Superfluid Rayleigh criterion

Carlo F. Barenghi
Division of Applied Mathematics, Department of Mathematics and Statistics, University of Newcastle upon Tyne, Newcastle upon Tyne, NE1 7RU, England
(Received 31 March 1995)

The stability of the flow of superfluid helium II between two corotating cylinders is investigated. The result is compared with the corresponding classical flow of helium I. The validity of Rayleigh's stability criterion and the relation between a superfluid and a classical inviscid fluid are discussed.

I. INTRODUCTION

One of the most classical problems of fluid dynamics is the determination of the stability of the motion of a liquid between rotating concentric cylinders (Couette flow). This problem was first studied by Lord Rayleigh, who, using simple physical arguments, derived the celebrated *Rayleigh criterion.*[1] The great importance of the Rayleigh criterion is that it provides simple physical insight into the stability of a rotating flow. The criterion states that the necessary and sufficient condition for the stability of axisymmetric disturbances is that the square of the circulation does not decrease anywhere. The viscosity of the fluid, which was neglected in Rayleigh's original argument, was later taken into account by Taylor.[2] Taylor investigated the stability of Couette flow with respect to the growth of infinitesimal disturbances. He found that when the outer cylinder is held fixed, there exists a finite range of rotation velocities of the inner cylinder in which viscosity has a stabilizing effect. Taylor's analysis showed also that when the two cylinders rotate fast in the same direction, a situation referred to as corotation, the viscous correction is small and the stability boundary becomes asymptotically close to Rayleigh's line. This second result vindicates the power of the simple, inviscid argument of Rayleigh.

The aim of this paper is to investigate what happens to the Rayleigh criterion if the fluid contained between the cylinders is superfluid helium (helium II) rather than a classical Navier-Stokes fluid, such as helium I. It was Chandrasekhar and Donnelly[3,4] in the late 1950's who first recognized the importance of the helium Couette problem. The motivations were the interest in the quantized vortex lines which are present when helium rotates, the issue of establishing the equations of motion of helium II, and the need to understand the operation of a Couette viscometer. However, contact between theory and experiments has happened only recently, when the measurements of Swanson and Donnelly[5] confirmed the theoretical predictions of Barenghi and Jones[6] and Barenghi[7] of the critical angular velocity of the inner cylinder at which Couette flow becomes unstable and Taylor vortex flow appears. The strong temperature dependence of the transition was also observed by Bielert and Stamm.[8] These results led Henderson, Barenghi, and Jones[9] and then Henderson and Barenghi[10] to investigate the nonlinear development of the flow above the transition. The latter computed the torque which helium exerts on the cylinders in the nonlinear Taylor vortex flow regime and compared it successfully with existing measurements. All these investigations refer to the simple case in which the outer cylinder is held stationary. The case in which the cylinders counter-rotate is being investigated.[11] What happens when the outer cylinder is allowed to rotate in the same direction of the inner cylinder is the subject of the present study.

II. THE MODEL AND THE EQUATIONS

Liquid helium at temperature T is contained between two concentric cylinders of inner radius R_1 and outer radius R_2, which rotate at constant angular velocities Ω_1 and Ω_2. The usual simplifying assumption[1] is made that the cylinders have infinite length. The calculations reported in this paper refer to the radius ratio $R_1/R_2=\eta=0.976\,28$ and the gap width $R_2-R_1=\delta=0.047$ cm of the helium Couette apparatus at the University of Oregon. The Reynolds numbers $\mathrm{Re}_1=\Omega_1 R_1\delta/\nu_n$ and $\mathrm{Re}_2=\Omega_2 R_2\delta/\nu_n$ are defined in terms of the kinematic viscosity ν_n of the normal fluid. If $T=T_\lambda$, then ν_n becomes equal to the kinematic viscosity of helium I, which is a classical Navier-Stokes liquid, and the more usual definitions of Re_1 and Re_2 are recovered.

