SU(2) coherent-state path integral for the Heisenberg ferromagnet

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The SU(2) coherent-state path-integral representation is used to derive a fairly explicit expression for the partition function $Z_0[\vec{\psi}]$ of noninteracting quantum spins in the presence of an arbitrary time-dependent external field $\vec{\psi}$. By performing then the standard Gaussian average of $Z_0[\vec{\psi}]$ over $\vec{\psi}$ the partition function of the ferromagnetically coupled Heisenberg model is represented in a way which allows one to simplify the derivation of results such as obtained by Leibler and Orland [Ann. Phys. (N.Y.) **132**, 277 (1981)] and to generalize them to arbitrary spin quantum number.

I. INTRODUCTION

Consider the quantum Heisenberg Hamiltonian

$$H = -\frac{1}{2}\sum_{i,j} J_{ij}\vec{S}_i\vec{S}_j ,$$

where the spin operators $\vec{S}_i$ are SU(2) generators of dimension $2S+1$. The exchange constants J_{ij} between the spins at (lattice) sites i and j are assumed to fulfil $J_{ij} \geq 0$ and $J_{jj} = 0$, corresponding to purely ferromagnetic coupling.

Following, e.g., Ref. 1 it is possible to represent the statistical operator $e^{-\beta H}$ of the Hamiltonian H at inverse temperature $\beta > 0$ as a suitable normalized Gaussian average over a "time"-dependent real-valued field $\vec{\psi} \equiv \{\vec{\psi}_i(\tau)\}$ $(i = 1,2,\ldots;\ \tau \in [0,\beta])$ of the statistical operator $\rho_\beta[\vec{\psi}]$ of noninteracting spins experiencing $\vec{\psi}$ as an external field:

$$e^{-\beta H} = \int D\vec{\psi}\exp\Bigg(-\frac{1}{2}\int_0^\beta d\tau \times \sum_{ij}(J^{-1})_{ij}\vec{\psi}_i(\tau)\cdot\vec{\psi}_j(\tau)\Bigg)\rho_\beta[\vec{\psi}] , \tag{1}$$

$$\rho_\beta[\vec{\psi}] \equiv T\exp\left\{\int_0^\beta d\tau\sum_i \vec{\psi}_i(\tau)\vec{S}_i\right\} . \tag{2}$$

Before proceeding, three remarks are in order. First, in Eq. (2) T denotes the time-ordering symbol. It is necessary to include, because the operator sum in the exponent does not commute for different values of τ. Second, strictly speaking, one should require the symmetric matrix $\{J_{ij}\}$ to be positive definite. Otherwise the functional integral (1) can only be understood in a formal way. Finally, it should be noted that no periodic boundary conditions are imposed on the field $\vec{\psi}$. Instead, the pair of random vectors $\vec{\psi}_i(0)$ and $\vec{\psi}_i(\beta)$ are (stochastically) independent for all sites i.

In order to derive the equilibrium properties of the Heisenberg ferromagnet, one has to calculate the partition function $Z \equiv \operatorname{tr} e^{-\beta H}$. By interchanging the trace operator with the functional integration one obtains

$$Z = \int D\vec{\psi}\exp\left(-\frac{1}{2}\int_0^\beta d\tau\sum_{ij}(J^{-1})_{ij}\vec{\psi}_i(\tau)\cdot\vec{\psi}_j(\tau)\right) \times Z_0[\vec{\psi}] , \tag{3}$$

where

$$Z_0[\vec{\psi}] \equiv \operatorname{tr}\rho_\beta[\vec{\psi}]. \tag{4}$$

At this stage it would be desirable to derive a fairly explicit closed-form expression for the unperturbed partition functional $Z_0[\vec{\psi}]$. As is well known, e.g., Ref. 1, such an expression is easily available for the Ising model, whereas for the Heisenberg model the noncommutativity of the components of $\vec{S}_i$ poses a severe problem.

To overcome this problem, Leibler and Orland[1] developed for $S = 1/2$ a perturbative technique based on a representation of the spins in terms of bosons or fermions.[2] In doing so, they succeeded in deriving from (3) a systematic expansion around the stationary mean field, that is, the stationary solution of the saddle-point equation for the functional integral. The major drawback of this approach is that the total number of bosons (or fermions) has to be kept fixed to 1, a fact which is rather inconvenient in calculations. For higher spin quantum numbers the constraint is even more complicated.

