Landau theory of quantum spin glasses of rotors and Ising spins

N. Read and Subir Sachdev
Department of Physics, P.O. Box 208120, Yale University, New Haven, Connecticut 06520-8120
and Department of Applied Physics, P.O. Box 208284, Yale University, New Haven, Connecticut 06520-8284

J. Ye
Department of Physics, Harvard University, Cambridge, Massachusetts 02138
(Received 6 December 1994)

We consider quantum rotors or Ising spins in a transverse field on a d-dimensional lattice, with random, frustrating, short-range, exchange interactions. The quantum dynamics are associated with a finite moment of inertia for the rotors, and with the transverse field for the Ising spins. For a suitable distribution of exchange constants, these models display spin-glass and quantum paramagnet phases and a zero-temperature (T) quantum transition between them. An earlier exact solution for the critical properties of a model with infinite-range interactions can be reproduced by minimization of a Landau effective-action functional for the model in finite d with short-range interactions. The functional is expressed in terms of a composite spin field which is bilocal in time. The mean-field phase diagram near the $T = 0$ critical point is mapped out as a function of T, strength of the quantum coupling, and applied fields. The spin-glass phase has replica symmetry breaking; but, as in the classical Ising spin glass, the order parameter becomes replica symmetric as $T \to 0$. Next we examine the consequences of fluctuations about the mean-field for the critical properties. Above $d = 8$, and with certain restrictions on the values of the Landau couplings, we find that the transition is controlled by a Gaussian fixed point with mean-field critical exponents. For couplings not attracted by the Gaussian fixed point above $d = 8$, and for all physical couplings below $d = 8$, we find runaway renormalization-group flows to strong-coupling. General scaling relations that should be valid even at the strong coupling fixed point are proposed and compared with Monte Carlo simulations.

I. INTRODUCTION

Models of quantum rotors or the Ising model in a transverse field on a d-dimensional lattice are among the simplest systems which exhibit a zero-temperature (T) quantum transition.[1–3] This is particularly true in the absence of quenched disorder: then the critical properties map directly onto those of the corresponding classical spin model in $d+1$ dimensions, with the quantum time behaving just like another spatial dimension. However, in the presence of quenched disorder, the situation is more complicated. The disorder has no dynamic fluctuations, so the corresponding classical spin models have randomness which is constant along the "time" direction; unfortunately, there is only a rather limited existing body of knowledge on such classical systems that one can draw on. Nevertheless, these models still constitute an attractive setting for the study of the complicated interplay of quenched disorder, interactions, and quantum mechanics. One can quite reasonably hope that the insight obtained from their analysis might prove profitable in other systems which involve the same ingredients—these include the metal-insulator and superfluid-insulator transitions and have been the focus of a large number of experimental investigations. Direct experimental studies of the models studied in this paper have been much more limited, although a recent investigation of a system which can be well described by the Ising model in a transverse field is noteworthy.[4] Three component quantum rotors with quenched disorder may also be a reasonable starting point for understanding some of the spin-fluctuation properties of the doped cuprate compounds in a regime with localized holes.[5]

We consider the following Hamiltonian of quantum rotors on the site i of a regular, d-dimensional lattice:

$$\mathcal{H}_R = \frac{g}{2}\sum_i \hat{\mathbf{L}}_i^2 - \sum_{\langle ij\rangle} J_{ij}\hat{\mathbf{n}}_i\cdot\hat{\mathbf{n}}_j \ , \tag{1.1}$$

where the sum $\langle ij\rangle$ is over nearest neighbors, although our results should also apply to models with more general short-range interactions. The M-component vectors $\hat{\mathbf{n}}_i$, with $M \geq 2$, are of unit length, $\hat{\mathbf{n}}_i^2 = 1$, and represent the orientation of the rotors on the surface of a sphere in M-dimensional rotor space. The operators $\hat{L}_{i\mu\nu}$ ($\mu < \nu$, μ, $\nu = 1\ldots M$) are the $M(M-1)/2$ components of the angular momentum $\hat{\mathbf{L}}_i$ of the rotor: the first term in $\mathcal{H}_R$ is the kinetic energy of the rotor with $1/g$ the moment of inertia. The different components of $\hat{n}_i$ constitute a complete set of commuting observables and the state of the system can be described by a wave function $\Psi(n_i)$. The action of $\hat{\mathbf{L}}_i$ on Ψ is given by the usual differential form of the angular momentum

$$L_{i\mu\nu} = -i\left(n_{i\mu}\frac{\partial}{\partial n_{i\nu}} - n_{i\nu}\frac{\partial}{\partial n_{i\mu}}\right) . \tag{1.2}$$

0163-1829/95/52(1)/384(27)/$06.00

The commutation relations among the $\hat{\mathbf{L}}_i$ and $\hat{\mathbf{n}}_i$ can now be easily deduced. We emphasize the difference of the rotors from Heisenberg-Dirac quantum spins: the components of the latter at the same site do not commute, whereas the components of the $\hat{\mathbf{n}}_i$ do.

Let us also introduce the Hamiltonian of the Ising model in a transverse field, $\mathcal{H}_I$:

$$\mathcal{H}_I = -g \sum_i \sigma_i^x - \sum_{\langle ij \rangle} J_{ij} \sigma_i^z \sigma_j^z . \tag{1.3}$$

Here σ^x, σ^z are the x, z components of the three Pauli spin operators, with the Pauli operators on different sites commuting with each other. Each site, therefore, has an Ising degree of freedom, represented by the eigenvalues of the σ_i^z. The σ_i^x is the kinetic energy term and induces on-site flips of the Ising spins.

The analogy between $\mathcal{H}_R$ and $\mathcal{H}_I$ should be quite clear. $\mathcal{H}_R$ has a global $O(M)$ symmetry, while $\mathcal{H}_I$ has a global Z_2, spin-flip symmetry associated with the unitary transformation $U = \prod_i \sigma_i^x$. In both models, the interaction terms proportional to the J_{ij} would, if g were zero, prefer a ground state in which each rotor or Ising spin has a definite orientation which minimizes the exchange energy, and quantum fluctuations are absent: any such choice will break the global $O(M)$ or Z_2 symmetry in a given sample (although the spin-glass phase preserves a statistical spin symmetry after averaging, discussed in Sec. II B 2). A small g will not necessarily destroy this phase, in sufficiently high spatial dimension. In the opposite limit $J_{ij} = 0$, both $\mathcal{H}_R$ and $\mathcal{H}_I$ possess nondegenerate ground states which preserve the global symmetry. For the rotors, each site is in the spherically symmetric "s-wave" state (using the language of $M = 3$). Similarly, each Ising spin is in the eigenstate of σ^x with eigenvalue $+1$: $= (|\uparrow\rangle + |\downarrow\rangle)/\sqrt{2}$ which is Z_2 invariant. In both cases, the $J_{ij} = 0$ ground state is separated from the first excited state by a gap of order g. It is therefore reasonable to consider the J_{ij}'s as perturbations in this limit (though the random nature of the J_{ij} causes some problems which we will discuss later) and to expect that this "quantum-disordered" phase persists at finite values of J_{ij}. Thus we expect both ordered and disordered phases to exist at zero temperature.

The above discussion should make it clear that it is natural to consider the transverse-field Ising model as simply the $M = 1$ case of the quantum rotors. As in the classical and nonrandom quantum cases, the main difference between $M = 1$ and $M > 1$ is that the latter has a continuous symmetry and possesses gapless spin-wave excitations in magnetically ordered phases. For ease of the subsequent discussion, it is convenient to introduce a notation which allows simultaneous discussion of the rotor and Ising models. We therefore introduce an M-component spin $S_{i\mu}$, with $M \geq 1$, such that

$$S_{i\mu} = \begin{cases} \hat{n}_{i\mu} \text{ for } M \geq 2 \\ \sigma_i^z \text{ for } M = 1. \end{cases} \tag{1.4}$$

We will use the general $S_{i\mu}$ notation in most of the remainder of the paper.

We now want to describe the possible phases of $\mathcal{H}_R$ or $\mathcal{H}_I$ at zero temperature. The J_{ij} are assumed to be statistically independent between different links and to possess the following first and second moments:

$$[J_{ij}] = J_0; \quad \left[(J_{ij} - J_0)^2\right] = J^2. \tag{1.5}$$

Here, and henceforth, the square brackets will represent an average over the quenched disorder. Averages over quantum or thermal fluctuations will be represented by angular brackets. On bipartite lattices, the properties of the system are invariant under the global sign change $J_{ij} \to -J_{ij}$: this is because the sign of the exchange energy can be reversed by a global spin-flip $S_{i\mu} \to -S_{i\mu}$ on one sublattice. On such lattices we can therefore assume without loss of generality that $J_0 > 0$, and we will so assume. Note that this property is not shared by models where $S_{i\mu}$ is replaced by quantum Heisenberg spins whose components do not commute.

It is also possible to modify $\mathcal{H}_R$ and $\mathcal{H}_I$ by allowing g to have random fluctuations about its mean value. This does not affect the analysis of the phases of the model. It does represent an important type of randomness that will be considered in our discussion of the Landau theory.

We have now introduced three energy scales, g, J_0, and J, and the nature of the ground state becomes especially clear when one of the three scales is much larger than the other two. The three phases so obtained are the ferromagnet ($J_0 \gg g, J$), the spin-glass ($J \gg J_0, g$), and the quantum paramagnet ($g \gg J_0, J$). We review the structure of the three phases in turn (analogous to finite-temperature phases in classical models):

(a) *Ferromagnet:* Each site acquires a static moment, and further the average of the moments on the different sites is nonzero:

$$\langle S_{i\mu} \rangle \neq 0; \quad [\langle S_{i\mu} \rangle] = M_{0\mu} \neq 0. \tag{1.6}$$

The ferromagnetic order parameter is $M_{0\mu}$; note however that the first, quantum expectation value will have fluctuations from site to site about $M_{0\mu}$. The reader may be aware of mappings between the low-energy properties of clean quantum Heisenberg spin models and quantum rotors:[6] we note here that it is the *anti*ferromagnetically-ordered Néel phase of Heisenberg spin that maps onto the ferromagnetic phase of quantum rotors.

(b) *Spin-glass:* Each site now has a random static moment,[7,8] and the average moment is therefore zero:

$$\langle S_{i\mu} \rangle \neq 0; \quad [\langle S_{i\mu} \rangle] = 0. \tag{1.7}$$

The Edwards-Anderson order parameter, q_{EA} is given by

$$q_{\text{EA}} = \frac{1}{M} \lim_{\tau \to \infty} [\langle S_{i\mu}(0) S_{i\mu}(\tau) \rangle] . \tag{1.8}$$

[We are using here the Einstein summation convention on the $O(M)$ vector index μ; this convention will be implicitly assumed throughout the paper.] One would expect that q_{EA} is also equal to

$$q = \frac{1}{M} \left[\langle S_{i\mu} \rangle^2 \right] . \tag{1.9}$$

In classical spin glasses, $q \neq q_{\rm EA}$ when replica symmetry breaking takes place,[7,8] because the $\langle\,\rangle$ average must be carefully defined. In spite of the quantum fluctuations that are present, the appearance of the ferromagnetic and spin-glass phases is quite similar to that of their classical analogues at $T=0$ or $T\neq 0$.

(c) *Quantum paramagnet*: This phase has no static moment:

$$\langle S_{i\mu}\rangle = 0. \tag{1.10}$$

It differs in one important respect from the strong-coupling, $J_{ij}=0$, picture discussed earlier. The gap in the excitation spectrum is filled in by contributions from rare regions in which the local values of J_{ij} are such as to place the system in one of the magnetically ordered phases. A brief discussion of such "Griffiths effects" is presented in Appendix B.

There has been some earlier work on the ferromagnetic and paramagnetic phases and the transition between them. In one dimension, many exact properties have been determined for the transverse field Ising model.[9–11] In $d=1$, the $M=1$ spin-glass phase and its transitions are closely related to those of the ferromagnet, as the former can be related to the latter by a gauge transformation which makes all the J_{ij} positive. (The magnetically ordered phases do not exist for $M\geq 2$ in $d=1$.) In higher dimensions, there has been a discussion of the field-theoretic properties of the ferromagnetic-paramagnetic transition.[12–14] Boyanovsky and Cardy[13] identified $d=4$ as the upper-critical dimension and discussed the scaling properties of the correlation functions. A study of the critical properties below $d=4$ in an expansion in $\varepsilon = 4-d$ yields a flow to strong-coupling with no stable, physical fixed point (the only stable fixed-point has unphysical properties like a negative variance of some observables). However, Boyanovsky and Cardy[13] showed that a stable, physical, fixed point could be obtained in a theory with ϵ_τ time dimensions, in a double expansion in ε and ϵ_τ; it is not known whether this fixed point continues to be pertinent all the way to $\epsilon_\tau = 1$.

We will not make any further references to the ferromagnetic phase. It is convenient, therefore, to specialize henceforth to the case of J_{ij} distributions which are symmetric about $J_{ij}=0$ and in particular have

$$J_0 = 0. \tag{1.11}$$

In this case, it can be shown[8] that the disordered-averaged correlation functions are in fact invariant under a Z_2 gauge transformation $S_{i\mu}\to \eta_i S_{i\mu}$, with $\eta_i = \pm 1$ and site dependent. A zero temperature spin glass to quantum paramagnet phase transition will occur as the ratio g/J is increased and is the subject of most of the following.

The spin-glass phase and its transitions were studied in Refs. 15 and 16, with considerable additional interest in the last year.[17–23] Huse and Miller[17] studied the transverse-field Ising model with infinite-range interactions and determined exact critical properties of the transition separating the spin-glass and paramagnetic phases. We have studied[18] the quantum-rotor model with infinite-range interactions and obtained essentially identical results. The independence of the critical properties on the value of M was also understood.[18] We now highlight features of these results which will be important for our discussion.

A. Review of results for the infinite-range model

In the infinite-range version of $\mathcal{H}_R$ (1.1), or $\mathcal{H}_I$ (1.3), the interactions J_{ij} are taken to be independent random variables with distribution

$$P(J_{ij}) \propto \exp\left(-NJ_{ij}^2/2J^2\right) \tag{1.12}$$

for all pairs i,j of distinct sites; N is the number of sites. Note that the variance of $J_{ij}\sim J^2/N$ in the infinite-range model, but is $\sim J^2$ in the finite-range model. As $N\to\infty$ mean-field theory becomes exact, which means that one only need solve a problem of a single site with a self-interaction between different times τ_1,τ_2 of $J^2D(\tau_1-\tau_2)$, where D has to be determined self-consistently as

$$D(\tau_1-\tau_2) = \frac{1}{M}\left\langle S_\mu(\tau_1)S_\mu(\tau_2)\right\rangle_D \tag{1.13}$$

with the average being calculated with the same self-interaction D. This was done by self-consistency arguments on the spectral density by Miller and Huse, and by taking the limit $M\to\infty$ and then expanding in $1/M$ using similar arguments, by the present authors. D represents the disorder average

$$D(\tau_1-\tau_2) = \frac{1}{M}\left[\langle S_{i\mu}(\tau_1)S_{i\mu}(\tau_2)\rangle\right] \tag{1.14}$$

in the original random problem. It was found that a spin-glass to paramagnet transition occurred at $g=g_c\sim J$. At $g>g_c$, D decays exponentially with τ as $\tau\to\infty$, indicating a gap Δ in the corresponding spectral density; at $g=g_c$, D decays as $1/\tau^2$, and in the ordered phase, $D\to$ constant $= q_{\rm EA}$. The Fourier transform $D(\omega)$ of $D(\tau)$ has the form

$$D(\omega)\sim \text{const} - \sqrt{\omega^2+\Delta^2} \tag{1.15}$$

for $g\geq g_c$, which is responsible for the $1/\tau^2$ behavior at g_c and it turns out that the correlation time $\xi_\tau\sim\Delta^{-1}$ diverges as

$$\xi_\tau \sim \left[(g-g_c)^{-1}\ln(g-g_c)\right]^{1/2}. \tag{1.16}$$

Thus we can define an exponent $z\nu$, anticipating anisotropic scaling in space and time in the short-range model, which takes the value $z\nu = 1/2$ in the infinite-range model. The log correction originates from effects of the length constraint. Its significance will be fully explained in our later analysis. Since $D(\tau\to\infty)$ is the Edwards-Anderson order parameter, we may also define $q_{\rm EA} = (g_c-g)^\beta$ and it is found that $\beta = 1$. At $g=g_c$ one expects $D(\tau)\sim\tau^{-\beta/z\nu}$,[5] which is satisfied with the values already obtained.

It is perhaps surprising that the critical properties of the self-consistent single-site quantum problem can be obtained exactly for all M, and it remains to explain some features of the results, such as the logarithmic violations of scaling. One of our goals in this paper is to provide a conceptually simpler derivation of the results based on a Landau action functional and to explain the logarithms as due to the decay of a marginally irrelevant variable which we will identify. We also wish to calculate critical properties in short-range, finite-dimensional models, and to this end we will study mean-field theory and Gaussian fluctuations around it in the Landau theory framework. We will show that in this approximation, $z = 2$ and $\nu = 1/4$. We will also define correlation functions in the next subsection and obtain their scaling properties in the same approximation later, obtaining $\eta = 0$ by definition in mean-field theory. For sufficiently large d ($d > 8$) and for a certain range of values of Landau couplings, we will show that these mean-field theory results are valid and have the same critical properties as in the infinite-range model (where comparisons are possible, i.e., not for correlations at large spatial separation and not for the nonlinear susceptibility which has a singular infinite-range limit[24]). However outside this range, and for all couplings at $d < 8$, we find a renormalization-group flow to a region where perturbation theory breaks down.

B. Order parameter and observables

In classical finite-temperature transitions the equilibrium critical properties can be studied in terms of static fluctuations of some "order-parameter" field. For classical spin glasses in the replica formalism, this is a matrix q^{ab}, $a, b = 1 \dots n$ are replica indices and $n \to 0$. The *off*-diagonal components of q_{ab} can be related to the Edwards-Anderson order parameter in a somewhat subtle way we will not go into here.[7,8] In quantum ($T = 0$) phase transitions, time-dependent fluctuations of the order parameter must be considered (in "imaginary" Matsubara time τ) and in the spin-glass case it is found that the standard decoupling, analogous to the classical case introducing q^{ab} leads now to a matrix function of two times[15] (see also Appendix A) which we can consider to be

$$Q^{ab}_{\mu\nu}(x, \tau_1, \tau_2) = \sum_{i \in \mathcal{N}(x)} S^a_{i\mu}(\tau_1) S^b_{i\nu}(\tau_2) \ , \qquad (1.17)$$

where $\mathcal{N}(x)$ is a coarse-graining region in the neighborhood of x. The definition (1.8) of the Edwards-Anderson order parameter now implies

$$D(\tau_1 - \tau_2) \equiv \lim_{n \to 0} \frac{1}{Mn} \sum_a \langle\langle Q^{aa}_{\mu\mu}(x, \tau_1, \tau_2) \rangle\rangle \ , \qquad (1.18)$$

$$q_{\rm EA} = \lim_{\tau \to \infty} D(\tau), \qquad (1.19)$$

relating $q_{\rm EA}$ to the replica diagonal components of Q. [A reminder: the Einstein summation convention on the vector index μ is being used in (1.18). However, no such convention is used for the a, b replica indices.] It is important to note here that the definition (1.9) also relates $q_{\rm EA}$ to the replica off-diagonal components of Q,[8] whose expectation value will be time independent. We have introduced above double angular brackets to represent averages taken with the translationally invariant replica action (recall that single angular brackets represent thermal and/or quantum averages for a fixed realization of randomness, and square brackets represent averages over randomness).

