be written as

$$H=\frac{1}{2}\rho_0\int|\nabla S(\mathbf{x})|^2d\mathbf{x}$$
$$+\frac{1}{2}\rho_0\int\left[\nabla\left[\frac{1}{2\rho_0}\int\delta\rho(\mathbf{x}')g^{-1}(\mathbf{x}-\mathbf{x}')d\mathbf{x}'\right]\right]^2d\mathbf{x}\ . \tag{9}$$

The first term can be considered as the kinetic energy and the second one as the potential energy of fluctuations. The Hamilton's equations are just functional derivatives of the Hamiltonian

$$\frac{d\delta\rho(\mathbf{x},t)}{dt}=\frac{\partial H}{\partial S(\mathbf{x},t)}=-\rho_0\nabla^2S(\mathbf{x},t)\ , \tag{10}$$

$$-\frac{dS(\mathbf{x},t)}{dt}=\frac{\partial H}{\partial\delta\rho(\mathbf{x},t)}$$
$$=-\frac{1}{4\rho_0}\nabla^2\int\int g^{-1}(\mathbf{x}-\mathbf{x}')g^{-1}(\mathbf{x}'-\mathbf{x}'')$$
$$\times\delta\rho(\mathbf{x}'')d\mathbf{x}'d\mathbf{x}''\ , \tag{11}$$

where we have used the fact that for an isotropic system $g(\mathbf{x})$ depends only on $|\mathbf{x}|$, and omitted some boundary terms. If we recall that ∇S is the velocity field, we can easily recognize the first equation as the continuity equation, and the second one as the generalized Euler equation of hydrodynamics in a form that also takes into account the density correlations in the system.

We can try a plane-wave solution to these equations

$$S(\mathbf{x},t)=Ae^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)}\ ,$$
$$\delta\rho(\mathbf{x},t)=Be^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)}\ , \tag{12}$$

which immediately requires the conditions

$$-iB\omega=A\rho_0k^2\ , \tag{13}$$

$$iA\omega=B\frac{k^2}{4\rho_0}\left[\int g^{-1}(\mathbf{x})e^{i\mathbf{k}\cdot\mathbf{x}}d\mathbf{x}\right]^2$$
$$=B\frac{k^2}{4\rho_0\mathcal{S}^2(k)}\ , \tag{14}$$

where $\mathcal{S}(k)$ is the structure factor defined as the Fourier transform of $g(\mathbf{x})$. A nontrivial solution to these conditions is

$$\omega=\frac{k^2}{2\mathcal{S}(k)}\ , \tag{15}$$

with

$$B=2i\rho_0\mathcal{S}(k)A\ . \tag{16}$$

The first relation is the famous Bijl-Feynman formula for the collective excitation spectrum,[12] and the second gives the relation between the phase and amplitude fluctuations.

C. Effective Hamiltonian in the presence of a vortex

Thus we have shown that Eq. (3), when used as the Hamiltonian of the collective excitations, gives us the correct spectrum. Now we want to put a vortex into the system and obtain an effective Hamiltonian for the long-wavelength excitations. For simplicity we assume a single, straight, fixed vortex line at the origin. The many-body wave function in Eq. (1) can now be written as

$$\Psi=\exp\left[\sum_{i=1}^{N}\alpha(\mathbf{x}_i)+iS(\mathbf{x}_i)\right]\prod_{j=1}^{N}f(|\mathbf{x}_j|)e^{i\theta_j}\Psi_0\ , \quad (17)$$

where f is a function that becomes 1 outside the vortex core but goes to zero at the origin. θ_j is the polar angle at the position of particle j. A full treatment of the problem should take into account the core structure through the function f. However we will make the further simplification of taking f equal to 1 everywhere, thus disregarding the effects of the vortex core. Then the effect of the vortex is to simply cause a shift in the canonical phase variable by θ

$$\nabla S\rightarrow\nabla S+\frac{\hat{\theta}}{r}\ , \quad (18)$$

where θ is the polar angle and r is the distance to the vortex line. Under these circumstances, the Hamiltonian up to second-order terms in $\delta\rho(\mathbf{x})$ and $\nabla S(\mathbf{x})$ is