As for nonperturbative approaches, one could think to compute first the operator $\rho_\beta[\vec{\psi}]$ by some disentangling formula of the Baker-Campbell-Hausdorff or the Wei-Norman type[3,4] and then take the trace. However, as is well known,[5,6] it seems to be hopelessly complicated to find unambiguous and explicit solutions to the differential equations posed by the disentangling problem. On the other hand, since we are only interested in the reduced information contained in the trace of $\rho_\beta[\vec{\psi}]$, one may hope to proceed in another way. For example, by representing the spin operators in terms of bosons (or fermions) one is in principle able to write $Z_0[\vec{\psi}]$ as a bosonic (or fermionic) coherent-state path integral. However, the Lagrange multipliers being necessary in this approach for fixing the spin quantum number at each lattice site[7] give rise to unphysical diagrams in the computation of Z and, in particular, obscure its mean-field expansion.

In such a situation it is natural to ask whether one can make progress by using the SU(2) coherent-state path integral to compute the unperturbed partition functional $Z_0[\vec{\psi}]$ in a fairly explicit way. After all, by their very consruction, SU(2) coherent states are best adapted to spin systems. It is the main goal of this paper to show that one can indeed make progress in the computation of Z by using the SU(2) coherent-state path-integral representation of $Z_0[\vec{\psi}]$. As a consequence, it is possible to simplify the derivation of results obtained in Ref. 1 and generalize them to arbitrary spin quantum number by using a minimal number of basic variables.

II. SU(2) PATH INTEGRAL FOR NONINTERACTING SPINS

The partition function (4) of noninteracting spins in an external fluctuating field $\vec{\psi}(t)$ can be represented by the SU(2) path integral

$$Z_0[\vec{\psi}] = \int_{\alpha_i(0)=\alpha_i(\beta)} \prod_i D\mu_S(\alpha_i) \exp\left(S\int_0^\beta d\tau \frac{\dot{\bar{\alpha}}_i \alpha_i - \bar{\alpha}_i \dot{\alpha}_i}{1+|\alpha_i|^2} + S\int_0^\beta d\tau \psi_i^{(0)} \frac{1-|\alpha_i|^2}{1+|\alpha_i|^2} - S\int_0^\beta d\tau \frac{\bar{\psi}_i \alpha_i}{1+|\alpha_i|^2} - S\int_0^\beta d\tau \frac{\psi_i \bar{\alpha}_i}{1+|\alpha_i|^2} \right), \tag{5}$$

where we have put $\bar{\psi} \equiv \psi_x + i\psi_y$, $\psi \equiv \psi_x - i\psi_y$, and $\psi^{(0)} \equiv \psi_z$. The functional measure in Eq. (5) is formally defined as an infinite pointwise product of the SU(2) invariant measures,

$$D\mu_S(\alpha) \equiv \prod_t \frac{2S+1}{\pi} \frac{d^2\alpha}{(1+|\alpha|^2)^2}.$$

Representation (5) is well known in the literature; for details the reader is referred to the comprehensive review paper by Kuratsuji.[8] The different and important point is that this integral can be evaluated by the change

$$\alpha(\tau) \to \frac{u(\tau)\alpha(\tau)+v(\tau)}{-\bar{v}(\tau)\alpha(\tau)+\bar{u}(\tau)},$$

where

$$\begin{pmatrix} u(\tau) & v(\tau) \\ -\bar{v}(\tau) & \bar{u}(\tau) \end{pmatrix} \in \mathrm{SU}(2).$$

The result is

$$\ln Z_0[\vec{\psi}] = \sum_i \ln \frac{\sinh(S+\frac{1}{2}) \int_0^\beta d\tau \Omega_i(\tau)}{\sinh \frac{1}{2} \int_0^\beta d\tau \Omega_i(\tau)}, \tag{6}$$

where

$$\Omega_i(\tau) = \psi_i^{(0)}(\tau) + \frac{1}{2}\bar{\psi}_i(\tau) z_i(\tau) + \frac{1}{2}\psi_i(\tau)\bar{z}_i(\tau)$$

and the functions $z = v/\bar{u}$ and $\bar{z} = \bar{v}/u$ depend upon $\vec{\psi}$ via the Riccati differential equations

$$\left(\frac{d}{d\tau} - \psi^{(0)}\right) z - (\bar{\psi}/2) z^2 + \psi/2 = 0, \quad z(0) = z(\beta), \tag{7}$$

$$\left(\frac{d}{d\tau} + \psi^{(0)}\right) \bar{z} + (\psi/2) \bar{z}^2 - \bar{\psi}/2 = 0, \quad \bar{z}(0) = \bar{z}(\beta). \tag{8}$$

To avoid a possible confusion the following remark is in order. Due to the Euclidean character of the action in Eq. (5) dynamical variables $z(\tau)$ and $\bar{z}(\tau)$ are related to each other by the so-called canonical conjugation rather than by ordinary complex conjugation.[9] What is important is that under this conjugation the time derivative changes its sign.

