Superfluid turbulence in the low-temperature limit

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Dissipationless relaxation kinetics of the superfluid turbulence is analyzed. The possibility and scenario of a specific Kolmogorov-like regime are revealed. Qualitative considerations are corroborated by numerical study of a simplified model of a self-crossing vortex line.

I. INTRODUCTION

A. Statement of the problem

The essential difference between the superfluid turbulence in He II (for an introduction see Ref. 1, and references therein for an extended review) and the turbulence in an ordinary fluid follows from the quantization of the circulation of the superfluid velocity resulting in the fact that the only possible turbulent motion is a motion of tangled vortex lines. One of the basic problems of superfluid turbulence is the relaxation kinetics of the vortex tangle. Ultimately, the decay of the tangle (line-length decrease) can be related to the dissipative interaction of the vortex lines with the normal component. This interaction is characterized by a dimensionless coefficient α, which measures the drag force exerted by the normal component on the vortex line element (in the units of corresponding Magnus force). The microscopic picture of the motion of the vortex tangle at not so small α's is well understood, in particular due to the direct numerical simulations by Schwarz,[2] which are in very good agreement with experiments. It is natural that in this case kinetic constants are essentially α dependent.

The purpose of the present paper is to consider the relaxation kinetics of superfluid turbulence in the limit of $\alpha \to 0$ [that is the low-temperature limit, $\alpha \propto T^5$ as $T \to 0$ (Ref. 4)] and to demonstrate the possibility of the existence of such a regime when relaxation kinetics does not depend explicitly on α. From general point of view this regime is analogous to the Kolmogorov regime in the ordinary turbulence where energy is transferred along the scales of length without dissipation rendering relaxation properties viscosity independent. It is worthwhile to note that a direct analog of the Kolmogorov scenario was initially adopted phenomenologically by Vinen in his pioneering works on the superfluid turbulence.[5] However, as it was shown later by Schwarz[3,2] relaxation dynamics of superfluid turbulence differs in an essential way from that of an ordinary fluid, and the direct analog of Kolmogorov regime generally speaking does not take place.

The particular scenario of the superfluid Kolmogorov regime (KR) turns out to be rather nontrivial owing to the conservation laws imposing serious restrictions on the free (without friction and reconnections) motion of the vortex lines. For example, these restrictions prohibit such "pure" scenarios as (i) decay of the vortex lines into vortex rings with subsequent decay of each ring into pair of rings of smaller radii, and so on (this picture was proposed by Feynman[6] and was referred to by Vinen[5] as a possible element of the mechanism of superfluid turbulence decay); (ii) transport of energy (equals vortex-line length) in the wave-length-scale space of the Kelvin waves towards larger wave vectors. On the other hand, most of the conservation laws are broken by reconnections which thus may lead to and play an essential part in KR. In particular, one may think of the following picture.

Reconnections of the vortex lines at the scale of the characteristic interline spacing, R_0, permanently transfer line length to somewhat lower (but of the same order of magnitude as R_0) scale of distance R_1. (In Sec. II we show that this occurs via the mechanism of generation of Kelvin waves which accompanies relaxation of the vortex angle formed in the process of reconnection.) Since conservation laws prohibit line-length transport to lower scales, line curvature increases until it reaches the order of R_1^{-1} and self-crossings become important. Any self-intersection at the scale R_1 with subsequent reconnection leads to a creation of a vortex ring with the radius $\sim R_1$. Relaxation of the two vortex angles (one on the line and the other on the ring) transfers in its turn line length to the lower scale R_2, adjacent to R_1. When the scale of curvature reaches R_2^{-1} the picture is repeated, and so on. Ultimately, vortex lines turn out to be fractalized in such a way that the curvature of the line smoothed out up to some scale R is close to R^{-1}, to support the above-mentioned mechanism of the line-length transport. Since the characteristic time of evolution at the scale R is $\propto R^2/\ln(R/a_0)$ (a_0 is the core radius of the vortex filament) and approaches zero as $R \to 0$, the time of fractalization of any initial tangled vortex state is finite [$\propto R_0^2/\ln(R_0/a_0)$]. Thus, after a short transient period (if necessary) evolution of superfluid turbulence at any scale $R \ll R_0$ should take on a quasi-steady-state form, being governed by the evolution at the main scale R_0. The only parameter characterizing the whole picture should be the line-length flux in the curvature space produced by the reconnections at the main scale.

The vortex rings created as a result of self-crossings do not seem to play any special part in the relaxation

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kinetics: some time after its emission a ring should be absorbed by the vortex tangle with the only effect that this produces one more reconnection and thus enhances to some extent the line-length transport.