$$H=\frac{1}{2}\rho_0\int|\nabla S(\mathbf{x})|^2d\mathbf{x}+\int\delta\rho(\mathbf{x})\frac{\hat{\theta}}{r}\cdot\nabla S(\mathbf{x})d\mathbf{x}+\frac{1}{2}\rho_0\int\left[\nabla\left[\frac{1}{2\rho_0}\int\delta\rho(\mathbf{x}')g^{-1}(\mathbf{x}-\mathbf{x}')d\mathbf{x}'\right]\right]^2d\mathbf{x}\ , \quad (19)$$

where we have ignored some boundary terms and terms second order in the vortex field. As discussed above, the first and last terms describe the collective excitations. The scattering of excitations is caused by the second term which, in classical terms, looks like a Doppler shift in the energy. The correlation function $g(\mathbf{x})$ is unchanged by the inclusion of the vortex because the function f representing the vortex core was taken to be 1 everywhere. For the same reason the vortex couples to the kinetic term but not the potential energy term. This is why the vortex interacts with the excitations in quite a universal way, as a result of which a simple and general expression for the scattering cross section will be obtained in the next section.

III. CALCULATION OF THE SCATTERING AMPLITUDE

Now we can analyze Hamilton's equations, and calculate the scattering amplitudes. The Hamilton's equations in this case differ from the previous ones only by the appearance of the vortex terms

$$\frac{d\delta\rho(\mathbf{x},t)}{dt}=-\rho_0\nabla^2S(\mathbf{x},t)-\frac{\hat{\theta}}{r}\cdot\nabla\delta\rho(\mathbf{x},t)\ , \quad (20)$$

$$-\frac{dS(\mathbf{x},t)}{dt}=\frac{\hat{\theta}}{r}\cdot\nabla S(\mathbf{x},t)-\frac{1}{4\rho_0}\nabla^2\int\int g^{-1}(\mathbf{x}-\mathbf{x}')g^{-1}(\mathbf{x}'-\mathbf{x}'')\times\delta\rho(\mathbf{x}'')d\mathbf{x}'d\mathbf{x}''\ . \quad (21)$$

The time dependence of the solutions can still be factored by $e^{-i\omega t}$, then the equations read

$$-i\omega\delta\rho(\mathbf{x})=-\rho_0\nabla^2S(\mathbf{x})-\frac{\hat{\theta}}{r}\cdot\nabla\delta\rho(\mathbf{x})\ , \quad (22)$$

$$i\omega S(\mathbf{x})=\frac{\hat{\theta}}{r}\cdot\nabla S(\mathbf{x})+\frac{1}{4\rho_0}\nabla^2\int\int g^{-1}(\mathbf{x}-\mathbf{x}')g^{-1}(\mathbf{x}'-\mathbf{x}'')\times\delta\rho(\mathbf{x}'')d\mathbf{x}'d\mathbf{x}''\ . \quad (23)$$

The vortex terms have position-dependent coefficients and prevent an exact solution. We will use a Born approximation, in which we will assume that we can replace the $S(\mathbf{x})$ and $\delta\rho(\mathbf{x})$ in the vortex terms by the free solutions obtained previously:

$$S_0(\mathbf{x})=Ae^{+i\mathbf{q}\cdot\mathbf{x}}\ ,\qquad \delta\rho_0(\mathbf{x})=Be^{i\mathbf{q}\cdot\mathbf{x}}\ , \quad (24)$$

with the same dispersion relation. We will also restrict ourselves to two dimensions in the plane perpendicular to the vortex line, since there is no scattering parallel to the line. The dynamical equations now contain inhomogeneous terms

$$-i\omega\delta\rho(\mathbf{x})+\rho_0\nabla^2S(\mathbf{x})=-iB\frac{\hat{\theta}}{r}\cdot\mathbf{q}e^{i\mathbf{q}\cdot\mathbf{x}}\ , \quad (25)$$

$$i\omega S(\mathbf{x})+\frac{1}{4\rho_0}\nabla^2\int\int g^{-1}(\mathbf{x}-\mathbf{x}')g^{-1}(\mathbf{x}'-\mathbf{x}'')\times\delta\rho(\mathbf{x}'')d\mathbf{x}'d\mathbf{x}''=iA\frac{\hat{\theta}}{r}\cdot\mathbf{q}e^{i\mathbf{q}\cdot\mathbf{x}}\ . \quad (26)$$