The equations which govern the incompressible flow of helium II are[12]

$$\frac{\partial \mathbf{v}_n}{\partial t}+(\mathbf{v}_n\cdot\nabla)\mathbf{v}_n=-\frac{1}{\rho}\nabla p_n+\nu_n\nabla^2\mathbf{v}_n+\frac{\rho_s}{\rho}\mathbf{F}\ , \tag{1}$$

$$\frac{\partial \mathbf{v}_s}{\partial t}+(\mathbf{v}_s\cdot\nabla)\mathbf{v}_s=-\frac{1}{\rho}\nabla p_s-\nu_s\mathbf{T}-\frac{\rho_n}{\rho}\mathbf{F}\ , \tag{2}$$

$$\nabla\cdot\mathbf{v}_n=0\ , \tag{3}$$

$$\nabla\cdot\mathbf{v}_s=0\ , \tag{4}$$

where $\mathbf{v}_n$ and $\mathbf{v}_s$ are the normal fluid and superfluid velocities, ρ_n and ρ_s are the normal fluid and superfluid densities, $\rho=\rho_n+\rho_s$ is the total density of helium, $\boldsymbol{\omega}_s=\nabla\times\mathbf{v}_s$ is the superfluid vorticity, $\hat{\boldsymbol{\omega}}_s=\boldsymbol{\omega}_s/|\boldsymbol{\omega}_s|$, p_n and p_s are effective pressures, $\nu_s=(\Gamma/4\pi)\ln(b_0/a_0)$ is the vortex tension parameter, a_0 is the vortex core radius, b_0 is the intervortex spacing, $\Gamma=h/m=0.997\times10^{-3}$ cm^2/sec is the quantum of circulation, h is Plank's constant, and m is the mass of the helium atom. The mutual friction force is[13]

$$\mathbf{F}=\frac{B}{2}\hat{\boldsymbol{\omega}}_s\times[\boldsymbol{\omega}_s\times(\mathbf{v}_n-\mathbf{v}_s-\nu_s\nabla\times\hat{\boldsymbol{\omega}}_s)]+\frac{B'}{2}\boldsymbol{\omega}_s\times(\mathbf{v}_n-\mathbf{v}_s-\nu_s\nabla\times\hat{\boldsymbol{\omega}}_s)\ , \qquad (5)$$

where B and B' are the mutual friction coefficients, and the vortex tension force $-\nu_s\mathbf{T}$ is given by $\mathbf{T}=\boldsymbol{\omega}_s\times(\nabla\times\hat{\boldsymbol{\omega}}_s)$.

Equations (1) and (2) have three interesting limits. First, if $T\to T_\lambda$, then $\rho_s\to 0$ and the normal fluid equation (1) reduces to the classical Navier-Stokes equation for a viscous fluid:

$$\frac{\partial\mathbf{v}}{\partial t}+(\mathbf{v}\cdot\nabla)\mathbf{v}=-\frac{1}{\rho}\nabla p+\nu\nabla^2\mathbf{v}\ . \qquad (6)$$

Second, if $T\to 0$ then $\rho_n\to 0$ and Eq. (2) describes a pure superflow:

$$\frac{\partial\mathbf{v}}{\partial t}+(\mathbf{v}\cdot\nabla)\mathbf{v}=-\frac{1}{\rho}\nabla p-\nu_s\boldsymbol{\omega}\times(\nabla\times\hat{\boldsymbol{\omega}})\ . \qquad (7)$$

Finally, if Plank's constant is set equal to zero, then $\nu_s=0$ and Eq. (7) reduces to the classical Euler equation for an inviscid fluid:

$$\frac{\partial\mathbf{v}}{\partial t}+(\mathbf{v}\cdot\nabla)\mathbf{v}=-\frac{1}{\rho}\nabla p\ . \qquad (8)$$

The Rayleigh inviscid stability criterion applies to Eq. (8) and states that a stratification of angular momentum about the axis is stable if and only if it increases monotonically outwards, that is to say, if $\Omega_1R_1^2<\Omega_2R_2^2$. This condition, expressed by the Rayleigh line $\mathrm{Re}_1=\mathrm{Re}_2/\eta$ showed in Fig. 1, divides the Re_1 vs Re_2 plane into a lower stable region and an upper unstable one.