We saw in the infinite-range model that the behavior of D at long times in mean-field theory changed significantly at the transition and it is clear that the theory of fluctuations must include the whole matrix function Q, for *both* its diagonal and off-diagonal components. Strictly speaking the replica diagonal Q at finite $\tau_1 - \tau_2$ is not an order parameter because its expectation is nonzero on both sides of the transition; nonetheless it does play the role of the order-parameter field in the Landau theory.

In the finite-dimensional model, where J_{ij} couples nearest neighbors only, $D(\tau)$ is expected to acquire a long-time limit only at the transition to, and in, the spin-glass phase. However its behavior in the paramagnetic phase may differ from that in mean-field theory (the infinite-range model) for the following reason. Since the J_{ij}'s are random, there is some chance of any given region having all J_{ij}'s large in magnitude and thus resembling a patch of the ordered phase. Such an event would be statistically rare, but can contribute significantly to the long-time behavior of D (which is the average on-site correlation function) because the finite region will have slow overturns of its instantaneous moment. The effects of these "Griffiths singularities" on D has been discussed recently by Thill and Huse[23] for $M = 1$; the extension of their argument to $M > 1$ is presented in Appendix B; for $M > 1$, we find only weak essential singularities at $\omega = 0$ in the spectral function for D. An important question is whether similar fluctuations affect the critical properties, an effect presumably not included in the treatment that will be given later.

It is also helpful to introduce here a number of correlation functions of the order parameter whose scaling properties will be described in the paper. A quantity intimately related to the spin-glass long-range order is the quantum-mechanical disconnected correlation function

$$G(i - j, \tau_1 - \tau_2, \tau_3 - \tau_4)$$

$$\equiv [\langle S_{i\mu}(\tau_1) S_{j\mu}(\tau_2) \rangle \, \langle S_{i\nu}(\tau_3) S_{j\nu}(\tau_4) \rangle] \, . \qquad (1.20)$$

Note that $[\langle S_{i\mu}(\tau_1) S_{j\mu}(\tau_2) \rangle] = 0$ for $i \neq j$ because of the Z_2 gauge symmetry, and no subtraction of products of disorder averages is necessary, as a subtraction analogous to that in (1.22) below will vanish for this case. After coarse graining both i and j over their respective averaging regions in the neighborhoods of x and y, we obtain the correlator of the order parameter Q

$$G(x-y,\tau_1-\tau_2,\tau_3-\tau_4) = \lim_{n\to 0}\frac{1}{n(n-1)}\sum_{a\neq b}\langle\langle Q^{ab}_{\mu\nu}(x,\tau_1,\tau_3)Q^{ab}_{\mu\nu}(y,\tau_2,\tau_4)\rangle\rangle. \qquad (1.21)$$

G will be found later to behave as the propagator for fluctuations of the Q order parameter field about the mean-field theory, and is directly analogous to a corresponding object in the classical spin glass.

A second correlator arises upon considering fluctuations (due to the randomness) of the on-site spin-correlation function $\langle S_{i\mu}(\tau_1)S_{i\mu}(\tau_2)\rangle$. The second cumulant of these fluctuations can be obtained from a quantum mechanically disconnected correlation function

$$G^d(i-j,\tau_1-\tau_2,\tau_3-\tau_4) \equiv [\langle S_{i\mu}(\tau_1)S_{i\mu}(\tau_2)\rangle\,\langle S_{j\nu}(\tau_3)S_{j\nu}(\tau_4)\rangle] - [\langle S_{i\mu}(\tau_1)S_{i\mu}(\tau_2)\rangle]\,[\langle S_{j\nu}(\tau_3)S_{j\nu}(\tau_4)\rangle]\ . \qquad (1.22)$$

After coarse graining this becomes another two-point correlation function of the order parameter Q,

$$G^d(x-y,\tau_1-\tau_2,\tau_3-\tau_4) = \lim_{n\to 0}\frac{1}{n(n-1)}\sum_{a\neq b}\langle\langle Q^{aa}_{\mu\mu}(x,\tau_1,\tau_2)Q^{bb}_{\nu\nu}(y,\tau_3,\tau_4)\rangle\rangle - D(\tau_1-\tau_2)D(\tau_3-\tau_4) \qquad (1.23)$$

obtained as before by averaging over i and j in neighborhoods of x and y. The analog of G^d in a classical spin glass is trivial, since $S^2_{i\mu}=1$. Higher moments of the on-site correlation function can also be constructed, and the entire set is expected to have rather nontrivial scaling properties near the quantum phase transition.[25]

Finally, to exhaust the set of different two-point correlators of the Q field, we consider the connected correlation function G^c

$$\begin{aligned}G^c_{\mu\nu\rho\sigma}(i-j,\tau_1-\tau_4,\tau_2-\tau_4,\tau_3-\tau_4) = {} & [\langle S_{i\mu}(\tau_1)S_{i\nu}(\tau_2)S_{j\rho}(\tau_3)S_{j\sigma}(\tau_4)\rangle] \\ & -\frac{1}{M^2}\delta_{\mu\nu}\delta_{\rho\sigma}\,[\langle S_{i\alpha}(\tau_1)S_{i\alpha}(\tau_2)\rangle\,\langle S_{j\beta}(\tau_3)S_{j\beta}(\tau_4)\rangle] \\ & -\frac{1}{M^2}\delta_{\mu\rho}\delta_{\nu\sigma}\,[\langle S_{i\alpha}(\tau_1)S_{j\alpha}(\tau_3)\rangle\,\langle S_{i\beta}(\tau_2)S_{j\beta}(\tau_4)\rangle] \\ & -\frac{1}{M^2}\delta_{\mu\sigma}\delta_{\nu\rho}\,[\langle S_{i\alpha}(\tau_1)S_{j\alpha}(\tau_4)\rangle\,\langle S_{i\beta}(\tau_2)S_{j\beta}(\tau_3)\rangle]\,, \end{aligned} \qquad (1.24)$$

where we used the symmetry properties. After coarse graining, we obtain

$$\begin{aligned}G^c_{\mu\nu\rho\sigma}(x-y,\tau_1-\tau_4,\tau_2-\tau_4,\tau_3-\tau_4) = {} & \lim_{n\to 0}\frac{1}{n}\sum_a\langle\langle Q^{aa}_{\mu\nu}(x,\tau_1,\tau_2)Q^{aa}_{\rho\sigma}(y,\tau_3,\tau_4)\rangle\rangle \\ & -\frac{1}{M^2}\delta_{\mu\nu}\delta_{\rho\sigma}\left\{G^d(x-y,\tau_1-\tau_2,\tau_3-\tau_4)+D(\tau_1-\tau_2)D(\tau_3-\tau_4)\right\} \\ & -\frac{1}{M^2}\delta_{\mu\rho}\delta_{\nu\sigma}G(x-y,\tau_1-\tau_3,\tau_2-\tau_4) \\ & -\frac{1}{M^2}\delta_{\mu\sigma}\delta_{\nu\rho}G(x-y,\tau_1-\tau_4,\tau_2-\tau_3). \end{aligned} \qquad (1.25)$$

Any scaling theory of the transition should obtain the scaling dimensions and functions of D, G, G^d, and G^c, and the possibilities appear rather varied.

Two important susceptibilities can also be related to the correlators considered above. The Edwards-Anderson spin-glass susceptibility $\chi_{\rm sg}$ is given in the paramagnetic phase by

$$\chi_{\rm sg} = \sum_j [\chi^2_{ij}]\,, \qquad (1.26)$$

where

$$\chi_{ij} = \int d\tau\,\langle S_{i\mu}(0)S_{j\mu}(\tau)\rangle \qquad (1.27)$$

and is analogous to the corresponding object used in the classical theories. After coarse graining we have the expression in terms of G

$$\chi_{\rm sg} = \int d^dx\, d\tau_1 d\tau_2 G(x,\tau_1,\tau_2). \qquad (1.28)$$

The second susceptibility associated with the spin-glass order is the nonlinear susceptibility, $\chi_{\rm nl}$, which is given by

$$\chi_{\rm nl} = \int d^dx\, d\tau_1 d\tau_2 d\tau_3 G^c_{1111}(x,\tau_1,\tau_2,\tau_3). \qquad (1.29)$$

The outline of the remainder of the paper is as follows. In Sec. II A we begin by setting up the Landau action functional for the finite-dimensional models near the zero-temperature phase transition, on which most of our results will be based. In Sec. II B, this functional is minimized to yield mean-field theory, including for the first time external fields; correlation functions are discussed at the Gaussian level. In Sec. III, a renormalization-group method is used to examine the stability of the mean-field results to fluctuations, and to search for nontrivial exponents below the upper critical dimension $d_u = 8$ (no accessible perturbative fixed points are found in $d < 8$, however). Section IV discusses general scaling theory and compares with results from numerical simulations. Sketches of a derivation of the Landau theory and of Griffiths singularities appear in the appendices.

II. LANDAU THEORY

A. Landau action functional

In order to write down the Landau theory for our model it will be useful to first review what is required in general of such a theory. The starting point of a Landau theory is a Landau (or Landau-Ginzburg-Wilson) functional which as a first step describes the free energy of the system near its critical point as a functional of its order parameter, for example the magnetization ϕ of an Ising-like ferromagnet near its classical finite temperature transition. In principle, the functional arises by considering the free energy in the presence of a field, say $h(x)$, that is thermodynamically conjugate to ϕ (so that $-\int h\phi d^dx$ appears in the Hamiltonian), finding the expectation of ϕ for each $h(x)$, and then writing the free energy as a functional of ϕ through a Legendre transformation. In the present example this would be assumed to take the form

$$\mathcal{F} = \int d^dx \left[\frac{1}{2}(\nabla\phi)^2 + \frac{1}{2}r\phi^2 + \frac{1}{4}\lambda\phi^4 + \cdots\right] , \quad (2.1)$$

where the dots denote terms with higher powers of ϕ and/or more derivatives. Mean-field (or Landau) critical behavior can be obtained by taking $\phi(x) = \phi$ independent of x and minimizing with respect to ϕ. Note that $r \sim T - T_c$ appears linearly. Then for $r > 0$, $\phi = 0$ and for $r < 0$, $\phi = \pm(|r|/\lambda)^{1/2}$, and the usual critical behavior of Landau theory can be obtained.

The key assumptions of Landau's approach are that, since ϕ is zero on the disordered (paramagnetic) side, then we can consider ϕ small and expand in powers, and the lowest powers will dominate near the critical point (odd powers are dropped by symmetry in this particular example). Similarly dependence of the coefficients on $T-T_c$ is analytic and all except the leading one $r = T-T_c$ can be dropped. In considering position dependent fluctuations, $\mathcal{F}$ can be used as the action in a functional integral over ϕ, and the interactions of ϕ at different positions are represented by the expansion in powers of gradients, corresponding to the short-range interactions in the original physical problem.

Turning to the quantum spin glasses discussed in Sec. I, we notice as mentioned there that the "order parameter" $\langle\langle Q_{\mu\nu}(x,\tau,\tau')\rangle\rangle$ is found in mean-field theory to be a *function* $\delta_{\mu\nu}D(\tau-\tau')$, independent of x, which is nonzero for all finite $\tau - \tau'$ even in the paramagnetic phase. We therefore cannot expand in powers of Q as Q is typically not small. If instead we attempt to expand in powers of $Q - D$, it turns out that the action obtained contains very nonanalytic frequency dependence near $g = g_c$ and so is unsatisfactory according to the criteria above.

The solution is found by examining the Fourier transform $D(\omega)$ of $D(\tau)$, Eq. (1.18). $D(\tau)$ is a positive function that decreases monotonically with $|\tau|$, and its Fourier transform has similar properties in ω. The constant in Eq. (1.15) is thus of order 1, and in fact can contain dependence on $g - g_c$ and on ω, but these are analytic and negligible compared with the more singular dependence contained in $\sqrt{\omega^2 + \Delta^2}$. The latter part is nonanalytic in both ω and $g - g_c$ as $g \to g_c$ and $\omega \to 0$, but it *is* small in this region, and it is this part in which we wish to expand the free action. That is, we expand in $Q(x, \omega_1, \omega_2) - \text{const}\,\delta(\omega_1 + \omega_2)$ where the constant is chosen so that $\langle\langle Q\rangle\rangle - \text{const}$ is zero (only) at $\omega_1 = \omega_2 = 0$ and $g = g_c$. The nonanalytic behavior of $\langle\langle Q\rangle\rangle - \text{const}$ away from this point should emerge from minimizing the action that should be analytic in form in both frequency (or τ derivatives) and $g - g_c$. The $\ln(g - g_c)$ corrections to power-law scaling should also emerge in this way.

In Appendix A we sketch a direct derivation of the action functional in a microscopic model. Here we will proceed by the alternative procedure of writing down all terms allowed by the above considerations, together with simple symmetry requirements. We use the same notation $Q(x, \tau_1, \tau_2)$ for the new, shifted field. As it differs only by a $\delta(\tau_1 - \tau_2)$ from the original field, which has little effect on the scaling of critical correlations, we can usually ignore the change in definition.

Explicitly, the terms allowed in the action must satisfy:

(1) The action is an integral over space of a local operator which can be expanded in gradients of powers of Q evaluated at the same position x.

(2) Q is bilocal (*i.e.* is a matrix) in time, and each time is associated with one of the two replica indices and with one of the two $O(M)$ vector indices [see definition Eq. (1.17)]. All these "indices" can appear more than once in a term and are summed over freely subject to the following rules before summations:

(a) Each distinct replica index appears an even number of times.[8]

(b) $O(M)$ vector indices appear twice each, and only when the corresponding replica indices are the same, due to the $O(M)$ symmetry of the randomness.

(c) Repetition of a time "index" corresponds to interaction of spins, which must be local in time and accordingly can be expanded as terms with times set equal plus the same with additional derivatives; it occurs when the corresponding replica indices are the same, and only then.

(3) The action should be invariant under space and

time translations, and under space and time inversions (under which Q is invariant).

(The rules for the indices may be best appreciated diagrammatically or from the microscopic approach in Appendix A.)

This procedure yields the Landau functional [recall that we are using the Einstein summation convention for the $O(M)$ vector indices]:

$$\mathcal{A} = \frac{1}{t}\int d^dx \left\{ \frac{1}{\kappa}\int d\tau \sum_a \left[\frac{\partial}{\partial\tau_1}\frac{\partial}{\partial\tau_2} + r\right] Q^{aa}_{\mu\mu}(x,\tau_1,\tau_2)\Big|_{\tau_1=\tau_2=\tau} + \frac{1}{2}\int d\tau_1 d\tau_2 \sum_{a,b}\left[\nabla Q^{ab}_{\mu\nu}(x,\tau_1,\tau_2)\right]^2 \right.$$
$$-\frac{\kappa}{3}\int d\tau_1 d\tau_2 d\tau_3 \sum_{a,b,c} Q^{ab}_{\mu\nu}(x,\tau_1,\tau_2)Q^{bc}_{\nu\rho}(x,\tau_2,\tau_3)Q^{ca}_{\rho\mu}(x,\tau_3,\tau_1)$$
$$\left. +\frac{1}{2}\int d\tau \sum_a \left[u\; Q^{aa}_{\mu\nu}(x,\tau,\tau)Q^{aa}_{\mu\nu}(x,\tau,\tau) + v\; Q^{aa}_{\mu\mu}(x,\tau,\tau)Q^{aa}_{\nu\nu}(x,\tau,\tau)\right]\right\}$$
$$-\frac{1}{2t^2}\int d^dx \int d\tau_1 d\tau_2 \sum_{a,b} Q^{aa}_{\mu\mu}(x,\tau_1,\tau_1)Q^{bb}_{\nu\nu}(x,\tau_2,\tau_2) + \cdots \, . \tag{2.2}$$

There are four terms in $\mathcal{A}$ whose coefficients contain products of powers of only two coupling constants, κ and t; this form can be reached without loss of generality by suitably rescaling the space and time coordinates. The reasons for our rather peculiar choices for these couplings will only become evident when the structure of perturbation theory is discussed. We have only retained the terms which our later power counting will tell us are relevant or marginal in high space dimension d, together with the leading irrelevant term. The exception to this statement is a quadratic term

$$\int d^dx d\tau_1 d\tau_2 \sum_{a,b}\left[Q^{ab}_{\mu\nu}(x,\tau_1,\tau_2)\right]^2, \tag{2.3}$$

which appears to be highly relevant in all d. However, it is a "redundant" operator as it can be removed by a further transformation $Q \to Q - C\delta^{ab}\delta_{\mu\nu}\delta(\tau_1-\tau_2)$ for a suitable choice of C. This relies on the presence of the cubic term with coefficient κ/t, and on the implicit $n \to 0$ (replica) limit to eliminate the contribution of the last $1/t^2$ term. The net effect is that if r is redefined to absorb a constant, the action has the form given in (2.2). We will see that this choice of definition of Q to eliminate the term in (2.3) also makes $\langle\langle Q\rangle\rangle = 0$ at $\omega = 0$ and $g = g_c$. This leaves r, the coupling in the term linear in Q, as the parameter expected to drive the system through its transition.

The possibility of eliminating a quadratic term by such a shift in the field variable arises generally in field theories containing a cubic term. The simplest example is for a theory with a single scalar field ϕ, which arises physically in the Yang-Lee[26] edge of a classical Ising-like ferromagnet in an imaginary magnetic field. The field induces an imaginary expectation of ϕ, which when eliminated by redefining ϕ by a shift, generates, due to the real ϕ^4 coupling in the Ising system, an imaginary ϕ^3 term in the new ϕ. In the Landau theory of this critical point, due to Fisher,[27] it is again convenient to shift away the quadratic term and leave a linear term in ϕ in the action. The coefficient, r, of this term controls the distance from the critical point, where power-law correlations of ϕ appear. Since r appears in the same form as an (additional) external field, there is only a single scaling field (ϕ) in the critical theory,[27,28] and there is a scaling relation between the exponent η governing correlations of ϕ at the critical point, and the correlation length exponent ν. In our spin-glass model, one might similarly expect an exponent relation, since the operatori $Q^{aa}_{\mu\mu}(x,\tau,\tau)$ is the "thermal" operator whose coefficient in $\mathcal{A}$ drives one across the transition, while $Q^{ab}_{\mu\nu}(x,\tau_1,\tau_2)$ is the basic order parameter field. However due to the "trace" over replica, $O(M)$ vector, and imaginary time indices in the thermal operator, it is far from clear that these two will in fact have the same scaling dimension in general. We will address this point further in Sec. IV. However at mean-field level, i.e., at the Gaussian fixed point, such a relation holds and we will find $\nu = 1/4$, the same value as for the Yang-Lee problem in mean-field, this being due in both cases to the linear plus cubic form of the action, in contrast to the value $\nu = 1/2$ found in most mean-field theories. We note that for the classical spin glass, though the leading coupling is cubic, the replica diagonal components Q^{aa} are not critical and are omitted from the Landau theory, so the same mechanism does not apply, and $\nu = 1/2$ in mean-field theory.[8]

A few last remarks on the action $\mathcal{A}$: the terms with coefficients u/t and v/t arise from quartic couplings of the spins within a single replica (see Appendix A). They are the only terms retained that break replica $O(n)$ symmetry to S_n, the permutational symmetry. If they were omitted, the action would describe randomly coupled simple harmonic oscillators which is definitely an unstable system in finite dimensions, and so anharmonic terms (and hence u and v) are expected to be necessary for stability. The last term with coefficient $1/t^2$ represents randomness in r (with variance $= +1$) which could originate from randomness in g (see Appendix A) but is also generated by the random J_{ij}'s. It turns out to play a central role.

