# Cavity-fields approach to quadrupolar glass

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The cavity-fields approach recently proposed and used in spin-glass theory is extended to study the infinite-range quadrupolar-glass model and the three-state Potts spin-glass model. This approach, which avoids the replica method, relies on well-known mathematical grounds and deals with obvious physical quantities; the Thouless-Anderson-Palmer equations are obtained for both models. The results are in agreement with the previous calculations based on the replica method.

## I. INTRODUCTION

The term quadrupolar glass (QG) was introduced by Sullivan *et al.*[1] for the low-temperature phase of randomly diluted mixed crystals. If the concentration of molecules bearing quadrupole moments is high enough, the system exhibits an order-disorder phase transition from a high temperature freely rotated quadrupole phase into a long-range orientationally ordered phase. The situation is changed when the concentration of molecules without quadrupole moments is increased. Such a dilution leads to the appearance of a new phase with quadrupole moments frozen in random directions at low temperature. Well-known examples of such systems are $o$-$p$-$H_2$ and $o$-$p$-$D_2$ crystals with low concentrations of $o$-$H_2$ or $p$-$D_2$ molecules with rotation moment $J = 1$, Ar diluted with $N_2$ molecules, and a series of $(KBr)_{1-x}(KCN)_x$-type mixed crystals (see, e.g., Refs. 2, 3). The presence of randomness and frustration makes these systems similar to spin glasses. However, the absence of reflection symmetry gives rise to a number of peculiarities in the behavior of quadrupolar glasses and puts them nearer to real glasses. Recently this subject was considered from a field theoretical and geometrical point of view by G. Parisi.[4]

The development of an analytic approach to quadrupolar glass theory was started in 1984 in Ref. 5. We have proposed a simple "solvable" model for the quadrupolar glass in $o$-$p$-$H_2$ in the spirit of the well-known and widely used model of Sherrington and Kirkpatrick[6] (SK). Using a SK-like replica-symmetric (RS) approach we have considered a kind of model effective random Hamiltonian for electrostatic quadrupole-quadrupole (EQQ) interactions between axial quadrupoles, and we have obtained[5] an expression for the free energy and the equations for the order parameters. Even in the frame of this simple model some of the peculiarities mentioned above could be seen. The algebra of quadrupole moment operators and symmetry properties different from the SK case have manifested themselves in nonzero spontaneous quadrupolarization even at high temperature and zero external field. The main result obtained in Ref. 5 was the absence of QG transitions in the simple SK sense: The equations for the order parameters have no trivial solution even at high temperature; the long-range orientational order parameter and the QG order parameter increase smoothly with decreasing temperature. This fact, which from the naive point of view contradicts the physical intuition, correlates, however, with the experimental data on $o$-$p$-$H_2$ and $N_2$-Ar systems (see, e.g., Ref. 7).

Another model for a uniaxial quadrupole glass has been considered by Goldbart and Sherrington.[8] They have studied quadrupoles with biquadratic exchange interactions and quenched frustrated exchange constants distributed by Gauss' law. In their model a paraquadrupolar phase exists at high temperature and for low temperature the QG phase was assumed. Some attempts were made to go beyond the simple models and to consider realistic Hamiltonians (see, e.g., Refs. 11, 12). Unfortunately, these Hamiltonians were too complicated, so that no useful results were obtained. The interesting treatment of axial QG's through the Thouless-Anderson-Palmer[9] (TAP) approach has been given in Ref. 10.

Later we used the ideas of the work[5] in Refs. 13–19. We have considered QG models with axial and nonaxial interactions between quadrupoles. From equations for four order parameters it follows that in the general case the system does not exhibit a phase transition to a QG phase; long-range orientational along with QG orders exist in the system in the whole range of temperatures. The exception is the "isotropic" case when axial and nonaxial constants are distributed with the same parameters and the system is isomorphic to the three-state Potts spin glass. We will discuss these results in detail in Sec. II.

Recently interest in QG's was renewed (see, e.g., Refs. 20–25). However, the results of the treatment of these papers based mainly on the equations obtained in our earlier works[5,13,14,16] do not go beyond the RS approach and lead only to progress in numerical solutions, and a number of problems beyond the RS approximation remains unsolved and we do not know what really the above-mentioned glasses do have in common with the well-known Sherrington-Kirkpatrick (SK) spin glass. Particularly, the answer to one of the main questions—whether the familiar picture of metastable states known from the results of Parisi,[26] Tanaka and Edwards,[27] Thouless, Anderson, and Palmer,[9] and Bray and Moore[28] on SK spin glass remains unchanged in some other more complicated cases—is far from being complete, although some steps in this direction have been done (see, e.g., Refs. 10, 17–19, 29–31) and the problem of the possibility of using the replica-symmetry-breaking (RBS) scheme of Parisi[26] in the case of the absence of reflection symmetry has been discussed since the beginning of 1985.[32]

Recently a method named the "cavity-fields approach"[33–35] has been proposed for the SK model. This approach avoids the replica trick, relies on well-known mathematics, and deals with obvious physical quantities, such as magnetizations and random magnetic fields. It seems to us that the extension of the method and the study of quadrupolar glasses in the frame of the cavity-fields approach may be useful for progress in the understanding of the physics of non-Ising spin glasses. This program is partially realized in the present paper.

The paper is organized as follows. In Sec. II, we discuss some of the main results of our previous study of QG and Potts glasses. These results form a logical basis for the use of the cavity-fields approach. In Sec. III, the cavity-fields approach is used to study the quadrupolar glass model with an effective EQQ Hamiltonian containing axial and non-axial quadrupole interactions. TAP-like equations are obtained and discussed. In Sec. IV, the case of "isotropic" QG's, which is equivalent to the three-state Potts spin glass, is considered in the frame of the cavity-fields approach. TAP-like equations are obtained and the transition temperature to the spin-glass phase is calculated.

