Confinement of spinons in the CP^{M-1} model

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We use the $1/M$ expansion for the CP^{M-1} model to study the long-distance behavior of the staggered spin susceptibility in the commensurate, two-dimensional quantum antiferromagnet at finite temperature. At $M = \infty$ this model possesses deconfined spin-1/2 bosonic spinons (Schwinger bosons), and the susceptibility has a branch cut along the imaginary k axis. We show that in all three scaling regimes at finite T, the interaction between spinons and gauge-field fluctuations leads to divergent $1/M$ corrections near the branch cut. We identify the most divergent corrections to the susceptibility at each order in $1/M$ and explicitly show that the full static staggered susceptibility has a number of simple poles rather than a branch cut. We compare our results with the $1/N$ expansion for the O(N) sigma model.

I. INTRODUCTION

Numerous studies of the two-dimensional (2D) quantum Heisenberg antiferromagnet (QHAF) undertaken in the past few years have significantly improved our understanding of the behavior of these systems at low temperatures.[1-3] There is no ordered state at any finite T (and hence, no phase transitions), but there are nevertheless three distinct low-T regimes depending on the relative values of the temperature and the coupling constant, g. These three regimes are[1] (i) renormalized-classical (RC) regime, where T is smaller than the spin stiffness in the ordered ground state, ϱ_s; (ii) quantum-disordered (QD) regime, when g is larger than the critical coupling for the $T = 0$ disordering transition, and T is smaller than the gap Δ between the singlet ground state and the lowest excited state with $S = 1$; (iii) quantum-critical (QC) regime, which lies between the other two, and in which temperature is the largest infrared cutoff, $k_BT \gg \varrho_s, \Delta$.

It is very likely that the low-energy physics of a 2D QHAF, at least in the vicinity of the disordering transition, is adequately described by the O(3) nonlinear sigma model (see Sec. I). Some information about the properties of this model can be obtained from the Bethe ansatz solution,[4,5] but most of the thermodynamic properties have been studied using several available perturbative techniques. The perturbative approach to the O(3) sigma model was initiated many years ago by Polyakov.[6] In this approach, one departs from the ordered state at $T = 0$, and applies renormalization-group theory which accounts for the effects of classical fluctuations at small but finite T. The expansion parameter for the RG studies is $T/2\pi\varrho_s$. It has to be small, i.e., the system should be in the RC regime. The RG approach allows one to obtain a scale at which the fluctuation corrections to the spin-wave coupling constant become comparable to its bare value. This scale is then identified with the correlation length in a system. Calculations along these lines yielded $\xi = A(\hbar c/2\pi\varrho_s)\exp(2\pi\varrho_s/k_BT)$ (Ref. 7) and $S(q) = N_0^2\ (k_BT/2\pi\varrho_s)^2\xi^2 f(q\xi)$.[1] Here A is a constant [$A \approx 0.34$ (Ref. 5)], $S(q)$ is the static structure factor, N_0 is the staggered magnetization, and $f(x)$ tends to a finite value at $x = 0$.

Another widely used approach is the $1/N$ expansion for the O(N) sigma model. The physical results are obtained in this approach by extending the perturbation series in $1/N$ to $N = 3$. The advantage of the $1/N$ expansion is that it works in all three scaling regimes, and the point of departure is a *disordered* state at any finite temperature. The weak point of the theory is the absence of the physically motivated small parameter for the physical case $N = 3$. In the RC regime, however, the series of $1/N$ terms can be explicitly summed up, and for the physical case of $N = 3$ one obtains *exactly* the same results as in the RG approach. At arbitrary N, one finds

$$\xi = A_N(\hbar c/k_BT)[(N-2)k_BT/2\pi\varrho_s]^{1/(N-2)} \times \exp\{2\pi\varrho_s/[(N-2)k_BT]\}$$

and

$$S(q) = N_0^2\ (k_BT/2\pi\varrho_s)^{(N-1)/(N-2)} \times\xi^2 f_N(q\xi) \quad [A_3 \equiv A,\ f_3(x) \equiv f(x)].$$

The results of $1/N$ expansion for all three scaling regimes are collected in Ref. 2

Finally, the third approach is the Schwinger-boson theory. In this theory, the staggered magnetization field, $\vec{n}(r_i) = \epsilon_i\vec{S}_i/|S|$, where $\epsilon_i = +1$ or (-1), depending on the sublattice, is expressed as a bilinear combination of two Bose operators (CP^1 representation),

$$n_a = z^\dagger_\alpha \sigma^a_{\alpha\beta} z_\beta, \quad (1.1)$$

where α, β are SU(2) indexes, $a = x, y, z$, and σ^a are the Pauli matrices. This representation also introduces a U(1) gauge degree of freedom, because $\vec{S}$ remains invariant under the transformation of the bosonic fields, $z_\alpha(\mathbf{r}, t) \rightarrow z_\alpha(\mathbf{r}, t) e^{i\varphi(\mathbf{r}, t)}$. Each z-field quantum carries $S = 1/2$ and is therefore a bosonic spinon. The condition $\vec{n}^2 = 1$, however, imposes a local constraint $z^\dagger_\alpha z_\alpha = 1$, which implies that spinons can appear only in pairs. A mean-field version of the Schwinger-boson theory has recently received a lot of attention.[8] In the mean-field approximation, one reduces the on-site constraint to a constraint imposed on the averaged quantities, and decouples the term that is quartic in z in the spin Hamiltonian. From the solution of the self-consistent equations in the RC regime one then obtains the spin-correlation length, $\xi \sim (k_B T/\varrho_s) \exp(2\pi \varrho_s / k_B T)$, and the static structure factor, $S(q) \sim (k_B T / 2\pi \varrho_s)^2 \xi^2 \tilde{f}(q\xi)$, where $\tilde{f}(x)$ has the same asymptotic behavior as $f(x)$.[8] The temperature dependence of $S(q)$ and the exponent in the expression for the correlation length are the same as in other approaches but the correlation length acquires an extra power of T in the prefactor. Because of this incorrect prefactor in ξ, the validity of the Schwinger-boson mean-field theory has been questioned.[9] More recently, however, the Schwinger-boson approach to a 2D antiferromagnet has been applied in a systematic way, by means of a controllable $1/M$ expansion.[10,11] To generate this expansion, each z field was assumed to have M components rather than two (this changes the symmetry of the underlying sigma model to CP^{M-1}). The $1/M$ computations have been performed for field-theory[10] and condensed-matter[11] applications. It has been shown that the extra power of T in the prefactor is indeed an artifact of the mean-field, $M = \infty$, approximation: the correct prefactor is $T^{1-2/M}$, and it reduces to a constant for the physical case of $M = 2$. At the same time, the $1/M$ corrections to the static structure factor do not change the power of T, and the mean-field result for $S(q)$ is therefore valid for all M. Clearly then, for physical spins, the n-field and the Schwinger-boson approaches yield the same temperature dependence of the observables in the RC regime, as they indeed should. One can therefore safely use any of these perturbative techniques. Notice however, that the series of regular $1/M$ corrections are poorly convergent, while the regular $1/N$ corrections are usually small. This makes the Schwinger boson approach less reliable for practical purposes than the $1/N$ expansion for the O(N) sigma model.