The partition function thus becomes

$$Z = \int D\vec{\psi} \exp A[\vec{\psi}], \tag{9}$$

$$A[\vec{\psi}] = -\frac{1}{2}\sum_{ij} \int d\tau \vec{\psi}_i(\tau) J_{ij}^{-1} \vec{\psi}_j(\tau) + \ln Z_0[\vec{\psi}]. \tag{10}$$

Representation (9) and (10) is a result which has not been stated earlier to our knowledge. In the case of the Ising model ($\psi = \bar{\psi} = 0$) the only solutions to Eqs. (7) and (8) are $z = \bar{z} = 0$. Equation (6) then reads

$$\ln Z_0[\vec{\psi}] = \sum_i \ln \frac{\sinh(S+\frac{1}{2}) \int_0^\beta d\tau \psi_i^{(0)}(\tau)}{\sinh \frac{1}{2} \int_0^\beta d\tau \psi_i^{(0)}(\tau)},$$

which being inserted into (9) results in the Ising-model partition function.[1,10] For the general case of the Heisenberg model Eqs. (7) and (8) cannot be solved explicitly. Nevertheless, the representation (9) and (10) in combination with (7) and (8) turns out to be a convenient starting point to generate the mean-field spin diagrammatic technique introduced by Izyumov, Kassan-Ogly, and Skryabin in the operator formalism.[10] The latter is based on the Wick theorem for spin operators; however, it differs from the conventional Wick theorem for bosonic (fermionic) operators and results in more complicated diagrams. It is therefore of interest to develop an independent approach not based on operators. We present here a direct path-integral formulation of the spin-diagrammatic technique. This representation can also be used to describe the system in the critical region, with $\vec{\psi}$ being the order parameter.

III. MEAN-FIELD THEORY

To begin with, let us note that Eqs. (7) and (8) imply

$$\frac{d}{d\tau}\ln(1+|z|^2)=\frac{1}{2}(\bar{\psi}z-\psi\bar{z}).$$

This allows one to rewrite Ω in the nonsymmetrical form

$$\Omega_i(\tau)=\psi_i^{(0)}(\tau)+\bar{\psi}_i(\tau)z_i(\tau).$$

In what follows, it is therefore sufficient to use only the first equation (7).

Let the stationary mean field $\vec{\psi}\,|_0\equiv\vec{\Phi}$ be chosen in the z direction: $\vec{\Phi}_i=(0,0,\Phi_i)$. The saddle-point equation

$$\left.\frac{\delta A}{\delta\bar{\psi}_j(\tau)}\right|_0=0$$

(the subscript $|_0$ denotes a quantity at the stationary point) reads

$$\Phi=J_0\,b(\beta\Phi), \qquad (11)$$

where $b(x)\equiv SB_S(Sx)$ and

$$B_S(x)\equiv\left(1+\frac{1}{2S}\right)\coth\left(1+\frac{1}{2S}\right)x-\frac{1}{2S}\coth\frac{x}{2S}$$

is the Brillouin function. To derive (11), we have put $J_0\equiv\sum_j J_{ij}$ and $\Phi_i=\Phi$ assuming translation invariance for the system. Expanding $A[\vec{\psi}]$ around Φ up to the second order, one is led to ($\vec{\eta}=\vec{\psi}-\vec{\psi}\,|_0$)

$$Z\approx\exp\{A[\Phi]\}\int D\vec{\eta}\exp\Bigg\{-\frac{1}{2}\sum_{ij}\int_0^\beta d\tau d\sigma\eta_i^{(0)}(\tau)\times(J^{-1}_{\text{eff;ln}})_{ij}(\tau,\sigma)\eta_j^{(0)}(\sigma)-\frac{1}{2}\sum_{ij}\int_0^\beta d\tau d\sigma\bar{\eta}_i(\tau)(J^{-1}_{\text{eff;tr}})_{ij}(\tau,\sigma)\eta_j(\sigma)\Bigg\}, \qquad (12)$$