An outline of the rest of the paper is as follows. In Sec. I B we discuss in detail the restrictions to the free motion of a vortex line which prohibit scenarios (i) and (ii). In Sec. II we consider relaxation of the vortex angle formed as a result of a reconnection of two vortex lines. We show that in the intrinsic variables (curvature and torsion) this process can be described by the well-known self-similar solution of the Schrödinger equation. From this solution (as well as from the general analysis of the constants of motion) it is seen that the relaxation of the angle is followed by the emission of Kelvin waves, in agreement with previous numerical observation of the formation of a helical structure on a relaxing anglelike vortex line.[7] Incidentally, we demonstrate that a reconnection of two lines at a small enough angle leads to a creation of a hierarchy of vortex rings. This effect, however, should not play a considerable role in the kinetics of superfluid turbulence since the critical angle $\gamma_* \approx 0.152$ is rather small.

In Sec. III we estimate the main characteristics of the fractalized vortex tangle under the conditions of KR.

In Sec. IV we present numerical evidence in favor of the existence of KR and of the scenario proposed. Direct numerical simulation of well-developed KR seems to be a rather difficult problem. First, because of the necessity to deal with different scales of distance and correspondingly with different scales of characteristic time. Second, due to the necessity of revealing and processing a considerable amount of reconnections. We propose a simplified approach which is free of the second difficulty. It is based on Hamiltonian equations of vortex-line motion which are exact while the vortex line can be parametrized as $x = x(z)$, $y = y(z)$ (x, y, and z are Cartesian coordinates). When the functions $x(z)$ and/or $y(z)$ are going to become non-single-valued at some point $z = z_*$ our model deviates from the real picture of vortex-line evolution: demonstrates a jump in $x(z)$ and/or in $y(z)$ at $z = z_*$. However, the model has two remarkable features which make it essentially relevant: (i) the total length of the line (smooth part plus amplitudes of the jumps) is an exact constant of motion, and (ii) relaxation of a jump is qualitatively equivalent to that of a vortex angle (is accompanied by the emission of Kelvin waves). Hence the process of creation and relaxation of a jump can be considered as a model for the process of self-crossing with creation and subsequent absorption of a vortex ring with diameter on the order of the amplitude of the jump. Our numerical model clearly demonstrates KR and corroborates the scenario proposed.

Section V contains some remarks concerning the effect of nonlocal interaction, neglected in our treatment, and transitory behavior of superfluid turbulence at not so small α's which should demonstrate rather weak dependence on α (Refs. 8 and 9) and thus may take place in a considerable temperature region. We also notice the expedience of the reproduction of the experiment[8] at lower temperatures for unambiguous observation of KR.

B. Equations and restrictions

The limit $\alpha \to 0$ implies that the vortex lines are the only excited degrees of freedom whose motion is described by the Biot-Savart law (see, e.g., Ref. 1)

$$\dot{\mathbf{s}} = \frac{\kappa}{4\pi} \int (\mathbf{s}_0 - \mathbf{s}) \times d\mathbf{s}_0 / |\mathbf{s}_0 - \mathbf{s}|^3 \; . \tag{1}$$

Here $\mathbf{s} = \mathbf{s}(\xi, t)$ describes the position of the vortex line at the instant t in a parametric form, $\mathbf{s}_0$ is the same as $\mathbf{s}$ but as an integration variable, and κ is the quantum of circulation. The dot denotes derivative with respect to time. The integral is over all of the lines. Equation (1) has two obvious constants of motion:

$$E = \iint ds\, ds_0 / |\mathbf{s} - \mathbf{s}_0| \tag{2}$$

and

$$\mathbf{P} = \int \mathbf{s} \times d\mathbf{s} \tag{3}$$

which are equal within a dimensional coefficient to the kinetic energy and the momentum of the fluid, respectively. The integrals in Eqs. (1) and (2) are divergent and imply a cutoff at $|\mathbf{s} - \mathbf{s}_0| \sim a_0$.

In most cases of interest Eq. (1) may be replaced by a much more simple equation, so-called localized-induction approximation[10] (see also Ref. 11)

$$\dot{\mathbf{s}} = \beta \mathbf{s}' \times \mathbf{s}'' \; , \tag{4}$$

where $\beta = (\kappa/4\pi)\ln(R/a_0)$, and R is some typical curvature radius which is treated as a constant. Equation (4) implies that parameter ξ is chosen to be the arc length. Primes denote derivatives with respect to ξ. Corrections to Eq. (4) are small in the parameter $1/\ln(R/a_0)$. Equation (4) exactly conserves $\mathbf{P}$ while the constant of motion analogous to E now is the total line length, L.[11]