## II. MODEL HAMILTONIAN AND PREVIOUS STUDIES

In recent years there has been a significant growth of interest in non-Ising spin glasses.[20–25,30,31] In this connection we recall the main of results we obtained earlier.[5,13–19] These results form a logical basis for the further investigations presented in Sec. III and Sec. IV. Some of these results were reproduced (or rederived?) in Refs. 22–24 nearly a decade later.

In Refs. 14, 16 we investigated in the replica-symmetric approximation the system of quadrupoles with the reduced Hamiltonian

$$H = H_1 + H_2, \tag{1}$$

where $H_1$ and $H_2$ represent axial and nonaxial parts, respectively:

$$H_1 = -\frac{1}{2}\sum_{i\neq j} J_{ij}Q_iQ_j, \tag{2}$$

$$H_2 = -\frac{1}{2}\sum_{i\neq j} G_{ij}V_iV_j. \tag{3}$$

Here $Q = 3J_z^2-2, V = \sqrt{3}(J_x^2-J_y^2), J=1, J_z = 1,0,-1$. The molecular quadrupolar moment is a second-rank tensorial operator with five independent components. In the principal axis frame only two of them remain: $Q$ and $V$. The coupling constants $J_{ij}$ and $G_{ij}$ are quenched random interactions of infinite range, independently distributed with Gaussian probabilities

$$P(J_{ij}) = (\sqrt{2\pi}J)^{-1}\exp[-(J_{ij}-J_0)^2/2J^2], \tag{4}$$

$$P(G_{ij}) = (\sqrt{2\pi}G)^{-1}\exp[-(G_{ij}-G_0)^2/2G^2]. \tag{5}$$

The scaling $J_0 = \tilde{J}_0/N$, $G_0 = \tilde{G}_0/N$, $J = \tilde{J}/N^{1/2}$, $G = \tilde{G}/N^{1/2}$ ensures a sensible thermodynamic limit.

Using the replica method it is easy to obtain the free energy of the system and the equations for the order parameters $m = \langle\langle Q\rangle_T\rangle_{J,G}$, $n = \langle\langle V\rangle_T\rangle_{J,G}$, $q_1 = \langle\langle Q\rangle_T^2\rangle_{J,G}$, $q_2 = \langle\langle V\rangle_T^2\rangle_{J,G}$. In the RS approximation the free energy has the form

$$F = -\frac{\tilde{J}^2}{(2kT)^2}(q_1+m-2)(q_1-m-2) + (\tilde{J}_0/2kT)m^2\frac{\tilde{G}^2}{(2kT)^2}(q_2+m-2)(q_2-m-2) + (\tilde{G}_0/2kT)n^2 - \iint \frac{dz_1dz_2}{2\pi}e^{-(z_1^2+z_2^2)/2}\ln[e^{\theta_1}(e^{\theta_2}+e^{-\theta_2})+e^{-2\theta_1}], \tag{6}$$

with

$$\theta_1 = (1/2)(\tilde{J}/kT)^2(m+q_1-2) + (1/2)(\tilde{G}/kT)^2(m-q_2+2) + (\tilde{J}/kT)z_1\sqrt{q_1} + (\tilde{J}_0/kT)m, \tag{7}$$

$$\theta_2 = (\tilde{G}/kT)z_2\sqrt{q_2} + (\tilde{G}_0/kT)n, \tag{8}$$

where $m$, $n$, $q_1$, and $q_2$ are solutions of the equations

$$m = \iint \frac{dz_1dz_2}{2\pi}e^{-(z_1^2+z_2^2)/2}\frac{e^{\theta_1}(e^{\theta_2}+e^{-\theta_2})-2e^{-2\theta_1}}{e^{\theta_1}(e^{\theta_2}+e^{-\theta_2})+e^{-2\theta_1}}, \tag{9}$$

$$n = \iint \frac{dz_1dz_2}{2\pi}e^{-(z_1^2+z_2^2)/2}\sqrt{3}\frac{e^{\theta_1}(e^{\theta_2}-e^{-\theta_2})}{e^{\theta_1}(e^{\theta_2}+e^{-\theta_2})+e^{-2\theta_1}}, \tag{10}$$

$$q_1 = \iint \frac{dz_1dz_2}{2\pi}e^{-(z_1^2+z_2^2)/2}\cdot\left[\frac{e^{\theta_1}(e^{\theta_2}+e^{-\theta_2})-2e^{-2\theta_1}}{e^{\theta_1}(e^{\theta_2}+e^{-\theta_2})+e^{-2\theta_1}}\right]^2, \tag{11}$$

$$q_2 = \iint \frac{dz_1dz_2}{2\pi}e^{-(z_1^2+z_2^2)/2}3\left[\frac{e^{\theta_1}(e^{\theta_2}-e^{-\theta_2})}{e^{\theta_1}(e^{\theta_2}+e^{-\theta_2})+e^{-2\theta_1}}\right]^2. \tag{12}$$

Performing a high-temperature expansion of Eqs. (9)–(12) it is easy to see that there is no trivial solution for the parameters $m$ and $q_1$ even at high temperature. The exception is the case of the isotropic model when $G^2 = J^2$. In this case the system was shown[14,16] to exhibit a phase transition from the disordered paraquadrupolar phase ($m = n = 0,\ q_1 = q_2 = 0$) to the quadrupolar glass phase ($m = n = 0,\ q_1 = q_2 \neq 0$). The critical temperature is $kT_c = 2\tilde{J}$. Such a situation is to be expected on a cubic lattice when the operators $Q$ and $V$ belong to the same irreducible representation of the rotation group subgroup. In this case the model describes the transition to the QG phase in mixed crystals of $(\mathrm{KBr})_x\ (\mathrm{KCN})_{1-x}$ type, where the glass phase is observed on a cubic lattice.[36,37] This is in contrast to the case $J \neq G$ without a phase transition, which describes the situation on a hexagonal lattice ($o$-$p$-$H_2$).