the effective inverse longitudinal and transverse interactions being given by

$$(J^{-1}_{\text{eff;ln}})_{ij}(\tau,\sigma)=J_{ij}^{-1}\delta(\tau-\sigma)-\left.\frac{\delta^2\ln Z_0}{\delta\psi_i^{(0)}(\tau)\delta\psi_j^{(0)}(\sigma)}\right|_0$$

$$=J_{ij}^{-1}\delta(\tau-\sigma)-b'(\beta\Phi)\delta_{ij}, \qquad (13)$$

$$(J^{-1}_{\text{eff;tr}})_{ij}(\tau,\sigma)=J_{ij}^{-1}\delta(\tau-\sigma)-2\left.\frac{\delta^2\ln Z_0}{\delta\bar{\psi}_i(\tau)\delta\psi_j(\sigma)}\right|_0$$

$$=J_{ij}^{-1}\delta(\tau-\sigma)-2b(\beta\Phi)\left.\frac{\delta z_i(\tau)}{\delta\psi_j(\sigma)}\right|_0. \qquad (14)$$

In deriving (12) one should keep in mind that

$$\left.\frac{\delta^2\ln Z_0}{\delta\psi\delta\psi}\right|_0=\left.\frac{\delta^2\ln Z_0}{\delta\bar{\psi}\delta\bar{\psi}}\right|_0=\left.\frac{\delta^2\ln Z_0}{\delta\bar{\psi}\delta\psi^0}\right|_0=\left.\frac{\delta^2\ln Z_0}{\delta\psi\delta\psi^0}\right|_0=0,$$

which follows from Eq. (7). One can also convince oneself that

$$\left(\frac{d}{d\tau}-\Phi\right)\left.\frac{\delta z_i(\tau)}{\delta\psi_j(\sigma)}\right|_0=-\frac{1}{2}\delta_{ij}\delta(\tau-\sigma),$$

$$\left.\frac{\delta z_i(\tau=0)}{\delta\psi_j(\sigma)}\right|_0=\left.\frac{\delta z_i(\tau=\beta)}{\delta\psi_j(\sigma)}\right|_0,$$

which results in

$$K_{ij}(\tau-\sigma)\equiv2\left.\frac{\delta z_i(\tau)}{\delta\psi_j(\sigma)}\right|_0=\delta_{ij}\exp[\Phi(\tau-\sigma)]\left\{n_\Phi\theta(\tau-\sigma)+(1+n_\Phi)\theta(\sigma-\tau)\right\}, \qquad (15)$$

where $n_\Phi=(e^{\beta\Phi}-1)^{-1}$ is the Bose function. Note that

$$b(\beta\Phi)K_{ij}(\tau-\sigma)=\frac{1}{2}\langle TS_i^-(\tau)S_j^+(\sigma)\rangle_0\equiv G^{(0)}_{ij,\text{tr}}(\tau-\sigma|\Phi)$$

represents the transverse temperature Green function for noninteracting spins with $H=H_0=\Phi\sum S_i^{(z)}$. The factor 1/2 is related to the spin normalization chosen.

Turning back to Eq. (14), one finds

$$(J^{-1}_{\text{eff;tr}})_{ij}(\tau-\sigma)=J_{ij}^{-1}\delta(\tau-\sigma)-G^{(0)}_{ij,\text{tr}}(\tau-\sigma|\Phi). \qquad (16)$$

In the energy-momentum representation this reads

$$J_{\text{eff;tr}}(\omega_n,\vec{q})=\frac{J(\vec{q})}{1-J(\vec{q})G^{(0)}_{\text{tr}}(i\omega_n)}. \qquad (17)$$

In view of Eq. (16), the Dyson equation for the whole propagator

$$G_{\text{tr}}=G^{(0)}_{\text{tr}}+G^{(0)}_{\text{tr}}J_{\text{eff;tr}}G^{(0)}_{\text{tr}} \qquad (18)$$

gives $[G^{(0)}_{\text{tr}}(i\omega_n)=b/(i\omega_n+\Phi);\ \omega_n\equiv2\pi n/\beta]$

$$G^{(0)}_{\text{tr}}(i\omega_n,\vec{q})=\frac{b(\beta\Phi)}{\epsilon_{\vec{q}}+i\omega_n}, \qquad (19)$$

where

$$\epsilon_{\vec{q}}=b(\beta\Phi)[J(0)-J(\vec{q})]$$

is the temperature-dependent energy of the spin wave excitation.