B. Phases of the model

In this subsection we will solve the action $\mathcal{A}$ in the mean-field, or more accurately, tree approximation, i.e., without any momentum loop integrations for fluctuations. For the infinite-range model, where spatial dependence of Q can be dropped to obtain either thermodynamic quantities as averages over the whole system, or on-site correlation functions, this is exact and constitutes a simpler rederivation of results obtained earlier.[18] For the short-range finite-dimensional quantum models, as in the classical case,[8] mean-field theory should be a useful starting point towards understanding the overall phase diagram and properties of the phases. For the critical properties of the quantum transition, the mean-field theory is an approximation whose validity as an attractive weak-coupling fixed point under renormalization group in sufficiently high dimensions will be examined in the next section. Here we will concentrate on the correlators defined in Sec. I B, and study first in Sec. II B 1 the paramagnetic phase and critical point. We will then, in Sec. II B 2, study the spin-glass phase and the appearance of replica symmetry breaking at finite temperatures. The properties of these phases in a longitudinal magnetic field will be discussed in Sec. II B 3 and in a field coupling to the conserved angular momentum (defined only for $M > 1$) in Sec. II B 4.

1. Quantum paramagnet and the critical point

The saddle-point and perturbative analysis are most conveniently performed in momentum (k) and frequency (ω) space. We will work at a finite, but small, temperature $T = 1/\beta$, and ω will therefore take values at the discrete Matsubara frequencies. The normalization of the Fourier transform is set by

$$Q(k,\omega_1,\omega_2) = \int d^dx \int_0^\beta d\tau_1 d\tau_2 Q(x,\tau_1,\tau_2) \times e^{i(kx-\omega_1\tau_1-\omega_2\tau_2)}. \tag{2.4}$$

In these Fourier transformed variables we expect the saddle-point value of Q to obey the following ansatz in the paramagnet and at the critical point

$$Q^{ab}_{\mu\nu}(k,\omega_1,\omega_2) = \beta\delta_{\mu\nu}\delta^{ab}(2\pi)^d\delta^d(k)\delta_{\omega_1+\omega_2,0}D(\omega_1). \tag{2.5}$$

The momentum and frequency structure of the right-hand side follows from the Fourier transform of (1.18). An explicit factor of β has been inserted to make $D(\omega)$ finite in the zero-temperature limit. The structure in the $O(M)$ spin space follows from spin rotation invariance, while the replica-diagonal structure follows from the absence of a static moment in the paramagnetic phase. Inserting this into $\mathcal{A}$ in (2.2), we obtain for the free-energy density $\mathcal{F}/n$ (as usual, $\mathcal{F}/n$ represents the physical disorder averaged free energy)

$$\frac{\mathcal{F}}{n} = \frac{M}{\beta t}\sum_\omega\left[\frac{\omega^2+r}{\kappa}D(\omega) - \frac{\kappa}{3}D^3(\omega)\right] + M\frac{u+Mv}{2t}\left[\frac{1}{\beta}\sum_\omega D(\omega)\right]^2 . \tag{2.6}$$

The contribution from the last $1/t^2$ term in $\mathcal{A}$ vanishes in the replica limit $n \to 0$ and is therefore absent. The stationary point with respect to variations in $D(\omega)$ gives us the result

$$D(\omega) = -\frac{1}{\kappa}(\omega^2+\tilde{r})^{1/2}, \tag{2.7}$$

where $\tilde{r}$ is given implicitly by

$$\tilde{r} = r - (u+Mv)\frac{1}{\beta}\sum_\omega(\omega^2+\tilde{r})^{1/2}. \tag{2.8}$$

The sign of $D(\omega)$ is determined by the fact that the Fourier transform $D(\tau)$ is positive. This solution for $D(\omega)$ is well defined for $\tilde{r} \geq 0$, while no sensible paramagnetic solution exists for $\tilde{r} < 0$, suggesting that the critical line in the r, T plane between the paramagnetic and spin glass phases is $\tilde{r} = 0$. The local density of excitations $\chi''(\omega)$ can be obtained by analytic continuation of $D(\omega)$ to real frequencies and is therefore

$$\chi''(\omega) = \mathrm{sgn}(\omega)\frac{(\omega^2-\tilde{r})^{1/2}}{\kappa}\theta(|\omega|-\sqrt{\tilde{r}}). \tag{2.9}$$

There is a gap, $\sqrt{\tilde{r}}$, in the spectral density which vanishes at the critical point $\tilde{r} = 0$. This gap is expected to be filled in at finite temperatures by loop corrections involving inelastic effects; in addition, Griffiths effects (see Appendix B) will lead to subgap absorption at both zero and finite temperatures.

From (2.8) we determine that the critical point $\tilde{r} = 0$ occurs when

$$r = r_c(T) \equiv \frac{u+Mv}{\beta}\sum_\omega|\omega|. \tag{2.10}$$

The frequency summation is obviously divergent, and the result will depend upon the nature of the ultraviolet cutoff. However the temperature dependence of the result is entirely in the subleading term, which turns out to be cutoff-independent (provided the cutoff is smooth on the scale of T). The summation can be evaluated by the Poisson summation formula, which yields

$$r_c(T) = r_c - (u+Mv)\frac{\pi T^2}{3}, \tag{2.11}$$

where for a high-frequency cutoff around $\Lambda_\omega \sim g$

$$r_c \equiv r_c(0) = \frac{(u+Mv)\Lambda_\omega^2}{2\pi}. \tag{2.12}$$

The line between the paramagnetic and spin-glass phases is shown in Fig. 1 in the r, T plane.

We now turn to determining the behavior of the "gap" parameter $\tilde{r}$ close to $r = r_c$ and at finite T, by solving (2.8). First we evaluate the frequency summation, using the following identity:

$$\begin{aligned}\frac{1}{\beta}\sum_{\omega}(\omega^2+\Delta^2)^{1/2} &= \frac{\Lambda_\omega^2}{2\pi}+\frac{\Delta^2}{2\pi}\ln(c_1\Lambda_\omega/\Delta) \\ &\quad +\mathcal{O}(e^{-\Delta/T}) \quad \text{for } \Delta \gg T, \\ &= \frac{\Lambda_\omega^2}{2\pi}-\frac{\pi T^2}{3}+T\Delta+\frac{\Delta^2}{2\pi}\ln(c_2\Lambda_\omega/T) \\ &\quad +\mathcal{O}(\Delta^3/T) \quad \text{for } \Delta \ll T, \end{aligned} \tag{2.13}$$

where c_1, c_2 are constants of order unity. Then, upon examining the solution of (2.8), we find that there are three different regimes in the paramagnet phase, labeled as I, II, and III in Fig. 1. The conditions defining the regimes, and the corresponding results for $\tilde{r}$ are given below:

$$\tilde{r} = \begin{cases} \frac{4\pi}{(u+Mv)}\frac{(r-r_c)}{\ln[\Lambda_\omega^2/(r-r_c)]}, & \text{Regime I, } (r-r_c)^{1/2} \gg T \\ \frac{2\pi^2 T^2}{3\ln(\Lambda_\omega/T)}, & \text{Regime II, } T \gg |r-r_c|^{1/2} \\ \frac{[r-r_c(T)]^2}{T^2}, & \text{Regime III, } [r-r_c(T)]^{1/2} \ll T. \end{cases} \tag{2.14}$$

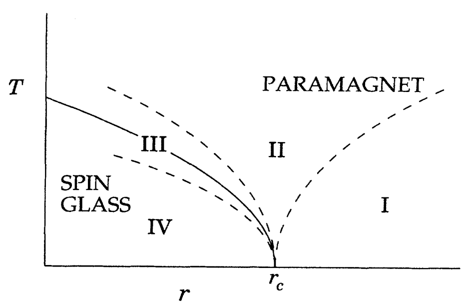

FIG. 1. Phase diagram of the action $\mathcal{A}$ [Eq. (2.2)] as a function of temperature T and the Landau parameter r which is a measure of the strength of quantum fluctuations (for $M = 1$, r is proportional to the transverse field). There is no significance to the position of the y axis, i.e., $r = 0$ does not correspond to zero quantum fluctuations which for (1.1) and (1.3) is $g = 0$. The full line is the only phase transition and dashed lines denote crossovers between different regimes. The position of the phase transition is given by $r = r_c(T)$ where $r_c(T) - r_c(0) \sim T^{1/z\nu}$ [in mean-field theory $z\nu = 1/2$ and the position is given by (2.11)]. We now list the characteristics of the regimes, and the conditions which bound them [note $r_c \equiv r_c(0)$]: (I) $(r - r_c)^{z\nu} \gg T$, quantum paramagnet: thermal effects are secondary; (II) $|r - r_c|^{z\nu} \ll T$, quantum-critical: the critical ground state at $r = r_c$ and its thermal excitations determine the physics; (III) $|r - r_c(T)|^{z\nu} \ll T$, classical: the behavior similar to that of the classical, finite-temperature, spin-glass; and (IV) $(r_c - r)^{z\nu} \gg T$, quantum spin-glass: as in I, thermal effects are secondary, but the ground state now has long-range, spin-glass order.

Regime I is that of the quantum paramagnet, where the properties are those of the quantum-disordered ground state, and thermal effects are not important. Regime II is "quantum-critical:" here the system behaves as if it is at the critical point $r = r_c$, and the properties reflect those of the critical ground state and its excitations; it is the analog of the quantum-critical region discussed in Refs. 2 and 3 for quantum rotors in the absence of disorder. Finally, in regime III, close enough to the finite-temperature phase transition, classical effects take over completely, and the behavior is that of the usual finite-temperature spin-glass–paramagnet transition in the classical model.[8] Regime III also extends into the spin-glass phase, although here we have only obtained results for its paramagnetic portion. Notice that in regime III, $\tilde{r}$ now depends upon the *square* of the distance from the transition $r - r_c(T)$: as will become clear from later results, $\tilde{r}^{1/4}$ plays the role of an inverse correlation length, so this is just what is needed to transform the quantum model with $\nu = 1/4$ to the classical model with $\nu = 1/2$ in mean-field theory. The results (2.9) and (2.14) for the local spectral weight and the asymptotic form of the gap, including the logarithmic correction are seen to be identical to those obtained in the infinite range model by different methods earlier,[17,18] as quoted in Sec. I A above [there was a factor of 2 error in the result for regime II in (2.14) in Ref. 18].

The asymptotic form of the free-energy density, $\mathcal{F}$ can be obtained from (2.6), (2.7), and (2.14). At zero temperature this yields

$$\begin{aligned}\frac{\mathcal{F}(T=0)}{nM} = \frac{1}{t\kappa^2}\Bigg\{&-\frac{\Lambda_\omega^4}{\pi}\left(\frac{1}{6}+\frac{u+Mv}{8\pi}\right)-(r-r_c)\frac{\Lambda_\omega^2}{2\pi} \\ &-(r-r_c)^2\frac{1}{2(u+Mv)} \\ &+\frac{(r-r_c)^2}{\ln[\Lambda_\omega^2/(r-r_c)]}\frac{2\pi}{(u+Mv)^2}+\cdots\Bigg\}.\end{aligned} \tag{2.15}$$

Note that the first three terms in $\mathcal{F}$ involve only integer powers of $r - r_c$. This does not immediately imply that these terms form a smooth background through the transition, as there could be discontinuities in the coefficients of $r - r_c$: a computation on the spin-glass side is required to determine this. Intuitively, we might guess that the first two cutoff-dependent terms will be analytic through the transition, while the coefficient of the $(r - r_c)^2$ term may have a discontinuity—we will see that even this discontinuity does not appear. The last term has a manifest logarithmic singularity at $r = r_c$: we will relate this singularity to marginal operators in Sec. III.

We also examined the temperature dependence of the free energy in regime II (Fig. 1), above the quantum critical point $r = r_c$, and found

$$\begin{aligned}&\frac{\mathcal{F}(r=r_c,T)-\mathcal{F}(r=r_c,T=0)}{nM} \\ &\quad = -\frac{4\pi^3 T^4}{45t\kappa^2}\{1+\mathcal{O}[1/\ln^{1/2}(\Lambda_\omega/T)]\}+\cdots.\end{aligned} \tag{2.16}$$

This predicts a low-temperature specific heat $\sim T^3$. A scaling interpretation of the power of T will be given later in Sec. III.

We now turn to a tree-level determination of the correlation functions that were introduced in Sec. I B. To do this we must expand Q about its saddle-point value

$$Q^{ab}_{\mu\nu}(k,\omega_1,\omega_2) = \beta\delta_{\mu\nu}\delta^{ab}(2\pi)^d\delta^d(k)\delta_{\omega_1+\omega_2,0}D(\omega_1) + \tilde{Q}^{ab}_{\mu\nu}(k,\omega_1,\omega_2) \quad (2.17)$$

and evaluate correlators of $\tilde{Q}$. Expanding $\mathcal{A}$ to order $\tilde{Q}^2$ we can obtain the propagator of the $\tilde{Q}$ field. It is easy to see that when $a \neq b$, this propagator is in fact $1/M$ times the G correlator [Eq. (1.21)]

$$G(k,\omega_1,\omega_2) = \frac{Mt}{k^2 + \sqrt{\omega_1^2 + \tilde{r}} + \sqrt{\omega_2^2 + \tilde{r}}}. \quad (2.18)$$

Note that this propagator has a factor t in the numerator and is independent of κ—the factors of κ were placed judiciously in $\mathcal{A}$ to achieve this. From the form of (2.18) we can deduce that at the critical point $\tilde{r} = 0$, $|\omega| \sim k^2$, so the dynamic exponent $z = 2$, while for $\tilde{r} \neq 0$ there is a length scale $\xi \sim \tilde{r}^{-1/4}$ so that the exponent defined by $\xi \sim (r - r_c)^{-\nu}$ in regime I is $\nu = 1/4$.

It is useful for the subsequent considerations to develop a diagrammatic representation of the $\tilde{Q}$ propagator and the interactions in $\mathcal{A}$. As Q is a matrix field, we will use a double-line representation for the Q propagator, G [Fig. 2(a)]. Each line can be considered to be one of the S field components of the composite Q [Eq. (1.17)], and carries with it a replica index, spin index, and frequency. The momentum is however carried by the pair of lines. The u/t and v/t terms now become two-point interactions [Fig. 2(b)] at which frequency is transferred between the four lines. The κ/t interaction is the three-point vertex shown in Fig. 2(c), while the $1/t^2$ term is in Fig. 2(d): no frequency is exchanged between the lines in the κ/t and $1/t^2$ vertices.

The connected Green's function is given by the sum of all diagrams with repeated two-point u and v interactions. Such diagrams can be easily summed and yield

$$G^c_{\mu\nu\rho\sigma}(k,\omega_1,\omega_2,\omega_3) = -\frac{1}{M^2 t}G(k,\omega_1,\omega_2)G(k,\omega_3,\omega_4)\left[\frac{u(\delta_{\mu\rho}\delta_{\nu\sigma}+\delta_{\mu\sigma}\delta_{\nu\rho})/2}{1+uL(\omega_1+\omega_2,k,\tilde{r})} + \frac{v\delta_{\mu\nu}\delta_{\rho\sigma}}{[1+uL(\omega_1+\omega_2,k,\tilde{r})][1+(u+Mv)L(\omega_1+\omega_2,k,\tilde{r})]}\right], \quad (2.19)$$

where the three frequencies ω_1, ω_2, ω_3 arise from the Fourier transform of the three time arguments in (1.24) and $\omega_1 + \omega_2 + \omega_3 + \omega_4 = 0$, and

$$L(\omega,k,\tilde{r}) \equiv \frac{1}{\beta}\sum_{\Omega}\frac{1}{k^2 + \sqrt{\Omega^2+\tilde{r}} + \sqrt{(\Omega-\omega)^2+\tilde{r}}} \approx \frac{1}{2\pi}\ln\left(\frac{\Lambda_\omega}{\max\,(k^2,|\omega|,\sqrt{\tilde{r}},T)}\right) \quad (2.20)$$

is the frequency integral that appears between two two-point vertices (we have assumed that we are not in the classical regime III). Note that the connected Green's function is proportional to the quantum-mechanical interactions u, v as it should be. Further note that $G^c \sim t$ as the two G's carry a factor of t.

Finally we can also evaluate the disconnected Green's function G^d. This is obtained by attaching G^c and G on the ends of a $1/t^2$ vertex [Fig. 2(d)], and carrying out the intermediate frequency integral (it is not necessary to include graphs with repeated $1/t^2$ insertions as they all vanish in the replica limit $n \to 0$); this gives

$$G^d(k,\omega_1,\omega_2) = \frac{1}{t^2}\frac{G(k,\omega_1,-\omega_1)G(k,\omega_2,-\omega_2)}{[1+(u+Mv)L(0,k,\tilde{r})]^2}. \quad (2.21)$$

Note that the factor of $1/t^2$ with the vertex will cancel against the two t's in the G's, and G^d is independent of t. It is seen that, even neglecting the denominators that contain L, G^c, and G^d both vary as G^2, so are more strongly divergent at long wavelengths than G.

The results for the correlation functions now allow us to define the correlation exponent η. The basic correlation function is G, and we define η by

$$G(k,0,0) \sim k^{-2+\eta}, \quad (2.22)$$

so that $\eta = 0$ in mean-field theory, as is conventional. In real space, G then decays as $x^{-(d+2z-2+\eta)}$. Our definition, though conventional in its relation to mean-field theory, differs from that used in two recent papers.[19,20] Their η, which we call η', is related to ours by $\eta' = \eta + z$. We can also define another exponent for G^d, in analogy with the random-field Ising model, by

$$G^d(k,0,0) \sim k^{-4+\bar{\eta}}, \quad (2.23)$$

so that $\bar{\eta}$ is also zero in the Gaussian approximation. In real space $G^d \sim x^{-(d+2z-4+\bar{\eta})}$. Discussion of these exponents will be continued in Sec. III and IV.