The isotropic model was shown to be equivalent to the three-state Potts spin glass (considered, e.g., in Ref. 38), this equivalence following from the identity

$$\delta(\alpha,\beta) = \frac{1}{6}(Q_\alpha Q_\beta + V_\alpha V_\beta + 2). \tag{13}$$

The low-temperature behavior of $q$ was found[16] to be

$$q = 2 - \frac{3\sqrt{3\pi}}{4}t^{-1} + O(t^{-3}),\ t = \tilde{J}/kT\ , \tag{14}$$

and the entropy at $T = 0$, $S = -27Nk/32\pi$.

The obtained RS solution was shown to be unstable at low temperature.[16] If we make an attempt to break up the replica symmetry, we must obtain the free energy in the vicinity of $T_c$ in the general case. This effective free energy[17] differs from that of the SK case in two respects. First, it contains an extra cubic term $\sum(q^{\alpha\beta})^3$ due to the absence of reflection symmetry and nonzero values of the averages of $(Q^\alpha)^3$ and $Q^\alpha(V^\alpha)^2$. Second, the presence of the negative term $-8\sum(q^{\alpha\beta})^2 q^{\alpha\gamma}q^{\beta\gamma}$ makes the "effective" sign of the fourth-order term negative. [The change of sign does not cause stability of the RS solution, because in our case it is unstable already at first order in $(T - T_c)$.] These terms were shown to make the RBS scheme of Parisi fail. A solution of the step function type was obtained,[17] which was stable in the vicinity of the transition point (see also Refs. 29–31).

One of the main characteristic features of spin glasses is the existence of many minima of the free energy separated by very-high-energy barriers. It is interesting to study those configurations which are local minima of the Hamiltonian in the sense that the energy increases when we flip a spin. These local minima are very important in the dynamics outside equilibrium because at low temperatures the system may be trapped for a very long time in these minima. The number of such metastable states at zero temperature is well known for the SK model.[27,28,39] As to non-Ising systems the low-temperature local minima structure has not been analyzed (see, however, Ref. 10). We have used the method of Tanaka and Edwards[27] to count the number of metastable states in the Potts model with $p = 3$ in Ref. 18, where we have exploited the equivalence of the three-state Potts model to the isotropic glass model.[14,16]

In what follows we shall put $J_0 = G_0 = 0$.

Following Ref. 27 we have defined the number of metastable states as the number of states with positive local one-particle excitation. The condition of positivity of local one-particle excitation through the properties of the operators $Q$ and $V$ in the subspace $J = 1$ can be brought up to the condition of the positivity of the quantity (see Ref. 18)

$$Q_i \sum_j J_{ij} Q_j + V_i \sum_j J_{ij} V_j\ . \tag{15}$$

The number of the metastable states can be written as

$$\langle N_s \rangle = \left\langle \mathrm{Tr} \prod_i \Theta\left( Q_i \sum_j J_{ij} Q_j + V_i \sum_j J_{ij} V_j \right) \right\rangle_J .$$

The obtained number of such states is macroscopically large:

$$\langle N_s \rangle = \exp[-N\Omega_3], \tag{16}$$

with $\Omega_3 = -(z^*/2) - \ln\Phi(z^*) - \ln 3$. Here

$$\Phi(x) = \int_{-\infty}^{x} \frac{\exp[-t^2/2]}{\sqrt{2\pi}} dt.$$

$z^* = 0.5061$,[27] $\Phi(z^*) = 0.6936$, so that

$$\Omega_3 = \Omega_2 - \ln(3/2) = -0.6047,$$

with $\Omega_2 = -0.199\,23$ being the well-known[27] quantity for the SK model (Potts spin glass with $p = 2$). This means that the "relative" number of metastable states (the part of all possible $p^N$ states) is the same in both models:

$$\frac{\exp[-N\Omega_3]}{3^N} = \frac{\exp[-N\Omega_2]}{2^N}. \tag{17}$$

In an analogous manner one can obtain the distribution function of the local-minima energies:[19]

$$P(E) = \left\langle \mathrm{Tr}\,\delta\left( E - \frac{1}{2}\sum J_{ij}(Q_i Q_j + V_i V_j) \right) \times \prod \Theta\left( Q_i \sum J_{ij} Q_j + V_i \sum J_{ij} V_j \right) \right\rangle_J . \tag{18}$$

The obtained normalized distribution is $N(\varepsilon) \simeq n(\varepsilon)^N$ for the dimensionless energy per particle $\varepsilon = E/N\tilde{J}$ with

$$n(\varepsilon) = 3\Phi(\tau^*)\exp\{[\tau^* - \varepsilon/(2\sqrt{2})]^2 - \tfrac{1}{2}\tau^{*2} - 0.6047\}$$

[compare with Eq. (5.13) of Ref. 27]. Here $\tau^* = \tau^*(\varepsilon)$ is the solution of the equation

$$\tau^* + \frac{\Phi'(\tau^*)}{\Phi(\tau^*)} = \varepsilon/\sqrt{2}.$$

The energies of the local minima are distributed similarly to those of the SK model. It is possible, however, that the states considered here present only a part of all metastable states. The other part connected with false vacua can be large, too.