By taking Fourier transform of both sides we turn the coupled equations into a linear matrix equation

$$\begin{bmatrix} i\omega & \dfrac{k^2}{4\mathcal{S}^2(k)\rho_0} \\ -k^2\rho_0 & i\omega \end{bmatrix}\begin{bmatrix}\tilde{S}(\mathbf{k})\\ \tilde{\delta\rho}(\mathbf{k})\end{bmatrix}=-\mathcal{F}\left\{i\frac{\hat{\theta}}{r}\cdot\mathbf{q}e^{i\mathbf{q}\cdot\mathbf{x}}\right\}\begin{bmatrix}A\\B\end{bmatrix}\ , \quad (27)$$

where a tilde denotes the Fourier transformed quantities. It is simple algebra to perform the necessary Fourier integral on the right-hand side of the equation. The result is

$$\mathcal{F}\left\{i\frac{\hat{\theta}}{r}\cdot\mathbf{q}e^{i\mathbf{q}\cdot\mathbf{x}}\right\}=2\pi\frac{\hat{z}\cdot(\mathbf{q}\times\mathbf{k})}{|\mathbf{q}-\mathbf{k}|^2}\ , \quad (28)$$

where $\hat{z}$ is the unit vector in the direction of the circulation. We invert Eq. (27), replace $\mathcal{S}(q)$ by the Feynman excitation spectrum, $q^2/[2\omega(q)]$, and then obtain the Fourier transform of the scattering amplitudes

$$\begin{bmatrix}\tilde{S}(\mathbf{k})\\ \tilde{\delta\rho}(\mathbf{k})\end{bmatrix}=\frac{A2\pi}{-\omega^2(k)+\omega^2(q)}\frac{\hat{z}\cdot(\mathbf{q}\times\mathbf{k})}{|\mathbf{q}-\mathbf{k}|^2}\times\begin{bmatrix} i\omega & -\dfrac{\omega^2(k)}{\rho_0k^2} \\ \rho_0k^2 & \omega\end{bmatrix}\begin{bmatrix}1\\ i\rho_0\dfrac{q^2}{\omega(q)}\end{bmatrix}\ . \quad (29)$$

The transformation back to normal space variables is given by

$$\begin{bmatrix} S(\mathbf{x}) \\ \delta\rho(\mathbf{x}) \end{bmatrix} = \int e^{i\mathbf{k}\cdot\mathbf{x}} \begin{bmatrix} \tilde{S}(\mathbf{k}) \\ \tilde{\delta}\rho(\mathbf{k}) \end{bmatrix} \frac{d^2k}{(2\pi)^2} . \tag{30}$$

The integrals cannot be evaluated exactly. Since our purpose is to calculate the cross section we only need the amplitudes in the radiation zone, $kr \gg 1$. A separation of the **k** integral over components, $k_\parallel$ and $k_\perp$, parallel and perpendicular components of **k** to **x**, respectively, is helpful. Then specifically for $S(\mathbf{x})$ we get

$$S(\mathbf{x},t) = Aie^{-i\omega t}\frac{2\pi}{(2\pi)^2}\int \frac{dk_\parallel dk_\perp}{\omega(k)-\omega(q)}\frac{e^{ik_\parallel r}}{\omega(k)+\omega(q)} \times \frac{\omega^2(k)q^2+\omega^2(q)k^2}{\omega(q)k^2} \times \frac{\hat{z}\cdot(\mathbf{q}\times\mathbf{k})}{|\mathbf{q}-\mathbf{k}|^2} . \tag{31}$$

We will first keep $k_\perp$ constant, and evaluate the $k_\parallel$ integration. We can take care of the boundary conditions at infinity by supplying a small imaginary part wherever necessary. The boundary conditions require that the outgoing scattering amplitudes should behave as $e^{ikr}/\sqrt{r}$, so the relevant poles are at the values of $k_{\parallel 0} = \sqrt{K^2-k_\perp^2} + i\eta$, where K satisfies

$$\omega(q) = \omega(K) , \tag{32}$$

and η is a positive infinitesimal constant.

We have to note that K is not necessarily equal to q, as might be expected from energy conservation, since the spectrum may exhibit a behavior such that several k values can correspond to the same energy. A very well-known example is the roton spectrum, which we will investigate in the next section. For the time being we will work with the case $K = q$.