2. Spin-glass phase

We will now look at the mean-field behavior of $\mathcal{A}$ for $r < r_c(T)$ where spin-glass order appears. This phase

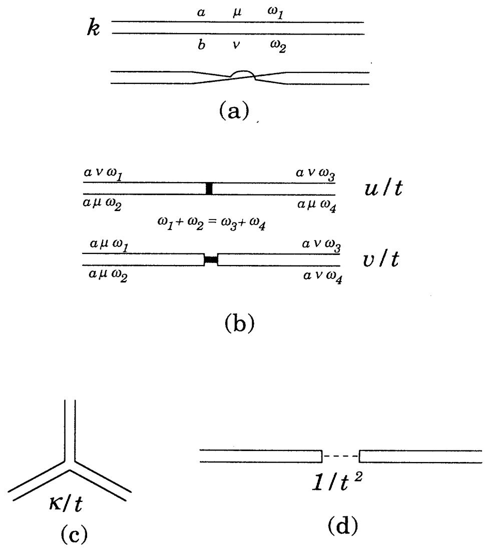

FIG. 2. (a) The double-line representation for the propagator of the Q field; each line represents one of its constituent S fields, and carries its own replica (a, b), vector $O(M)$ (μ, ν) and frequency (ω_1, ω_2) indices. Momentum (k) is however only carried by the composite double-line propagator. Also shown is a twisted partner which can replace the untwisted propagator in all the Feynman diagrams in Figs. 6 and 7. (b) The quantum-mechanical interactions u/t and v/t with replica, $O(M)$ and frequency indices shown explicitly. Note that the single lines do preserve their replica and $O(M)$ labels, but can exchange frequency. All double-line propagators carry the same momentum. (c) The cubic coupling κ/t. Each single line now preserves replica, $O(M)$ and frequency labels through the vertex. The double lines however do exchange momenta, with the total momentum being conserved. (d) The randomness-induced $1/t^2$ quadratic coupling. Again, each single line preserves replica, $O(M)$ and frequency labels.

was studied in Ref. 18 for the infinite-range model only at $M = \infty$, and a replica-symmetric solution for the Q order parameter was obtained at all temperatures. The absence of replica-symmetry breaking was not surprising as the classical model is also known to preserve replica symmetry at $M = \infty$.[29] The present mean-field theory is being carried out at finite M, and so one expects, first, that at finite temperature replica symmetry breaking should occur since the system maps onto the classical system, and second that at zero temperature the ordered ground state resembles that of a classical system, However, this leaves open the question how replica symmetry breaking behaves as $T \to 0$. This turns out to be quite subtle already in the classical spin glass,[7,30,31] with which we will compare and contrast our results after they have been described.

We will begin this section with a mean-field treatment of $\mathcal{A}$ [Eq. (2.2)], which yields a replica-symmetric solution for the spin-glass phase at all temperatures. We will then consider the consequences of adding higher-order terms to $\mathcal{A}$, terms which are formally irrelevant under the subsequent renormalization-group analysis of Sec. III; all such terms will continue to be innocuous at zero temperature, but one of them becomes dangerous at any finite temperature and leads to replica symmetry breaking, whose strength is proportional to temperature. The mechanism of the replica symmetry breaking at low temperatures will be discussed within the framework of our Landau theory.

The generalization of the saddle-point ansatz (2.5) to the spin-glass phase is

$$\begin{aligned} Q^{ab}_{\mu\nu}(k,\omega_1,\omega_2) &= (2\pi)^d\delta^d(k)\delta_{\mu\nu}[\beta D(\omega_1)\delta_{\omega_1+\omega_2,0}\delta^{ab} \\ &\quad +\beta^2\delta_{\omega_1,0}\delta_{\omega_2,0}q^{ab}], \\ D(\omega) &\equiv \bar{D}(\omega) + \beta\tilde{q}\delta_{\omega,0}. \end{aligned} \tag{2.24}$$

We have introduced the parameters q^{ab}, $\tilde{q}$, and $\bar{D}(\omega)$, which have to be varied to determine the stationary point of the free energy. As before, factors of β have been judiciously placed to ensure that all these parameters are finite in the limit $\beta \to \infty$. Note that the replica off-diagonal components of Q at the saddle-point are nonzero only when both frequencies are zero: this is because they represent averages of the type $[\langle S(\tau_1)\rangle\,\langle S(\tau_2)\rangle]$ which must be time independent by time translational invariance of the quantum-mechanical averages. Without loss of generality we may assume that the diagonal components of q^{ab} are zero, and that $\bar{D}(\omega = 0) = 0$, as both of these can be absorbed into $\tilde{q}$. We expect a solution in which $\bar{D}(\omega)$ is finite and continuous (except perhaps at $\omega = 0$) as $\beta \to \infty$, in which case (1.19) implies that at zero temperature $\tilde{q}$ is just the Edwards-Anderson order parameter $q_{\rm EA}$; the equality between $q_{\rm EA}$ and $\tilde{q}$ does not hold at nonzero temperatures, although we always have $q_{\rm EA} = \max_{a\neq b} q^{ab}$.[8] We now insert this ansatz into $\mathcal{A}$ and find for the free-energy density $\mathcal{F}$

$$\begin{aligned} \frac{\mathcal{F}}{Mn} &= \frac{1}{\beta\kappa t}\sum_\omega(\omega^2+r)\bar{D}(\omega) + \frac{r}{\kappa t}\tilde{q} - \frac{\kappa}{3\beta t}\sum_\omega \bar{D}^3(\omega) \\ &\quad + \frac{(u+Mv)}{2t}\left(\tilde{q} + \frac{1}{\beta}\sum_\omega \bar{D}(\omega)\right)^2 \\ &\quad - \frac{\kappa\beta^2}{3t}\left(\tilde{q}^3 + 3\tilde{q}\frac{{\rm Tr}q^2}{n} + \frac{{\rm Tr}q^3}{n}\right). \end{aligned} \tag{2.25}$$

All factors of β have been written explicitly and all other variables are expected to be finite in the limit $\beta \to \infty$. Let us now examine this limit in $\mathcal{F}$. In the first four terms, we only have factors of $1/\beta$ associated with frequency summations, and the combinations should be finite as $\beta \to \infty$. The remaining terms all arise from the static contributions to the Q^3 term in $\mathcal{A}$, and appear to have a dangerous divergence $\sim \beta^2$ as $\beta \to \infty$. We will determine the saddle point of $\mathcal{F}$ for finite β below, and find that in the replica limit of $n \to 0$, all the terms of order β^2 (and also of order β) in fact cancel with each other at the saddle point, yielding a finite zero-temperature free-energy density.

We now find the saddle point of $\mathcal{F}$ with respect to variations in $\tilde{q}$, the function $\bar{D}(\omega)$, and the space of ultrametric matrices q^{ab}. This can be done by a straightforward extension of the classical methods[8] and we simply quote the final results, valid for arbitrary β

$$\tilde{q} = \frac{1}{\beta\kappa}\sum_{\omega}|\omega| - \frac{r}{\kappa(u+Mv)}$$
$$= \frac{1}{(u+Mv)\kappa}[r_c(T) - r], \quad (2.26)$$

$$q^{ab} = \tilde{q} \quad \text{for } a \neq b, \quad (2.27)$$

$$\bar{D}(\omega) = -\frac{|\omega|}{\kappa}, \quad (2.28)$$

where $r_c(T)$ was defined in (2.11). Note that this solution is replica symmetric as all off-diagonal matrix elements of q^{ab} are equal. The order parameter q^{ab} has to be positive in the spin-glass phase, which is therefore restricted to $r < r_c(T)$. At zero temperature, the properties of D are a little more transparent in the time domain,

$$D(\tau) = \tilde{q} + \frac{1}{\pi\kappa\tau^2} \quad \text{at } T = 0, \quad (2.29)$$

where it decays to a finite, positive value ($= \tilde{q} = q_{\rm EA}$) at large time. Note also that the classical definition[8] $q_{\rm EA} = \max_{a\neq b} q^{ab}$, and (2.27) give the same value of $q_{\rm EA}$; this equality between the two approaches to $q_{\rm EA}$ is also easily seen to lead to a cancellation of all terms of order β^2 in $\mathcal{F}$. The $1/\tau^2$ decay in $D(\tau)$ also holds at the critical point $r = r_c, T = 0$ where the decay is to zero. The power-law decay is related to the gaplessness of the spectral density in the entire spin-glass phase [from (2.28) $\chi''(\omega) = \omega/\kappa$].

The results [(2.26)–(2.28)] are actually valid in both regimes III and IV of Fig. 1 in the spin-glass phase. As a result, the order-parameter exponent $\beta = 1$ in both the classical and quantum transitions. Crossovers between regimes III and IV will presumably appear upon considering fluctuations about the present mean-field theory.

We also quote the result for the free-energy density in the spin-glass phase at zero temperature:

$$\frac{\mathcal{F}(T=0)}{nM} = -\frac{\Lambda_\omega^4}{\pi\kappa^2 t}\left(\frac{1}{6} + \frac{u+Mv}{8\pi}\right) - (r-r_c)\frac{\Lambda_\omega^2}{2\pi\kappa^2 t} - (r-r_c)^2\frac{1}{2(u+Mv)\kappa^2 t}. \quad (2.30)$$

This differs from the expression (2.15) for the paramagnetic phase only by the absence of the term with a logarithmic singularity. The terms in (2.30) thus *do* constitute an analytic background to the singular part.

Replica symmetry breaking. It is known from the Landau theory of the classical spin glass that replica symmetry breaking does not appear until terms of order Q^4 have been included in the action.[8] So to address the stability of the replica symmetric mean-field solution (2.26)–(2.28), we have to extend $\mathcal{A}$ to include at least such terms. Upon examining these terms, a second obstacle immediately confronts us: one of the quartic terms has four time integrals, so the contribution of the static moments to the free energy appears to diverge as β^3 in the zero-temperature limit. It is also easy to see that this obstacle gets worse with terms of higher order, which appear to contribute even higher powers of β to the free energy. It might therefore appear that it is necessary to resume these strongly divergent terms and that the whole Landau theory framework is breaking down in the spin-glass phase. We now argue that this is not the case, and that enough factors of β cancel out to yield finite results for $\tilde{q}$, q^{ab} and $\bar{D}(\omega)$ in the $\beta \to \infty$ limit. One can proceed by the usual Landau theory framework to develop an expansion for all physical quantities in powers of $[r_c(T) - r]$, and all dangerous factors of β will cancel out order by order. We illustrate this cancellation explicitly for the quartic terms. We consider three of the many quartic terms which can be added to (2.2) (we drop all vector indices μ, ν and treat only the case $M = 1$, as the vector nature plays no role in the following):

$$\mathcal{A} = \cdots - \frac{1}{6t}\int d^dx\Bigg\{y_1\int d\tau_1 d\tau_2 \sum_{a,b}\left[Q^{ab}(x,\tau_1,\tau_2)\right]^4 + y_2\int d\tau_1 d\tau_2 d\tau_3 \sum_{a,b,c}[Q^{ab}(x,\tau_1,\tau_2)]^2[Q^{bc}(x,\tau_2,\tau_3)]^2$$
$$+ y_3\int d\tau_1 d\tau_2 d\tau_3 d\tau_4 \sum_{a,b,c,d} Q^{ab}(x,\tau_1,\tau_2)Q^{bc}(x,\tau_2,\tau_3)Q^{cd}(x,\tau_3,\tau_4)Q^{da}(x,\tau_4,\tau_1)\Bigg\}, \quad (2.31)$$

where the initial dots denote the terms already in (2.2). Let us now insert the mean-field ansatz (2.24) into the extended action $\mathcal{A}$. The analysis for arbitrary ultrametric matrices q^{ab} is rather complicated—fortunately the general principles become clear by allowing for a small amount of replica symmetry breaking. We will begin by considering the replica symmetric case, and then add a perturbation to allow for replica symmetry breaking. For the replica symmetric case $q^{ab} = q$ for all $a \neq b$, we find for the free-energy density:

$$\mathcal{F} = -\frac{n}{t}\Bigg[\frac{\beta^2\kappa}{3}(\tilde{q}-q)^2(\tilde{q}+2q) + \frac{\beta y_1}{6}(\tilde{q}-q)(\tilde{q}+q)(\tilde{q}^2+q^2) + \frac{\beta^2 y_2}{6}(\tilde{q}-q)^2(\tilde{q}+q)^2 + \frac{\beta^3 y_3}{6}(\tilde{q}-q)^3(\tilde{q}+3q)\Bigg] + \cdots. \quad (2.32)$$

We have explicitly written down terms which depend only upon the static moments; all omitted terms are explicitly finite in the limit $\beta \to \infty$ and play no role in the following discussion. The key observation that we make from (2.32) is that, term by term, powers of β are always paired with an equal number of powers of $(\tilde{q}-q)$: a mean-field solution with $(\tilde{q}-q) \sim 1/\beta$ will therefore give a finite limit as $\beta \to 0$. That this actually occurs can be verified by obtaining an explicit expression for q in terms of $\tilde{q}$: we take the derivative of $\mathcal{F}$ with respect to q [all terms not explicitly displayed in (2.32) are independent of q], equate it to zero, and obtain an expansion for q in terms of $\tilde{q}$; this yields

$$q = \tilde{q} + \frac{y_1\tilde{q}^2}{3\beta\kappa}\left(1 - \frac{2y_2}{3\kappa}\tilde{q} + \frac{y_3}{3\kappa^2}\tilde{q}^2\right), \qquad (2.33)$$

with omitted terms being higher order in both $\tilde{q}$ and $1/\beta$. Here we see an example of our earlier claim that the Landau-type expansion in powers of the order parameter, $\tilde{q} \sim q$, is well behaved, with no singular powers of β appearing. We also see that $\tilde{q}-q \sim 1/\beta$ as desired. Note that the most important contribution to $\tilde{q}-q$ comes from y_1, the term associated with the fewest powers of β in the original expansion for $\mathcal{F}$. Terms associated with y_2 and y_3, had higher powers of β in $\mathcal{F}$, make contributions $\tilde{q}-q$ which are higher order in $\tilde{q}$. This happens because

$$\beta(\tilde{q}-q) \sim \tilde{q}^2; \qquad (2.34)$$

so each factor of $\beta(\tilde{q}-q)$ actually ends up behaving like two powers of the order parameter. Thus contrary to naive expectations, it is the terms with the fewest powers of β which are most important in the Landau expansion.

Let us now add some replica symmetry breaking to the mean-field ansatz

$$q^{ab} = q + \alpha^{ab}, \quad a \neq b, \qquad (2.35)$$

where α^{ab} is a small perturbation along the direction in replica space where symmetry breaking is expected. We choose α^{ab} to lie along the "replicon" direction of de Almeida and Thouless,[33] proportional to an eigenvector along which they found an instability of the replica symmetric solution: $\alpha^{ab} = \alpha$ for $a, b > 2$, $\alpha^{ab} = \frac{1}{2}(3-n)\alpha$ for $a = 1, 2$ and $b > 2$, $\alpha^{12} = \frac{1}{2}(3-n)(2-n)\alpha$, and $\alpha^{ba} = \alpha^{ab}$. For this form of q^{ab}, we expand the free energy to order α^2, and obtain in addition to the terms in (2.32)

$$\mathcal{F} = \ldots - \frac{6\beta\alpha^2}{t}\left[\beta\kappa(\tilde{q}-q) + y_1q^2 + \frac{\beta y_2}{3}(\tilde{q}^2 - q^2) + \beta^2 y_3(\tilde{q}-q)^2\right] + \cdots . \qquad (2.36)$$

Again factors of β are paired with $(\tilde{q}-q)$ and the combination will be finite as $\beta \to \infty$. There is however an overall prefactor of $\beta\alpha^2$ which needs attention. By analogy with the low temperature properties of the solution to the classical spin-glass, we may expect[7] that the magnitude of the replica symmetry breaking will be proportional to $1/\beta$, in which case $\alpha \sim 1/\beta$. The contribution of the replica broken component of q to $\mathcal{F}$ is now seen to be $\sim 1/\beta$, as is also the case in the classical limit.[7] We will verify these expectations in a more complete analysis of the replica symmetry breaking below. For now we note that if we insert the expansion (2.33) in (2.36) we find the α-dependent contribution to the free energy

$$\mathcal{F} = \cdots - 4\beta\alpha^2 y_1\tilde{q}^2/t + \cdots . \qquad (2.37)$$

The couplings κ, y_2, and y_3 have canceled out and there is an instability towards replica symmetry breaking driven solely by y_1; the analogous result for the classical model is well known.[7]

Let us now reiterate the rather simple conclusion to which the above chain of reasoning has lead us: the low-temperature replica symmetry-broken state of the spin-glass phase can be determined simply by adding to $\mathcal{A}$ the single quartic term proportional to y_1, and solving the saddle-point equations. All corrections from other terms will involve higher powers of $\tilde{q}$, and hence $(r_c - r)$, and no dangerous powers of β will appear in this expansion. The q^{ab} contribution to $\mathcal{F}$, including the y_1 term, for general q^{ab} has the form

$$\mathcal{F} = \cdots - \frac{\kappa\beta^2}{3t}\left(\mathrm{Tr}q^3 + 3\tilde{q}\mathrm{Tr}q^2\right) - \frac{y_1\beta}{t}\sum_{a\neq b}(q^{ab})^4 + \cdots . \qquad (2.38)$$

It is a straightforward matter[8] to obtain the optimum saddle point of $\mathcal{F}$ with respect to arbitrary ultrametric matrices q^{ab}. Such matrices are characterized by a function $q(s)$ on the interval $0 \le s \le 1$, which is found to be of the form[8]

$$q(s) = \begin{cases} (s/s_1)q(1) & \text{for } 0 < s < s_1 \\ q(1) & \text{for } s_1 < s < 1 \end{cases} \qquad (2.39)$$

with (expanding to the appropriate order in $\tilde{q}$):

$$q(1) = \tilde{q} + \frac{y_1\tilde{q}^2}{\beta\kappa}, \qquad s_1 = \frac{2y_1 q(1)}{\beta\kappa}. \qquad (2.40)$$

To leading order in powers of $r - r_c(T)$, we may use the value of $\tilde{q}$ from (2.26) in the above. Note that $q(s)$ differs from a constant over a region of length s_1 which is therefore a measure of the strength of replica symmetry breaking; as expected this vanishes as $1/\beta$ at low temperature. Alternatively we may compute, from the above and (2.26), the "broken ergodicity" order parameter Δ_q (Ref. 8)

$$\Delta_q \equiv q(1) - \int_0^1 q(s)ds = \frac{y_1}{(u+Mv)^2\kappa^3}T[r_c(T) - r]^2; \qquad (2.41)$$

notice the prefactor of T indicating suppression of replica symmetry breaking at low temperatures. A renormalization-group interpretation of the dependencies on T and $r_c - r$ will be given in Sec. III.

The meaning of the statement that replica symmetry breaking disappears as $T \to 0$ in the spin-glass phase is as follows. For the infinite-range classical spin glass, it is well known that Parisi's order-parameter function $q(s)$ is related to the existence of many minima in the energy landscape, separated by infinite energy barriers in the ordered phase.[7,8,34] Thus "ergodicity is broken" and (in each realization of the disorder) configuration space breaks up into disjoint regions, labeled by α, on each of which a Gibbs weight $e^{-\beta f_\alpha}$ can be computed by summing $e^{-\beta H}$ over all configurations in the region. The partition function is then $Z = \sum_\alpha e^{-\beta f_\alpha}$ and $P_\alpha = e^{-\beta f_\alpha}/Z$ is the Gibbs probability for each of the "pure phases" α, for a given set of random bonds. Each phase α has a thermal average spin m_i^α for each site i; $q^{\alpha\beta} = (1/N)\sum_i m_i^\alpha m_i^\beta$ is the overlap of different phases, and

$$P_J(q) = \sum_{\alpha\beta} P_\alpha P_\beta \delta(q - q^{\alpha\beta}) \tag{2.42}$$

is the Gibbs probability of finding overlap equal to q. The disorder average $P(q)$ of $P_J(q)$ is given in terms of Parisi's $q(s)$ by $P(q) = (dq(s)/ds)^{-1}$. A further key result is that the free energies f_α are independently exponentially distributed random variables (due to the randomness in the J_{ij}'s): their distribution is

$$P(f) \propto e^{\rho f} \; , \tag{2.43}$$

where ρ is a function of T and any external fields. Thus ρ is the inverse spacing of the low-lying f_α's. It turns out that $\rho = \beta s_1$, where s_1 is the value of s such that $q(s) = q(1)$ for $s > s_1$,[34] and that similar results hold for the distributions of *clusters* of pure phases.