There is, however, another discrepancy between the Schwinger-boson and the n-field approaches, which in our opinion has not been fully clarified in the literature. Namely, in the n-field approach, the staggered static spin susceptibility $\chi_s(k, \omega = 0)$ is proportional to the static n-field propagator which has a simple pole along the imaginary k axis, at $k = \pm i\xi^{-1}$. The residue of the pole is finite at $N = \infty$, and remains finite in the physical case of $N = 3$. This result is valid in all three scaling regimes. On the other hand, in the Schwinger-boson formalism, the staggered spin susceptibility is a convolution of two spinon propagators. At the mean-field level, spinons behave as free particles, and elementary calculations show that the staggered susceptibility has only a branch cut singularity at $k = 2im_0$, where m_0 is the mass (inverse correlation length) of a Schwinger boson. Since the behavior near the singularity in $\chi(ik, 0)$ determines the long-distance properties of the spin-correlators, the difference in the type of the singularity in the two models leads to different predictions about the long-distance behavior of the correlation function. Obviously, one of these predictions must be wrong.

In this paper, we show that the branch-cut behavior of $\chi(ik, 0)$ is also an artifact of the mean-field Schwinger-boson formalism. We will see that $1/M$ corrections are divergent near the branch cut, and eventually transform the branch cut into a simple pole. This phenomenon is closely related to the confinement of spinons in the CP^{M-1} model in $2+0$ dimensions, studied by Witten.[12] He found that massless gauge fluctuations in the CP^{M-1} model give rise to a linear confining potential between spinons, and this yields a bound state with a mass m which is a nonanalytical function of $1/M$: $m = 2m_0[1 + O(1/M^{2/3})]$. This result was reproduced in a number of more recent papers (for a review see, e.g., Ref. 13 and references therein). However, to the best of our knowledge, the effect of the $1/M$ corrections on the staggered susceptibility has not been studied in detail. The results of such study will be reported in this paper. Besides the RC regime, we will also study staggered susceptibility in the QD and QC regimes.

Before we proceed to the description of our calculations, it is useful to specify which $1/M$ corrections are essential to our analysis. The point is that in the RC regime, there exists a self-energy correction to the spinon propagator of the form $1/M \ln \ln(k_B T/m_0)$. Since m_0 is itself exponential in T, the double logarithm in fact reduces to $1/M \ln(\varrho_s/k_B T)$. A series of these logarithmic terms give rise to the above-mentioned change in the power of temperature in the preexponential factor in ξ from T to $T^{1-2/M}$. Below we will assume that these corrections are *already* included into the expression for the Schwinger-boson mass. We will therefore consider only *regular* $1/M$ corrections which, as we will show, are responsible for the confinement.

The paper is organized as follows. In Sec. II, we will briefly review the large M expansion for the CP^{M-1} sigma model and present $M = \infty$ results for the correlation length, spin susceptibility, and gauge-field propagator. In Sec. III, we consider the static staggered susceptibility in the RC regime. We first compute the lowest-order $1/M$ corrections, select the most divergent ones, and then sum up the ladder series of divergent $1/M$ terms by reducing the problem to a Schrodinger equation. Discrete solutions of this Schrodinger equation will correspond to the *poles* in the staggered susceptibility. In Secs. IV and V we report analogous calculations for QD and QC phases, respectively. Finally, in Sec. VI we state our conclusions and discuss open questions.

II. THE SIGMA MODEL

Our starting point is the partition function for the O(3) nonlinear sigma model in Euclidean space

$$\mathcal{Z} = \int Dn_l \delta(n_l^2 - 1) \times \exp\left\{ -\frac{\varrho_s^0}{2\hbar} \int_0^{1/T} d\tau \int d^2\mathbf{r} \left[\frac{1}{c_0^2} (\partial_\tau n_l)^2 + (\partial_i n_l)^2 \right] \right\}, \quad (2.1)$$

where $i = x, y$; $l = 1, 2, 3$, and ϱ_s^0 and c_0 are the bare spin stiffness and spin-wave velocity. For simplicity, throughout the paper we choose the units where $\hbar = 1$ and $c_0 = 1$. Vector $\mathbf{n}$ describes local staggered magnetization. In the CP^1 representation (1.1), $\mathcal{Z}$ transforms into

$$\mathcal{Z} = \int D\bar{z}\, Dz\, \delta(|z|^2 - 1) \times \exp\left\{ -2\varrho_s^0 \int_0^{1/T} d\tau \int d^2\mathbf{r} [|\partial_\mu z|^2 + |\bar{z}\partial_\mu z|^2] \right\}. \quad (2.2)$$

Here $\mu = \tau, x, y$. Introducing the Hubbard-Stratonovich vector field A_μ to decouple the quartic term, we obtain

$$\mathcal{Z} = \int D\bar{z}\, Dz\, DA_\mu \delta(|z|^2 - 1) \times \exp\left\{ -2\varrho_s^0 \int_0^{1/T} d\tau \int d^2\mathbf{r} |(\partial_\mu - iA_\mu)z|^2 \right\}. \quad (2.3)$$

Now we generalize the doublet z to the M-component complex vector, rescale the z field to $z \to \sqrt{M}z$, and introduce the coupling constant $g = M/2\varrho_s^0$. Introducing then the constraint into the action in a standard way, we obtain for the partition function

$$\mathcal{Z} = \int D\bar{z}\, Dz\, DA_\mu D\lambda \times \exp\left\{ -\frac{1}{g} \int_0^{1/T} d\tau \int d^2\mathbf{r} [|(\partial_\mu - iA_\mu)z|^2 + i\lambda(|z|^2 - M)] \right\}. \quad (2.4)$$

This is the partition function for the CP^{M-1} model.[14] The action in (2.4) is quadratic in the $\bar{z}, z$ fields, and we can therefore integrate them out. This generates an effective action for the A_μ and λ fields, which contains M only as a prefactor.[6,14] At large M, this effective action can be well approximated by the quadratic fluctuations of A_μ and λ around their saddle-point values[6]

$$i\langle\lambda\rangle = m^2, \langle A_\mu \rangle = 0. \quad (2.5)$$

The value of the spinon mass m can be obtained in the $1/M$ expansion by solving the constraint equation to any given order in $1/M$. The nonzero solution for m implies that the rotational symmetry is unbroken. A more detailed discussion of the derivation of the CP^{M-1} model can be found in Ref. 15.

At $M = \infty$, the theory is particularly simple, because it describes free massive spinons. The spinon Green's function $G_0(\vec{p}, \omega_n)$ is given by

$$G_0(\vec{p}, \omega_n) = \frac{1}{p^2 + \omega_n^2 + m_0^2}, \quad (2.6)$$

and the constraint equation reads

$$T \sum\nolimits_{\omega_n} \int \frac{d\vec{p}}{(2\pi)^2} G_0(\vec{p}, \omega_n) = \frac{1}{g}. \quad (2.7)$$

Using the Pauli-Villars regularization of ultraviolet divergencies, we obtain

$$\frac{m_0}{4\pi} + \frac{k_B T}{2\pi} \ln\left(1 - e^{-m_0/k_B T}\right) = 2\left(\frac{1}{g_c} - \frac{1}{g}\right), \quad (2.8)$$

where $g_c = 8\pi/\Lambda$, and Λ is the ultraviolet regulator. Depending on the values of g/g_c and the temperature, the solutions of (2.8) are[2,8] (i) RC regime ($\varrho_s > k_B T$): $m_0 = k_B T \exp -(4\pi\varrho_s/MT)$, where $\varrho_s = (M/2)\,(1/g - 1/g_c)$ is the renormalized stiffness; (ii) QD regime ($m_0 > k_B T$): $m_0 = \Delta + O(\exp -\Delta/T)$, where $\Delta = 8\pi(1/g_c - 1/g)$; (iii) QC regime ($T > \varrho_s, \Delta$): $m_0 = \Theta k_B T$, where $\Theta = 2\ \ln[(\sqrt{5} + 1)/2]$.