## III. CAVITY-FIELDS APPROACH TO THE QG MODEL

Now we shall formulate the cavity-fields method for the infinite-range QG model and then (in Sec. IV) we shall use it to describe the three-state Potts spin-glass model. Thouless-Anderson-Palmer- (TAP-) like equations will be obtained.

The basic idea of the cavity-fields approach[33–35] is to compare the behavior of the model moving from $N$ to $N+1$ particles. The system reacts to the inclusion of the newcomer by reshuffling the various levels in the ultrametric topology of states. The connected correlation functions for different states inside a pure state are neglected in the thermodynamic limit ($N \to \infty$).

Following the idea of the cavity-fields approach we add one quadrupole with moment components $Q$ and $V$ at site 0. Now we discuss the problem on the level of pure states,[34] so that the reaction of the system will be a reshuffling of the configurations inside a pure state. We consider a model of $N$ quadrupoles with Hamiltonian (1) in a pure state denoted by $\alpha(N)$. The number of configurations belonging to this state with energy in the interval $(E_N, E_N + dE_N)$ is (the definition of the entropy)

$$dN(E_N) = e^{S(E_N)} dE_N \; . \tag{19}$$

At a fixed temperature the relevant configurations inside the state $\alpha(N)$ have energies near its internal energy $E_{\alpha(N)}$. Therefore, assuming $|E_N - E_{\alpha(N)}|/|E_{\alpha(N)}|$ to be small, we can write

$$dN(E_N) = e^{-\beta F_{\alpha(N)}} e^{\beta E_N} dE_N, \tag{20}$$

where $F_{\alpha(N)} = E_{\alpha(N)} - \beta^{-1} S(E_{\alpha(N)})$ is the free energy of the $N$-quadrupole system in the state $\alpha(N)$.

Now we define the cavity fields[34] as the fields $\{h_k\}$ ($k$=1,2) produced by a given configuration of $N$ quadrupoles and acting on the new site 0 once the corresponding quadrupole has been removed:

$$h_1 = \sum_{i=1}^{N} J_{0i} Q_i \; , \tag{21}$$

$$h_2 = \sum_{i=1}^{N} G_{0i} V_i \; . \tag{22}$$

By construction, the random couplings $J_{0i}$ and $G_{0i}$ are uncorrelated with variables $Q_i$ and $V_i$ ($i = 1, ..., N$), which are in equilibrium among themselves but not with the newcomer. Therefore, considering the ensemble of relevant configurations inside the $N$-quadrupole system equilibrium state, the fields $h_1$ and $h_2$ are random variables controllable through the central limit theorem and hence obeying the two-dimensional Gaussian probability distribution with the means (compare with Refs. 40, 41)

$$h_1^0 = \langle h_1 \rangle_{\alpha(N)} = \sum_{i=1}^{N} J_{0i} \langle Q_i \rangle_{\alpha(N)} = \sum_{i=1}^{N} J_{0i} m_{i\alpha(N)} \; , \tag{23}$$

$$h_2^0 = \langle h_2 \rangle_{\alpha(N)} = \sum_{i=1}^{N} G_{0i} \langle V_i \rangle_{\alpha(N)} = \sum_{i=1}^{N} G_{0i} n_{i\alpha(N)} \; , \tag{24}$$

and variances

$$\chi_{kk'} = \beta \left[ \frac{1}{N} \sum_{i=1}^{N} \langle A_i^k A_i^{k'} \rangle_{\alpha(N)} - q^{kk'} \right] , \tag{25}$$

where

$$A_i^k = \begin{cases} Q_i, & k = 1, \\ V_i, & k = 2, \end{cases}$$

$$q^{kk'} \equiv \frac{1}{N} \sum_{i=1}^{N} \langle A_i^k \rangle_{\alpha(N)} \langle A_i^{k'} \rangle_{\alpha(N)},$$

and

$$\sum_{k''=1}^{2} (\chi^{-1})_{kk''} \chi_{k''k'} = \delta_{kk'},$$

where $\langle \cdots \rangle_{\alpha(N)}$ denotes the averages over the configurations inside the state $\alpha(N)$ of the $N$-quadrupole system.

Thus, the probability distribution for $\{h_k\}$ has the form

$$P(\{h_1, h_2\}) = \frac{\beta}{2\pi} \det{}^{-1/2} \left\| \frac{\chi_{kk'}}{J_k J_{k'}} \right\| \exp\left[ -\frac{\beta}{2} \sum_{k,k'=1}^{2} \frac{1}{J_k J_{k'}} \times (\chi^{-1})_{kk'} (h_k - h_k^0)(h_{k'} - h_{k'}^0) \right] . \tag{26}$$

Here $J_1 = \tilde{J}$ and $J_2 = \tilde{G}$.

The probability distribution (26) for cavity fields is statistically independent of the energy value, since $h_k$ are functions of the new couplings $J_{0i}, G_{0i}$, whereas the energy is a function of the old $J_{ij}, G_{ij}$. Therefore, the number of configurations of $N$ quadrupoles with energy in the interval $(E, E + dE)$ and cavity fields in $(h_k, h_k + dh_k)$ is given by

$$dN(E_N, h_1, h_2) = e^{-\beta F_{\alpha(N)}} e^{\beta E_N} \times P(\{h_1, h_2\}) \, dE_N dh_1 dh_2 \; . \tag{27}$$

To each one of them there correspond three configurations of the $(N+1)$-quadrupole system with energy

$$E_{N+1} = E_N - (h_1 Q_0 + h_2 V_0), \tag{28}$$

where $Q_0$ and $V_0$ are quadrupole variables referred to the new site 0.