The path integral (12) is easily calculated to yield the contribution to the partition function coming from the Gaussion fluctuations around the mean field. Denoting by b' the derivative of b, the result can be written as

$$\beta F \equiv -\ln Z = -A[\Phi] + \frac{1}{2}\ln\det(JJ^{-1}_{\text{eff;ln}}) + \ln\det(JJ^{-1}_{\text{eff;tr}})$$
$$= N\frac{\beta\Phi}{2}b(\beta\Phi) - N\ln\frac{\sinh(S+1/2)\beta\Phi}{\sinh(\beta\Phi/2)} + \frac{1}{2}\sum_{\vec q}\ln[1-\beta b'(\beta\Phi)J(\vec q)] + \sum_{\vec q}\ln\frac{\sinh\frac{\beta\Phi}{2}[1-J(\vec q)/J_0]}{\sinh\frac{\beta\Phi}{2}}.$$

As it should do, it coincides at $S = 1/2$ with the result of Ref. 1.

One of the advantages of the representation (9) and (10) is that it gives in principle the spin-correlation functions by taking appropriate functional derivatives of Z_0. In view of Eq. (6), these derivatives are calculated explicitly in terms of the functional derivatives of the solution of the Riccati Eq. (7). In the mean-field approximation the latter derivative appears as multiple products of Green functions of a linear first-order differential equation. Higher-order derivatives can then be expressed through those of lower order. For illustration the spontaneous magnetization and the transverse Green function will be derived below.

In principle, the antiferromagnetic case could be examined along similar lines, provided the spins on one of the sublattices are canonically transformed to

$$S_j^{\pm} \to S_j^{\mp}, \quad S_j^z \to -S_j^z.$$

In this case it would be of particular interest to examine wether the representation (9) could be adjusted to study nonperturbative effects in antiferomagnets, e.g., the difference between integer and half-integer spins found by Haldane in one dimension.[11]

IV. SPONTANEOUS MAGNETIZATION

Spontaneous magnetization is given by

$$\langle S_j^z(\sigma)\rangle = \frac{\int D\vec\psi e^{A[\vec\psi]}S_j^z(\sigma|\vec\psi)}{\int D\vec\psi e^{A[\vec\psi]}} \equiv \langle\!\langle S_j^z(\sigma|\vec\psi)\rangle\!\rangle, \tag{20}$$

where

$$S_j^z(\sigma|\vec\psi) \equiv \frac{1}{Z_0(\vec\psi)}\frac{\delta Z_0(\vec\psi)}{\delta\psi_j^{(0)}(\sigma)} \tag{21}$$

is an average spin in an external field $\vec\psi$ in the absence of spin-spin interaction. A direct calculation yields

$$\langle S_j^z(\sigma)\rangle = \left\langle\!\!\left\langle b_j + \sum_i b_i\int_0^\beta d\tau\bar\psi_i(\tau)\frac{\delta z_i(\tau)}{\delta\psi_j^{(0)}(\sigma)}\right\rangle\!\!\right\rangle, \tag{22}$$
$$b_i \equiv b\left(\int_0^\beta d\tau(\psi_i^{(0)} + z_i\bar\psi_i)\right).$$

In order to calculate corrections to the average spin coming from the Gaussian fluctuations over the mean field, one has to expand $A[\vec\psi]$ up to the second order in $\vec\eta$ as is done in Eq. (12). Expanding then the functional in $\langle\!\langle\cdots\rangle\!\rangle$ of Eq. (22) in powers of $\vec\eta$ and calculating all integrals, one will finally reach the goal. As is seen from Eq. (12), corrections will appear in powers of the effective interactions $J_{\text{eff;ln}}$ and $J_{\text{eff;tr}}$, namely