To take viscosity into account one must study the Navier-Stokes equation (6) rather than the Euler equation (8). Equation (6), which describes the motion of a classical viscous fluid such as helium I, admits the Couette solution $\mathbf{v}=V(r)\hat{\mathbf{e}}_\phi=(Ar+B/r)\hat{\mathbf{e}}_\phi$, where $\hat{\mathbf{e}}_\phi$ is the unit

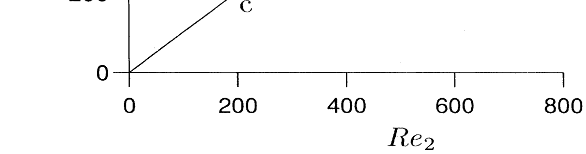

FIG. 1. Regions of stable and unstable flow in the Re_1 vs Re_2 plane: a, stability boundary of helium II, b, the Taylor curve (helium I), and c, the Rayleigh line.

vector in the azimuthal direction and the parameters $A=-\Omega_1\eta^2(1-\mu/\eta^2)/(1-\eta^2)$ and $B=\Omega_1R_1^2(1-\mu)/(1-\eta^2)$, where $\mu=\Omega_2/\Omega_1$, are chosen to satisfy the noslip boundary conditions $v_\phi\ (r=R_1)=\Omega_1R_1$ and $v_\phi\ (r=R_2)=\Omega_2R_2$. The analysis of the stability of the Couette solution with respect to both axisymmetric and nonaxisymmetric perturbations results in the Taylor stability boundary showed in Fig. 1. The most significant difference between the Taylor curve and the Rayleigh line is that when the outer cylinder is fixed there is a nonzero critical velocity, $\mathrm{Re}_{1c}=268$, at this value of radius ratio.

In the case of helium II, the stability is determined by the full set of equations (1) and (2). The basic state is assumed to be Couette flow again, $\mathbf{v}_n=\mathbf{v}_s=V(r)\hat{\mathbf{e}}_\phi=(Ar+B/r)\hat{\mathbf{e}}_\phi$, which corresponds to a uniform array of vortices aligned along the axis of rotation with areal density $n_0=2|A|/\Gamma$. If it is only the inner cylinder which rotates, the assumption that the basic state is Couette flow is justified by the experimental observation[5] that the attenuation of second sound is proportional to Ω_1. The case in which both cylinders rotate together has not been studied experimentally yet, but a similar assumption is mathematically convenient and physically plausible. Nevertheless, one should be careful and understand the approximations involved in this assumption. First, it neglects the existence of a vortex free strip[14] near the walls. Second, it neglects the possible existence of remnant vortex lines,[15] which can be created when helium is cooled through the λ point and then be present between the cylinders when still at rest. The Couette flow assumption is therefore valid only if the remnant vortices have very small density or if they realign along the axis of rotation when the cylinders are spun up for the first time. Third, the Couette state is only an approximate solution because the finite height of the apparatus must necessarily induce a weak vertical circulation. This limitation exists in both the helium II and the classical Couette problem. Finally, it is important to note that if the rotations of the cylinders are such that A is exactly zero, then $n_0=0$ and there are no vortices in the system. The superfluid is in a state of irrotational motion corresponding to a virtual vortex on the axis having strength $N\Gamma$, where N is the closest integer to $2\pi B$. For the theory to be valid A must be either slightly positive or negative, corresponding to vortices oriented along the $+z$ or $-z$ direction.