Expanding around the pole to linear order in $(k_\parallel - k_{\parallel 0})$ gives

$$\omega(k) \approx \omega(q) + \omega'(q)\frac{dk}{dk_\parallel}\Big|_{k_{\parallel 0}}(k_\parallel - k_{\parallel 0}) . \tag{33}$$

Then we place it into the integral and obtain the simple result

$$S(\mathbf{x},t) = -Ae^{-i\omega t}\mathrm{Res}(q) , \tag{34}$$

with

$$\mathrm{Res}(q) = -\int dk_\perp e^{ik_{\parallel 0} r}\frac{1}{\omega'(q)(dk/dk_\parallel)|_{k_{\parallel 0}}}\frac{q(k_{\parallel 0}\sin\theta - k_\perp\cos\theta)}{k_{\parallel 0}^2 + k_\perp^2 + q^2 - 2qk_{\parallel 0}\cos\theta - 2qk_\perp\sin\theta} , \tag{35}$$

and keeping in mind that $k_{\parallel 0} = \sqrt{q^2 - k_\perp^2}$. Now we have to evaluate the integral over $k_\perp$. In the radiation zone where we take $r \to \infty$, the integral will be dominated by the stationary point of the exponential, which is located at $k_\perp = 0$. Assuming that the integrand other than the exponential behaves smoothly, we can set $k_\perp = 0$ everywhere except in the exponential. A stationary phase approximation gives

$$\mathrm{Res}(q) = -e^{-i(\pi/4)}\sqrt{\pi/2}\frac{e^{iqr}}{\sqrt{r}}\frac{\sqrt{q}}{\omega'(q)}\frac{\sin\theta}{1-\cos\theta} . \tag{36}$$

Combining with Eq. (34) we get the desired result

$$S(\mathbf{x},t) = Ae^{-i\omega t}e^{-i(\pi/4)}\frac{e^{iqr}}{\sqrt{r}}\sqrt{\pi/2}\frac{\sqrt{q}}{\omega'(q)}\frac{\sin\theta}{1-\cos\theta} . \tag{37}$$

The cross section is given by the square of the amplitude of the outgoing wave

$$\sigma(q,\theta) = \frac{\pi}{2}\frac{q}{[\omega'(q)]^2}\cot^2\frac{\theta}{2} , \tag{38}$$

which is the central result of this article. Note that the angular dependence is universal, which is symmetric with respect to $\theta \to -\theta$, and diverges at small angles. The spectrum function enters in a simple form, and only affects the q-dependent factor.

One can evaluate the integrals in the general case where there may be more than one possible value of K. A similar calculation leads to the scattering amplitude given by the sum of residues satisfying Eq. (32)

$$S(\mathbf{x},t) = Ae^{-i\omega t}\sum_K e^{-i(\pi/4)}\frac{e^{iKr}}{\sqrt{r}}\sqrt{\pi/2}\frac{q(K^2+q^2)}{\sqrt{K}} \times \frac{1}{\omega'(K)}\frac{\sin\theta}{K^2+q^2-2Kq\cos\theta} . \tag{39}$$

However we must note that one has to pay more attention to the degenerate case when $\omega'(K) = 0$. In that case the idea is still the same, only we need to expand to quadratic order in Eq. (33), which forces us to evaluate the derivative of the integrand in Eq. (35).

The Born approximation is valid if the scattered amplitude is small when compared with the incoming amplitude. We can expect this to be violated most where the potential is strongest, which is the origin in our case. A simple calculation after setting $\mathbf{x} = 0$ in Eq. (31) leads to the requirement

$$\frac{q}{\omega'(q)} \ll 1 . \tag{40}$$

Thus if the excitation spectrum vanishes with a power less than 2 as $q \to 0$ then Born approximation is valid in this limit.

Classical Hamiltonian

The other approach which is mostly used for rotons is the classical equations of motion for the collective excitations. It is possible to show that one can obtain the classical Hamiltonian, thus the classical equations, from Eqs. (20) and (21). These equations are like wave equations for the amplitudes S and $\delta\rho$. The classical equations follow from the geometrical optics approximation or the WKB. We replace the amplitudes S and $\delta\rho$ in Eqs. (20) and (21) by

$$S(\mathbf{x},t)=Ae^{i\phi(\mathbf{x},t)} ,\quad \delta\rho(\mathbf{x},t)=Be^{i\phi(\mathbf{x},t)} , \tag{41}$$

where A and B in general have spatial variation, but in the spirit of the WKB approximation they can be taken as constant. Then from Eqs. (20) and (21) we obtain

$$i\left[\frac{\partial\phi}{\partial t}+\frac{\hat{\theta}}{r}\cdot\nabla\phi\right]B=\rho_0|\nabla\phi|^2A , \tag{42}$$

$$-i\left[\frac{\partial\phi}{\partial t}+\frac{\hat{\theta}}{r}\cdot\nabla\phi\right]A=\frac{|\nabla\phi|^2}{4\rho_0\mathcal{S}^2(\nabla\phi)}B . \tag{43}$$