We assume that similar results hold for the quantum spin-glass (with sums over states replaced by traces). Then our result that $s_1 \to 0$ as $T \to 0$ does not mean that many pure phases do not exist, but rather, since by (2.40) $\rho \to$ const, that the typical energy differences approach a constant, and so only the lowest free-energy phase is significant in this limit. This differs from what has been found in the *classical* Ising spin glass, where $q(s) \to 1$ as $T \to 0$ for each s, but $s_1 \to 1/2$.[30,31] In the latter model, the nature of the $T \to 0$ behavior of $q(s)$ affects the calculation of, e.g., the energy at $T = 0$ and although we have not done the corresponding calculation in our system, we expect that the same is true here. Thus the $T = 0$ behavior *cannot* be obtained by setting $T = 0$ and doing a replica symmetric calculation, as the factors of β in the free-energy calculation make clear.

Finally, we comment in passing on claims in the literature[32] that the infinite-range, transverse-field Ising spin-glass has a region of stability for the replica-symmetric solution just below the finite T transition to the paramagnet, i.e., for r close to but below $r_c(T)$ and T finite. These claims disagree with our Landau theory which finds broken replica symmetry at all finite T in the spin-glass phase. We believe the earlier results are artifacts of the approximations used therein.[32]

3. Phases in a longitudinal field

We now consider the behavior of the system in a field, h, which couples linearly to the on-site spin field. In such a field, the Hamiltonian of the Ising model (1.3) is modified by

$$\mathcal{H}_I \to \mathcal{H}_I - h\sum_i \sigma_i^z, \tag{2.44}$$

while for the rotor Hamiltonian (1.1) we have

$$\mathcal{H}_R \to \mathcal{H}_R - h\sum_i n_{1i} \tag{2.45}$$

with the field pointing along the 1 direction. Carrying h through the derivation of the effective action in Appendix A, we find the following modification to the Landau action:

$$\mathcal{A} \to \mathcal{A} - \frac{h^2}{2t}\int d^dx\, d\tau_1 d\tau_2 \sum_{ab} Q_{11}^{ab}(x,\tau_1,\tau_2) - \frac{\beta}{2}\chi_{hb}h^2, \tag{2.46}$$

where χ_{hb} is a background, local contribution to the linear susceptibility. We will now extend the above mean-field theory to include the additional term. It turns out to be useful to consider the cases $M = 1$ and $M > 1$ separately, as there are significant differences between them.

(a) *M=1, the Ising spin-glass.* The field term acts like a source for the off-diagonal components of Q, which are therefore always nonzero. We will first find the complete mean-field solution in the absence of Q^4 terms, in which case, replica symmetry will be unbroken and all the off-diagonal components of q^{ab} will be equal. We will then proceed to examine the consequences of a small quartic y_1 coupling, and will find that replica symmetry breaking occurs at finite temperature for small enough h and $r < r_c(T)$. The instability line towards replica symmetry breaking is of course the quantum analog of the well-known de Almeida-Thouless (or AT) line.[33]

First, the solution at $y_1 = 0$. We insert the ansatz (2.24) into $\mathcal{A}$, determine stationary points with respect to variations in $\tilde{q}$, q^{ab}, and $\bar{D}(\omega)$ and find the following solution, valid everywhere provided $h \neq 0$:

$$\begin{aligned}\bar{D}(\omega) &= -\frac{1}{\kappa}(\omega^2+\Delta^2)^{1/2}, \quad \omega \neq 0,\\ q^{ab} &= q = \frac{h^2}{4\Delta}, \quad a \neq b,\\ \tilde{q} &= \frac{h^2}{4\Delta} - \frac{\Delta}{\kappa\beta},\end{aligned} \tag{2.47}$$

where we have as before $\bar{D}(0) = 0$, and the parameter Δ is determined implicitly by the equation

$$\Delta^2 = r + (u+v)\left(\frac{\kappa h^2}{4\Delta} - \frac{1}{\beta}\sum_\omega(\omega^2+\Delta^2)^{1/2}\right). \quad (2.48)$$

Small h behavior. For small h, the solution to these equations depends strongly on the sign of $r - r_c(T)$, and we consider the different cases separately.

In the paramagnetic phase $r > r_c(T)$, and we see by comparing (2.48) and (2.8) that $\Delta^2 \to \tilde{r}$ as $h \to 0$. This value can be inserted in (2.47) to obtain the low-field dependence of q and $\tilde{q}$. We also expanded the free energy to order h^4 and thus obtained the linear susceptibility, χ_h

$$\chi_h = \chi_{hb} - \sqrt{\tilde{r}}\kappa \quad (2.49)$$

and the nonlinear susceptibility $\chi_{\rm nl}$

$$\chi_{\rm nl} = \frac{1}{4\tilde{r}}\frac{u+v}{1+(u+v)L(0,0,\tilde{r})}, \quad (2.50)$$

where L was defined in (2.20), and $\tilde{r}$ is given by (2.14) for small $r - r_c(T)$; note that the result for $\chi_{\rm nl}$ is consistent with (1.29) and (2.19). As expected, at the transition where $\tilde{r} = 0$, χ_h remains finite, while $\chi_{\rm nl}$ diverges. The singularity of $\chi_{\rm nl}$ at the $T = 0$ quantum transition is $\chi_{\rm nl} \sim 1/(r - r_c)$, while at the finite T classical transition we have $\chi_{\rm nl} \sim 1/[r - r_c(T)]$. The mean-field result for the $T = 0$ singularity in $\chi_{\rm nl}$ as $r \to r_c$ [$\chi_{\rm nl} \sim (r - r_c)^{-1}$] differs from that in the infinite-range model[17] [$\chi_{\rm nl} \sim (r - r_c)^{-1/2}$ up to logarithms].[24]

Precisely at the $T = 0$ paramagnetic–spin-glass phase boundary, we find the following low-field dependencies

$$\tilde{q} = q = \frac{h^{4/3}\ln^{1/3}(\Lambda_\omega/\lambda_\omega)}{(32\pi\kappa)^{1/3}}, \quad (2.51)$$

where $\lambda_\omega = (\kappa h^2)^{1/3}$, while the free energy behaves as

$$\mathcal{F}(h) - \mathcal{F}(h=0) = n\frac{3}{4t}\left(\frac{\pi}{16\kappa^2\ln(\Lambda_\omega/\lambda_\omega)}\right)^{1/3} h^{8/3}. \quad (2.52)$$

Above the zero-field spin-glass phase we find

$$\Delta = (u+v)\kappa h^2/[4(r_c(T) - r)] + \mathcal{O}(h^6),$$

which gives from (2.47)

$$q = \frac{1}{(u+v)\kappa}[r_c(T) - r] + \mathcal{O}(h^4),$$
$$\tilde{q} = q - \frac{h^2}{4q\kappa\beta}. \quad (2.53)$$

The absence of any h^2 term in q leads to the vanishing of the Q-field contribution to χ_h, which is now given only by its background value

$$\chi_h = \chi_{hb}. \quad (2.54)$$

Recall that the classical model[7] also has a constant linear susceptibility in the spin-glass phase. The nonlinear susceptibility can be obtained by expanding the free energy to order h^4, and we get

$$\chi_{\rm nl} = \frac{u+v}{4[r_c(T) - r]}. \quad (2.55)$$

Finite h properties. The main phenomenon at finite h is the appearance of the AT surface in the phase diagram; replica symmetry breaking occurs at fields below this surface (see Fig. 3). Determination of the position of the initial instability towards replica symmetry breaking closely parallels the $h = 0$ calculation of Sec. II B 2. For nonzero h, the generalization of (2.33) is

$$q = \tilde{q} + \frac{y_1\tilde{q}^2}{3\beta\kappa} + \frac{h^2}{4\tilde{q}\beta\kappa}, \quad (2.56)$$

where we have turned on quartic y_1 coupling, but not the unimportant y_2, y_3 terms [this equation is consistent with (2.53)]. We now perform a replica symmetry-breaking deformation of q as in (2.35), and evaluate the change in the free energy to order α^2. This generalizes (2.37) to

$$\mathcal{F} = \cdots + \frac{6\beta\alpha^2}{t}\left(\frac{h^2}{4\tilde{q}} - \frac{2y_1\tilde{q}^2}{3}\right) + \cdots . \quad (2.57)$$

There is an instability towards replica symmetry breaking only above the spin-glass region, $r < r_c(T)$, in which case we find from (2.53) the following final result for the position of the AT instability:

$$h_{\rm AT}^2 = \frac{8y_1}{3\kappa^3(u+v)^3}[r_c(T) - r]^3. \quad (2.58)$$

A sketch of $h_{\rm AT}$ as a function of r and T is shown in Fig. 3. An important feature of the above result is that $h_{\rm AT}$ has a nonzero limit as $T \to 0$. So, even though the strength of the replica symmetry breaking becomes vanishingly small as $T \to 0$, it requires a finite field strength to re-

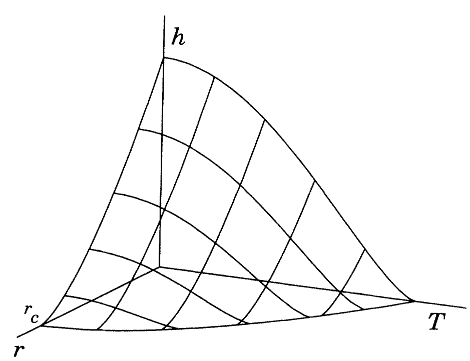

FIG. 3. Mean-field phase diagram for the $M = 1$ Ising spin-glass in a longitudinal field h. The surface shown is the analog of the de Almeida-Thouless (Ref. 33) line; Eq. (2.58) determines the surface position close to quantum critical point $r = r_c$ and $T = 0$. Replica symmetry breaking occurs for values of h below the surface. However the strength of the replica symmetry breaking vanishes both as h approaches the surface, and as $T \to 0$; the strength of the replica symmetry breaking is given by (2.60).

store replica symmetry even at an infinitesimal T. We can quantify this by obtaining an explicit solution for the replica symmetry breaking for h below $h_{\rm AT}$. The computation parallels that of Sec. II B 2 and that for the classical case:[8] the only modification of the $h = 0$ result for the order parameter $q(s)$ in (2.39) is that there is an additional plateau at $q(s) = q_0$ for $0 < s < s_0 = s_1 q_0/q(1)$ with

$$q_0 = \left(\frac{3h^2}{8y_1}\right)^{1/3} . \tag{2.59}$$

As in (2.41) we can also obtain the broken ergodicity order parameter

$$\Delta_q = \frac{(9y_1)^{1/3}}{4\kappa} T \left(h_{\rm AT}^{4/3} - h^{4/3}\right) \tag{2.60}$$

which is clearly nonzero for $h < h_{\rm AT}$. Notice however the prefactor of T, which indicates the weakness of the replica symmetry breaking.

$M > 1$, quantum rotor spin-glass. We now have to consider the distinct behavior of the components of Q which are longitudinal and transverse to the applied field. For a field along the 1 direction, the replica off-diagonal components of the longitudinal Q_{11} are always nonzero, while those of the transverse $Q_{\mu\mu}$, $\mu > 1$, do not couple linearly to the applied field, and need not be nonzero. The Gabay-Toulouse (GT) line[8] identifies the boundary along which transverse, replica off-diagonal components of Q turn on, and we will determine its position below.

We find it slightly more convenient to determine the GT boundary by approaching it from below, where the transverse, replica-off-diagonal components of Q are nonzero. We generalize the ansatz (2.24) by introducing the longitudinal parameters q_L^{ab}, $\tilde{q}_L$, and $\bar{D}_L(\omega)$ for $\mu = 1$, and the transverse parameters q_T^{ab}, $\tilde{q}_T$, and $\bar{D}_T(\omega)$ for $\mu > 1$. As the GT boundary is present even in the absence of replica symmetry breaking, we will work with the action $\mathcal{A}$ with the quartic terms omitted. In this case $q_L^{ab} = q_L$ for $a \neq b$ and similarly for q_T^{ab}. The generalization of the saddle-point equations (2.47) to the $M > 1$ case is

$$\begin{aligned} \tilde{q}_T &= q_T \\ \bar{D}_T(\omega) &= -\frac{1}{\kappa}|\omega|, \quad \omega \neq 0 \\ \bar{D}_L(\omega) &= -\frac{1}{\kappa}(\omega^2 + \Delta^2)^{1/2}, \quad \omega \neq 0, \\ q_L &= \frac{h^2}{4\Delta}, \\ \tilde{q}_L &= \frac{h^2}{4\Delta} - \frac{\Delta}{\kappa\beta}, \end{aligned} \tag{2.61}$$

where $\bar{D}_T(0) = \bar{D}_L(0) = 0$ and Δ and q_T are determined by solution of the equations

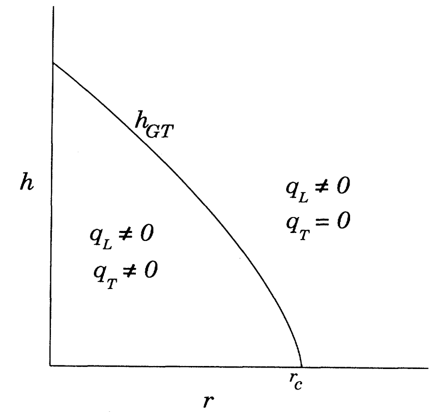

FIG. 4. Phase diagram for the $M > 1$ rotor spin-glass in a longitudinal field h at zero temperature. The boundary $h_{\rm GT}$ is the quantum analog of the Gabay-Toulouse line (Ref. 8) and is given by (2.63) in mean-field theory. The spin-glass order parameters q_L and q_T refer to replica off-diagonal components of Q_{11} and $Q_{\mu\mu}$, $\mu > 1$ respectively. (Replica off-diagonal components of $Q_{\mu\nu}$ with $\mu \neq \nu$ are zero everywhere.) The field h points along the 1 direction and couples linearly to the rotor coordinate $\hat{n}$. Replica symmetry breaking is expected to occur in the mean-field theory at nonzero T for all $h < h_{\rm GT}$.

$$\begin{aligned} \Delta^2 &= r + (u+v)\left(\frac{\kappa h^2}{4\Delta} - \frac{1}{\beta}\sum_\omega (\omega^2+\Delta^2)^{1/2}\right) \\ &\quad + (M-1)v\left(\kappa q_T - \frac{1}{\beta}\sum_\omega |\omega|\right), \\ 0 &= r + v\left(\frac{\kappa h^2}{4\Delta} - \frac{1}{\beta}\sum_\omega (\omega^2+\Delta^2)^{1/2}\right) \\ &\quad + (u + (M-1)v)\left(\kappa q_T - \frac{1}{\beta}\sum_\omega |\omega|\right). \end{aligned} \tag{2.62}$$

It is not difficult to solve these equations and determine the position of the GT boundary by imposing the condition $q_T = 0$. We give below the results of this procedure at $T = 0$

$$h_{\rm GT} = \left(\frac{2}{\pi\kappa}\right)^{1/2} \left(\frac{u(r_c - r)}{v}\right)^{3/4} \ln^{1/2}(\Lambda_\omega^2/\lambda_\omega^2), \tag{2.63}$$

where $\lambda_\omega = (r - r_c)^{1/2}$. The resulting $T = 0$ phase diagram is sketched in Fig. 4. The $T = 0$ singularity in $h_{\rm GT}$ as $r \to r_c$ is seen to be quite different from that in the classical model.

4. Phases in a field coupling to the conserved total angular momentum

For spin-glass models with a continuous, global symmetry, Noether's theorem implies that there is a conserved charge which commutes with the Hamiltonian. For the present quantum rotor models this charge is the total angular momentum. In this section, we will examine the properties of $\mathcal{H}_R$ in a field which couples to this total angular momentum:

$$\mathcal{H}_R \to \mathcal{H}_R - \frac{1}{2} H_{\mu\nu} \sum_i \hat{L}_{i\mu\nu}. \tag{2.64}$$

For $M = 3$, $H_{\mu\nu} = \epsilon_{\mu\nu\lambda} H_\lambda$, for the usual vector field H_λ. For the experimental situations in which the quantum rotor model may be realized in systems with short-range antiferromagnetic order,[5] H corresponds simply to a uniform external magnetic field. Obviously there is no analog of such a field for the $M = 1$ Ising case.

The properties of nonrandom quantum rotor models in the presence of a field H were examined recently in Ref. 35, where it was found that, in high enough dimensions, a strong H always induced magnetic long-range order in a plane perpendicular to the applied field. We will find an analogous phenomenon here for the spin-glass case.

The coupling to H in the effective action can be determined by general gauge-invariance arguments:[36,35] these are equivalent to the physical requirement that the only effect of an applied H is a uniform precession of all the rotors. By this method, we can deduce that the only effect of H is to replace the linear, time derivative terms in $\mathcal{A}$ in (2.2) by covariant derivatives:

$$\frac{1}{\kappa t} \int d^d x \int d\tau \sum_a \left(\frac{d}{d\tau_1} \delta_{\mu\nu} + i H_{\mu\nu} \right) \left(\frac{d}{d\tau_2} \delta_{\mu\rho} + i H_{\mu\rho} \right) Q^{aa}_{\nu\rho}(x, \tau_1, \tau_2) \Bigg|_{\tau_1 = \tau_2 = \tau}, \tag{2.65}$$

where $H_{\mu\nu} = -H_{\nu\mu}$. Note that unlike the longitudinal field, h, H does not couple directly to the replica off-diagonal components of Q. Related to this is the rather straightforward consequence of H in the classical limit. This can be seen by looking at the zero-frequency component of (2.65) (which dominates in the classical limit) from which we obtain a shift $-H_{\mu\nu}H_{\mu\rho}$ in the value of r controlling spin-glass ordering in the $\nu\rho$ plane; the only effect of H is therefore to break the $O(M)$ invariance by inducing anisotropic shifts in the critical point towards ordering in different directions.