A. Staggered susceptibility

The static staggered susceptibility is defined in continuum limit as

$$\chi(k, 0)\delta_{ab} = \frac{a^2}{N_s} \int_0^{1/k_B T} d\tau \int \frac{d^2\mathbf{r}}{(2\pi)^2} \langle S_a(\mathbf{r}, \tau) S_b(0, 0) \rangle \times \exp[-i(\vec{k} + \vec{Q}) \cdot \vec{r}], \quad (2.9)$$

where $\vec{k}$ is a small momentum, $\vec{Q} = (\pi/a, \pi/a)$, and N_s is the number of spins in the system. As each spin is a by product of z fields, the physical susceptibility is related to the polarization operator of z, $\Pi(k, 0) = \Pi(k) = T \sum_\omega \int (d^2 q/4\pi^2) G(\vec{q}, \omega) G(\vec{k} + \vec{q}, \omega)$. In the RC and QC regimes, the relation between $\chi(k, 0)$ and $\Pi(k)$ can be obtained in the same way as in Ref. 3. We find

$$\chi(k, 0) = \frac{N_0^2}{2\varrho_s^2} \Pi(k), \quad (2.10)$$

where N_0 and ϱ_s are the fully renormalized on-site magnetization and spin stiffness at $T = 0$. Notice that there is a factor of 2 difference with the analogous expression for frustrated systems.[3] In the QD regime, the rescaling factor between susceptibility and polarization operator can be related to the overall factor in the local susceptibility.[2] However, in this regime, we will only obtain $\Pi(k)$ up to an overall factor, so there is no need to discuss the exact relation between $\chi(k)$ and $\Pi(k)$ in the QD regime.

B. Polarization operator

We now present the expressions for the polarization operator at $M = \infty$. In the RC regime, the summation over frequency reduces to the $\omega = 0$ term, and one obtains

$$\Pi_0(k) = k_B T \int \frac{d^2p}{4\pi^2} \frac{1}{(\vec{p}-\vec{k}/2)^2 + m_0^2} \frac{1}{(\vec{p}+\vec{k}/2)^2 + m_0^2}. \tag{2.11}$$

The momentum integration yields[2,3,10]

$$\Pi_0(k) = \frac{k_B T}{4\pi\delta\sqrt{\delta^2 - m_0^2}} \ln\left[\frac{\sqrt{\delta^2 - m_0^2} + \delta}{m_0}\right], \tag{2.12}$$

where we introduced $\delta^2 = k^2/4 + m_0^2$. Clearly, $\Pi_0(k)$ has a branch-cut singularity at $\delta = 0$, i.e., at $k^2 = -4m_0^2$. Near the singularity, we obtain

$$\Pi_0(k) = \frac{k_B T}{8 m_0 \delta}. \tag{2.13}$$

The behavior of $\Pi(k)$ near the branch cut determines the long-distance behavior of the spin susceptibility in real space. Evaluating the Fourier transform of (2.13), we obtain $\chi(r) \sim \exp(-r/\xi)/r$, where $\xi = 1/2m_0$ at the mean-field level. At the same time, the susceptibility in the O(N) sigma model has a simple pole at $k^2 = -\xi^{-2}$, and its long-distance behavior is $\chi(r) \sim \exp(-r/\xi)/\sqrt{r}$.

For the QD regime, the leading term in $\Pi(k)$ is the $T = 0$ piece. The frequency summation is then replaced by the integration, and, to leading order in δ, one obtains

$$\Pi_0(k) = \frac{1}{16\pi m_0} \ln\left(\frac{m_0}{\delta}\right). \tag{2.14}$$

We see that the polarization operator has only a weak logarithmical singularity. In real space, the singularity in $\Pi_0(k)$ gives rise to $\chi(r) \sim \exp(-2m_0 r)/r^2$, which is again different from the mean-field result for the n-field model in this regime, $\chi(r) \sim \exp(-r/\xi)/r$.

Finally, for the QC regime, elementary considerations show that the most singular behavior in $\Pi(k)$ comes from the $\omega = 0$ term in the summation over frequency, and hence $\Pi(k)$ is still given by (2.13). This result is intuitively obvious as the QC regime is the interpolation regime between the classical and the quantum-disordered ones, and the singularity in $\Pi(k)$ is much stronger in the classical regime.

C. Gauge-field propagator

In the next sections, we will show that the mean-field ($M = \infty$) behavior of susceptibilities changes drastically at finite M. The $1/M$ corrections to susceptibility include the propagators of the gauge field, A, and the constraint field, λ. We will see that to study confinement, it is sufficient to know the propagators of λ and A at distances much larger then the spin-correlation length. At $k \ll \xi^{-1}$, the fluctuations of λ and of the temporal component of the gauge field are well screened.[2,11] However, there is no screening for the spatial part of the gauge field.[14,10,11] The spatial component of the gauge-field propagator is given by

$$\Pi_{\mu\nu}(q,\Omega) = 2\delta_{\mu\nu} k_B T \sum\nolimits_\omega \int \frac{d\vec{k}}{(2\pi)^2} G_0(k,\omega)$$
$$- k_B T \sum\nolimits_\omega \int \frac{d\vec{k}}{(2\pi)^2} G_0(k,\omega) G_0$$
$$\times(\vec{k}+\vec{q}, \omega+\Omega)(2k+q)_\mu (2k+q)_\nu, \tag{2.15}$$

where $k_0 = \omega, q_0 = \Omega$. For the RC and the QC regimes, $\xi^{-1} \leq k_B T$, and at $k \ll \xi^{-1}$ we can also restrict to the $\Omega = 0$ component of Π. The evaluation of integrals in the limit of $q \ll m_0$ is then straightforward, and we find

$$\Pi_{\mu\nu}(q,0) = \left(\delta_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}\right) \frac{q^2}{12\pi} T \sum_n \frac{1}{\omega_n^2 + m_0^2}. \tag{2.16}$$

Inverting now $\Pi_{\mu\nu}$ in the Coulomb gauge, we find for the gauge-field propagator $D_{\mu\nu}$ at small momenta

$$D_{\mu\nu}(q) = \left(\delta_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}\right) D(q) = \left(\delta_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}\right) \frac{12\pi A}{q^2}, \tag{2.17}$$

where $\mu(\nu) = x, y$, and the values of A are $A = m_0^2/k_B T$ (RC regime), $A = 2\Theta k_B T/\sqrt{5}$ (QC regime). In the QD regime at $T = 0$, Ω is a continuous variable, and q, Ω are the components of a 3D *momentum*. We found that Eq. (2.17) is still valid in this regime, $A = 2m_0$, and μ and ν are running over x, y, and τ.

We now consider separately $1/M$ corrections in the three scaling regimes.

III. RENORMALIZED-CLASSICAL REGIME

A. First order in $1/M$

As we said above, we only have to consider the $1/M$ corrections associated with the fluctuations of the gauge

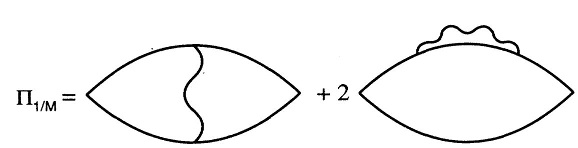

FIG. 1. The first-order $1/M$ corrections to the polarization operator of spinons.