Now the joint distribution is

$$dN(E_{N+1}, h_1, h_2, Q_0, V_0) = e^{-\beta F_{\alpha(N)}} e^{\beta E_{N+1}} P(\{h_1, h_2\}) \exp[\beta(h_1 Q_0 + h_2 V_0)] dE_{N+1} dh_1 dh_2 . \tag{29}$$

Integrating over $h_1$ and $h_2$ and tracing over $Q_0$ and $V_0$, we obtain an exponential distribution of the energy $E_{N+1}$,

$$dN(E_{N+1}) = e^{-\beta F_{\alpha(N+1)}} e^{-\beta E_{N+1}} dE_{N+1}. \tag{30}$$

The change of the normalization in (30) determines the free energy $F_{\alpha(N+1)}$ of the $(N+1)$-quadrupole system. It is not difficult to see that

$$F_{\alpha(N+1)} - F_{\alpha(N)} = -\beta^{-1} \ln \mathrm{Tr} \int [h_1 Q_0 + h_2 V_0] P(\{h_1, h_2\})\, dh_1 dh_2. \tag{31}$$

Thus, Eq. (29) can be rewritten as

$$dN(E_{N+1}, h_1, h_2, Q_0, V_0) = e^{-\beta F_{\alpha(N)}} e^{\beta E_{N+1}} \mathcal{P}(\{h_1, h_2, Q_0, V_0\})\, dE_{N+1} dh_1 dh_2, \tag{32}$$

where

$$\mathcal{P}(\{h_1, h_2, Q_0, V_0\}) = \mathcal{K} e^{\beta(F_{\alpha(N+1)} - F_{\alpha(N)})} P(\{h_1, h_2\}) \times \exp[\beta(h_1 Q_0 + h_2 V_0)] \tag{33}$$

is the probability distribution that in the $(N+1)$-quadrupole system the variables $Q_0, V_0$ and the fields $h_1, h_2$ have certain values at a fixed value of $E_{N+1}$. $\mathcal{K}$ is the normalization constant.

From Eq. (33) it is easy to see that the probability distribution for the fields $h_k$ in the presence of a new quadrupole is no longer Gaussian and is correlated with the values of $Q_0, V_0$.

Using Eq. (33) and Eqs. (21)–(24) one can obtain the average values $\langle Q_0\rangle_{\alpha(N+1)}$ and $\langle V_0\rangle_{\alpha(N+1)}$ and the average fields $\langle h_1\rangle_{\alpha(N+1)}$ and $\langle h_2\rangle_{\alpha(N+1)}$ in the $(N+1)$-quadrupole system:

$$m_0^{\alpha(N+1)} \equiv \langle Q_0\rangle_{\alpha(N+1)} = \sum_{Q_0, V_0} \int dh_1 dh_2 Q_0 \mathcal{P}(\{h_1, h_2, Q_0, V_0\}), \tag{34}$$

$$n_0^{\alpha(N+1)} \equiv \langle V_0\rangle_{\alpha(N+1)} \sum_{Q_0, V_0} \int dh_1 dh_2 V_0 \mathcal{P} \times (\{h_1, h_2, Q_0, V_0\}), \tag{35}$$

$$h_1^{\alpha(N+1)} \equiv \langle h_1\rangle_{\alpha(N+1)} = \sum_{i=1}^{N} J_{0i} \langle Q_i\rangle_{\alpha(N+1)} = \sum_{i=1}^{N} J_{0i} m_i^{\alpha(N+1)} = \sum_{Q_0, V_0} \int dh_1 dh_2 h_1 \mathcal{P}(\{h_1, h_2, Q_0, V_0\}), \tag{36}$$

$$h_2^{\alpha(N+1)} \equiv \langle h_2\rangle_{\alpha(N+1)} = \sum_{i=1}^{N} G_{0i} \langle V_i\rangle_{\alpha(N+1)} = \sum_{i=1}^{N} G_{0i} n_i^{\alpha(N+1)} = \sum_{Q_0, V_0} \int dh_1 dh_2 h_2 \mathcal{P}(\{h_1, h_2, Q_0, V_0\}), \tag{37}$$

with $\langle \cdots \rangle_{\alpha(N+1)}$ denoting the average over configurations inside the state $\alpha(N+1)$ of the $(N+1)$-quadrupole system.

Using the features of the operators $Q$ and $V$ it is not difficult to obtain the variances (25) entering the distribution (26):

$$\chi_{11} = (2 - \bar{m} - q_1)\beta,$$

$$\chi_{22} = (2 + \bar{m} - q_2)\beta,$$

$$\chi_{12} = \chi_{21} = \beta(\bar{n} - q_{12}),$$

$$(\chi^{-1})_{11} = \frac{1}{\beta} \frac{2 + \bar{m} - q_2}{\Delta},$$

$$(\chi^{-1})_{22} = \frac{1}{\beta} \frac{2 - \bar{m} - q_1}{\Delta},$$

$$(\chi^{-1})_{12} = (\chi^{-1})_{21} = \frac{1}{\beta} \frac{q_{12} - \bar{n}}{\Delta},$$

where

$$\Delta = (2 - \bar{m} - q_1)(2 + \bar{m} - q_2) - (\bar{n} - q_{12})^2,$$

$$\bar{m} = \frac{1}{N} \sum_{i=1}^{N} m_i,$$

$$\bar{n} = \frac{1}{N} \sum_{i=1}^{N} n_i.$$

Now we can write

$$\mathcal{P}(\{h_1, h_2, Q_0, V_0\}) = \mathcal{K} e^{\beta(h_1 Q_0 + h_2 V_0)} \exp\left\{ -\frac{(h_1 - h_1^0)^2}{2\tilde{J}^2 \beta} \frac{2 + \bar{m} - q_2}{\Delta} - \frac{(h_2 - h_2^0)^2}{2\tilde{G}^2 \beta} \frac{2 - \bar{m} - q_1}{\Delta} - \frac{2(h_1 - h_1^0)(h_2 - h_2^0)}{2\tilde{J}\tilde{G}\beta} \frac{q_{12} - \bar{n}}{\Delta} \right\}. \tag{38}$$