Conservation of E and $\mathbf{P}$ prohibits successive decay of a vortex ring into rings of smaller radii. Indeed, suppose first that the shape of the ring is close to a circle. Then the absolute value of $\mathbf{P}$ is close to the circle's area. Since the circle has the minimal possible length at a given area and the latter is a conserving quantity, a considerable deformation of the ring necessary for its self-crossing and decay into two smaller ones would mean a considerable increase of its length L. But this is inconsistent with the conservation of E. [From (2) one can estimate $E \approx 2L\ln(L/a_0)$, $L \gg a_0$.] If the initial ring is curved enough and its self-crossing is not in contradiction with Eqs. (2) and (3) the cascade of decays nevertheless should cease at some step, provided $\mathbf{P}$ is nonzero. This is seen from the scaling argument. Let L_m be the length of the largest ring at some stage of the decay cascade. Then from conservation of E and $\mathbf{P}$ it immediately follows that $L_m \geq P/L$ [here we assume $\ln(L/a_0) \sim \ln(L_m/a_0)$].

Another set of the restrictions follows from Eq. (4). As was shown by Betchov,[12] transformation of Eq. (4) to the intrinsic variables (curvature ζ and torsion τ) reveals two additional invariants. The most refined form of the intrinsic equation was proposed by Hasimoto[13] who found that Eq. (4) is equivalent to the nonlinear Schrödinger equation (time is measured in the units of β^{-1})

$$i\frac{\partial\psi}{\partial t} = -\frac{\partial^2\psi}{\partial\xi^2} - \frac{1}{2}|\psi|^2\psi \;, \tag{5}$$

where ζ and τ are related to $\psi(\xi,t)$ by

$$\zeta = |\psi| \;, \quad \tau = \frac{\partial\Phi}{\partial\xi} \;. \tag{6}$$

(Φ is the phase of ψ.) The nonlinear Schrödinger equation is known to have an infinite set of the constants of motion (see, e.g., Ref. 14). This makes any nontrivial relaxation kinetics impossible without participation of reconnections.

II. RELAXATION OF THE VORTEX ANGLE

Consider the following self-similar solution of the intrinsic equation of vortex-line motion (A is an arbitrary constant, time is measured in the units of β^{-1})

$$\zeta = A/\sqrt{t} \;, \tag{7}$$

$$\tau = \frac{1}{2}\xi/t \;. \tag{8}$$

In terms of Eq. (5) [see (6)] this is nothing else than the well-known self-similar solution of the Schrödinger equation. It is easy to see that Eqs. (7) and (8) describe the evolution of a vortex angle started at $t=0$, the parameter A being related to the value of the angle, γ. Indeed, at $\xi \to \pm\infty$ the curve (7)-(8) approaches a straight line with a small amplitude helical structure. Taking the Z axis along this line and choosing parametrization $x = x(z)$, $y = y(z)$, the asymptotic form of the curve (7)-(8) is ($\xi \to +\infty$ for definiteness)

$$x(z,t) + iy(z,t) = (4At^{3/2}/z^2)\exp(iz^2/4t) \;, \quad z \gg \sqrt{t} \;. \tag{9}$$

In the region $|\xi| \sim \sqrt{t}$ the spatial form of the curve can be restored from (7)-(8) numerically by Frenet-Seret formulas. The resulting shape is in qualitative agreement with that observed in the numerical study of the anglelike vortex filament evolution.[7]

The process of setting up self-similar regime (7)-(8) after a reconnection of two vortex filaments (which at small enough scales of distance may be treated as straight lines) should occur very rapidly: with a speed on the order of the velocity of a Kelvin wave with wave vector $\sim a_0^{-1}$.

With Eqs. (7) and (8) it can be directly seen how the reconnections remove the restrictions imposed by Eq. (5) to the free motion of the vortex lines. Consider a set of constants of motion of Eq. (5), given by the recurrence relations[14]

$$I_n = \int_{-\infty}^{\infty} \varphi_n(\xi)d\xi \;,$$
$$\varphi_1 = \frac{1}{4}|\psi|^2 \;, \tag{10}$$
$$\varphi_{n+1} = \psi\frac{d}{d\xi}(\varphi_n/\psi) + \sum_{n_1+n_2=n} \varphi_{n_1}\varphi_{n_2} \;.$$

For the solution (7)-(8) at $|\xi| \to \infty$ we have $\varphi_n \sim \xi^{n-1}$, so the integrals I_n are divergent. Hence conservation of I_n's is strongly violated even by one reconnection.

As is seen from (9), the process of vortex angle relaxation leads to Kelvin wave generation. At a time $t = t_*$ at $z \sim \sqrt{t_*}$ (which is of the order of the curvature radius at the curve's vertex) a helical Kelvin-wave structure of the wavelength $\sim \sqrt{t_*}$ is formed in a self-similar fashion (cf. Ref. 7). This structure moves along the Z axis away from the region $z \sim \sqrt{t_*}$ with the velocity $\sim 1/\sqrt{t_*}$, carrying an excessive line length $\sim \sqrt{t_*}$. Hence, upon relaxation of a vortex angle formed due to a reconnection of two lines of characteristic curvature radius R, a piece of line with the length $\sim R$ turns out to be transferred as a Kelvin wave to a larger scale of curvature, adjacent to R^{-1}.