The equations, written in cylindrical coordinates r,ϕ,z, are made dimensionless using the length scale δ and the time scale δ^2/ν_n. The stability of the Couette state is then analyzed with respect to infinitesimal perturbations of the form $\exp(ikz+im\phi+ipt)$, where k and m are the axial and azimuthal wave numbers and p is complex. If the growth rate $\sigma=-\mathrm{Im}(p)>0$, then the Couette state is unstable. The linearized equations for the perturbations are an eighth-order eigenvalue problem for p. The problem is solved at given η, δ, Re_2, k, and m for the value of Re_1 at which $\sigma=0$. The wave numbers k and m are then varied and the lowest value of Re_1 found, Re_{1c}, defines the state of marginal stability. The numerical technique used in the calculations is based on Chebyshev spectral expansions and has already been described.[6] The calculations are performed at $T = 2.16$ K, because it is in the high-temperature regime that contact between theory and experiments has been achieved for rotations of the inner cylinder only.

III. NUMERICAL RESULTS

The result of the calculation of the marginal states is the top curve in Fig. 1, which must be compared with the Taylor curve and the Rayleigh line. Within the range of parameters investigated, the marginal states are determined by axisymmetric perturbations ($m = 0$) as in the classical Couette problem. The critical axial wave numbers range from $k = 2.4$ when $\mathrm{Re}_2 = 0$ to $k = 2.9$ when $\mathrm{Re}_2 = 900$, which must be compared to the value $k = 3.1$ of the Taylor curve. Since the height of the Oregon Couette apparatus is $h = 9.398$ cm, there are about 74 Taylor cells along the axis when the instability sets in, a number which is large enough to justify the use of the infinite cylinder approximation. It is evident from Fig. 1 that the corotating flow of helium II is always more stable than the corresponding flow of helium I. The difference between helium I and helium II is larger if Re_2 is smaller. If the outer cylinder is fixed then helium II's critical velocity is $\mathrm{Re}_{1c} = 313$, which must be compared with the value $\mathrm{Re}_{1c} = 268$ in helium I. When both cylinders rotate fast in the same direction, however, Fig. 1 shows that the stability curves of both helium I and helium II tend towards the Rayleigh line. The Rayleigh criterion is therefore valid for helium II as well.

IV. A SIMPLIFIED MODEL

To understand the physical meaning of this result, which is hidden by the numerical calculation, it is instructive to consider a simplified model of Eqs. (1) and (2) which can be solved analytically. The model is the following. Equations (1) and (2) are simplified by introducing the narrow gap limit that $\delta \ll R_1$, neglecting the effects of vortex tension in the mutual friction force and dropping the transverse component of mutual friction proportional to B'. Since the cylinders rotate rapidly in the same direction at almost the same speed, the Couette flow's angular velocity $\Omega = V/r$ is approximated by $\Omega \approx (\Omega_1 + \Omega_2)/2 = \Omega_0$. Finally, the no-slip boundary conditions of the normal fluid are replaced by stress-free conditions.[1,3] The advantage of these modified boundary conditions is that the eigenfunctions and all their derivatives are simply proportional to $\sin(m\pi x)$ for $m = 1, 2, \ldots$, where $x = (r - R_1)/\delta$. Experience of similar boundary value problems in fluid dynamics suggests that, although the modified boundary conditions alter the critical Reynolds number, the dependence on the other parameters of the problem is essentially the same. Assuming axisymmetric perturbations, one can eliminate the pressures p_n and p_s from the z component of (1) and (2), and the axial velocity components v_{nz} and v_{sz} from the continuity equations (3) and (4). Setting $p = 0$ (exchange of stabilities), the equations for the remaining perturbations v_{nr}, $v_{n\phi}$, v_{sr}, and $v_{s\phi}$ are

$$\frac{(m\pi^2 + k^2)^2}{k^2} v_{nr} = 2\Omega_0 v_{n\phi} - \alpha_s B|A|(v_{nr} - v_{sr}) , \quad (9)$$