The reader is reminded that we are still using units $\hbar=m=1$. The momentum and energy of classical particles corresponding to wave packets of the amplitudes are given by

$$E=-\frac{\partial\phi}{\partial t} \quad\text{and}\quad \mathbf{p}=\nabla\phi . \tag{44}$$

Thus we obtain the relation

$$E=\omega(\mathbf{p})+\frac{\hat{\theta}}{r}\cdot\mathbf{p} , \tag{45}$$

where we identified the energy spectrum of the excitations, $\omega(\mathbf{p})$ as in the Bijl-Feynman formula. This is obviously the classical Hamiltonian for the collective excitations with the Doppler shift in the energy due to the vortex. The equations of motion that follow from this Hamiltonian have been used extensively in the literature for rotons,[1,2,5] and recently for scattering of phonons from a classical vortex.[7]

The WKB approximation is valid, if the spatial variation of $\phi(\mathbf{x},t)$ satisfies

$$\nabla^2\phi \ll |\nabla\phi|^2 . \tag{46}$$

For this approximation to be valid the wave packets have to satisfy

$$\nabla\cdot\mathbf{p} \ll |\mathbf{p}|^2 , \tag{47}$$

or as an estimate on the left-hand side we can use the uncertainties of momentum and position

$$\frac{\Delta p}{\Delta x} \ll |\mathbf{p}|^2 . \tag{48}$$

We can take $\Delta p \sim 1/\Delta x$ and $\Delta x \sim \xi$. The classical approximation is valid if

$$p\xi \gg 1 . \tag{49}$$

Thus WKB is appropriate for large momentum. This is the opposite limit of validity of the Born approximation given in Eq. (40).

IV. APPLICATIONS

The results of the previous section are quite general in the sense that as long as we know what the collective excitation spectrum of a system is, we can find the scattering cross section of these excitations from a vortex. In this section we will show their use in superfluid helium and superconductor thin films.

A. Superfluid He

1. Phonon scattering

In the phonon part of the spectrum—the long-wavelength low-energy part—the relation between k and ω is one to one, and is given by

$$\omega(q)=sq , \tag{50}$$

where s is the speed of sound. The only pole satisfying $\omega(q)=\omega(K)$ is at $K=q$. Thus from Eq. (38) we obtain

$$\sigma(q,\theta)=\frac{\pi}{2}q\left[\frac{\hbar}{ms}\right]^2\cot^2\frac{\theta}{2} . \tag{51}$$

This result was obtained by Fetter[10] using phase-shift analysis of the scattered wave in the hydrodynamic equations. The other results in the literature vary. Pitaevskii[8] and Sonin[5] have different results from the one above although they both use Born approximation and similar equations. However we have found that Pitaevskii's calculation suffers from an algebraic error. In his paper Eq. (25) does not follow from the substitution of Eq. (24) into Eq. (19). Instead when correctly done the result is the same as what we have obtained above. The calculation of Sonin is a more complicated one that it takes into account the curvature of the vortex line. His calculation must be also incorrect, however, as it gives the result of Pitaevskii in the limit of a straight vortex line.