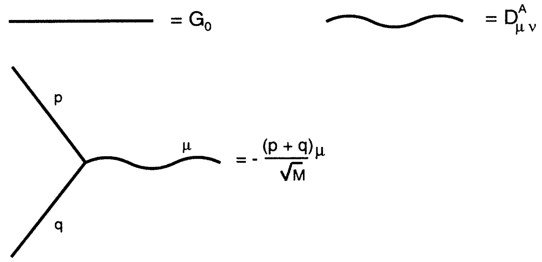

FIG. 2. The diagrammatic representation of the spinon and the gauge-field propagators and of the spinon-gauge-field interaction vertex.

field. Corrections related to the fluctuations of the constraint field renormalize the spinon mass, but these fluctuations are screened at $q < 2m_0$ and are therefore irrelevant for the confinement. The first-order $1/M$ corrections to the polarization operator $\Pi(k)$ are shown in Fig. 1, the propagators and vertex function which appear in the $1/M$ expansion are collected in Fig. 2. A simple power counting argument shows that the most singular corrections appear if we select a contribution proportional to the *external* momentum, k, at each interaction vertex with the gauge field. We then obtain near $k = \pm 2im_0$, and setting $G(\vec{p}) = G(\vec{p}, \omega = 0)$,

$$\Pi_{1/M}(k) = \frac{1}{M}(k_BT)^2 \int \frac{d^2p d^2l}{16\pi^4} G(\vec{p} - \vec{k}/2)G(\vec{p} + \vec{k}/2) \times G(\vec{p} + \vec{l} + \vec{k}/2) \times \left[2G(\vec{p} + \vec{k}/2) - G(\vec{p} + \vec{l} - \vec{k}/2)\right] k_\mu k_\nu \times \left(\delta_{\mu\nu} - \frac{l_\mu l_\nu}{l^2}\right) D(l), \tag{3.1}$$

or, in explicit form,

$$\Pi_{1/M}(k) = -\frac{4m_0^2}{M}(k_BT)^2 \int \frac{d^2p d^2l}{16\pi^4} D(l) \frac{\sin^2\psi}{(p^2+\delta^2)^2 + 4m_0^2p^2\cos^2\varphi} \times \left(2\frac{(\vec{p}+\vec{l})^2 + \delta^2 - i2m_0(p\cos\varphi + l\cos\psi)}{[(\vec{p}+\vec{l})^2+\delta^2]^2 + 4m_0^2(p\cos\varphi + l\cos\psi)^2} \frac{p^2+\delta^2 - i2m_0p\cos\varphi}{(p^2+\delta^2)^2 + 4m_0^2\cos^2\varphi} - \frac{1}{[(\vec{p}+\vec{l})^2+\delta^2]^2 + 4m_0^2(p\cos\varphi + l\cos\psi)^2}\right), \tag{3.2}$$

where φ and ψ are the angles between $\vec{p}$ and $\vec{k}$, and $\vec{l}$ and $\vec{k}$, respectively. The key point in the computation of $\Pi_{1/M}(k)$, as well as in other computations later in the paper, is that for $\delta^2 \ll m_0^2$ (which is the only one we consider), the angular integration is confined to a region where $\varphi, \psi \approx \pm\pi/2$.[16] For practical purposes, it is convenient to restrict the angular integration to the vicinity of $\pi/2$, but extend the integration over p and l from $-\infty$ to $+\infty$. Rescaling the angular variables, $\varphi \to \varphi/2m_0p, \psi \to \psi/2m_0l$, we then obtain

$$\Pi_{1/M}(k) = -\frac{1}{M}\left(\frac{k_BT}{4\pi^2}\right)^2 \int_{-\infty}^{\infty} dp \int_{-\infty}^{\infty} dl \int_{-\infty}^{\infty} d\varphi \int_{-\infty}^{\infty} d\psi \frac{1}{(p^2+\delta^2)^2+\varphi^2} \times \left(2\frac{(p-l)^2+\delta^2 - i(\varphi+\psi)}{[(p-l)^2+\delta^2]^2 + (\varphi+\psi)^2} \frac{p^2+\delta^2-i\varphi}{(p^2+\delta^2)^2+\varphi^2} - \frac{1}{[(p-l)^2+\delta^2]^2+(\varphi+\psi)^2}\right). \tag{3.3}$$

The angular integration extends up to $|\varphi|, |\psi| \sim O(1/\delta)$. The integrals are convergent in the ultraviolet, so we can safely set the limits of the angular integration equal to infinity. The integration is now trivial, and we find

$$\Pi_{1/M}(k) = -\frac{1}{M}\left(\frac{k_BT}{4\pi}\right)^2 \int_{-\infty}^{\infty} dp \int_{-\infty}^{\infty} dl\, D(l) \frac{1}{p^2+\delta^2}\left(\frac{1}{p^2+\delta^2} - \frac{1}{(p-l)^2+\delta^2}\right). \tag{3.4}$$

We emphasize at this point that only the momenta perpendicular to the external momentum $\vec{k}$ contribute to the angular integration. This implies that the evaluation of the $1/M$ correction to the polarization operator in $2+0$ dimensions in fact reduces to a one-dimensional problem. This dimensional transmutation will play a key role in our analysis of the series of $1/M$ terms. Finally, substituting $D(l)$ from (2.17) and performing the momentum integration, we obtain

$$\Pi_{1/M}(k) = -\frac{3\pi m_0^2 T}{M(k^2+4m_0^2)^2}. \tag{3.5}$$

Notice the absence of divergencies in the integration over l—this is the result of including both self-energy and vertex corrections into $\Pi_{1/M}(k)$.

In obtaining (3.5), we implicitly assumed that $\delta^2 = (k^2+4m_0^2)/4 \geq 0$. Calculations for $\delta^2 \leq 0$ proceed along the same lines, and the result is

$$\Pi_{1/M}(k) = i\frac{12m_0^2T}{M(k^2+4m_0^2)^2}, \quad k^2+4m_0^2 \to -0. \tag{3.6}$$

The $1/M$ corrections to the polarization operator in the RC regime were studied earlier by Campostrini and

Rossi.[10] They numerically evaluated the leading singularity in $\Pi_{1/N}(k)$ for $\delta^2 \to +0$ and found the same result as in (3.5), with the numerical prefactor 9.425 which, as we show, is in fact exactly 3π. However, their estimate for $\Pi_{1/M}(k)$ for negative δ^2 is inconsistent with (3.6).

Let us now discuss the $1/M$ results. Our first observation is that the $1/\delta^4$ divergence of $\Pi_{1/M}(k)$ is stronger than one could expect assuming that $\Pi(k)$ preserves its form (2.13), and the two-spinon mass has a regular $1/M$ expansion [the latter would correspond to $\Pi_{1/M}(k) \sim O(\delta^{-3})$]. A crude estimate that provides insight into this singularity may be obtained by absorbing the $1/M$ correction into a mass renormalization and solving a self-consistent equation for the mass.[10] One then obtains $\Pi \sim (k^2+m^2)^{-1/2}$, where $m = 2m_0[1+O(1/M^{2/3})]$, which is consistent with Witten's result for the mass renormalization. However, this self-consistent procedure is clearly not exact, even for large M, because near $k^2=-m^2$, $\delta \sim M^{-1/3}$, and the $1/M$ contribution to Π has the same order $O(1/M^{-1/3})$ as the leading term.[10] Moreover, simple estimates show that the higher-order $1/M$ corrections behave as $\delta^{-1}(M\delta^3)^{-l}$ and therefore all have the same order near $k^2=-m^2$.