$$-(m^2\pi^2 + k^2) v_{n\phi} = 2A v_{nr} + \alpha_s B|A|(v_{n\phi} - v_{s\phi}) , \quad (10)$$

$$0 = (2\Omega_0 + \beta\psi k^2) v_{s\phi} + \alpha_n B|A|(v_{nr} - v_{sr}) , \quad (11)$$

$$0 = [2A + \beta\psi(m^2\pi^2 + k^2)] v_{sr} - \alpha_n B|A|(v_{n\phi} - v_{s\phi}) , \quad (12)$$

where $\beta = \nu_s/\nu_n$, $\psi = A/|A|$, $\alpha_n = \rho_n/\rho$, and $\alpha_s = \rho_s/\rho$. Nontrivial solutions of the linear system (9), (10), (11), and (12) are determined by the roots of the characteristic equation

$$c_1\left[c_2 + \frac{c_3}{\mathrm{Ta}}\right] = \mathrm{Ta}\left[1 + \frac{c_4}{\mathrm{Ta}}\right]\left[c_5 + \frac{c_6}{\mathrm{Ta}}\right] , \quad (13)$$

where $\mathrm{Ta} = -4A\Omega_0$ is the Taylor number and the coefficients c_1, c_2, c_3, c_4, c_5, and c_6 are

$$c_1 = \frac{-1}{k^2}\left[(m^2\pi^2 + k^2)^2 + k^2 B|A| + \beta\frac{(m^2\pi^2 + k^2)^3}{2|A|} + \beta\frac{\alpha_s B|A|k^2}{2|A|}(m^2\pi^2 + k^2)\right] , \quad (14)$$

$$c_2 = -m^2\pi^2 - k^2 - B|A| , \quad (15)$$

$$c_3 = +2|A|\beta k^2(m^2\pi^2 + k^2 + \alpha_s B|A|) , \quad (16)$$

$$c_4 = \alpha_n B|A|\frac{(m^2\pi^2 + k^2)^2}{k^2} - 2|A|\beta k^2 , \quad (17)$$

$$c_5 = 1 + \beta\frac{(m^2\pi^2 + k^2)}{k^2} , \quad (18)$$

$$c_6 = \alpha_n B|A|(m^2\pi^2 + k^2) . \quad (19)$$

The quadratic equation (13) determines Ta as a function of $|A|$. The roots $\mathrm{Ta} = -Q \pm \sqrt{Q^2 - 4P}/2c_5$, where $Q = c_4c_5 + c_6 - c_1c_2$ and $P = c_5(c_4c_6 - c_1c_3)$, form two families Ta^- and Ta^+, corresponding to the choice of sign, where the index $m = 1, 2, \ldots$ labels the members of each family. The existence of two families of solutions in this problem was recognized by Chandrasekhar and Donnelly in their pioneering paper,[3] although at the time the exact form of the mutual friction force was not clear and the vortex tension was not known. Chandrasekhar and Donnelly argued that two separate families originate, respectively, from the Taylor instability of a classical viscous fluid (the normal fluid) and from the Rayleigh instability of an ideal fluid (the superfluid). To see this it suffices to consider Eq. (13) in the limit $T \to T_\lambda$ by setting $\beta = 0$, $B = 0$, $\alpha_s = 0$, and $\alpha_n = 1$. Then the two roots are $\mathrm{Ta}^+ = 0$ and $\mathrm{Ta}^- = (m^2 + k^2)^3/k^2$. The first root Ta^+ is equivalent to the condition $A = 0$, which is the Rayleigh line $\Omega_1 = \Omega_2/\eta^2$. Minimization of the second root Ta^- with respect to k and m yields the critical wave number $k_c = \pi/\sqrt{2}$ at $m = 1$ and the critical Taylor number $\mathrm{Ta}_c = 658$. This well-known number[1] appears at the place of 1708 when stress-free boundary conditions replace no-slip boundary conditions in the study of classical Couette flow and thermal convection. A finite value of Ta^- means that if $\Omega_0 \to \infty$, then $A \to 0$, and this is why the

Taylor curve approaches the Rayleigh line.