$$\langle S_j^z(\sigma)\rangle = b(\beta\Phi) + \sum_{pq}\int_0^\beta dudv\langle\!\langle\bar\eta_p(u)\eta_q(v)\rangle\!\rangle_0\left[\delta_{jp}\frac{\delta z_p(u)}{\delta\psi_q(v)}\bigg|_0 b'(\beta\Phi)\right.$$
$$\left.+\frac{\delta^2 z_p(u)}{\delta\psi_j^{(0)}(\sigma)\delta\psi_q(v)}\bigg|_0 b(\beta\Phi)\right] + \frac{1}{2}\sum_{pq}\int_0^\beta dudv\langle\!\langle\eta_p^{(0)}(u)\eta_p^{(0)}(v)\rangle\!\rangle_0\delta_{jp}\delta_{jq}b'', \tag{23}$$

$$\langle\!\langle\ldots\rangle\!\rangle_0 \equiv \frac{\int D\vec\eta\cdots e^{\frac{1}{2}\delta^2 A}}{\int D\vec\eta e^{\frac{1}{2}\delta^2 A}}.$$

By virtue of Eq. (7) one has

$$\left(\frac{d}{d\tau} - \Phi\right)\frac{\delta^2 z_i(\tau)}{\delta\psi_j^{(0)}(\sigma)\delta\psi_q(v)}\bigg|_0 = \delta_{ij}\delta(\tau-\sigma)\frac{\delta z_i(\tau)}{\delta\psi_q(v)}\bigg|_0.$$

The periodic boundary condition in τ then immediately yields

$$\frac{\delta^2 z_i(\tau)}{\delta\psi_j^{(0)}(\sigma)\delta\psi_q(v)}\bigg|_0 = -\frac{1}{2}K_{ij}(\tau-\sigma)K_{jq}(\sigma-v). \tag{24}$$

In view of this we finally arrive at

$$\langle S_j^z(\sigma)\rangle = b + \sum_{pq}\int_0^\beta dudv\Bigg[\delta_{jp}b' J_{qp}^{\rm tr}(v-u)K_{pq}(u-v) -bK_{jq}(\sigma-v)J_{qp}^{\rm tr}(v-u)K_{pj}(u-\sigma) +\frac{b''}{2}\delta_{jp}\delta_{jq}J_{pq}^{\rm ln}(u-v)\Bigg] + O(J^2), \quad (25)$$

where we have dropped the subscript "eff" in J. Formula (25) gives the expression for the spontaneous magnetization up to order $(a/R)^3$, where R is the radius of the exchange interaction and a is a lattice constant. The analytic result (25) coincides with the expression of the one-loop approximation obtained in Ref. 10 via the operator approach.

The semiclassical spin-wave calculation of $\langle S^z\rangle$ to first order in $1/S$ in the spirit of the Anderson paper[12] coincides with the low-temperature limit of Eq. (25). However, as is seen from this formula the coherent-state path-integral method takes into account the longitudinal spin excitations as well as the transverse ones and is certainly superior at higher orders. In fact the exact expression (22) generates all one-irreducible diagrams for the spontaneous magnetization.

V. GREEN FUNCTIONS

For definiteness, we shall consider the transverse Green function

$$G_{ij;{\rm tr}}(\tau,\sigma) \equiv \frac{1}{2}\langle T\, S_i^-(\tau)S_j^+(\sigma)\rangle = \langle\!\langle G_{ij}^{(0)}(\tau,\sigma|\vec{\psi}(t))\rangle\!\rangle, \quad (26)$$

where $G_{ij}^{(0)}(\tau,\sigma|\vec{\psi}(t))$ is the Green function of noninteracting spins in the external fluctuating field $\vec{\psi}(t)$:

$$G_{ij}^{(0)}(\tau,\sigma|\vec{\psi}(t)) \equiv \frac{2}{Z_0}\frac{\delta^2 Z_0(\vec{\psi})}{\delta\bar{\psi}_i(\tau)\delta\psi_j(\sigma)}.$$

One-irreducible diagrams are generated by

$$G_{ij;{\rm tr}}(\tau,\sigma) = 2\Bigg\langle\!\!\Bigg\langle \frac{\delta z_j(\sigma)}{\delta\psi_i(\tau)}b_j + \sum_l\int_0^\beta dt\bar{\psi}_l(t)\frac{\delta^2 z_l(t)}{\delta\psi_i(\tau)\delta\bar{\psi}_j(\sigma)}b_l + \sum_m\int_0^\beta dt\bar{\psi}_m(t)\frac{\delta z_m(t)}{\delta\psi_i(\tau)}\{\delta_{jm}z_m(\sigma)b'_m + z_j(\sigma)b_m b_j\} + \sum_{lm}\int_0^\beta dt_1dt_2\bar{\psi}_l(t_1)\bar{\psi}_m(t_2)\frac{\delta z_l(t_1)}{\delta\bar{\psi}_j(\sigma)}\frac{\delta z_m(t_2)}{\delta\psi_i(\tau)}\{b_l b_m + \delta_{lm}b'_l\}\Bigg\rangle\!\!\Bigg\rangle, \quad (27)$$

The last two terms contribute to orders higher than $O(J)$. In the low-temperature limit the representation (27) is simplified, since the terms depending on the derivatives of b will make no contribution.