2. Roton scattering

We can employ formula (39) to obtain the cross section of rotons. There are two values that satisfy Eq. (32). The spectrum near the roton minimum can be written in the Landau form

$$\omega(q)=\Delta+\frac{(q-q_0)^2}{2\mu} , \tag{52}$$

where Δ denotes the energy and q_0 denotes the wave vector at the roton minimum. The roton has two poles that satisfy $\omega(q)=\omega(K)$, which are

$$K_1=q \quad\text{and}\quad K_2=2q_0-q . \tag{53}$$

Now using Eq. (39)

$$S(\mathbf{x},t)=Ae^{-i\omega t}\sqrt{\pi/2}\frac{e^{iqr}}{\sqrt{r}}\frac{\sqrt{q}}{|q-q_0|}\mu\left\{\frac{\sin\theta}{1-\cos\theta}-e^{2i(q-q_0)r}\frac{\sqrt{q}}{2q_0-q}\frac{\sin\theta}{1-[2q(2q_0-q)/(2q_0-q)^2+q^2]\cos\theta}\right\}. \quad (54)$$

We immediately notice that the scattered wave has two parts, one cylindrical wave with momentum q and one echo with $2q_0-q$. The cross section will not only involve the individual contributions of each part but also the interference. For $|q-q_0|\gg(1/r)$ the interference can be ignored and the cross section will be given by

$$\sigma(q,\theta)=\pi\left[\frac{\mu}{m}\right]^2\frac{q}{(q-q_0)^2}\cot^2\frac{\theta}{2}\ . \quad (55)$$

This is also the same result obtained long ago by Hall and Vinen.[9]

B. Superconductor thin films

Another system where we can use the formula (38) is a superconductor thin film. The scattering cross section and mutual friction in superconductors were investigated on the basis of quasiparticle-vortex interactions. This is the dominant contribution for ordinary superconductors. However, at low temperatures and small core size, the quasiparticles are frozen, and interactions between vortices and collective excitations might become more important.

In a very broad sense, superconductivity is superfluidity of charged particles. The important distinction is that the vortex singularities are like thin flux tubes for bulk materials because of screening due to the diamagnetic currents set up in the sample. However, in thin films this screening is only effective beyond a length L_s given by[16]

$$L_s=\frac{\lambda_L^2}{d}\ , \quad (56)$$

where λ_L is the London penetration depth and d is the thickness of the film. This length scale can be quite large compared to the bulk value λ_L. Thus the vortex in a thin film is very much like a vortex in neutral ^{4}He. There is one difference that is related to pairing in the ground state.[17] This forces us to introduce a modified shift of $\hat{\theta}/2r$ in Eq. (18). The final effect of this on the cross section is a factor of $1/4$:

$$\sigma(q,\theta)=\frac{\pi}{8}\frac{q}{[\omega'(q)]^2}\cot^2\frac{\theta}{2}\ . \quad (57)$$

The collective excitations of a superconductor are the density fluctuations of cooper pairs, and have the same dispersion as the ordinary plasmon at zero temperature, which has the following form in two dimensions:

$$\omega(q)=\left[\frac{2\pi ne^2}{m_e^*}\right]^{1/2}q^{1/2}\ , \quad (58)$$

where n is the 2D density of electrons, e is the electron charge, and m_e^* is the effective electron mass. The resulting cross section is then

$$\sigma(q,\theta)=\pi\left[\frac{\hbar}{m_e^*c}\right]^2 L_s q^2\cot^2\frac{\theta}{2}\ , \quad (59)$$

where c is the speed of light.

As a particular application of the cross section we can calculate the frictional force on a superconductor vortex due to the collective excitations. In an analysis similar to the one presented in[6] we find that the perpendicular force is zero, and the parallel force per unit length of the line is given by

$$\mathbf{F}_{\parallel}=(\mathbf{v}_s-\mathbf{v}_L)\frac{\pi}{8}\frac{k_B^{10}\times 10!\times\zeta(10)}{m_e^{*2}a^{12}\hbar^7 d}T^{10}\ , \quad (60)$$

where $\mathbf{v}_L$ is the velocity of the vortex line, $\mathbf{v}_s$ is the velocity of the background supercurrent, k_B is the Boltzmann constant, a is the coefficient in front of the $q^{1/2}$ in the plasmon spectrum, d is the thickness of the film, and ζ is the Riemann zeta function. In order to get a feeling of the magnitude of this force, we use the typical values of the parameters for high-T_c thin films, $n=2\times10^{19}$ m^{-2}, $m_e^*=6m_e$, and $d=10^{-8}$ m. The resulting force per unit length is

$$\mathbf{F}_{\parallel}=(\mathbf{v}_s-\mathbf{v}_L)\eta_p=(\mathbf{v}_s-\mathbf{v}_L)4\times10^{-49}T^{10}\ , \quad (61)$$

where we have defined the coefficient of viscosity per unit length η_p.