In the case of helium II, the curves of the normal fluid family Ta^- vs k are higher than the curves of the superfluid family Ta^+ vs k. The region of interest in the Ω_1 vs Ω_2 plane is where $|A|$ is small above the Rayleigh line ($A<0$). Here the superfluid family attains its minimum in the long-wavelength limit $k \ll 1$, which is

$$\mathrm{Ta}^- \approx \left[1+\frac{\beta m^2\pi^2}{2|A|}\right]^{-1} \frac{(m^2\pi^2\alpha_n^2 B^2|A|^2)}{(\alpha_s B|A|+m^2\pi^2)} . \quad (20)$$

In the high-temperature range relevant to the experiments, $\alpha_s \ll 1$ and $\alpha_n \approx 1$ and Eq. (20) yields the critical Taylor number

$$\mathrm{Ta}^- \approx \frac{2B^2|A|^3}{\beta\pi^2} . \quad (21)$$

It follows from Eq. (21) and the definition of the Taylor number that $\Omega_0 = B^2|A|^2/(2\beta\pi^2)$, from which

$$\Omega_1 = \frac{\Omega_2}{\eta^2}\left[1+\frac{c}{2\eta^2\Omega_2}\right]\left[1\pm\left[1-\frac{1-c/\Omega_2}{(1-c/2\eta^2\Omega_2^2)^2}\right]^{1/2}\right] . \quad (22)$$

In the limit of rapid rotation $\Omega_2 \to \infty$, Eq. (22) reduces to the Rayleigh line $\Omega_1 \approx \Omega_2/\eta^2$ because $c=\beta\pi^2(1-\eta^2)^2/B^2$ is a small quantity. This result confirms the finding of the numerical calculation.

The fact that the stability of the flow of helium II is determined by the superfluid family Ta^+ rather than the normal fluid family Ta^- suggests a study of what happens in the case of a pure superflow. If helium's temperature is lowered from the λ region to the vicinity of absolute zero, Eqs. (1) and (2) reduce to Eq. (7). In this case, it is possible to do some analytic progress in the narrow gap limit by assuming axisymmetric perturbations without altering the boundary conditions. Care must be taken not to assume exchange of stabilities, however. The governing dimensional superfluid equation[16] can be reduced to

$$[\psi\nu_s k^2(2\Omega+\psi\nu_s k^2)-p^2]\frac{1}{\delta^2}\frac{d^2}{dx^2}v_r = 2Ak^2(2\Omega_0+\psi\nu_s k^2)v_r , \quad (23)$$

where the radial superfluid velocity component v_r must satisfy the boundary conditions $v_r(x=0)=v_r(x=1)=0$. The solution is $v_r=\sin(m\pi x)$, where n is an integer and the eigenvalue is

$$p^2=(2\Omega_0+\psi\nu_s k^2)\left[\psi\nu_s k^2+\frac{2Ak^2\delta^2}{(m^2\pi^2+k^2)}\right] . \quad (24)$$

If the outer cylinder rotates faster than the inner one, then $\psi>0$, $p^2>0$ and the flow is always stable. If the inner cylinder rotates faster than the outer one, then $\psi<0$ and the flow is stable only if $2\Omega_0<\nu_s k^2$. This means that Couette flow is unstable to the growth of long-wavelength ($k\to 0$) perturbations. The stability of corotating Couette superflow is therefore determined by the sign of A, that is to say, by the same Rayleigh condition $\mathrm{Re}_1<\mathrm{Re}_2/\eta$ that governs the stability of a classical inviscid fluid. The physical mechanism is, however, different: $\nu_s k^2$ is the angular frequency of vortex waves,[17] which are antiparallel to the vorticity. The superfluid instability, therefore, is not caused by an unstable stratification of angular momentum, but by the balance of centrifugal forces and vortex tension forces.