We will demonstrate in the next section that the self-consistent procedure yielding $\Pi \sim (k^2+m^2)^{-1/2}$ is actually incomplete because the $1/M$ corrections contribute to both the mass renormalization and to the renormalization of the spinon wave function. To see this, we will need to solve the problem exactly by summing up the series of the most divergent $1/M$ corrections (but we will still assume that $M \gg 1$ and neglect terms which have relative smallness in $1/M$).

B. Solution of ladder series

The relevant diagrams at each order in $1/M$ are presented in Fig. 3. These diagrams form a ladder series.

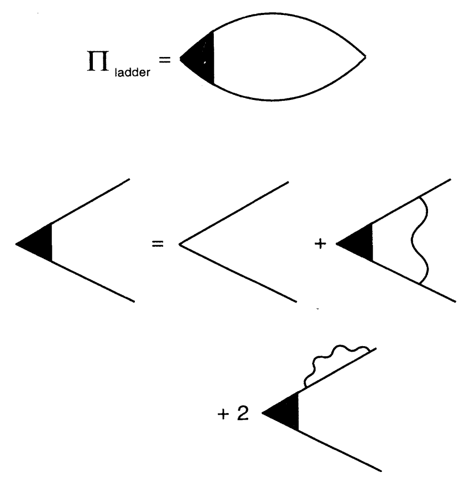

FIG. 3. The full polarization operator and the equation for the full vertex function. The most divergent corrections to the mean-field value of $\Pi(k)$ near $k=\pm 2im_0$ comes form ladder series. For convenience, both vertex and self-energy terms are absorbed into the vertex renormalization.

We will first discuss the solution of the ladder series and then show that other diagrams are relatively smaller at large M.

It is convenient to formally include both self-energy and vertex renormalization terms into an effective vertex renormalization, such that

$$\Pi(k) = k_BT \int \frac{d^2p}{4\pi^2}\Gamma(\vec{p},\vec{k})\; G(\vec{p}+\vec{k}/2)G(\vec{p}+\vec{k}/2). \tag{3.7}$$

The integral equation for the vertex can be written down as

$$\Gamma(\vec{p},\vec{k}) = 1 + \frac{k_BT}{M}\int \frac{d^2l}{4\pi^2}\;\Gamma(\vec{p}+\vec{l},\vec{k})\; G(\vec{p}+\vec{k}/2+\vec{l})G(\vec{p}-\vec{k}/2+\vec{l})(2\vec{p}+\vec{k}+\vec{l})_\mu\;(2\vec{p}-\vec{k}+\vec{l})_\nu\; D_{\mu\nu}(l)$$
$$+\frac{2k_BT}{M}\int \frac{d^2l}{4\pi^2}\;\Gamma(\vec{p},\vec{k})G(\vec{p}+\vec{k}/2)G(\vec{p}+\vec{k}/2+\vec{l})(2\vec{p}+\vec{k}+\vec{l})_\mu\;(2\vec{p}+\vec{k}+\vec{l})_\nu\; D_{\mu\nu}(l). \tag{3.8}$$

Again, the angular integration over both $\vec{p}$ and $\vec{l}$ is confined to a narrow region ($\sim\delta$) in which both internal momenta are nearly orthogonal to the external momentum $\vec{k}$. We assume that $\Gamma(\vec{p},\vec{k})$ and $\Gamma(\vec{p}+\vec{l},\vec{k})$ are nonsingular for these values of the angles. We then can set $\vec{p},\vec{l}$ to be orthogonal to $\vec{k}$ in Γ and perform angular integration in the Green functions. Restricting, as before, to the integration only near $\pi/2$, and extending the integration over p and l from $-\infty$ to $+\infty$, we obtain

$$\Pi(k) = \frac{k_BT}{4m_0}\int_{-\infty}^{\infty}\frac{dp}{2\pi}\,\frac{\Gamma(p,k)}{p^2+\delta^2}, \tag{3.9}$$

where

$$\Gamma(p) = 1 + \frac{m_0k_BT}{2\pi M}\int_{-\infty}^{\infty} dl\, D(l)\frac{\Gamma(p-l,k)}{(p-l)^2+\delta^2} - \frac{m_0k_BT}{2\pi M}\,\frac{\Gamma(p,k)}{p^2+\delta^2}\int_{-\infty}^{\infty} dl\, D(l). \tag{3.10}$$

Substituting now $\Gamma(p,k) = (p^2+\delta^2)\Psi(p)$, we can rewrite (3.10) as

$$(p^2+\delta^2)\Psi(p)$$
$$= 1 - \frac{m_0k_BT}{2\pi M}\int_{-\infty}^{\infty} dl\, D(l)\,[\Psi(p)-\Psi(p-l)]\,. \tag{3.11}$$

Finally, performing a Fourier transformation to real space,

$$\Psi(x) = \int_{-\infty}^{\infty} \frac{dp}{2\pi} e^{ipx} \Psi(p), \tag{3.12}$$

we obtain for $\Psi(x)$ a *one-dimensional* Schrödinger equation with a source

$$\left(-\frac{d^2}{dx^2} + V(x) + \delta^2\right) \Psi(x) = \delta(x). \tag{3.13}$$

The potential $V(x)$ is given by

$$V(x) = \frac{m_0 k_B T}{M} \int_{-\infty}^{\infty} \frac{dl}{2\pi} D(l) \left(1 - e^{-ilx}\right) = \frac{6\pi m_0^3}{M} |x|. \tag{3.14}$$

Notice that the integral over l is free from divergencies. This is again the result of including both self-energy and vertex corrections in the ladder series.

Going back to (3.9), we see that $\Pi(k)$ takes a simple form

$$\Pi(k) = \frac{k_B T}{4 m_0} \Psi(x = 0). \tag{3.15}$$

We emphasize at this point that $\Psi(x)$ is *not* a wave function of the 1D Schrödinger equation, but rather a Green's function of the inhomogeneous differential equation (3.13). This equation was solved earlier in the content of the weak localization theory,[17] and we simply quote the result:

$$\Psi(x = 0) = -\frac{1}{2} \left(\frac{6\pi m_0^3}{M}\right)^{-1/3} \frac{Ai(s)}{Ai'(s)}, \tag{3.16}$$

where $s = \delta^2 \, (6\pi m_0^3/M)^{-2/3}$, $Ai(s)$ is the Airy function, and Ai' is its derivative with respect to the argument. For positive s, both Ai and Ai' decay exponentially, but for negative s, they have zeros.[18] The singular contributions to $\Psi(x = 0)$ indeed come from the zeros of Ai'. Expanding near each of these zeros, we obtain

$$Ai'(s) \approx (s - s_n) \; Ai''(s_n) = -|s_n|(s - s_n) Ai(s_n), \tag{3.17}$$

where $s_n < 0$ is the nth zero of the derivative of the Airy function. Clearly, the polarization operator now has a set of simple poles at $\delta^2 = s_n (6\pi m_0^3/M)^{2/3}$.