The consequences of H in the quantum model are a little more interesting, as we now see in a mean-field treatment of $\mathcal{A}$ in the presence of H. For simplicity, we will present the analysis of the $M = 3$ case only. Let us choose H to lie along the three axis, i.e., $H_{12} = -H_{21} = H$, with other components zero. We will also restrict ourselves to the paramagnetic phase in which case we can use the ansatz (2.5), with the modification that D has to be replaced by a general tensor $D_{\mu\nu}$ in $O(3)$ space. It turns out that the following parametrization of $D_{\mu\nu}$, in a "circularly-polarized" basis is most convenient:

$$D_{11} = D_{22} = \frac{1}{2}(D_{+-} + D_{-+}),$$
$$D_{12} = -D_{21} = \frac{i}{2}(D_{+-} - D_{-+}), \tag{2.66}$$

with all other components, except D_{33}, set equal to 0. Inserting this result in $\mathcal{A}$, we find that the result (2.6) for the free-energy density is modified to

$$\begin{aligned}\frac{\mathcal{F}}{n} = &\frac{1}{t\kappa\beta} \sum_\omega \{ (\omega^2 + r) D_{33}(\omega) + [(\omega + iH)^2 + r] D_{+-}(\omega) + [(\omega - iH)^2 + r] D_{-+}(\omega) \} \\ &- \frac{\kappa}{3t\beta} \sum_\omega [D_{33}^3(\omega) + D_{+-}^3(\omega) + D_{-+}^3(\omega)] \\ &+ \frac{u}{2t} \left[\frac{1}{\beta} \sum_\omega D_{33}(\omega) \right]^2 + \frac{u}{t} \left[\frac{1}{\beta} \sum_\omega D_{+-}(\omega) \right] \left[\frac{1}{\beta} \sum_\omega D_{-+}(\omega) \right] \\ &+ \frac{v}{2t} \left[\frac{1}{\beta} \sum_\omega [D_{33}(\omega) + D_{+-}(\omega) + D_{-+}(\omega)] \right]^2 . \end{aligned} \tag{2.67}$$

The saddle-point equations for D_{33}, D_{+-}, and D_{-+} can now be obtained and solved. The solution is

$$\begin{aligned} D_{33}(\omega) &= -\frac{1}{\kappa}(\omega^2 + \tilde{r}_1)^{1/2}, \\ D_{+-}(\omega) &= -\frac{1}{\kappa}[(\omega + iH)^2 + \tilde{r}_2]^{1/2}, \\ D_{-+}(\omega) &= -\frac{1}{\kappa}[(\omega - iH)^2 + \tilde{r}_2]^{1/2}, \end{aligned} \tag{2.68}$$

where $\tilde{r}_1$ and $\tilde{r}_2$ are to be determined from the solutions to the equations

$$\begin{aligned} \tilde{r}_1 &= r - \frac{u+v}{\beta} \sum_\omega (\omega^2 + \tilde{r}_1)^{1/2} \\ &\quad - \frac{2v}{\beta} \sum_\omega [(\omega + iH)^2 + \tilde{r}_2]^{1/2}, \\ \tilde{r}_2 &= r - \frac{v}{\beta} \sum_\omega (\omega^2 + \tilde{r}_1)^{1/2} \\ &\quad - \frac{u + 2v}{\beta} \sum_\omega [(\omega + iH)^2 + \tilde{r}_2]^{1/2}, \end{aligned} \tag{2.69}$$

which generalize (2.8). The only effect of H has been to shift frequency ω by iH (corresponding to transforming to a rotating reference frame in Euclidean time) for fluctuations in the 12 plane.

It is useful to have a better understanding of the H-dependent frequency summations above. The following identity, obtained from the Poisson summation formula, is helpful:

$$\frac{1}{\beta}\sum_{\omega}[(\omega+iH)^2+\Delta^2]^{1/2} = \int_{-\Lambda_\omega}^{\Lambda_\omega}\frac{d\omega}{2\pi}(\omega^2+\Delta^2)^{1/2} - \frac{1}{\pi}\int_{\Delta}^{\infty}d\Omega\sqrt{\Omega^2-\Delta^2} \times\left(\frac{1}{e^{\beta(\Omega-H)}-1}+\frac{1}{e^{\beta(\Omega+H)}-1}\right) \quad (2.70)$$

valid for $\Delta > H$. The physical meaning of the second term in the above equation should also be clear: this is the contribution of the thermally excited, circularly polarized states with zero-field energy Ω, and density of states $\sqrt{\Omega^2-\Delta^2}$; these states have been split into states with energy $\Omega \pm H$ in the presence of a field. Notice that at $T = 0$, (provided $H < \Delta$), the second term in (2.70) vanishes, and the entire contribution comes from the first term which is independent H. This allows us to immediately obtain an important $T = 0$ solution to the Eqs. (2.69):

$$\tilde{r}_1 = \tilde{r}_2 = \tilde{r} \quad \text{at } T=0, \text{ provided } H < \sqrt{\tilde{r}}, \quad (2.71)$$

where $\tilde{r}$ is the solution of the $H = 0$ equation (2.8). This solution could have been anticipated *a priori*—at $H = 0$ we had an $O(3)$ singlet paramagnet ground state with a gap $\sqrt{\tilde{r}}$, and the energy and wave function of such a state will not be modified by a uniform external field. However, when $H > \sqrt{\tilde{r}}$, the energy of an excited spin-1 state will go below that of the ground state. By analogy with the nonrandom case,[35] we then expect condensation into the spin-1 state, and appearance of spin-glass order in the plane perpendicular to the field— $q_{11}^{ab} = q_{22}^{ab} \neq 0$ for $a \neq b$. The phase boundary to the appearance of this spin-glass ordering is $H = \sqrt{\tilde{r}}$, which at $T = 0$ gives us from (2.14)

$$H = \left(\frac{4\pi}{(u+Mv)}\frac{(r-r_c)}{\ln[\Lambda_\omega^2/(r-r_c)]}\right)^{1/2}. \quad (2.72)$$

This is sketched in the $T = 0$, finite H phase diagram of Fig. 5. This finite H spin-glass–paramagnet quantum transition is in a different universality class from the $H = 0$ transition. Some of its critical properties can be deduced from the results above, using methods very similar to those described in this paper for the $H = 0$ transition. We will not go into any details on this issue in this paper, apart from noting that examining the low ω behavior of (2.68) tells us immediately that the finite H quantum transition has the mean-field critical exponents $z = 4$ and $\nu = 1/4$.

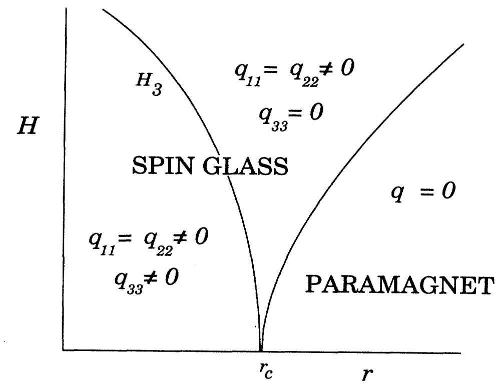

FIG. 5. Phase diagram for the $M = 3$ quantum rotor spin-glass at $T = 0$ in the presence of a field H which couples to the conserved total angular momentum in the 1, 2 plane. The position of the boundary of the paramagnetic phase is given by (2.72) in mean-field theory. The finite H spin-glass–paramagnet quantum transition is in a different universality class from the $H = 0$ transition. As in Fig. 4, the spin-glass order parameter q denotes replica off-diagonal components of Q, and replica off-diagonal components of $Q_{\mu\nu}$ with $\mu \neq \nu$ are zero everywhere. The boundary H_3 is given by (2.73) in mean-field theory. Replica symmetry breaking is expected to occur in the mean-field theory at nonzero T everywhere in the spin-glass phase.

A second phase boundary in Fig. 5 at $H = H_3(r)$ marks the appearance of spin-glass order in q_{33}^{ab}. Ordering in q_{33} is clearly present for $r < r_c$ at $H = 0$, and should therefore also be present for small enough $H < H_3(r)$. Determining H_3 requires computations in the spin-glass phase, with replica off-diagonal components of Q nonzero. As the computation is rather similar to those already carried out, we will not present any details and simply state the final result. We find

$$H_3 = \left(\frac{u}{2v}(r_c - r)\right)^{1/2}. \quad (2.73)$$

We close our discussion of H by examining the special point $r = r_c$. At finite H and T, we expect a transition to a spin-glass phase when H/T is greater than a universal number[35] [which can again be determined from (2.69)]. For $H \ll T$ however, the paramagnetic phase is stable and we can perform an expansion in powers of H to obtain the finite-T spin susceptibility of the quantum-critical point. An expansion of the free-energy density in the paramagnetic phase in powers of H yields

$$\frac{\mathcal{F}(H)-\mathcal{F}(H=0)}{n} = -\frac{\beta H^2}{\pi\kappa^2 t}\int_{\sqrt{\tilde{r}}}^{\infty}d\Omega\frac{\Omega(\Omega^2-\tilde{r})^{1/2}}{\sinh^2(\beta\Omega/2)}, \quad (2.74)$$

where $\tilde{r}$ is determined by the solution of $H = 0$ equation (2.8). Evaluating the integral at $r = r_c$ and at low T (regime II of Fig. 1) we find

$$\frac{\mathcal{F}(H) - \mathcal{F}(H=0)}{n} = -\frac{4\pi H^2 T^2}{3\kappa^2 t}. \quad (2.75)$$

The susceptibility, χ_H associated with H therefore behaves as $\sim T^2$ at low temperatures. We may combine this result with T dependence of the specific heat in (2.16) to obtain the dimensionless, universal Wilson ratio.[35] W, characterizing the quantum-critical regime II of Fig. 1:

$$W = \frac{T\chi_H}{C_V} = \frac{5}{6\pi^2}. \quad (2.76)$$

III. RENORMALIZATION-GROUP ANALYSIS

This section will attempt to go beyond the mean-field results of Sec. II by subjecting the finite-dimensional theory to a Wilson-style renormalization group (RG) which integrates out degrees of freedom at high energies and large momenta.

We will study the properties of $\mathcal{A}$ under the rescaling transformation

$$x' = x/b, \quad \tau' = \tau/b^z, \quad (3.1)$$

where z is the dynamic critical exponent. The coupling t will play a special role in the following, and we find it convenient to give it its own independent rescaling transformation

$$t' = tb^{-\theta}. \quad (3.2)$$

The exponent $-\theta$ will be the scaling dimension of t near a fixed point with $t = 0$; all fixed points found below in fact will have $t = 0$, and $\theta > 0$, making t a (dangerously) "irrelevant" coupling. As is conventional, we define the anomalous dimension η such that $\eta = 0$ in the approximation in the previous section; in RG this means that the coefficient of the $(\nabla Q)^2$ term is not rescaled at tree-level. This imposes the field rescaling

$$Q' = Qb^{(d-\theta+2z-2+\eta)/2}. \quad (3.3)$$

The exponents z and θ will be fixed by demanding that the transformations of the $1/\kappa t$ linear term with time derivatives, and the $1/t^2$ quadratic coupling, are consistent with (3.2) and the RG equation for the cubic κ/t coupling.

It is now a simple matter to determine the tree-level rescalings of the remaining couplings (we determine the rescaling of κ from the cubic term):

$$r' = rb^{2z}, \quad \kappa' = \kappa b^{(6+\theta-d-3\eta)/2},$$
$$u' = ub^{2-z-\eta}, \quad v' = vb^{2-z-\eta}. \quad (3.4)$$

The full action remains invariant at tree level under these transformations if we choose

$$z = 2, \quad \eta = 0, \quad \theta = 2. \quad (3.5)$$

We see then from (3.4) that u and v are always marginal, while κ becomes relevant below $d = 8$. As θ is positive, t is irrelevant and flows towards $t = 0$.

Before we can reach any conclusions on the meaning of these scaling dimensions, we need to understand the role of the coupling t in the loop corrections. Recall that we determined in Sec. II B 1 that the propagator G came with a prefactor t, the cubic interaction was κ/t, and the quadratic couplings are u/t, v/t and $1/t^2$ (Fig. 2). The presence of inverse powers of t leaves open the possibility that the loop corrections acquire even higher powers of $1/t$. In fact, it is not difficult to check that this does *not* happen. The argument is similar to that in the classical problem of a Ising ferromagnet in a random field:[37,38] there are always enough t factors in the numerator from G to cancel those from the couplings. In particular, there can be only one insertion of the $1/t^2$ quadratic coupling in any propagator, as two of them will involve a vanishing trace over replica indices. This discussion can be summarized in terms of a simple result: all loop corrections to any coupling or Green's function are no more singular as $t \to 0$ than the results at tree level. The coefficient of this leading singular power of t will however contain an infinite number terms with positive powers of the couplings u, v, and κ. These three couplings, therefore, do indeed measure the strength of the loop corrections. Above $d = 8$, the cubic κ coupling is irrelevant, and we expect that the mean-field and tree-level results of Sec. II B are related to those of a Gaussian fixed point.

Now we turn to determining loop corrections to the renormalization-group equations. We will obtain the one loop flow equations by integrating out degrees of freedom with frequencies ω such that $\Lambda_\omega/b^z < \omega < \Lambda_\omega$ *or* momenta k such that $\Lambda_k/b < k < \Lambda_k$, where Λ_k is a cutoff in momentum space. We determine the saddle-point of the action with respect to variations in Q fields in this range of momenta and frequencies *only*, and obtain fluctuation corrections to one loop. This is followed by the rescaling transformation discussed above. It is also useful to keep track of the free-energy density $\mathcal{F}$ in this procedure: we introduce an overall constant to $\mathcal{A}$

$$\mathcal{A} \to \mathcal{A} + \int d^dx\, d\tau f, \quad (3.6)$$

and keep track of the flow equation of f. We will obtain the differential form of the scaling equations with $b = e^\ell$.

The diagrams which contribute at one loop order are shown in Figs. 6 and 7. There are two types of diagrams at this order. The first have a loop with an internal momentum [Figs. 6(a), 6(c), and 7] which must be integrated over, with all frequencies pinned to the external ones: these are associated with a momentum space factor of $K_k\ell \equiv S_d\ell/(2\pi)^d$ where S_d is the surface area of a sphere in d dimensions. The second [Fig. 6(b)] are

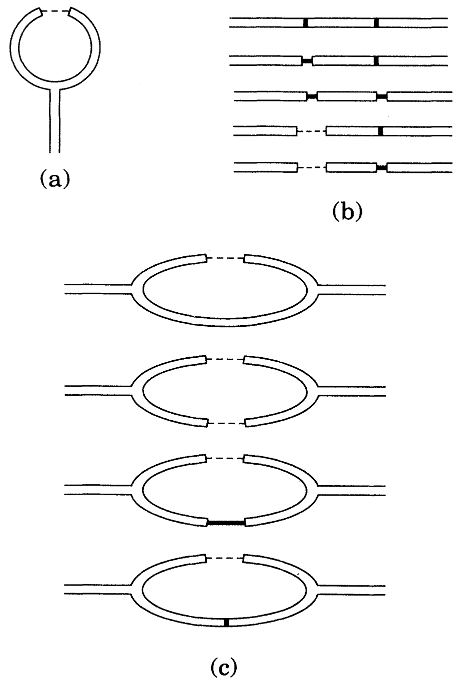

FIG. 6. Diagrams contributing to the one-loop renormalization group equations. The diagrams contribute to the renormalization of (a) $r/\kappa t$, (b) u/t, v/t, v/t, $1/t^2$, and $1/t^2$, respectively, and (c) η, $1/t^2$, v/t, and u/t, respectively. Not shown are the diagrams with double-line propagators replaced by their twisted partners [see Fig. 2(a)].

really tree-level diagrams but have a frequency integration because of quantum-mechanical interactions: these have a frequency space factor of $K_\omega \ell \equiv 2z\ell/(2\times 2\pi)$ (the factor of 2 in the numerator comes from the frequency shells at $\pm\Lambda_\omega$, and the 2 in the denominator is from the propagator which at zero momentum is $1/2\sqrt{\omega^2+r}$). It

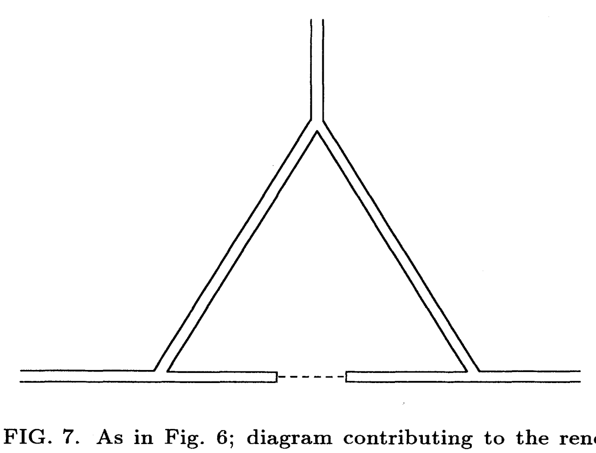

FIG. 7. As in Fig. 6; diagram contributing to the renormalization of κ/t.

is convenient to absorb these factors into the coupling constants by the transformations $K_\omega u \to u$, $K_\omega v \to v$, $K_k\kappa^2 \to \kappa^2$, $r/\Lambda_\omega^2 \to r$ and $f \to K_\omega \Lambda_\omega^4 f$. Evaluating the diagrams by completely standard methods we obtain the following flow equations (dropping innocuous factors of Λ_k^{d-8} in the terms with a momentum integration):

$$\begin{aligned} z &= 2+4\kappa^2, \\ \eta &= 2\kappa^2, \\ \theta &= 2-\frac{2(u+Mv)}{\sqrt{1+r}}, \\ \frac{dt}{d\ell} &= -\theta t, \\ \frac{dr}{d\ell} &= 2zr-2(u+Mv)\sqrt{1+r}-\frac{\Lambda_k^4}{\Lambda_\omega}\kappa^2 , \\ \frac{d\kappa}{d\ell} &= \frac{8-d}{2}\kappa+9\kappa^3-\frac{(u+Mv)\kappa}{\sqrt{1+r}}, \\ \frac{du}{d\ell} &= -2u\kappa^2-\frac{u^2}{\sqrt{1+r}}, \\ \frac{dv}{d\ell} &= -2v\kappa^2-\frac{2uv+Mv^2}{\sqrt{1+r}}, \\ \frac{df}{d\ell} &= (d+z)f-\frac{4}{3t\kappa^2}(1+r)^{3/2}. \end{aligned} \tag{3.7}$$

[As an aside, we note here that the dependence of the flow equations (and therefore the critical exponents) on M is illusory. The couplings u and v enter into the equations for the other couplings only in the combination $U \equiv u + Mv$. The equation for U is in fact M independent:

$$\frac{dU}{d\ell} = -2U\kappa^2 - \frac{U^2}{\sqrt{1+r}}. \tag{3.8}$$

We may trade the couplings u, v for U, u, and the equations for the latter are M independent. This independence on M is similar to that found in the infinite-range model[18].]

We begin by determining all the fixed points of these flow equations. The parameter r is obviously associated with the thermal operator, and has relevant flows from all the fixed points we find. The value of r at these fixed points is of order $\epsilon = 8-d$, and to leading order in ϵ this value of r does not feed into the positions of the other couplings. We can therefore set $r=0$, and focus on the other couplings. We find five fixed points; their positions to leading order in ϵ are listed below, along with the three eigenvalues controlling flows near the fixed point in κ, u, v space (there is, of course, also the omnipresent relevant eigenvalue associated with r). All the fixed points have $t=0$ and the eigenvalue controlling flow in the t direction is $-\theta$:

$$\begin{aligned}
&(A) \quad \kappa = 0, u = 0, v = 0 \quad (\epsilon/2, 0, 0)\\
&(B) \quad \kappa^2 = -\epsilon/22, u = \epsilon/11, v = 0 \quad [-(5 \pm \sqrt{14})\epsilon/11, -\epsilon/11]\\
&(C) \quad \kappa^2 = -\epsilon/22, u = 0, v = \epsilon/11M \quad [-(5 \pm \sqrt{14})\epsilon/11, \epsilon/11]\\
&(D) \quad \kappa^2 = -\epsilon/18, u = 0, v = 0 \quad (-\epsilon, \epsilon/9, \epsilon/9)\\
&(E) \quad \kappa^2 = -\epsilon/18, u = \epsilon/9, v = -\epsilon/9M \quad (-\epsilon, \epsilon/9, -\epsilon/9).
\end{aligned} \tag{3.9}$$

The properties and stabilities of these fixed points depend strongly on the sign of ϵ, and we will consider $\epsilon < 0$, $\epsilon = 0$ and $\epsilon > 0$ separately.