Note that Kelvin-wave generation accompanying the evolution of any anglelike vortex configuration is an inevitable consequence of the conservation of I_n's rather than a sort of instability, as it was though in Ref. 7. Taking into account the increasing characteristic curvature radius it is easy to estimate that the conservation of, say, I_1 would be impossible without emission of Kelvin waves.

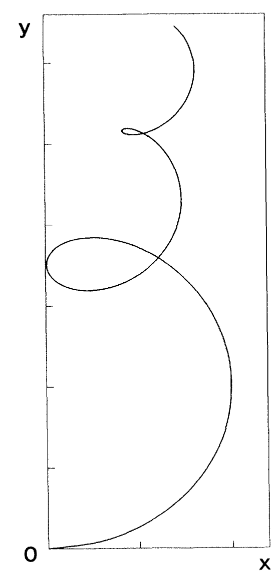

FIG. 1. The shape of the relaxing vortex angle (projection on the $z = 0$ plane) at the critical value of $\gamma \approx 0.152$. The origin is at the $\xi = 0$ point (vertex of the curve), the X, Y, and Z axes are taken along the tangent, normal, and binormal vectors at the vertex, respectively. The line is symmetric with respect to $x = 0$ plane, so its contact with this plane means self-intersection.

It is interesting to trace the variation of the shape of the curve (7)-(8) with γ. The limit $A \to 0$ obviously corresponds to $\gamma = \pi$. Increasing of A means decreasing γ. From (9) it thus follows that decreasing γ leads to the increasing of the amplitude of the helical wave structure. At the critical angle $\gamma_* \approx 0.152$ (obtained numerically) the self-contact arises (see Fig. 1), and the solution (7)-(8) is no longer relevant. Physically this should mean that the evolution of the vortex angle with $\gamma \le \gamma_*$ will lead to the generation of a cascade of vortex rings. This process, however, can hardly have a considerable effect on the kinetics of superfluid turbulence as the critical angle is rather small.

III. QUANTITATIVE CONSIDERATIONS

In this section we introduce and estimate some important characteristics of the vortex tangle under the conditions of KR. Superfluid turbulence in the case of not so small friction coefficient α can be quantitatively characterized either by the interline spacing, R_0, or by the line-length density, L, both parameters being easily related to each other: $L \sim 1/R_0^2$. In the low-temperature case, however, the situation is different. Now we deal with the hierarchy of scales in the Kelvin-wave-length space. As we shall see, contributions to L come from different scales of wavelength, so the most complete characteristic of the tangle is $\tilde{L}(\lambda_0)$, which is the same as L, but with all short-wave line structure smoothed out up to some scale λ_0. In particular, $\tilde{L}(\lambda_0 \to 0) = L$. The function $\tilde{L}(\lambda_0)$ thereby describes the distribution of the line length over the scales of wavelength of the Kelvin wave structure. It obeys the relation

$$\ln \tilde{L}(\lambda_0) = \ln(1/R_0^2) + \int_{\lambda_0}^{R_0} (\rho_\lambda/\lambda)^2 d\lambda/\lambda \,, \tag{11}$$

where ρ_λ is the effective amplitude of the Kelvin wave structure at the wavelength λ.

Kinetics of the tangle are characterized by the number of self-intersections at the scale λ per unit time in the unit volume, N_λ. (A self-intersection at the scale λ leads to a creation and subsequent absorption of a ring with the radius $\sim \lambda$.) N_λ can be estimated as

$$N_\lambda \sim \Omega(\lambda)[\tilde{L}(\lambda)/\lambda]\,\omega_\lambda \,, \tag{12}$$

where $\Omega(\lambda)$ is the probability that (at a given point and a given instant) the amplitude of the Kelvin wave structure of the wavelength $\sim \lambda$ is also of the order of λ (which is necessary for a self-intersection to occur), $\omega_\lambda \sim \beta/\lambda^2$ is the frequency of a Kelvin wave with wavelength $\sim \lambda$ and hence is the inverse correlation time for a turbulent state of the Kelvin waves at the scale λ, and the factor $\tilde{L}(\lambda)/\lambda$ comes from the fact that the characteristic correlation length at the wavelength scale λ is $\sim \lambda$. For the estimation of $\Omega(\lambda)$ we use random phase approximation which leads to a Gaussian distribution for the amplitude of the Kelvin wave turbulence at a given point and thus yields

$$\Omega(\lambda) \sim (\lambda/\rho_\lambda) \exp\left[-(\lambda/\rho_\lambda)^2\right] \,. \tag{13}$$