Finally, we remind that we are actually interested in the long-distance behavior of the spin-correlation function. At long distances, only the pole with the smallest mass is relevant. Near this pole, we can approximate the polarization operator and, hence, the staggered susceptibility as

$$\chi(k, 0) = \frac{N_0^2}{\varrho_s} \frac{k_B T}{2\pi \varrho_s} \left(\frac{2}{M}\right)^{1/3} \frac{Z}{k^2 + m^2}, \tag{3.18}$$

where $Z = (3\pi^4)^{1/3}/2|s_1|$, $s_1 \simeq -1.02$,[18] and the renormalized mass m^2 is given by

$$m^2 = 4 m_0^2 \left[1 - s_1 \left(\frac{6\pi}{M}\right)^{2/3}\right]. \tag{3.19}$$

This last result coincides with the expression for the mass obtained by Witten.[12]

Equation (3.18) is the key result of this section. We see that the series of divergent $1/M$ corrections near the branch cut not only produce nonanalytical renormalization of the two-spinon mass, but also change the branch-cut behavior of the staggered susceptibility to a set of poles at discrete values of k. In real space, we then have $\chi(r) \sim M^{-1/3} \; \exp(-mr)/\sqrt{r}$ at very large distances which agrees with the result for the n-field sigma model. Notice, however, that the residue of each pole contains a factor $M^{-1/3}$ and vanishes in the limit of $M = \infty$. In this limit, the spacing between neighboring poles vanishes, and one recovers the mean-field branch-cut solution for $\Pi(k)$, and hence $\chi(r) \sim \exp(-2m_0 r)/r$. It is not difficult to show that at finite but large M, the asymptotic behavior associated with the lowest pole exists only at very large distances, $mr > M^{1/3}$, while at smaller distances the behavior of spin correlations remains the same as in the mean-field theory. Indeed, at large M, the asymptotic behavior is of no practical relevance, since at $mr \sim M^{1/3} \gg 1$, the spin correlation function is already negligibly small. However, for the physical case of $M = 2$, we expect the behavior associated with the pole to dominate at all scales larger than the correlation length.

Notice, however, that the existence of the nonanalytical corrections to the spin correlation length makes it very difficult to obtain the exact expression for $\xi = m^{-1}$ in the CP^{M-1} model. The exact value of ξ in the O(N) model is known at arbitrary N.[2,13] The $M = 2$ value of the overall factor Z in the staggered susceptibility (3.18) ($Z = 3.26$) is also substantially larger than $Z \sim 2$ expected for the O(3) model from the $1/N$ expansion.[2]

Before concluding this section, we show that the terms not included in the ladder series do not contribute to the renormalization of the polarization operator to leading order in $1/M$. The key point is that if we allow the gauge-field propagators to cross each other even once, the integral over the internal momentum of the two Green functions located between the propagators gives zero because the poles are located in the same half-plane. To illustrate this, consider the second, "umbrella"-like, diagram in Fig. 4. The analytical expression for this diagram is

$$\begin{aligned}\Pi_{\text{ext}} = \frac{16 m_0^4 (k_B T)^2}{M^2} \int\!\!\int\!\!\int \frac{d^2p d^2l d^2q}{64\pi^6} \; D(l) D(u) \; \frac{\sin^2\psi \sin^2\gamma}{(p^2+\delta^2)^2 + 4m_0^2 p^2 \cos^2\varphi} \\ \times \frac{1}{[(\vec{p}+\vec{q})^2+\delta^2]^2 + 4m_0^2 (p\cos\varphi + q\cos\gamma)} \frac{1}{[(\vec{p}+\vec{l})^2+\delta^2] + 2im_0(p\cos\varphi + l\cos\psi)} \\ \times \frac{1}{[(\vec{p}+\vec{l}+\vec{u})^2+\delta^2] + 2im_0(p\cos\varphi + l\cos\psi + u\cos\gamma)}.\end{aligned} \tag{3.20}$$

Here φ, ψ, and γ are the angles between the external momentum $\vec{k}$ and the internal ones $\vec{p}$, $\vec{l}$, and $\vec{q}$, respectively. As before, the relevant $M^{-2}(k^2+\delta^2)^{-7/2}$ contribution from this diagram is confined to the integration over internal momenta which are nearly orthogonal to $\vec{k}$. Expanding the angles around $\pi/2$, rescaling them in the same way as in (3.3), and shifting the variable $\psi \to \psi - \gamma$, we find that the angular integration in (3.20) reduces to

$$\int_{-\infty}^{\infty} d\gamma \int_{-\infty}^{\infty} d\varphi \frac{1}{(p^2+\delta^2)^2+\varphi^2} \frac{1}{[(p-q)^2+\delta^2]^2+(\varphi+\gamma)^2} \times \int_{-\infty}^{\infty} d\psi \frac{1}{[(p-l)^2+\delta^2]+i\psi} \frac{1}{[(p-l-q)^2+\delta^2]+i(\varphi+\psi)} . \quad (3.21)$$

The integration over ψ then gives zero. The same is true for all other "crossing" diagrams, as well as for the "rainbow"-like graphs for the self-energy (see Fig. 4). Note that this result does not depend on the dimensionality.

IV. QUANTUM-DISORDERED REGION

We now perform the same type of analysis for the QD regime. This is the low-temperature regime at $g > g_c$, when long-range order in the ground state is destroyed by quantum fluctuations. The ground state is then a total spin singlet, and there is a gap Δ towards the lowest excited triplet state with $S = 1$. The temperature corrections to the staggered susceptibility in the QD regime are exponentially small, and we therefore restrict our analysis in this section to $T = 0$. At zero-temperature frequency becomes a continuous variable, and perturbative $1/M$ expansion has to be performed in a three-dimensional space time. The higher dimensionality has already shown up in the computation of the polarization operator at $M = \infty$, which in the QD regime has only a weak, logarithmic singularity at $k^2 = -4m_0^2$ [see Eq. (2.14)].

A. $1/M$ corrections

The computation of the leading $1/M$ corrections proceeds exactly in the same way as in the RC regime. One can easily verify that the typical internal momenta in the 3D analog of (3.2) are of the order of δ and nearly orthogonal to the external momentum $\vec{k}$. This, in turn, implies that all internal momenta are confined to the plane perpendicular to $\vec{k}$, which makes the problem effectively *two dimensional.*

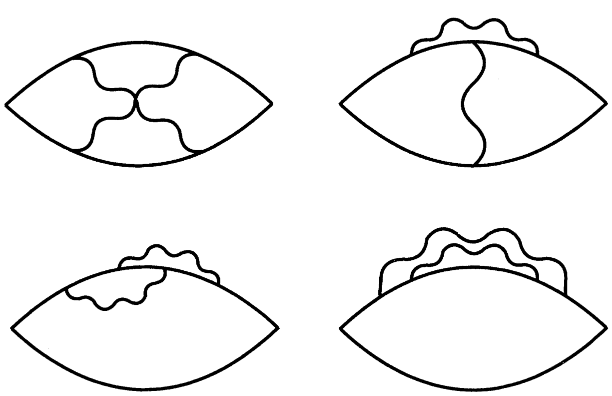

FIG. 4. Examples of the second-order diagrams which do not contribute to the ladder series. All these diagrams are less divergent near the branch cut singularity than those included in Fig. 3.

Technically this can be seen as follows. Consider 3D vectors $\vec{p}$ and $\vec{l}$. Let them form angles φ and ψ with $\vec{k}$, respectively. Then $\vec{p}\,\vec{l} = pl[\cos\varphi\cos\psi + \sin\varphi\sin\psi\cos\theta]$, where θ is the azimuthal angle ($0 \le \theta \le 2\pi$). In the spirit of our approximation, we expand φ and ψ around $\pi/2$. Then immediately $\vec{p}\,\vec{l} = pl[\cos\theta + O(\delta^2)] \approx pl\cos\theta$, which is the scalar product in 2D.