A. $d > 8$

While all five fixed points have real values of κ, the only stable fixed point is the Gaussian fixed point A. Related to the two zero eigenvalues, the couplings u and v are both marginal. The renormalization-group flows are shown in the $\kappa, U = u + Mv$ plane in Fig. 8. Observe that the basin of attraction of A is limited to a portion of the $U > 0$ quadrant (the basin of attraction is actually also restricted to $u > 0$, $v > 0$). For initial values of the couplings outside this basin, we have runaway flows to strong couplings and are therefore unable to make any firm predictions. The rest of the discussion in this subsection will therefore be limited to initial values within the basin attraction of A. Some speculations on the properties of the remainder of the phase space will be made in Sec IV.

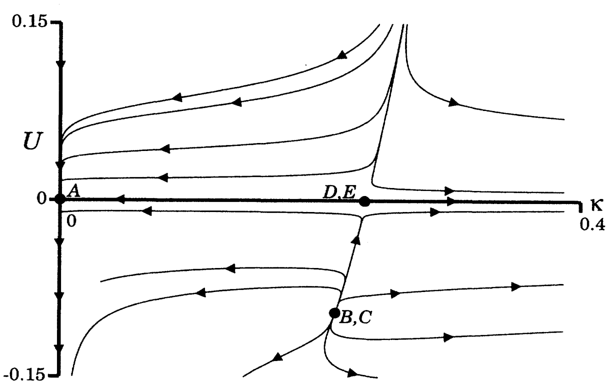

FIG. 8. Renormalization-group flows in the U, κ plane (recall $U \equiv u + Mv$) for $d > 8$ (we chose $\epsilon = 8 - d = -1$). The filled circles represent fixed points, and are labeled in the notation of Eq. (3.9). The fixed points B, C coincide in this plane, but have different fixed points values of u (similarly for D, E). The only stable fixed point is A and its basin of attraction is restricted to a portion of the $U > 0$ quadrant. The remainder of the quadrant, and all $U < 0$, eventually have runaway flows to strong coupling. The flow into the stable fixed point A remains marginal in the U direction for *all* $d > 8$.

The properties of A are obviously related to those of the mean-field theory of Sec. II B. The logarithmic corrections found there can be attributed to the couplings u, v which are marginal near A for *all* $d > 8$; this will be shown below. First we focus on the scaling properties, modulo logarithms. The "thermal" coupling r has eigenvalue 4, leading to the critical exponent

$$\nu = \frac{1}{4}. \tag{3.10}$$

We also know from the fixed-point values of κ, u, and v, and (3.7) that

$$z = 2, \quad \eta = 0, \quad \theta = 2 \tag{3.11}$$

as found in the mean-field theory of Sec II B 1. There are two irrelevant couplings, t and κ, associated with eigenvalues $-\theta$, and $-\theta_\kappa/2$, respectively, with

$$\theta_\kappa = d - 8. \tag{3.12}$$

Now we turn to the Green's functions of Sec. I B. Their scaling dimensions will be given by those of Q in (3.3) and any dependence they have on the irrelevant t, κ couplings. From the results of Sec. II B 1 we see that $D \sim 1/\kappa$ [Eq. (2.7)], $G \sim t$ [Eq. (2.18)], $G^c \sim t$ [Eq. (2.19)], and $G^d \sim t^0$ [Eq. (2.21)]. Combining this information with the scaling dimension of Q [Eq. (3.3)], we obtain the following results for the scaling dimensions of the Green's functions *after* their arguments have been Fourier transformed to k and ω:

$$\begin{aligned}
\dim(D) &= (d - \theta - \theta_\kappa - 2 + \eta)/2,\\
\dim(G) &= -2 + \eta,\\
\dim(G^d) &= -2 - \theta + \eta,\\
\dim(G^c) &= -2 - z + \eta.
\end{aligned} \tag{3.13}$$

Note that these results are consistent with those in (2.18)–(2.23) and that

$$\bar{\eta} = \eta + 2 - \theta. \tag{3.14}$$

The free-energy also has a singular dependence upon κ and t: $\mathcal{F} \sim 1/t\kappa^2$ [see Eq. (2.15)]. Related to this is the singular $1/t\kappa^2$ term in the renormalization-group flow equation for f in (3.7). From this flow equation it is easily seen that

$$\dim(\mathcal{F}) = d + z - \theta - \theta_\kappa. \tag{3.15}$$

Thus hyperscaling is violated; notice that θ does not vanish as d approaches 8 from above, so violation of hyperscaling continues to occur even in the upper critical dimension. This result, with the exponent values in (3.10), (3.11), and (3.12), predicts a d-independent singular $(r-r_c)^2$ term in $\mathcal{F}$, consistent with (2.15).

Next, we examine the form of the logarithmic corrections to scaling which are present in all $d>8$. For $d-8$ of order unity, the irrelevant couplings κ,t decrease rapidly, and can therefore be set equal to 0 in the renormalization-group equations (3.7), (3.8) (except in the equation for f, where they appear in the denominator). For small r, we can then integrate (3.8) and find

$$U(\ell) \sim \frac{1}{\ell} \qquad (3.16)$$

for large ℓ. We have to now insert this into the equation for r in (3.7), and integrate up to the scale $\ell=\ell^*$ at which $r(\ell^*)\approx 1$. Doing this to leading logarithmic accuracy we find that

$$\ell^* e^{-2z\ell^*} \approx r(\ell=0) - r_c. \qquad (3.17)$$

The typical frequency scale near the critical point is $e^{-z\ell^*}$ which from the above is of order $\{(r-r_c)/\ln[1/(r-r_c)]\}^{1/2}$; this is identical to that obtained in (2.14) in the mean-field theory of Sec. II B 1. A similar analysis can be undertaken for the free-energy.

Finally, recall that in the mean-field results within the spin-glass phase in Secs. II B 2 and II B 3, all replica symmetry-breaking effects were associated with the formally irrelevant coupling y_1 in (2.31). The scaling dimension of the y_1 perturbation is $-(d+2z-\theta-4+2\eta)$; knowing this and the scaling dimensions above, the mean-field critical properties of the replica symmetry-breaking effects can be given a standard scaling interpretation.

B. $d=8$

The flows are shown in Fig. 9. The only stable fixed point remains A and its basin of attraction is the region

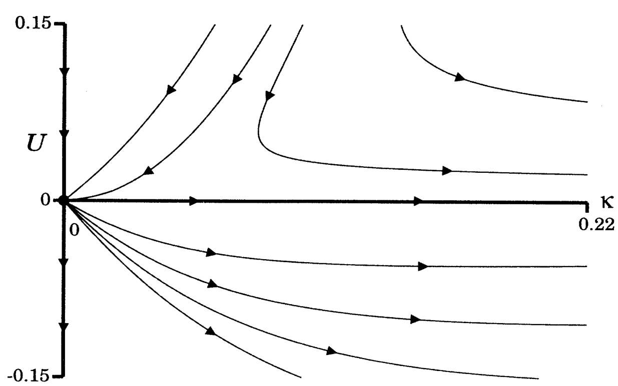

FIG. 9. As in Fig. 8 but for $d=8$ ($\epsilon=0$). The only fixed point is $\kappa=U=0$, and its domain of attraction is $\kappa^2\le U/20$. For other initial values we have runaway flow to strong coupling. All the fixed points in Fig. 8 coalesce into the $y=\kappa=0$ fixed point as d approaches 8 from above.

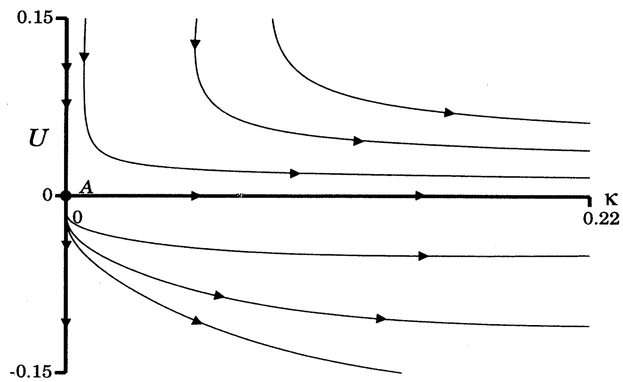

FIG. 10. As in Fig. 8 but for $d<8$ (we chose $\epsilon=8-d=0.2$). Only the fixed point A is now physical as the others have imaginary value of κ. However, A is unstable for any nonzero κ, and all physical couplings flow to strong coupling.

$\kappa^2\le U/20$. Now the flow of κ into this fixed point is also marginal (in addition to u,v) and so there will be new logarithmic corrections to scaling. As before, we know little about the system outside the basin of A.

C. $d<8$

Now A is the only fixed point with κ real, but is unstable. The flows are shown in Fig. 10. Thus over the entire, physical region we have flows to strong coupling. There is no simple expansion of the critical properties in powers of ϵ. No definite results can thus be obtained, and we resort to some speculations using phenomenological scaling ideas in the next section.

IV. SCALING HYPOTHESES

In this section we discuss the scaling properties that may be expected at the transition in our models; as far as possible this will be done without reference to the Landau theory or to replicas. We begin by collecting the definitions of exponents and the scaling hypotheses that have been mentioned at various points earlier in the paper.

Because of the importance of the dangerously irrelevant (DI) variable t in the Landau theory and to gain greater generality, we will assume that such a variable, also denoted t with scaling dimension $-\theta<0$, exists in the scaling theory. We assume that there is only this one DI variable; the coupling κ that is also DI in Landau theory for $d>8$, will not be for $d<8$, as in ordinary critical phenomena. The extension to include more DI variables is straightforward, and of course if none are present, one can simply set $\theta=0$. Moreover, it will be assumed that t appears in a similar way as in the Landau theory, which contains a $1/t^2$ term as well as $1/t$

terms. This means that in each realization of disorder, t plays a role similar to $\hbar$ in quantum field theory (or T in classical statistical mechanics): when it is small, certain types of disorder-induced fluctuations (those directly responsible for determining the local position of the critical point) dominate. (The analogy is not exact because of the internal frequency integrals that can occur even in tree diagrams in Q within Landau theory.)

Defining G^d as in (1.22), we can then define the exponent $\bar{\eta}$ by

$$G^d(x,\tau,\tau) \sim x^{-(d+2z-4+\bar{\eta})} \tag{4.1}$$

for fixed τ/x^z at criticality; there is no dependence on t on the right-hand side. Thus the dimension of $Q \sim SS$ when the spins are separated in time is $(d+2z-4+\bar{\eta})/2$. Therefore also [recall (1.18)]

$$D \sim \tau^{-(d+2z-4+\bar{\eta})/2z}. \tag{4.2}$$

The spin-glass correlator G [recall (1.21)] vanishes if $t = 0$, so is proportional to t (as $t \to 0$) and thus behaves as

$$G(x,0,0) \sim x^{-(d+2z-4+\bar{\eta}+\theta)}. \tag{4.3}$$

Comparing with the definition of η, $G \sim x^{-(d+2z-2+\eta)}$ yields

$$\theta = 2 + \eta - \bar{\eta}. \tag{4.4}$$

From these we may obtain other scaling relations, e.g., the spin-glass susceptibility $\chi_{SG} \sim (r-r_c)^{-\gamma}$ with

$$\gamma = (2-\eta)\nu, \tag{4.5}$$

and the order parameter $q = [\langle S\rangle^2] \sim (r_c - r)^\beta$ with

$$\beta = (d + 2z - \theta - 2 + \eta)\frac{\nu}{2}. \tag{4.6}$$

In general, the usual scaling relations are obeyed, except that hyperscaling involves $d + 2z - \theta$ in place of d for classical critical phenomena whenever the bilocal field is involved (hence the $2z$) and the θ is due to the DI variable t. On the other hand, a field that is local in time behaves normally (on including z and θ). Thus the free-energy density scales as $(r-r_c)^{(d+z-\theta)\nu}$ as $r \to r_c$, and the specific heat at finite temperature $T \to 0$ at $r = r_c$ behaves as $T^{(d-\theta)/z}$; the values $z = 2$, $\theta = 2$ yield T^3 at $d = d_u = 8$ in the Gaussian theory, as was obtained directly in the infinite-range model[18] and from the mean-field theory in (2.16). The response of the system to an external field coupling to the conserved angular momentum scales as $T^{(d-\theta-z)/z}$ (Ref. 35) in the quantum-critical region II of Fig. 1—at $d = 8$ this is T^2, in agreement with (2.75).

A local variable, which has not been introduced so far, but is important for the present scaling considerations, is the "thermal" operator $\psi(x,\tau)$. This variable couples to the control parameter r that tunes the system across the quantum transition:

$$\psi(x,\tau) = S_\mu(x,\tau)S_\mu(x,\tau) \tag{4.7}$$

in a single replica of the system. For the disconnected correlator of ψ we may define at criticality

$$[\langle\psi(x,\tau_1)\rangle\,\langle\psi(0,\tau_2)\rangle] \sim x^{-(d-4+\bar{\eta}_\psi)} \tag{4.8}$$

(so that the Fourier transform $\sim k^{-4+\bar{\eta}_\psi}$; note the correlator is independent of τ_1, τ_2) so that the scaling dimension of ψ is $(d-4+\bar{\eta}_\psi)/2$. The connected correlation function then goes as $\sim t$ and so

$$\begin{aligned}[\langle\psi(x,0)\psi(0,0)\rangle] - [\langle\psi(x,0)\rangle\,\langle\psi(0,0)\rangle] &\sim x^{-(d-4+\theta+\bar{\eta}_\psi)}\\ &\equiv x^{-(d+z-2+\eta_\psi)}\end{aligned} \tag{4.9}$$

(from $k^{-2+\eta_\psi}$ in Fourier space) and hence

$$\theta = 2 + z + \eta_\psi - \bar{\eta}_\psi. \tag{4.10}$$

As usual the dimension of ψ determines that of $r - r_c$, and hence the value of ν, through modified hyperscaling, that is

$$\begin{aligned}\frac{1}{\nu} &= d + z - \theta - \frac{1}{2}(d - 4 + \bar{\eta}_\psi)\\ &= \frac{1}{2}(d + z - \theta + 2 - \eta_\psi).\end{aligned} \tag{4.11}$$

An interesting rigorous inequality was proved by Schwartz and Soffer[39] for the exponents η_ψ, $\bar{\eta}_\psi$ in any system where a local field ψ couples to Gaussian disorder, as is the case for our ψ. Extending their proof to the case where the disorder is time independent, we obtain

$$\bar{\eta}_\psi \le 2\eta_\psi \tag{4.12}$$

and hence from (4.10) $\theta \ge 2 + z - \eta_\psi$. Using (4.11) this implies

$$\nu \ge \frac{2}{d}. \tag{4.13}$$

This inequality was proved by Chayes *et al.*[41] The present approach appears easier but rests upon the use of a scaling relation to obtain ν.

For the classical random-field Ising model, it has been claimed that $\bar{\eta}_\psi = 2\eta_\psi$ is satisfied as an equality.[40] This would imply that the correlation length at $T = T_c$ due to a uniform field h would go as $\xi \sim h^{-2/d}$. However we find the proof unconvincing, though series results do seem to show that the equality is accurately obeyed in $d = 3, 4, 5$ in that problem.

Since $\psi = SS$ it is tempting to equate the scaling dimensions of Q and ψ ($d + 2z - 4 + \bar{\eta} \stackrel{?}{=} d - 4 + \bar{\eta}_\psi$) and obtain using (4.4) and (4.10)

$$\eta_\psi \stackrel{?}{=} z + \eta. \tag{4.14}$$

However ψ involves bringing spins S to the same time as well as position (and summing over spin indices) and so may have different renormalizations than Q. Thus we do *not* expect this relationship to hold.

Finally, let us consider the connected correlation function G^c defined in Eq. (1.24). Similar scaling ideas suggest the following form at criticality (taking the $M = 1$ case for simplicity)

$$G^c(x,\tau,0,\tau) \sim \left(x^{2z}+\tau^2\right)^{-(d+2z-2+\eta)/2z}\Gamma^c(\tau/x^z) \tag{4.15}$$

with $\Gamma^c(y)$ a universal function. Taking either of the limits $\tau \to 0$, or $x \to 0$ has the consequence of bringing two of the four S fields in G^c to the same space time point, so that G^c reduces to a correlator of ψ. Knowing the scaling dimension of ψ, this procedure fixes the asymptotic limits of Γ^c:

$$\Gamma^c(y) \sim \begin{cases} y^{(\eta_\psi - z-\eta)/z} & \text{as } y \to 0 \\ y^{(\eta+z-\eta_\psi)/z} & \text{as } y \to \infty. \end{cases} \tag{4.16}$$

Similar statements hold for the general $G^c(x,\tau_1,\tau_2,\tau_3)$, and for G^d with a different scaling function Γ^d.

We now compare the above relations to Monte Carlo results for $M = 1$ in $d = 2$ (Ref. 20) and $d = 3$.[19] Their results are $\nu^{-1} \approx 1.3$, $z \approx 1.3$ ($d = 3$) and $\nu^{-1} \approx 1$, $z \approx 1.5$ ($d = 2$). They examined scaling of several susceptibilities, most of which are related to G and hence involve η in straightforward ways; however their definition of η is what we denote $\eta' = \eta + z$. In our notation their results are $\eta \approx -0.2$ ($d = 3$), $\eta \approx -1.0$ ($d = 2$). (We have corrected an arithmatical error in the paper Guo, Bhatt, and Huse:[19] $\eta' = 1.1$, not 0.9.) They also examined $\chi_{\rm nl}$ which is related to G^c [Eq. (1.29)] and find its scaling dimension is consistent with the assumption that the scaling forms like (4.15) can be integrated so that η_ψ drops out and $\chi_{\rm nl} \sim L^{z+2-\eta}$, and using the same η as the other susceptibilities. Note that the negative value of η is quite reasonable in a disordered system.[5]

The numerical results so far give no test of hyperscaling, but Guo, Bhatt, and Huse[19] also studied D, obtaining $D(\tau) \sim \tau^{-1.3}$. Using the scaling relations and values of exponents in $d = 3$ we obtain $\theta \approx 0.0$. This may mean that conventional hyperscaling is obeyed, though because of uncertainties in exponents, a small positive θ cannot be ruled out. Clearly more work on this point, and tests of other scaling relations, would be welcome. (If $\theta = 0$ then $\eta_\psi = z + \eta$ is ruled out.) It is interesting that in both $d = 2$ and 3 , $\nu^{-1} \approx d/2$ and this also holds *exactly* in the $d = 1$ model[11]—this raises the question whether the inequalities (4.12), (4.13) are saturated; at present we have no argument why this might be so.