Now we introduce smoothed-line-length flux in the wavelength space at the wavelength scale λ in a unit volume, $Q_\lambda(\lambda_0)$ [where λ_0 has the same meaning as in $\tilde{L}(\lambda_0)$]. By definition, for a (quasi-)steady-state Kolmogorov regime under consideration this quantity should have the same value for any λ, so we omit the subscript λ from now on. $Q(\lambda_0)$ may be related to N_λ:

$$Q(\lambda_0) \sim N_\lambda l_\lambda(\lambda_0) \,, \tag{14}$$

where $l_\lambda(\lambda_0)$ is a characteristic smoothed out up to the scale λ_0 (in above-mentioned sense) length of a vortex ring of the radius $\sim \lambda$.[15]

In analogy with (11),

$$\ln l_\lambda(\lambda_0) \sim \ln \lambda + \int_{\lambda_0}^{\lambda} (\rho_{\lambda_1}/\lambda_1)^2 d\lambda_1/\lambda_1 \,. \tag{15}$$

Combining (12), (13), and (14) and noticing from (11) and (15) that $\tilde{L}(\lambda) l_\lambda(\lambda_0) \sim \tilde{L}(\lambda_0)\lambda$, we obtain

$$(\beta/\lambda\rho_\lambda) \exp\left[-(\lambda/\rho_\lambda)^2\right] \sim Q(\lambda_0)/\tilde{L}(\lambda_0) \,. \tag{16}$$

The left-hand side of Eq. (16) formally depends only on λ while the right-hand side depends only on λ_0. This means that actually they are independent of λ and λ_0. Their order of magnitude can be estimated setting in Eq. (16) $\lambda \sim R_0$, where $\rho_\lambda \sim \lambda \sim R_0$. This yields

$$(\rho_\lambda/\lambda)^2 \sim [1 + \ln(R_0/\lambda)]^{-1} \,. \tag{17}$$

Substituting this into (11), we obtain

$$\tilde{L}(\lambda_0) \sim R_0^{-2} [1 + \ln(R_0/\lambda_0)]^\nu \,, \tag{18}$$

where ν is some constant of the order unity which cannot be found from our simple estimates. It is seen that despite the fact that the relative amplitude of the Kelvin wave structure, ρ_λ/λ, decreases with λ, all scales of wavelength contribute to the total line length, so that $\tilde{L}(\lambda_0)$ diverges at $\lambda_0 \to 0$.

To extract from (18) the full line-length density L one should introduce the parameter λ_*, the lowest value of λ_0 at which the above considerations are relevant. Then

$$L \sim R_0^{-2} [\ln(R_0/\lambda_*)]^\nu \,. \tag{19}$$

At not so small temperatures λ_* is the scale of wavelength where dissipative line-length loss becomes comparable to the Kolmogorov line-length flux: $\lambda_* \sim R_0\sqrt{\alpha}$. At $T \to 0$ it might seem reasonable to set $\lambda_* = a_0$. However, nonlocal effects neglected in our considerations may change the picture of KR at smaller scales and render λ_* much greater than a_0 (see the remark in Sec. V).

In conclusion of this section we write the equation describing the free decay of superfluid turbulence under the conditions of KR which readily follows from the above considerations. This equation has the most simple form if expressed in terms of R_0:

$$\dot{R}_0 \sim \beta/R_0 \tag{20}$$

[where to a good approximation $\beta = (\kappa/4\pi)\ln(R_0/a_0)$]. Being written in terms of L it takes on the Vinen-like form.[5] In particular, at not so small α's when $\lambda_* \sim R_0\sqrt{\alpha}$ it reads

$$\dot{L} \sim -\frac{\beta}{[\ln(1/\sqrt{\alpha})]^{\nu}} L^2 . \tag{21}$$

Note that in contrast to the description of the tangle relaxation in terms of R_0, Eq. (21) contains large (especially if ν turns out to be not small as compared to unity) α-dependent denominator. Clearly, this is no more than a reflection of the fractalization of the vortex lines. Nevertheless, if one watches the free decay of the low-temperature superfluid turbulence by measuring the quantities related to L rather than to R_0, which is actually the case in most of the experiments, he will see an effective deceleration of the process as compared to the case of $\alpha \sim 1$.