After calculating the constant $\mathcal{K}$ from the normalization condition

$$\sum_{Q_0,V_0}\int dh_1 dh_2 \mathcal{P}\left(\{h_1,h_2,Q_0,V_0\}\right)=1$$

and performing an integration in the Eqs. (34)–(37) we obtain

$$m_0=\frac{-2+2\cosh\varphi_1\exp\varphi_2}{1+2\cosh\varphi_1\exp\varphi_2}, \quad (39)$$

$$n_0=\frac{2\sqrt{3}\sinh\varphi_1\exp\varphi_2}{1+2\cosh\varphi_1\exp\varphi_2}, \quad (40)$$

$$h_1^{\alpha(N+1)}=h_1^0+\beta\tilde{J}^2 m_0(2-\bar{m}-q_1)-\beta n_0\tilde{J}\tilde{G}(q_{12}-\bar{n}), \quad (41)$$

$$h_2^{\alpha(N+1)}=h_2^0+\beta\tilde{G}^2 n_0(2+\bar{m}-q_2)-\beta m_0\tilde{J}\tilde{G}(q_{12}-\bar{n}), \quad (42)$$

where

$$\varphi_1=\sqrt{3}\beta[h_2^0-\beta\tilde{J}\tilde{G}(q_{12}-\bar{n})],$$

$$\varphi_2=3\beta[h_1^0-\tfrac{1}{2}\beta\tilde{J}^2(2-\bar{m}-q_1)+\tfrac{1}{2}\beta\tilde{G}^2(2+\bar{m}-q_2)].$$

If we write $h_1^0$ and $h_2^0$ in terms of the quantities averaged over the $\alpha(N+1)$ state using Eqs. (41),(42) and substitute the result in Eqs. (39),(40), then Eqs. (39),(40) become just what one can call TAP-like equations for a quadrupolar glass.

## IV. ISOTROPIC CASE ($\tilde{J}=\tilde{G}$): POTTS SPIN GLASS

As we have already mentioned above our system in the general case cannot exhibit a phase transition to a glasslike state with the order parameters $m_1=0$, $m_2=0$, $q_1\neq 0$, $q_2\neq 0$. This can be seen[14,16] from the high-temperature expansion of Eqs. (9)–(12): There are no trivial solutions for $m$ and $n$ if $G\neq J$. If $G=J$, we have the three-state Potts spin glass,[38] and for this case we can assume that, in the resulting equations of the previous section, $\bar{m}=0$, $\bar{n}=0$, $q_{12}=0$, $q_1=q_2=q$. Now the TAP-like equations for the three-state Potts glass can be written in the following form:

$$m_0=\frac{-2+2e^{3\beta h_1^0}\cosh\sqrt{3}\beta h_2^0}{1+2e^{3\beta h_1^0}\cosh\sqrt{3}\beta h_2^0}, \quad (43)$$

$$n_0=\frac{2\sqrt{3}e^{3\beta h_1^0}\sinh\sqrt{3}\beta h_2^0}{1+2e^{3\beta h_1^0}\cosh\sqrt{3}\beta h_2^0}, \quad (44)$$

where

$$h_1^0=h_1^{\alpha(N+1)}-\beta m_0\tilde{J}^2(2-q), \quad (45)$$

$$h_2^0=h_2^{\alpha(N+1)}-\beta n_0\tilde{J}^2(2-q). \quad (46)$$

To make the difference between the two cases clearer let us consider the high-temperature expansion of the TAP-like equations (39)–(42) for a quadrupole system in general case. They can be written as follows:

$$m_0=2\beta\sum_j J_{0j}m_j-2\beta^2\tilde{J}^2 m_0(2-\bar{m}-q_1)-2\beta^2\tilde{J}\tilde{G}n_0(q_{12}-\bar{n})$$
$$-\beta^2\tilde{J}^2(2-\bar{m}-q_1)+\beta^2\tilde{G}^2(2+\bar{m}-q_2)-\beta^2\left(\sum_j J_{0j}m_j\right)^2-\beta^2\left(\sum_j G_{0j}n_j\right)^2, \quad (47)$$

$$n_0=2\beta\sum_j G_{0j}n_j-2\beta^2\tilde{G}^2 n_0(2+\bar{m}-q_2)-2\beta^2\tilde{J}\tilde{G}m_0(q_{12}-\bar{n})+2\beta^2\sum_j J_{0j}m_j\sum_j G_{0j}n_j\ . \quad (48)$$