The 2D approximation breaks down at wavelengths shorter than the thickness of the film. The plasmon energy corresponding to this wavelength is close to the bulk plasma frequency which is roughly 10^4 K. The validity of the Born approximation can also be demonstrated as follows. The condition (40) requires the wave vector q of the plasmons to be smaller than the critical value

$$q_c=\left[\frac{m_e^*a}{2\hbar}\right]^{2/3}\ . \quad (62)$$

This condition is also well satisfied even at the transition temperature.

A comparison to the Bardeen-Stephen (BS) theory of dissipation reveals the smallness of the effect. In this theory dissipation is caused mainly in the core region where electronic states are normal, and the excitation spectrum is assumed to be continuous. The coefficient of viscosity per unit length is given by $\eta_{\rm BS}=2\pi\hbar H_c2/\rho_n ec$.[18] High-$T_c$ materials can have quite high transition temperatures (≈100 K), and also high H_{c_2} (122 T for YBa_2CuO_7). The normal-state resistivity ρ_n is around 10^{-6} Ωm, and viscosity per unit length is on the order of 2×10^{-7} N sec/m^2. Therefore, near the transition temperature the plasmon-induced viscosity is at least 20 orders of magnitude smaller than the viscosity of the core.

The approximation of continuous spectrum in the BS model breaks down when the temperature is smaller than the energy spacing of the excitation spectrum in the core. In that range of temperatures we can simply extend the BS model with an exponential factor $e^{-\delta E/k_B T}$, where δE is the energy spacing in the core given by[19]

$$\delta E = \hbar \frac{eH_{c2}}{4m_e^* c} . \tag{63}$$

This energy separation is around 10 K with the typical values used above. Assuming this exponential behavior for the BS model, the plasmon viscosity will dominate the BS viscosity at temperatures lower than 10^{-2} K, although they will be too small to be measurable.

The other possibility of a scattering event is with the BCS quasiparticles.[20,21] According to the results of the latter reference the viscosity due to quasiparticle scattering depends exponentially on the superconductor gap Δ. We use a simple expression to extrapolate between the low- and high-temperature expressions given in that reference, $\eta_{\rm BCS} \approx m_e^* p_F \Delta/\hbar^2 e^{-\Delta/k_B T}$. The order of magnitude of this viscosity near $T_c \approx 100$ K is 10^{-12} N sec/m^2. Obviously this is smaller than the core viscosity by five orders of magnitude, but is still 15 orders of magnitude larger than the plasmon viscosity.

Our calculations clearly show that the plasmon scattering in superconductors is insufficient to produce any appreciable effect in vortex motion and dissipation. The reason for the ineffectiveness is that plasmon viscosity has a high power dependence on the temperature, and temperature is scaled by the high plasma frequency.

V. CONCLUSIONS

In summary, we have investigated the interaction between collective excitations and vortices in superfluid systems. Starting from a Feynman-type many-body wave function, general dynamic equations for the collective excitations in the presence of a vortex have been derived, in which the specific properties of a given system enters only through the density-correlation function or the static structure factor. The scattering cross section has been derived in the Born approximation for long wavelengths. The angular dependence has the universal form $\cot^2(\theta/2)$, with a prefactor only involving the energy spectrum of the collective excitations given by the Feynman formula in terms of the static structure factor. For short wave lengths, the geometrical optics of the WKB approximation to the dynamical equations has been shown to yield the classical equations of motion for wave packets.

We have applied the Born approximation cross section formula to phonons and rotons in ^{4}He, and showed that the results are consistent with previous calculations. As a new result, we have also obtained the cross section of plasmons from vortices in superconductor thin films. These cross sections, when combined with the formulas in Ref. 6, can be used to find the frictional forces on a vortex. For the case of superconductor thin films, the longitudinal dissipative force on a moving vortex is shown to be proportional to a high power of temperature T^{10}, which supports the idea that vortices move freely at low temperatures.

The transverse force due to scattering vanishes as long as the Born approximation is valid, which is the case for phonon and plasma scattering. However, the dynamics of rotons in superfluid helium falls rather into the validity range of the WKB approximation, and can yield a transverse force on vortices.[1,2,5] This is very similar to the scattering of electrons by a magnetic flux line. The classical calculation valid at short wavelengths yields a transverse force, whereas the Born approximation valid in the long-wavelength limit does not.

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