IV. NUMERICAL SIMULATION

Now we derive a Hamiltonian equation of motion for a vortex line which will turn out to be very convenient for the numerical simulation of KR. Suppose that a vortex line can be parametrized as $x = x(z)$, $y = y(z)$, where x, y, and z are Cartesian coordinates, the functions $x(z)$ and $y(z)$ being single valued. Then the position of the line can be unambiguously described by the two-dimensional vector $\boldsymbol{\rho}(z,t) = [x(z,t), y(z,t)]$. Note that in contrast to the earlier-defined vector $\mathbf{s}(\xi,t)$ [see Eq. (1)] which at a given ξ follows the real motion of the corresponding element of liquid containing the vortex core (in accordance with Kelvin's theorem), vector $\boldsymbol{\rho}(z_0,t)$ determines only the geometrical point of intersection of the vortex line with the plane $z = z_0$ at the instant t. Purely geometrically it is seen that

$$\dot{\boldsymbol{\rho}} = \dot{\mathbf{s}} - (\dot{\mathbf{s}}\hat{z})(\hat{z} + \boldsymbol{\rho}') , \tag{22}$$

where $\hat{z}$ is a unit vector along Z axis. From now on a prime denotes the derivative with respect to z. Substituting for $\dot{\mathbf{s}}$ the right-hand side (rhs) of either Eq. (1) (for the exact description) or the rhs of Eq. (4) (for the localized-induction approximation), expressing then $\mathbf{s}$ in terms of $\boldsymbol{\rho}$, and introducing the variable $w(z,t) = x(z,t) + iy(z,t)$, we get

$$i\dot{w} = \frac{\delta H[w]}{\delta w^*} , \tag{23}$$

where the Hamiltonian functional $H[w]$ is equal either to $(\kappa/4\pi)E$ (exact description) or to $2\beta L$ (localized-induction approximation) expressed as a functional of w:

$$H_{\text{exact}} = \frac{\kappa}{4\pi} \int\int dz_1 dz_2 \frac{1 + [w'(z_1)w^{*\prime}(z_2) + \text{c.c.}]/2}{\sqrt{(z_1 - z_2)^2 + |w(z_1) - w(z_2)|^2}} , \tag{24}$$

$$H_{\text{local}} = 2\beta \int dz \sqrt{1 + |w'(z)|^2} . \tag{25}$$

Equation (23) is exact until the genuine function $w(z)$ becomes non single valued. Since $w(z)$ in (23) is single-valued by definition, it is clear that if as a result of evolution the true $w(z)$ becomes many valued, the solution of Eq. (23) should demonstrate some peculiarity. Numerical analysis of Eq. (23) with the local Hamiltonian (25) shows that this peculiarity is a jump at some point inside the region of multivaluedness of the true $w(z)$. Once emerged, the jump lives some time and then relaxes, the picture of the relaxation being qualitatively reminiscent of the process of a vortex angle evolution considered in Sec. II: the relaxation is accompanied by emission of Kelvin waves (see Fig. 2).

Hence, in the case of strong nonlinearity, Eq. (23) generally speaking is not adequate because of the jumps. However, there is an argument to believe that it is essentially relevant for a qualitative description of superfluid turbulence with self-intersections taken into account automatically. Indeed, consider a discrete analog of the model (23), (25) (here and after we deal only with the localized-induction approximation):

$$i\dot{w}_n = \frac{\partial H}{\partial w_n^*} , \quad H = \sum_{n=0}^{N-1} \sqrt{1 + |w_{n+1} - w_n|^2} . \tag{26}$$

(Periodic boundary conditions are assumed: $w_N = w_0$.) In this model the length of the broken line defined by the points $(x_0, y_0), (x_1, y_1), \ldots, (x_N, y_N)$ ($w_n = x_n + iy_n$, n plays the role of z coordinate) is an exact constant of motion equal to the value of the Hamiltonian function H.

The jumps therefore do not lead to the loss of the line length. But they should contribute to the line-length transport in the curvature space due to the mechanism of Kelvin wave emission. Qualitatively, the kinetic role of a jump in the model (26) is equivalent to that of a vortex ring (in the real picture) created as a result of a self-intersection and then absorbed by the vortex tangle. Moreover, within an order-of-magnitude accuracy this analogy is quantitative: the amplitude of the jump