The zeroth approximation gives

$$G_{ij;{\rm tr}}^{(0)} = 2\frac{\delta z_j(\sigma)}{\delta\psi_i(\tau)}\bigg|_0 b(\beta\Phi) = G_{ij;{\rm tr}}^{(0)}(\tau,\sigma|\Phi)$$

as it should. To first order in $J_{\rm eff}$ one has

$$\begin{aligned} G_{ij;{\rm tr}}(\tau,\sigma) = 2\sum_{pq}\int_0^\beta \langle\!\langle\bar{\eta}_p(u)\eta_q(v)\rangle\!\rangle_0 \Bigg[&\delta_{ip}b'\frac{\delta z_i(\tau)}{\delta\psi_j(\sigma)}\bigg|_0\frac{\delta z_p(u)}{\delta\psi_q(v)}\bigg|_0 + b\frac{\delta^3 z_i(\tau)}{\delta\psi_j(\sigma)\delta\bar{\psi}_p(u)\delta\psi_q(v)}\bigg|_0 \\ &+b\frac{\delta^3 z_p(u)}{\delta\psi_j(\sigma)\delta\bar{\psi}_i(\tau)\delta\psi_q(v)}\bigg|_0 \\ &+\delta_{ip}b'\frac{\delta z_p(u)}{\delta\psi_j(\sigma)}\bigg|_0\frac{\delta z_p(\tau)}{\delta\psi_q(v)}\bigg|_0 + b^2\frac{\delta z_p(u)}{\delta\psi_j(\sigma)}\bigg|_0\frac{\delta z_i(\tau)}{\delta\psi_q(v)}\bigg|_0\Bigg] + \sum_{pq}\int_0^\beta\langle\!\langle\eta_p^{(0)}(u)\eta_q^{(0)}(v)\rangle\!\rangle_0 \\ &\times\Bigg[\delta_{ip}\delta_{iq}b''\frac{\delta z_i(\tau)}{\delta\psi_j(\sigma)}\bigg|_0 + 2\delta_{iq}b'\frac{\delta^2 z_i(\tau)}{\delta\psi_j(\sigma)\delta\psi_p^{(0)}(u)}\bigg|_0 + b\frac{\delta^3 z_i(\tau)}{\delta\psi_j(\sigma)\delta\psi_p^{(0)}(u)\delta\psi_q^{(0)}(v)}\bigg|_0\Bigg]\ . \end{aligned} \quad (28)$$

From the Riccati equation (7) it follows that

$$\left(\frac{d}{dx}-\Phi\right)\frac{\delta^3 z_\alpha(x)}{\delta\bar{\psi}_{\beta_1}(y_1)\delta\psi_{\beta_2}(y_2)\delta\psi_{\beta_3}(y_3)}\bigg|_0 -\delta\alpha\beta_1\delta(x-y_1)\frac{\delta z_\alpha(x)}{\delta\psi_{\beta_2}(y_2)}\bigg|_0\frac{\delta z_\alpha(x)}{\delta\psi_{\beta_3}(y_3)}\bigg|_0 = 0$$

and

$$\left(\frac{d}{dt}-\Phi\right)\frac{\delta^3 z_i(t)}{\delta\psi_j(\sigma)\delta\psi_q^{(0)}(v)\delta\psi_p^{(0)}(u)}\bigg|_0 -\delta ip\,\delta(t-u)\frac{\delta^2 z_i(t)}{\delta\psi_j(\sigma)\delta\psi_q^{(0)}(v)}\bigg|_0 -\delta iq\,\delta(t-v)\frac{\delta^2 z_i(t)}{\delta\psi_j(\sigma)\delta\psi_p^{(0)}(u)}\bigg|_0 = 0$$

by employing the periodic boundary conditions. Taking into account Eqs. (15) and (24) one finds