Replacing the sums $\sum_j J_{0j}m_j=2\tilde{J}m_0$ and $\sum_j G_{0j}n_j=2\tilde{G}n_0$ we can obtain from Eqs. (47) and (48) the following equations for $\bar{m}$ and $\bar{n}$, respectively:

$$\bar{m}=4\beta\tilde{J}\bar{m}-2\beta^2\tilde{J}^2\bar{m}(2-\bar{m}-q_1)-2\beta^2\tilde{J}\tilde{G}\bar{n}(q_{12}-\bar{n})$$
$$-2\beta^2(\tilde{J}^2-\tilde{G}^2)+\bar{m}\beta^2(\tilde{J}^2+\tilde{G}^2)+\beta^2(\tilde{J}^2 q_1+\tilde{G}^2 q_2)-4\beta^2\tilde{G}^2 q_1-4\beta^2\tilde{J}^2 q_2\ , \quad (49)$$

$$\bar{n}=4\beta\tilde{G}\bar{n}-2\beta^2\tilde{G}^2\bar{n}(2+\bar{m}-q_2)-2\beta^2\tilde{J}\tilde{G}\bar{m}(q_{12}-\bar{n})+8\beta^2\tilde{J}\tilde{G}q_{12} \quad (50)$$

$$(q_1=\overline{m^2},\ q_2=\overline{n^2},\ q_{12}=\overline{mn})\ .$$

After multiplying Eqs. (47) and (48) by $m_0$ and $n_0$, respectively, and taking the sums over sites we obtain

$$q_1=4\beta\tilde{J}q_1-2\beta^2\tilde{J}^2 q_1(2-\bar{m}-q_1)-2\beta^2\tilde{J}\tilde{G}q_{12}(q_{12}-\bar{n})$$
$$-\beta^2\tilde{J}^2\bar{m}(2-\bar{m}-q_1)+\beta^2\tilde{G}^2\bar{m}(2+\bar{m}-q_2)-4\beta^2\tilde{J}^2\overline{m^3}-4\beta^2\tilde{G}^2\overline{mn^2}, \quad (51)$$

$$q_2 = 4\beta\tilde{G}q_2 - 2\beta^2\tilde{G}^2 q_2(2+\bar{m}-q_2) - 2\beta^2\tilde{J}\tilde{G}q_{12}(q_{12}\bar{n}) + 8\beta^2\tilde{J}\tilde{G}\overline{mn^2}\ , \tag{52}$$

$$q_{12} = 4\beta\tilde{J}q_{12} - 2\beta^2\tilde{J}^2 q_{12}(2-\bar{m}-q_1) - 2\beta^2\tilde{J}\tilde{G}q_2(q_{12}-\bar{n}) - \beta^2\tilde{J}^2\bar{n}(2-\bar{m}-q_1) + \beta^2\tilde{G}^2\bar{n}(2+\bar{m}-q_2) - 4\beta^2\tilde{J}^2\overline{nm^2} - 4\beta^2\tilde{G}^2\overline{n^3}\ . \tag{53}$$

It is easy to see that Eqs. (49), (50) have no trivial solution $\bar{m}=0$, $\bar{n}=0$ if the coefficient $(\tilde{J}^2-\tilde{G}^2)$ is nonzero. In the case of nonzero $\bar{m}$ and $\bar{n}$, Eqs. (51) and (52) also have no trivial solution for $q$. This means that there is no traditional phase transition to the spin-glass-like state in the case $\tilde{J}^2 \neq \tilde{G}^2$ (compare with Refs. 5, 7, 13, 42). It is interesting of course to look for a crossover to spin-glass-like behavior in the next stage of the cavity-fields approach.

In the case of the Potts model (that is, the isotropic quadrupole glass model with $\tilde{J}^2=\tilde{G}^2$) there exists a kind of traditional phase transition to the spin-glasslike state with $\bar{m}=\bar{n}=0$ and nonzero spin-glass order parameter. The temperature of the transition can be found from the Eqs. (43)–(46) in a manner used in Ref. 9.

In this case for $T$ near $T_c$ we expect $m_0$ and $n_0$ to be small. The same can be said about the eigenvectors $M_i$ and $N_i$ belonging to the largest eigenvalue $(J_\lambda)_{\max}=2\tilde{J}$ of the matrix $J_{ij}$:

$$\sum_j J_{ij}M_j = 2\tilde{J}M_i,\quad \sum_j J_{ij}N_j = 2\tilde{J}N_i. \tag{54}$$

Now

$$h_1^0 = \frac{1}{6\beta}\ln\frac{(2+m_0)^2-3n_0^2}{4(1-m_0)^2}\ , \tag{55}$$

$$h_2^0 = \frac{\sqrt{3}}{6\beta}\ln\frac{2+m_0+n_0\sqrt{3}}{2+m_0-n_0\sqrt{3}}\ . \tag{56}$$

With the use of these equations we can obtain from (45),(46), respectively,

$$\sum_j J_{0j}m_j = \beta\tilde{J}^2(2-q)m_0 + \frac{1}{2\beta}\left[m_0 + \frac{1}{4}(m_0^2-n_0^2) + \frac{1}{4}m_0(m_0^2+n_0^2)\right]\ , \tag{57}$$

$$\sum_j J_{0j}n_j = \beta\tilde{J}^2(2-q)n_0 + \frac{1}{2\beta}\left[n_0 - \frac{1}{2}m_0n_0 + \frac{1}{4}n_0(m_0^2+n_0^2)\right]\ . \tag{58}$$

Keeping in mind Eq. (54) and taking the scalar product of Eqs. (55) and (56) with $M_i$ and $N_i$, respectively, we obtain the linearized equation

$$q[4(\beta\tilde{J})^2 - 4\beta\tilde{J} + 1] = 0\ , \tag{59}$$

so that the phase transition temperature is $kT_c = 2\tilde{J}$, in accordance with previous results.[38,16]

## V. CONCLUSIONS

To summarize, a first stage description of the quadrupolar glass model (and Potts spin-glass model with $p=3$) in the frame of the cavity-fields approach is obtained. The rearrangement of the configurations of quadrupole moments inside a pure state is discussed. The TAP-like equations are derived.