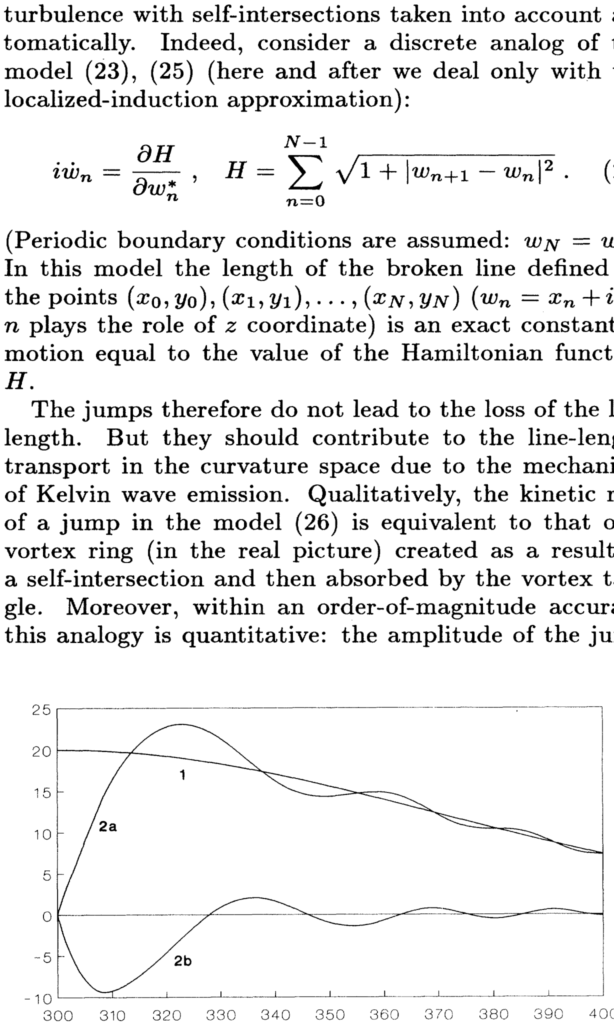

FIG. 2. Relaxation of a jump in w. Evolution is governed by Eqs. (26) with the initial condition $\text{Im}w_n = 0$, and $\text{Re}w_n = 20\,\text{sgn}(\chi_n)e^{-\chi_n^2}$, where $\chi_n = 0.001(n - N/2)$, and total number of points $N = 600$. Curve 1 is $\text{Re}w$ at the initial moment $t = 0$. Curves 2a and 2b are $\text{Re}w$ and $\text{Im}w$, respectively, at the instant $t = 300$. Abscissa axis corresponds to n. The results are represented by solid lines to emphasize the smoothness.

corresponds to the radius of the ring.

The model (26) may be used for a simplified numerical simulation of superfluid turbulence. Its apparent advantage is that the jumps occur and disappear as a result of self-evolution, while in a strict model self-intersections should be specially revealed and processed,[2] which requires a large amount of CPU time.

Our numerical procedure was arranged as follows. The self-evolution of $N = 600$ complex numbers $\{w_n\}$ was governed for a certain period by the equations (26) solved by Runge-Kutta method. Then a Fourier analysis was made:

$$q_m = \sum_{n=0}^{N-1} e^{-i\frac{2\pi mn}{N}} w_n \, , \tag{27}$$

and the $\{w_n\}$ were modified in such a way that slow ($|m| \le m_{\rm slow}$) and fast ($|m| \ge m_{\rm fast}$) harmonics were removed. After this the $\{w_n\}$ were "pumped" with additional random phase harmonics in the region $m_{\rm slow} < |m| \le m_{\rm pump}$. (We used $m_{\rm slow} = 8$, $m_{\rm pump} = 20$, and $m_{\rm fast} = 150$.) Then the whole procedure was repeated, and so on. The idea is that after some transient period a steady-state regime should set in (provided the intervals between the pumpings are not too large), and one may expect that there will take place one of the following alternatives. (i) If KR is impossible for this or that reason, almost all additional line length injected by the pumping will be removed by the elimination of slow harmonics. (ii) In the case of KR a considerable part of line length will pass in the momentum space through the inertial region $m_{\rm pump} < |m| < m_{\rm fast}$ and disappear as a

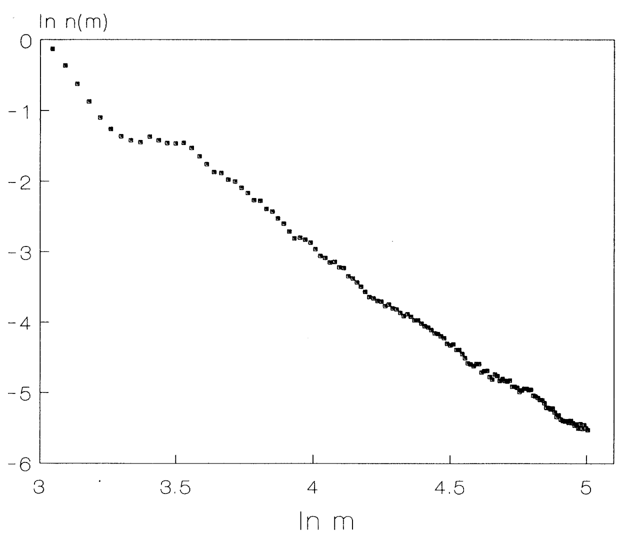

FIG. 3. Fourier-harmonic distribution in the inertial region ($20 < |m| < 150$) of the steady-state Kolmogorov regime for the numerical model. Here $n(m) = \frac{1}{2}\left[\langle |q_m|^2\rangle + \langle |q_{-m}|^2\rangle\right]$, averaging being performed over 360 arbitrarily selected instants. Apart from the region $m \le 33$ where the picture is affected presumably due to the proximity to the pumping region, the plot is close to a straight line with the slope 3. This is just what follows from the considerations of Sec. III, neglecting logarithmically slow corrections [see Eq. (17)].

result of elimination of fast harmonics. Thus, comparing the rates of line-length decrease in the $|m| \le m_{\rm slow}$ and $|m| \ge m_{\rm fast}$ regions, it is possible to determine whether or not KR takes place.