V. CONCLUSIONS

In this section we summarize our main results and discuss their relationship to other open problems. A comparison of our results with some recent work has already been presented in Sec IV.

We have studied models of quantum rotors or Ising spins in a transverse field with random, short-range, frustrating exchange interactions. We examined properties of these systems in the vicinity of a zero-temperature quantum transition between a spin-glass and a quantum paramagnet phase. We characterized this transition by an order-parameter field $Q^{ab}_{\mu\nu}(x,\tau_1,\tau_2)$ which is a matrix in spin components (μ, ν) and replica indices (a, b), and is *bilocal* in Matsubara time (τ_1, τ_2). The expectation value of Q is the on-site two-spin correlation function which becomes long-range in time at the onset of spin-glass order. We then introduced a Landau effective-action functional for Q: the functional was written down as the most general one consistent with a set of symmetry criteria and the usual Landau theory requirements of locality in space and time. The functional involves *both* the replica diagonal and off-diagonal components of Q—contrast this with classical spin glass[7,8] for which the Landau theory involves only the replica off-diagonal components of an order parameter matrix field which is also independent of time.

A mean-field functional minimization of the Landau action yielded a great deal of structure. For parameters favoring large quantum fluctuations, we found a paramagnetic phase whose properties (as well as those of the quantum-critical point at which the paramagnetic phase terminates) were identical to those obtained in an earlier exact solution of a model with infinite-range interactions.[17,18] In systems with weaker quantum fluctuations, we found a replica-symmetric spin-glass ground state, with replica symmetry-breaking appearing at any nonzero temperature. We were able to study systematically the behavior of replica symmetry breaking at small T, in contrast to the classical Sherrington-Kirkpatrick model where the order parameter is of order unity as $T \to 0$, and no Landau expansion exists (see however Ref. 31). In the present situation, we used the proximity to a quantum-critical point to our advantage, and developed a Landau expansion valid even at $T = 0$. The response of the spin-glass and paramagnetic phases to a variety of external fields was also studied.

Next we examined consequences of fluctuations about mean-field for the critical properties of the quantum transition. We identified $d = 8$ as the upper critical dimension. Above $d = 8$, and with certain restrictions on the values of the Landau couplings, we found that the transition was controlled by a Gaussian fixed point with mean-field critical exponents. For couplings not attracted by the Gaussian fixed point above $d = 8$, and for all physical couplings below $d = 8$, we found runaway renormalization-group flows to strong coupling. An important feature of the renormalization-group analysis was the appearance of a dangerously irrelevant coupling (even below $d = 8$), which played a role similar, though *not* identical, to Planck's constant $\hbar$. As a result certain disconnected correlation functions measuring fluctuations due to quenched disorder were found to be more singular than the corresponding connected correlations which contain quantum-mechanical fluctuations. The structure implied by this dangerously irrelevant coupling was used to motivate a general scaling scenario for the quantum transition, conjectured to be valid even in the region of runaway flows to strong coupling.

We conclude by discussing implications for some related problems. It would clearly be desirable to extend our results to the case of insulating spin-glasses of true quantum Heisenberg spins. Unlike the case for quantum rotors, the different vector components of a Heisenberg spin do not commute with each other—this leads to Berry phase terms in the path integral which cannot be removed by any obvious choice of variables, and complicates the problem substantially. A Heisenberg spin model with random infinite-range interactions was studied recently[42] and dramatically different behavior was found even at this "mean-field" level—the quantum paramagnetic phase was gapless, unlike the fully gapped quantum paramagnet in infinite-range quantum rotor model.[17,18] A starting point for further analysis might be to obtain a suitable Landau action functional whose minimization reproduces the properties of the infinite-range quantum Heisenberg model.

A great deal of work[43] has appeared recently on another quantum transition in the large dimensionality limit: the metal-insulator transition in Mott-Hubbard type models. Like the spin-glass problem, the order parameter for this transition is the long-time limit of a correlation function—at short times the correlation is nonzero on both sides of the transition. The techniques developed in this paper could perhaps be helpful in extending the infinite dimensionality results to the metal insulator transition in finite dimension systems. While this paper was being written, we learned of the work of Kirkpatrick and Belitz[44] on the Landau theory of a metal-insulator transition in random electronic systems. While we do not understand the details of their analysis, there does appear to be at least a passing resemblance of their methods to ours—like us, they find it necessary to consider a linear term in the order parameter, to which randomness couples most effectively.

ACKNOWLEDGMENTS

The research was supported by NSF Grants No. DMR-91-57484 (N.R.), DMR-92-24290 (S.S. and J.Y.), and DMR-91-15491 (J.Y.). We are pleased to thank D.S. Fisher, D.A. Huse, and A.P. Young for useful discussions.

APPENDIX A: DERIVATION OF THE LANDAU EFFECTIVE ACTION

In this appendix we outline an explicit derivation of the Landau effective action $\mathcal{A}$ in (2.2) from the Hamiltonians $\mathcal{H}_R$ and $\mathcal{H}_I$. It is slightly more convenient to work with soft spins rather than the fixed-length quantum rotors or Ising spins (although the derivation below can be extended to these cases). We begin with the path integral for these spins in the presence of fixed realization of the disorder

$$Z = \int \mathcal{D}S_{i\mu} \exp\left(-\int d\tau \left\{\mathcal{L}_0(S_{i\mu}) - \sum_{<ij>} J_{ij} S_{i\mu} S_{j\mu}\right\}\right),$$
$$\mathcal{L}_0 = \sum_i \left[\frac{1}{2g}\left(\partial_\tau S_{i\mu}\right)^2 + \frac{m^2}{2} S_{i\mu}^2 + \frac{\tilde{u}}{2}\left(S_{i\mu}^2\right)^2\right]. \tag{A1}$$

This action may be interpreted as that of M-component harmonic oscillators on the sites i of a lattice, with a nonlinear self-coupling u and a random interaction J_{ij}. We now introduce replicas and average over symmetric distribution of the J_{ij}. Neglecting all but the second cumulant of the J_{ij}, we obtain the replicated, translationally invariant result

$$[Z^n] = \int \mathcal{D}S_{i\mu}^a \exp\left(-\int d\tau \sum_a \mathcal{L}_0(S_{i\mu}^a) - \frac{J^2}{2}\sum_{\langle ij\rangle}\int d\tau_1 d\tau_2 \sum_{ab} S_{i\mu}^a(\tau_1) S_{j\mu}^a(\tau_1) S_{i\mu}^b(\tau_2) S_{j\mu}^b(\tau_2)\right). \tag{A2}$$

Now, as in classical spin glasses,[8] we decouple the quartic term by a Hubbard-Stratonovich transformation

$$[Z^n] = \int \mathcal{D}Q_{i\mu\nu}^{ab} \exp\left(-\frac{1}{2J^2}\int d\tau_1 d\tau_2 \sum_{i,j}\sum_{ab} Q_{i\mu\nu}^{ab}(\tau_1,\tau_2) K_{ij}^{-1} Q_{j\mu\nu}^{ab}(\tau_1,\tau_2)\right) Z_S[Q],$$
$$Z_S[Q] = \int \mathcal{D}S_{i\mu}^a \exp\left(-\int d\tau \sum_a \mathcal{L}_0(S_{i\mu}^a) - \int d\tau_1 d\tau_2 \sum_i \sum_{ab} Q_{i\mu\nu}^{ab}(\tau_1,\tau_2) S_{i\mu}^a(\tau_1) S_{i\nu}^b(\tau_2)\right), \tag{A3}$$

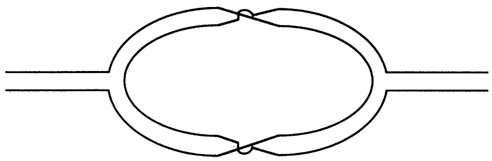

FIG. 11. Feynman diagram which generates an effective coupling $1/t^2$ even if only the cubic coupling κ/t is present originally.

where K_{ij} is the connectivity matrix of the lattice. It is now straightforward to expand $Z_S[Q]$ in powers of $Q^{ab}_{\mu\nu}(x,\tau_1,\tau_2) - C\delta^{ab}_{\mu\nu}\delta(\tau_1-\tau_2)$ for a suitably chosen value of C, as explained in Sec. II A; the constant C can then be absorbed into $\mathcal{L}_0$ and becomes part of the quadratic $S^2_{i\mu}$ term which can be shown to remain stable at small $\tilde{u}$. It is easy to see that this procedure will generate all terms consistent with the criteria enumerated in Sec. II A. In particular, the term linear in Q appears immediately. The rather unfamiliar looking time derivatives in this linear term can be seen to follow from a gradient expansion of a term like

$$\int d^dx d\tau_1 d\tau_2 Q^{aa}_{\mu\mu}(x,\tau_1,\tau_2) f(\tau_1-\tau_2)$$

where $f(\tau)$ is an even function of τ which falls rapidly to zero within a τ of order $1/g$. All the terms in $\mathcal{A}$ in (2.2) will be generated at higher orders, with the exception of the last $1/t^2$ term. There are two routes to generating such a term:

(i) Renormalize the functional integral over Q itself. At order κ^2 one observes from the Feynman diagram in Fig. 11 that an effective $1/t^2$ term is generated. Thus, even though it is absent in the bare Q action, it will eventually appear. For the renormalization-group analysis, it is therefore advantageous to include the $1/t^2$ term at the starting point.

(ii) Introduce additional *on-site* sources of randomness in $\mathcal{L}_0$. Randomness in the value of m^2 couples to the $[S^a_{i\mu}(\tau)]^2$, an operator which has the same quantum numbers as $Q^{aa}(x,\tau,\tau)$. Integrating over the randomness will then generate the $1/t^2$ term. Randomness in g or $\tilde{u}$ has the same effect, eventually.

APPENDIX B: GRIFFITHS SINGULARITIES IN THE QUANTUM PARAMAGNET

Our mean-field result (2.9) for the local susceptibility in the paramagnetic phase has a gap at low frequencies. As already noted, this feature is an artifact of the mean-field theory and we expect random fluctuations in the J_{ij} to create local environments which will have excitations within this "gap." A careful formulation of this "Griffiths" effect has been presented by Thill and Huse for the $M=1$ case:[23] they found a power-law $\sim \omega^{\phi}$ absorption at low frequency, where the exponent ϕ varies continuously with microscopic couplings and need not be positive. Here we will extend their argument to $M>1$ and find a very different result: there is only a much weaker essential singularity in the absorption spectrum for rotors with a continuous internal symmetry.

We will be satisfied here by sketching the basic idea: it should not be too difficult to formalize the argument below along the lines of Ref. 23. The important contribution to the long-time limit of the average, local, spin-correlation function in the paramagnetic phase comes from regions whose local environment has couplings similar to those in the spin-glass phase. Let us examine the contribution of a region $\mathcal{R}_L$ of linear dimension L, whose coupling constants are those required to be well within the spin-glass or ferromagnetic phase. Such a region will occur with a probability $\sim \exp(-c_1 L^d)$ where the constant c_1 depends upon the precise criteria chosen. Apart from short-lived fluctuations, the spins in $\mathcal{R}_L$ will follow each other and behave like a single block spin evolving in imaginary time. Neglecting the coupling to the environment, the fluctuations of this block spin can be described by a one-dimensional (corresponding to imaginary time direction), classical, M-component spin chain with a ferromagnetic coupling $K \sim L^d$. Now the properties of such a spin chain are well known: it has a finite correlation "time" ξ_τ which scales with large K, for $M>1$, as

$$\xi_\tau \sim K \sim L^d. \tag{B1}$$

In contrast, for $M=1$ we have $\ln \xi_\tau \sim K \sim L^d$.[23] This much shorter correlation time for continuous spins ($M>1$) is responsible for the difference from the Ising case. So for a typical site i in $\mathcal{R}_L$ we will have

$$\langle \hat{n}_i(\tau)\cdot\hat{n}_i(0)\rangle \sim \exp(-c_2 L^{-d}|\tau|) \tag{B2}$$

for some constant c_2. The above argument is equivalent to the statement that the region $\mathcal{R}_L$ behaves like a single quantum rotor with coupling $g \sim L^{-d}$.

We can now add up the contribution of all regions like $\mathcal{R}_L$ and obtain a lower bound on the average, local, spin-correlation function:

$$[\langle \hat{n}_i(\tau)\cdot\hat{n}_i(0)\rangle] \geq c_3 \int dL \exp(-c_1 L^d)\exp(-c_2 L^{-d}|\tau|)$$
$$\sim c_4 \exp\left(-2\sqrt{c_1 c_2|\tau|}\right), \tag{B3}$$

where the integral over L was performed by the saddle-point method. By an inverse Laplace transform, it can be shown that this leads to a low-frequency contribution to $\chi''(\omega)$ of $\sim \mathrm{sgn}(\omega)\exp(-c_1c_2/|\omega|)$. So the gap in (2.9) is filled in by a weak absorption which has an essential singularity at $\omega=0$.

[1] J.B. Kogut, Rev. Mod. Phys. **51**, 659 (1979).
[2] S. Chakravarty, B.I. Halperin, and D.R. Nelson, Phys. Rev. B **39**, 2344 (1989).
[3] A. Chubukov, S. Sachdev, and J. Ye, Phys. Rev. B **49**, 11 919 (1994).
[4] W. Wu, B. Ellman, T.F. Rosenbaum, G. Aeppli, and D.H. Reich, Phys. Rev. Lett. **67**, 2076 (1991); W. Wu, D. Bitko, T.F. Rosenbaum, and G. Aeppli, *ibid.* **71**, 1919 (1993).
[5] S. Sachdev and J. Ye, Phys. Rev. Lett. **69**, 2411 (1992).
[6] F.D.M. Haldane, Phys. Lett. **93A**, 464 (1983); Phys. Rev. Lett. **50**, 1153 (1983); J. Appl. Phys. **57**, 3359 (1985).
[7] K. Binder and A.P. Young, Rev. Mod. Phys. **58**, 801 (1986).
[8] K.H. Fischer and J.A. Hertz, *Spin Glasses* (Cambridge University Press, Cambridge, 1991).
[9] B.M. McCoy and T.T. Wu, Phys. Rev. **176**, 631 (1968); B.M. McCoy, Phys. Rev. B **2**, 2795 (1970).
[10] R. Shankar and G. Murthy, Phys. Rev. B **36**, 536 (1987).
[11] D.S. Fisher, Phys. Rev. Lett. **69**, 534 (1992); (unpublished).
[12] S.N. Dorogovstev, Phys. Lett. **76A**, 169 (1980).
[13] D. Boyanovsky and J.L. Cardy, Phys. Rev. B **26**, 154 (1982).
[14] D. Lawrie and V.V. Prudnikov, J. Phys. C **17**, 1655 (1984).
[15] A.J. Bray and M.A. Moore, J. Phys. C **13**, L655 (1980).
[16] Y.Y. Goldschmidt and P.-Y. Lai, Phys. Rev. Lett. **64**, 2467 (1990); K.D. Usadel, G. Buttner, and T.K. Kopec, Phys. Rev. B **44**, 12583 (1990); B. Boechat, R.R. dos Santos, and M.A. Continentino, *ibid.* **49**, 6404 (1994); T.K. Kopec, *ibid.* **50**, 9963 (1994).
[17] D.A. Huse and J. Miller, Phys. Rev. Lett. **70**, 3147 (1993).
[18] J. Ye, S. Sachdev, and N. Read, Phys. Rev. Lett. **70**, 4011 (1993); J. Ye, Ph.D. thesis, Yale University, 1993.
[19] M. Guo, R.N. Bhatt, and D.A. Huse, Phys. Rev. Lett. **72**, 4137 (1994).
[20] H. Rieger and A.P. Young, Phys. Rev. Lett. **72**, 4141 (1994).
[21] A. Crisanti and H. Rieger (unpublished).
[22] R. Oppermann and A. Muller-Groeling, Nucl. Phys. **B401**, 507 (1993); R. Oppermann and M. Binderberger, Ann. Phys. **3**, 494 (1994).
[23] M.J. Thill and D.A. Huse (unpublished).
[24] D. Huse has pointed out that the behavior of $\chi_{\rm nl}$ in the limit of infinite d is different from that in the infinite-range model; our mean-field theory presumably describes the infinite d limit.
[25] B. Duplantier and A.W.W. Ludwig, Phys. Rev. Lett. **66**, 247 (1991); A.W.W. Ludwig, Nucl. Phys. **B330**, 639 (1990).
[26] C.N. Yang and T.D. Lee, Phys. Rev. **87**, 404 (1952); T.D. Lee and C.N. Yang, *ibid.* **87**, 410 (1951).
[27] M.E. Fisher, Phys. Rev. Lett. **40**, 1610 (1978).
[28] J.L. Cardy, Phys. Rev. Lett. **54**, 1354 (1985).
[29] J.R.L. de Almeida, R.C. Jones, J.M. Kosterlitz, and D.J. Thouless, J. Phys. C **11**, L871 (1978).
[30] G. Parisi and G. Toulouse, J. Phys. (Paris) Lett. **41**, L361 (1980)
[31] T. Temesvári, J. Phys. A **22**, L1025 (1989).
[32] D. Thirumalai, Q. Li, and T.R. Kirkpatrick, J. Phys. A **22**, 3339 (1989); P. Ray, B.K. Chakrabarti, and A. Chakrabarti, Phys. Rev. B **39**, 11 828 (1989).
[33] J.R.L. de Almeida and D.J. Thouless, J. Phys. A **11**, 983 (1978).
[34] M. Mezard, G. Parisi, N. Sourlas, G. Toulouse, and M. Virasoro, J. Phys. (Paris) **45**, 843 (1984); M. Mezard, G. Parisi, and M. Virasoro, J. Phys. (Paris) Lett. **46**, L217 (1985).
[35] S. Sachdev, Z. Phys. B **94**, 469 (1994).
[36] D.S. Fisher, Phys. Rev. B **39**, 11 783 (1989).
[37] G. Grinstein, Phys. Rev. Lett. **37**, 944 (1976).
[38] D.S. Fisher, Phys. Rev. Lett. **56**, 416 (1986).
[39] M. Schwartz and A. Soffer, Phys. Rev. Lett. **55**, 2499 (1985).
[40] M. Schwartz, M. Gofman, and T. Natterman, Physica A **178**, 6 (1991); M. Gofman, J. Adler, A. Aharony, A.B. Harris, and M. Schwartz, Phys. Rev. Lett. **71**, 1569 (1993), and references therein.
[41] A.B. Harris, J. Phys. C **7**, 1671 (1974); J.T. Chayes, L. Chayes, D.S. Fisher, and T. Spencer, Phys. Rev. Lett. **57**, 2999 (1986).
[42] S. Sachdev and J. Ye, Phys. Rev. Lett. **70**, 3339 (1993).
[43] G. Kotliar in *Strongly Correlated Electronic Materials*, edited by K.S. Bedell, Z. Wang, D. Meltzer, A. Balatsky, and E. Abrahams (Addison-Wesley, Reading, MA, 1993), and references therein.
[44] T.R. Kirkpatrick and D. Belitz, Phys. Rev. Lett. **74**, 1178 (1995).