$$\left.\frac{\delta^3 z_\alpha(x)}{\delta\bar{\psi}_{\beta_1}(y_1)\delta\psi_{\beta_2}(y_2)\delta\psi_{\beta_3}(y_3)}\right|_0$$

$$= -\frac{1}{4}K_{\alpha\beta_1}(x-y_1)K_{\beta_1\beta_2}(y_1-y_2)K_{\beta_1\beta_3}(y_1-y_3)$$

and

$$\left.\frac{\delta^3 z_i(t)}{\delta\psi_j(\sigma)\delta\psi_q^{(0)}(v)\delta\psi_p^{(0)}(u)}\right|_0$$

$$= \frac{1}{2}[K_{ip}(t-u)K_{pq}(u-v)K_{qj}(v-\sigma) +(p\leftrightarrow q; u\leftrightarrow v)].$$

The transverse Green function is then found to be

$$G_{ij;\mathrm{tr}}(\tau,\sigma) = \sum_{pq}\int_0^\beta dudv\Sigma^{\mathrm{tr}}_{ij;pq}(\tau,\sigma;u,v) \qquad (29)$$

with

$$\begin{aligned}\Sigma^{\mathrm{tr}}_{ij;pq}(\tau,\sigma;u,v) = {}& \delta_{ip}b'K_{ij}(\tau-\sigma)K_{pq}(u-v)J^{\mathrm{tr}}_{qp}(v-u) - bK_{ip}(\tau-u)K_{pq}(u-v)J^{\mathrm{tr}}_{qp}(v-u)K_{pj}(u-\sigma)\\ &-bK_{ij}(\tau-\sigma)K_{pi}(u-\tau)J^{\mathrm{tr}}_{qp}(v-u)K_{iq}(\tau-v)\\ &+\delta_{ip}b'J^{\mathrm{tr}}_{qp}(v-u)K_{pj}(u-\sigma)K_{pq}(\tau-v) + b^2K_{ip}(\tau-v)J^{\mathrm{tr}}_{qp}(v-u)K_{pj}(u-\sigma)^\star\\ &+\frac{1}{2}\delta_{ip}\delta_{iq}b''K_{ij}(\tau-\sigma)J^{\mathrm{ln}}_{qp}(v-u) - b'\delta_{iq}K_{ip}(\tau-u)J^{\mathrm{ln}}_{qp}(v-u)K_{pj}(u-\sigma)\\ &+bJ^{\mathrm{ln}}_{qp}(v-u)K_{ip}(\tau-u)K_{pq}(u-v)K_{qj}(v-\sigma) + O(J^2). \end{aligned} \qquad (30)$$

Each term in (30) corresponds to a particular one-irreducible diagram with a single interaction line. All of them are of first order in $(a/R)^3$ except for the term marked by $\star$ which is of zeroth order. The marked term together with $G^{(0)}_{\mathrm{tr}}$ makes up Eq. (18). As $(a/R)^3$ is usually regarded as a small parameter in a mean-field theory, the Green function (18) is to be considered as the zeroth-order or tree-level mean-field transverse Green function.

VI. CONCLUSION

We have presented a simplifying method for studying the thermodynamics of the quantum Heisenberg model based on the SU(2) coherent-state path integral. It provides a natural path-integral representation for interacting spin systems, which directly generates the conventional spin-diagrammatic expansion, usually obtained in the operator formalism, in a rather simple and transparent way.

This fact is of particular importance, because the technique can be extended to more complicated groups. For example, consider the Hamiltonian

$$H = \sum_{ab;ij} J^{ab}_{ij}Q^a_iQ^b_j$$

with Q^α being the generators of a Lie (super)group G. In this case also one is able to start from a Gaussian linearization similar to Eqs. (1) and (2). In the case of a supergroup it would involve Grassmann-valued fields as well. To proceed further one should employ a path integral over coherent states associated with irreducible representations of G. The integral is then again evaluated by a change of variables in accordance with the G action in the underlying phase (super)space. This technique results in an implicit analytical representation for Z which involves a set of auxiliary fields to be taken to satisfy a system of (super-) Riccati equations. The OSP(2|2) supergroup relevant for the Hubbard (t-J) model provides quite a nontrivial example of this kind,[13] Q^α representing the Hubbard operators. It would be desirable to investigate the Hubbard model along these lines. Such a study is currently being pursued.

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