Our simulation clearly demonstrated KR. To make sure that the scenario of KR is that described above, we examined $\{w_n\}$ in the process of evolution to reveal the jumps. We found out that the jumps really occurred from time to time. We also studied the statistics of the Fourier harmonics in the inertial region and found it to be consistent with our scenario (see Fig. 3).

V. CONCLUDING REMARKS

We have considered superfluid turbulence in the localized-induction approximation. A question thus may arise of the effect of the nonlocal interaction on the picture of KR. Since the nonlocal interaction is small in the parameter $1/\ln(R_0/a_0)$ this effect is considerable only if nonlocal terms lead to some qualitative difference in the process of the vortex-tangle evolution. This difference may take place if the above-discussed restrictions to the self-evolution of the Kelvin waves follow essentially from the localized-induction approximation and are removed by nonlocal terms. Anyway, it can be estimated that nonlocal interaction may be neglected at least for the scales of wavelength $\lambda > R_0/\ln(R_0/a_0)$, where the line-length flux supported by nonlocal terms (if it exists) is smaller than $Q(\lambda)$. We also notice that the effect of nonlocal interaction may in principle be studied numerically within the approach of Sec. IV. Equations (26) in this case should be replaced by nonlocal ones arising from the discrete analog of the Hamiltonian (24).

Considering superfluid turbulence in the low-temperature region we cannot leave without a discussion the results of Milliken, Schwarz, and Smith (MSS).[8] MSS studied experimentally the free decay of superfluid turbulence in He^4 in the temperature range over which friction constant α varied by a factor of 2 (from ≈ 0.05 to ≈ 0.1). No temperature dependence of the decay rate was observed. MSS also presented theoretical considerations accounting for this circumstance. These considerations are essentially different from those discussed in the present paper. Namely, MSS assume that the vortex tangle may be characterized only by two scales of distance: a typical interline separation R_0 and a typical curvature radius R ($R \ll R_0$ at $\alpha \ll 1$). Their scenario thus cannot be regarded as a Kolmogorov cascade since it does not imply a transport through a hierarchy of scales. However, in Ref. 9 it was pointed out that the theoretical considerations of MSS contain an error: MSS suggest that the kinetics of the tangle are determined by the local velocity corresponding to the curvature radius R, while it is obvious that this velocity characterizes only the precession of the kinks around the average position of the filament, and the velocity of the line element of the size R_0 as a whole is determined by the global radius of curvature $\sim R_0$ (see also Ref. 15). With this fact taken into account the treatment of Ref. 9 which is similar to that of MSS in the sense that it also assumes that the Kelvin wave structure of the vortex lines can be characterized by only the parameter R leads to the result that the decay rate of the superfluid turbulence is $\propto \alpha^{1/3}$. Hence an essentially dissipative scenario of Refs. 8 and 9 yields very weak dependence on α. In this connection a question arises whether the absence of temperature dependence in Ref. 8 indicates the onset of KR at $\alpha < 0.1$ or this is just the consequence of the very weak dependence on α which cannot be revealed within a rather small temperature interval because of the limited experimental accuracy. The numerical simulation of the superfluid turbulence in a steady-state regime[2] which is in very good agreement with experiments suggests that the dependence of the decay rate on α persists at least down to $\alpha \sim 0.01$ (and is rather close to that found in Ref. 9). Therefore it is likely that KR was not reached in Ref. 8.

It seems very reasonable to reproduce the experiment of MSS at lower temperatures to study the case of very small α's and to reveal unambiguously the onset of KR. Its open geometry may allow detection of the most characteristic feature of KR: the emission of vortex rings by the relaxing vortex tangle.

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[15] We take into account the fact that the short-wavelength structure does not change the global velocity of motion of a vortex arc, which can be illustrated in the following way. Consider a plane vortex ring formed by two arcs, the first one being smooth and the second one covered with a short-wavelength structure. Then to a good approximation, the momentum (3) is perpendicular to the plane of the ring since the net contribution of the wave structure is close to zero. If the two arcs had essentially different velocities, the momentum would acquire a component tangent to the plane of the initial ring. But this contradicts the momentum conservation.