V. CONCLUSIONS

The analytic solution of the simplified model equations confirms the numerical solution of the exact equations of motion, and clarifies the origin of the instability and the relation between a viscous fluid, an inviscid fluid, and a superfluid. The conclusion is that the Rayleigh stability condition, which was introduced in the context of an inviscid fluid and later proved valid for the description of a viscous fluid, is also relevant to the stability of a quantum fluid such as helium II. This result, which is apparently simple, is achieved in a subtle way.

To appreciate this conclusion it is worth noting that there is no reason to expect *a priori* that helium II should behave like a classical fluid. For example, the experiments of Heikkila and Hollis Hallet[18] at temperature lower than considered here showed that, if the inner cylinder is held at rest, the rotation of the *outer* cylinder can have a destabilizing effect. This phenomenon contradicts classical fluid mechanics, because when the outer cylinder rotates, the angular momentum is stratified in a stable way and there should be no instability. It is vortex tension which makes a pure superflow different from a classic inviscid flow: the observed instability is due to vortex waves and the growth of nonaxisymmetric perturbations.[16]

Finally, the similarity at high rotation rates between the flow of helium II and the flow of a classical fluid is interesting from the point of view of the attempts of using helium II at temperatures just below T_λ to study issues of classical fluid mechanics at high Reynolds numbers.[19]

ACKNOWLEDGMENTS

Thanks are due to Professor Chris Jones for stimulating this investigation and to Andrew Herdman for help in running the program.

[1]S. Chandrasekhar, *Hydrodynamics and Hydromagnetic Stability* (Oxford University Press, Oxford, 1961).
[2]G. I. Taylor, Philos. Trans. R. Soc. A **223**, 289 (1923).
[3]S. Chandrasekhar and R. J. Donnelly, Proc. R. Soc. London A **241**, 9 (1957).
[4]R. J. Donnelly, Phys. Rev. Lett. **3**, 507 (1959).
[5]C. J. Swanson and R. J. Donnelly, Phys. Rev. Lett. **67**, 1578 (1991).
[6]C. F. Barenghi and C. A. Jones, J. Fluid Mech. **197**, 551 (1988).
[7]C. F. Barenghi, Phys. Rev. B **45**, 2290 (1992).

[8]F. Bielert and G. Stamm, Physica B **194**, 561 (1994).
[9]K. L. Henderson, C. F. Barenghi, and C. A. Jones, J. Fluid Mech. **283**, 329 (1995).
[10]K. L. Henderson and C. F. Barenghi, Phys. Lett. A **191**, 438 (1994).
[11]C. F. Barenghi, C. E. Swanson, and R. J. Donnelly, J. Low Temp. Phys. (to be published).
[12]R. N. Hills and P. H. Roberts, Arch. Rat. Mech. Anal. **66**, 43 (1977).
[13]C. F. Barenghi, R. J. Donnelly, and W. F. Vinen, J. Low Temp. Phys. **52**, 189 (1983).
[14]P. J. Bendt, Phys. Rev. **153**, 280 (1967).
[15]D. D. Awschalom and K. W. Schwarz, Phys. Rev. Lett. **52**, 49 (1984).
[16]C. F. Barenghi and C. A. Jones, Phys. Lett. A **122**, 425 (1987).
[17]W. I. Glaberson and R. J. Donnelly, in *Progress in Low Temperature Physics,* edited by D. F. Brewer (North-Holland, Amsterdam, 1986), Vol. IX.
[18]W. J. Heikkila and A. C. Hollis Hallet, Can. J. Phys. **33**, 42 (1955).
[19]R. J. Donnelly, *High Reynolds Number Flows Using Liquid and Gaseous Helium* (Springer-Verlag, Berlin, 1991).