The obtained results are in agreement with previous calculations based on the replica method and this fact can be regarded as an additional support for the replica method predictions.

We believe that the cavity-fields approach may be very useful in putting spin-glass theory on firmer mathematical and physical grounds. The next step in exploring the potentialities of the cavity approach includes consideration of the rearrangement of the quadrupolar moment configurations inside a cluster. These results will be published elsewhere.

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[1] N. S. Sullivan, M. Devoret, B. P. Cowan, and C. Urbina, Phys. Rev. B **17**, 5016 (1978).
[2] U. T. Höchli, K. Knorr, and A. Loidl, Adv. Phys. **39**, 409 (1990).
[3] K. Binder and J. D. Reger, Adv. Phys. **41**, 547 (1992).
[4] G. Parisi (unpublished).
[5] E. A. Lutchinskaia, V. N. Ryzhov, and E. E. Tareyeva, J. Phys. C **17**, L665 (1984).
[6] D. Sherrington and S. Kirkpatrick, Phys. Rev. Lett. **32**, 1972 (1975).
[7] N. S. Sullivan, C. M. Edwards, and J. R. Brookeman, Mol. Cryst. Liq. Cryst. **139**, 365 (1986).

[8] P. M. Goldbart and D. Sherrington, J. Phys. C **18**, 1923 (1985).
[9] D. J. Thouless, P. W. Anderson, and R. G. Palmer, Philos. Mag. B **35**, 593 (1977).
[10] M. I. Vladimir, S. P. Kozhukhar', and V. A. Moscalenko, Theor. Math. Phys. (Moscow) **71**, 129 (1987).
[11] V. B. Kokshenev, Solid State Commun. **55**, 143 (1985).
[12] A. B. Harris and H. Meyer, Can. J. Phys. **63**, 3 (1985).
[13] E. A. Lutchinskaia, V. N. Ryzhov, and E. E. Tareyeva, Theor. Math. Phys. (Moscow) **67**, 463 (1986).
[14] E. A. Lutchinskaia and E. E. Tareyeva, Theor. Math. Phys. (Moscow) **70**, 477 (1987).
[15] E. A. Lutchinskaia, Theor. Math. Phys. (Moscow) **67**, 463 (1986).
[16] E. A. Lutchinskaia and E. E. Tareyeva, Theor. Math. Phys. (Moscow) **87**, 473 (1991).
[17] E. A. Lutchinskaia and E. E. Tareyeva, Theor. Math. Phys. (Moscow) **91**, 157 (1992).
[18] E. A. Lutchinskaia and E. E. Tareyeva, Europhys. Lett. **17**, 109 (1992).
[19] E. A. Lutchinskaia and E. E. Tareyeva, Phys. Lett. A **181**, 331 (1993).
[20] K. Walasek, Phys. Rev. B **46**, 14 480 (1992).
[21] T. K. Kopec, Phys. Rev. B **48**, 3698 (1993).
[22] T. K. Kopec, Phys. Rev. B **48**, 15 658 (1994).
[23] T. K. Kopec, Phys. Rev. B **48**, 16 792 (1994).
[24] K. Walasek and K. Lukerska-Walasek, Phys. Rev. B **49**, 9460 (1994).
[25] K. Walasek and K. Lukerska-Walasek, Phys. Rev. B **49**, 9973 (1994).
[26] G. Parisi, J. Phys. A **13**, 1101 (1980).
[27] F. Tanaka and S. F. Edwards, J. Phys. F **10**, 2769 (1980).
[28] A. Bray and M. A. Moore, J. Phys. C **12**, L477 (1979); **13**, L469 (1980).
[29] D. J. Gross, I. Kanter, and H. Sompolinsky, Phys. Rev. Lett. **55**, 304 (1985); I. Kanter and H. Sompolinsky, Phys. Rev. B **33**, 2073 (1986).
[30] G. Cwilich and T. R. Kirkpatrick, J. Phys. A **22**, 4971 (1989).
[31] G. Cwilich, J. Phys. A **23**, 5029 (1990).
[32] P. M. Goldbart and D. Elderfield, J. Phys. C **18**, L229 (1985).
[33] M. Mézard, G. Parisi, and M. A. Virasoro, Europhys. Lett. **1**, 77 (1986).
[34] M. Mézard, G. Parisi, and M. A. Virasoro, *Spin Glass Theory and Beyond*, Lecture Notes in Physics, Vol. 9 (World Scientific, Singapore, 1987).
[35] R. Brunetti and G. Parisi, Europhys. Lett. **11**, 281 (1990).
[36] H. O. Carmesin, J. Phys. A **22**, 297 (1989).
[37] H. Vollmayr, R. Kree, and A. Zippelius, Europhys. Lett. **12**, 235 (1990).
[38] A. Erzan and E. J. S. Lage, J. Phys. C **16**, L555 (1983); **16**, L873 (1983).
[39] C. De Dominisis, M. Gabay, T. Garel, and H. Orland, J. Phys. (Paris) **41**, 923 (1980).
[40] L. De Cesare, K. Lukerska-Walasek, I. Rubuffo, and K. Walasek, Phys. Rev. B **45**, 1041 (1992).
[41] L. De Cesare, K. Lukerska-Walasek, and K. Walasek, Phys. Rev. B **45**, 8127 (1992).
[42] E. A. Lutchinskaia, V. N. Ryzhov, T. I. Schelkatcheva, and E. E. Tareyeva, in *Proceedings of the 5th International Symposium on Selected Topics in Statistical Mechanics*, edited by A. A. Logunov *et al.* (World Scientific, Singapore, 1990), pp. 170–175.