Performing the angular integration in the same way as in the previous section, we obtain

$$\Pi_{1/M}(k) = -\frac{1}{4\pi^2 M} \int\int \frac{d^2p\,d^2l}{16\pi^4} \frac{D(l)}{p^2+\delta^2} \times \left(\frac{1}{p^2+\delta^2} - \frac{1}{(\vec{p}+\vec{l})^2+\delta^2} \right) . \quad (4.1)$$

Evaluating this integral, we find

$$\Pi_{1/M}(k) = -\frac{3m_0}{4\pi M\delta^2} \left(\ln 1/\delta + C\right) , \quad (4.2)$$

where C is a constant. As in the RC regime, the $1/M$ correction to Π has a stronger dependence on δ than is required for a regular $1/M$ expansion for the two-spinon mass in (2.14). However, contrary to the previous case, the $(\ln\delta)/\delta^2$ corrections come only from the self-energy term. Vertex corrections only contribute to a constant C term in (4.2). Meanwhile, if we look back on how the transformation of a branch cut into a pole was obtained in the previous section, we can see that this transformation is primarily due to vertex corrections. The inclusion of the self-energy terms only serves to make the Fourier transform of $D(l)$ infrared-finite, and allows the correct mass renormalization. By analogy, we can expect that the series of logarithmic terms will only contribute to the mass renormalization, but the mean-field form of the polarization operator will survive. We performed an explicit computation of the ladder series of the logarithmic $1/M$ corrections, and found that with the logarithmic accuracy, the polarization operator is given by

$$
\begin{aligned}
\Pi_{\ln}(k) &= \frac{1}{8\pi m_0}\int \frac{dp\cdot p}{p^2+\delta^2}\left[1 - \frac{12m_0^2}{M}\frac{\ln m_0/\delta}{p^2+\delta^2}\right.\\
&\quad \left. + \left(\frac{12m_0^2}{M}\frac{\ln m_0/\delta}{p^2+\delta^2}\right)^2 + \cdots\right]\\
&= \frac{1}{8\pi m_0}\int \frac{dp\cdot p}{p^2+\delta^2+\frac{12m_0^2}{M}\ln\frac{m_0}{\delta}}\\
&= \frac{1}{16\pi m_0}\ln\left(\frac{m_0^2}{k^2+\bar{m}^2}\right),
\end{aligned}
\tag{4.3}
$$

where

$$
\begin{aligned}
\bar{m}^2 &= 4m_0^2\left(1+\frac{6}{M}\ln\frac{m_0^2}{m^2-m_0^2}\right)\\
&= 4m_0^2\left(1+\frac{6}{M}\ln M + O(M)\right).
\end{aligned}
\tag{4.4}
$$

We see that $\Pi_{\ln}(k)$ still has a logarithmic singularity at $k^2=-\bar{m}^2$. However, this result is still incomplete, and, in fact, we have to go beyond the logarithmic accuracy and include vertex corrections, which are crucial for the renormalization of the functional dependence of the propagator. Therefore, we must again consider the full ladder equation for the polarization operator.

B. The ladder diagrams

As before, the ladder equation for the polarization operator is obtained by multiplying the equation for the vertex function $\Gamma(p)$ [which is a 3D analog of (3.8)] by $G(\vec{p}+\vec{k}/2)G(\vec{p}-\vec{k}/2)$ and integrating over 3D momentum $\vec{p}$. This procedure is completely equivalent to that in the RC regime, and we present only the result. Introducing $\Psi(p)=\Gamma(p)/(p^2+\delta^2)$, we obtain

$$
\begin{aligned}
\Psi(p)(p^2+\delta^2) &= 1+\frac{m_0}{M}\int\frac{d^2\vec{l}}{4\pi^2}D(l)\Psi(\vec{p}+\vec{l})\\
&\quad -\frac{m_0}{M}\int\frac{d^2\vec{l}}{4\pi^2}D(l)\Psi(\vec{p}).
\end{aligned}
\tag{4.5}
$$

In real space, this equation is again equivalent to the Schrödinger equation with a source

$$
\left(-\nabla^2+V(r)+\delta^2\right)\Psi(r)=\delta(r),
\tag{4.6}
$$

where $V(r)$ is the Fourier transform of the gauge-field propagator

$$
V(r)=+\frac{m_0}{M}\int\frac{d^2\vec{l}}{4\pi^2}D(l)\left(1-e^{i\vec{l}\vec{r}}\right)=\frac{12m_0^2}{M}\ln|r|.
\tag{4.7}
$$

The integral over $\vec{l}$ is again infrared finite. The dependence on M in this equation can be eliminated by a rescaling $r\to\left(\frac{12m_0}{M}\right)^{-1/2}\tilde{r}$, and we obtain

$$
\left(-\tilde{\nabla}^2+\ln|\tilde{r}|+\epsilon\right)\Psi(\tilde{r})=\delta(\tilde{r}),
\tag{4.8}
$$

where the derivatives are taken with respect to $\tilde{r}$, and we introduced

$$
\epsilon=\frac{M}{12m_0^2}\left(\delta^2-\frac{6m_0^2}{M}\ln\frac{12m_0^2}{M}\right)=\frac{M}{48m_0^2}(k^2+\bar{m}^2),
\tag{4.9}
$$

where $\bar{m}$ is given by (4.4). Polarization operator is again related to $\Psi(r=0)$ via $\Pi(k)=(\pi/2m_0)\Psi(r=0)$.

Surprisingly, to the best of our knowledge, no exact solution of the 2D Schrödinger equation with a logarithmic potential ("2D hydrogen atom") is known. However, as $V(r)\sim\ln|r|$ is unbounded from above, there exists a discrete set of energy levels for the homogeneous Schrödinger equation, and, just as in the RC regime, $\Psi(0)$ will have an infinite number of simple poles at some discrete $\epsilon_n\sim O(1)$:

$$
\Psi(0)\propto\sum_n\frac{A_n}{\epsilon-\epsilon_n}=\frac{48m_0^2}{M}\sum_n\frac{A_n}{k^2+\bar{m}^2-\frac{48m_0^2}{M}\epsilon_n}.
\tag{4.10}
$$

Here A_n is the (unknown) residue of the nth pole. Using the WKB approximation it is easy to estimate that at large n, $\epsilon_n\sim\ln(n)$, and $A_n\sim n^{-1}$. Also observe that momentum k in (4.10) is a vector in $2+1$ space-time dimensions. Hence, Eq. (4.10) in fact describes the dynamic staggered susceptibility.

Equation (4.10) is the key result of this section. We have found that, just as was the case in the RC region, the mean-field expression for the staggered susceptibility is also invalid at finite M in the QD region, although the corrections to the two-spinon mass and to the susceptibility have different dependence on the expansion parameter $1/M$ in the QD case. The staggered susceptibility computed to order $1/M$ contains an infinite number of simple poles. The lowest pole governs the behavior of the spin-correlation function at very large distances. Performing the Fourier transform of (4.10), we obtain for equal-time spin-spin correlator $\chi(r)\sim(1/M)\exp(-mr)/r$, compared to the mean-field expression $\chi(r)\sim\exp(-2m_0r)/r^2$. As in the RC regime, there is a crossover between the two regimes, which occurs at $mr\sim M$.

A confinement of spinons due to the logarithmic potential was qualitatively discussed by Wen.[19] A complimentary scenario of spinon confinement in the QD phase was considered by Read and Sachdev.[15] They argued that the long-wavelength action (2.4) contains an extra Berry-phase term (which we did not consider). This term is absent for any smooth configuration of the z field, but it is present for singular "hedgehog" spinon configurations. Due to the Berry term, spinons experience a *linear* confining potential at distances $\xi_C\sim\xi^{M\cdot\rho_1}$, $\rho_1\approx0.06$.[20] At large M, this scale is much larger than the typical confinement scale, $\sim M\xi$, which appears in our consideration. The two scales, however, may become comparable at small M.

V. QUANTUM-CRITICAL REGIME

Finally, we consider the QC regime, which is the intermediate regime between the RC and the QD regimes.

At $T = 0$, the CP^{M-1} model action describes D=3 critical theory which possess no confinement.[21] The $M = \infty$ susceptibility behaves as $1/k$, and the $1/M$ corrections from both constraint and gauge fluctuations are of the form $\ln(1/k)/k$. Evaluating the $1/M$ terms and exponentiating the result we obtain $\chi(k) \sim k^{-(2-\eta)}$, where $\eta = 1 - \frac{32}{\pi^2 M}$. Notice that the correction to the mean-field result for η is large. On the contrary, for O(N) sigma model at the critical point we have $\eta = 0$ which is much closer to the Monte Carlo result for the O(3) model—$\eta = 0.03$.[22]

At finite temperature, the confinement does exist, and we proceed in exactly the same way as in the two previous sections. In the QC regime, the typical frequencies are of the order of $k_B T$, so, in principle, one has to perform full frequency summation in the $1/M$ formulas for the polarization operator. However, the situation is simpler, as our earlier results show that the confining potential is much stronger in the RC regime than in the QD regime. As a result, the most divergent contributions in the $1/M$ series always come from the $\omega = 0$ terms in the frequency summations. This observation makes the analysis of the confinement in the QC regime very similar to that in the RC regime; the only difference is in the form of the gauge-field propagator. Using (2.17) and the results of Sec. III, we obtain after simple manipulations that the branch-cut behavior of the spin susceptibility is indeed replaced by a set of poles. Near the lowest pole, the polarization operator behaves as

$$\Pi_L(k) = \frac{1}{2\Theta|s_1|}\left(\frac{6\pi\Upsilon}{M}\right)^{1/3}\frac{k_B T}{k^2+\tilde{m}^2}, \tag{5.1}$$

where $\Upsilon = (2\Theta^2)/\sqrt{5} = 0.828471$, and the bound-state mass is given by

$$\frac{\tilde{m}^2}{(k_B T)^2} = 4\Theta^2\left[1 - s_1\left(\frac{6\pi\Upsilon}{M}\right)^{2/3}\right]. \tag{5.2}$$

VI. CONCLUSIONS

We summarize the key results of the paper. We considered here the form of the static staggered susceptibility in the CP^{M-1} model of an M-component *complex* unit vector in two spatial dimensions. The $M = 2$ limit of this model describes two-dimensional Heisenberg antiferromagnet, and the $1/M$ expansion we discuss in this paper is a systematic way to calculate various observables. The elementary excitations in the CP^{M-1} model are $S = 1/2$ bosonic spinons. The spin variables are bilinear in spinon fields, and all physical excitations are collective modes of two spinons. In the mean-field approximations, the spinons behave as noninteracting particles, and the static staggered spin susceptibility has a branch cut along the imaginary k axis, terminated at $k^2 = -4m_0^2$, where m_0 is the mass of a spinon.

We have shown that in all three scaling regimes at finite T, the fluctuations of the gauge-field give rise to divergent $1/M$ corrections near the branch cut. We selected the most divergent corrections and have shown explicitly that they form ladder series. We then found that the ladder problem is equivalent to a Schrödinger equation with the δ-functional source in dimension $D - 1$, where D is the space-time dimension for the original problem. Schrödinger equation has a discrete set of solutions, and expanding near each of the solutions, we obtained, instead of a branch cut, a sequence of simple poles. This in turn quantitatively changes the behavior of the spin-correlation function at large distances compared to the mean-field predictions.

The result that the staggered susceptibility has a pole, and not a branch cut, is consistent with the results of the alternative perturbative technique for 2D antiferromagnets, which is the $1/N$ expansion for the O(N) sigma model for an N-component *real* unit field. The latter model describes Heisenberg antiferromagnet at $N = 3$. Elementary excitations in the O(N) sigma model carry $S = 1$, and, at any N, the static staggered spin susceptibility has a well defined single pole at imaginary momentum $k = \pm i\xi^{-1}$, where ξ is the correlation length in the system.

Comparing now for the CP^{M-1} and the O(N) sigma models, we conclude that although the differences between them found at the mean-field level are now gone, in practical calculations it is more convenient to use the O(N) model. The reason is that the perturbative series in $1/N$ is regular for this model, and converges much better than the $1/M$ series for the CP^{M-1} model. The latter can even be nonanalytical in some cases, e.g., when calculating the spin-correlation length.

Our results still leave a minor discrepancy between the two approaches: in the O(N) model, the static mean-field susceptibility has a *single* pole, and $1/N$ corrections do not lead to the appearance of new singularities. On the contrary, in the CP^{M-1} model, we found, to leading order in $1/M$, an infinite set of poles. Indeed, O(N) and CP^{M-1} models are equivalent *only* for $N = 3$ and $M = 2$, when they both describe Heisenberg antiferromagnet. However, Bethe ansatz solution for the O(3) model also indicates that there is a single mass in the problem because the free energy as a function of the uniform magnetic field, h, has a single threshold at $h_c = m$.[4,5] We therefore expect that all the poles that we have found in the CP^{M-1} model, except for the lowest one, should disappear for $M = 2$. At present, we can only speculate how this may occur—the most likely possibility, in our opinion, is that the solutions of the Schrödinger equation with $n > 0$ simply acquire a finite lifetime due to higher-order corrections in $1/M$. If this is true, then the susceptibility in the CP^{M-1} model has a single stable pole at *any* M, while all other poles that we have found to first order in $1/M$, form an incoherent continuum where both real and imaginary part of susceptibility are present. This is corroborated by an observation that the n-field propagator of the O(3) model, evaluated at zero frequency and along imaginary k axis, also has an imaginary part for $|\tilde{k}| > 3m$ ($\tilde{k} = ik$) due to decay of a $S = 1$ quanta into three others [the simplest way to see this is to substitute a fixed-length constraint on $\vec{n}$ field by the $u(\vec{n}^2)^2$ interaction term and do a perturbative expansion in u (Ref. 23)]. We, however, emphasize that this issue is irrelevant for the long-distance behavior of the spin-correlation function as this behavior is associated with the lowest-energy ($n = 0$) pole, which is always stable.

A final remark. In this paper, we considered a field-theoretic description of the *commensurate* antiferromagnet. One can also study the anisotropic version of the CP^{M-1} model, with the extra factor $(1-\gamma)$ in front of $|\bar{z}\partial_\mu z|^2$ in (2.2) $(0 \le \gamma \le 1)$. For $\gamma \neq 0$ the gauge-field propagator acquires a mass, which prevents the effective potential $V(r)$ in the Schrödinger equation from becoming arbitrary large. As a result, even without damping, the number of poles in the staggered susceptibility decreases and finally, when γ exceeds some critical value, the disordered phase possesses deconfined $S = 1/2$ bosonic spinons.[13] The disordered phase with deconfined spinons was also obtained in Ref. 3 by a direct $1/M$ expansion at small $1-\gamma$. This limit is of particular interest as the z-field model with $\gamma \approx 1$ describes incommensurate quantum antiferromagnet near the critical point.

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