One-dimensional Ising model with long-range interactions: A renormalization-group treatment

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The one-dimensional Ising model with ferromagnetic interactions which decay as $1/r^\alpha$ is considered. Using a real-space renormalization group scheme (RG) we calculate the critical temperature and the correlation-length critical exponent as a function of α. General asymptotic properties are obtained for arbitrary values of the rescaling length b of the RG transformation. Several rigorous results are recovered exactly in the limit $b \to \infty$. We obtain a $b = \infty$ extrapolation of the critical temperature for arbitrary values of $\alpha > 1$, which we conjecture approximates with high precision the exact one. In particular, we obtain the value $T_c/J = \pi^2/12$ for the $1/r^2$ model.

The spin-1/2 Ising model (with arbitrary interactions) is perhaps the most successful one in statistical mechanics. Despite its (relative) simplicity (which has allowed the testing of an enormous quantity of new methods and theories), it has been used to model such a variety of interacting systems exhibiting cooperative phenomena (ranging from simple ferromagnetism to complex spin glasses) that it can be considered the paradigm of a model system in statistical mechanics, on an equal footing with the harmonic oscillator in quantum mechanics.

Even though there is an enormous amount of known results about this model, there exists some important properties about which little is known. One of these problems concern the critical behavior of Ising ferrromagnets with *long-range* interactions, which means systems described by the Hamiltonian

$$H = -J \sum_{(i,j)} \frac{1}{r_{ij}^{\alpha}} S_i S_j \quad (S_i = \pm 1, \forall\, i;\ J > 0;\ \alpha > 0), \tag{1}$$

where r_{ij} is the distance (in crystal units) between sites i and j, and where the sum $\sum_{(i,j)}$ runs over all distinct pairs of sites on a d-dimensional simple hypercubic lattice. The $\alpha \to \infty$ limit corresponds to the first-neighbor model. The $\alpha = 0$ limit corresponds to an infinite-range ferromagnet which, after a rescaling $J \to J/N$, yields basically the mean field approach. For $\alpha \to d^+$ the critical temperature $T_c(\alpha)$ diverges and remains infinite for $\alpha \leq d$[1]. In other words, all the thermodynamic functions diverge and the usual Boltzmann-Gibbs statistical formalism turns out to be inadequate for this problem. A very interesting proposal about an alternative thermodynamic formulation using Tsallis q statistics can be found in Ref. 1.

Besides their fundamental theoretical interest in physics, spin models with long-range interactions which decay slowly are of interest nowadays, in view of their relationship with neural systems modeling,[2] where far away localized neurons interact through a post-synaptic potential which decays slowly along the axon. Other related problems are spin systems with RKKY-like interactions, $1/r_{ij}^{\alpha}\ \cos(ar_{ij})$, which are present in spin glasses.[3]

In this work we address the one-dimensional problem (hence, $r_{ij} = \mid i - j \mid = 1, 2, 3, \ldots$). Let us first summarize the known results up to the present: (i) it has been proved that, for $1 < \alpha \leq 2$, this model can exhibit long-range order at *finite* temperatures,[4,5] while for $\alpha > 2$ it has no phase transition at *finite* temperature, more precisely, $T_c = 0$ (short-range interactions); (ii) for $\alpha = 2$ the spontaneous magnetization is discontinuous at $T = T_c \neq 0$ (the so-called Thouless effect);[5] (iii) for $1 < \alpha < 1.5$ the critical exponents are classical;[6] (iv) the region $1.5 < \alpha < 2$ shows nontrivial critical exponents, which are not known exactly. Approximate results in the latter region were obtained by finite-range scaling approximations[7] or by ϵ expansions around $\alpha = 1.5$ (Ref. 8) and $\alpha = 2$ (Ref. 9) where the critical behavior is of essentially singularity type.[10]

This problem is of considerable theoretical interest in order to understand the critical behavior of higher-dimensional spin models with long-range interactions. In particular, the $\alpha = 2$ case is of particular interest, because it can be mapped into the spin-1/2 Kondo problem.[11]

We use a real-space renormalization group (RG) method in order to calculate the critical temperature and the corresponding critical exponent (i.e., the universality class) of the (long-range) phase transition. This technique, based on a Kadanoff-block construction using the majority rule, is a very well known one and was introduced by Niemeijer and van Leeuwen[12].

First, we define Kadanoff blocks of length $b > 1$, as shown in Fig. 1 for the particular case $b = 3$; b is always an odd number which characterizes the *rescaling length* of the RG transformation. We will assign a block-spin variable $S'_I = \pm 1$ to every block I. Let us denote by S_i^I ($i = 1, 2, \ldots, b$; $I = 1, 2, \ldots, N/b$) the (site) spins belonging to the block I. Then, defining the dimensionless Hamiltonian

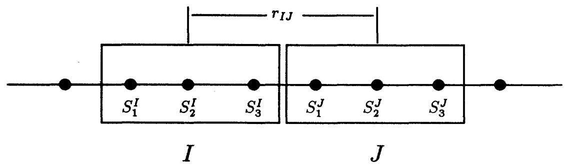

FIG. 1. Kadanoff blocks of length $b = 3$ in the one-dimensional lattice; r_{IJ} is the distance between the blocks I and J.

$$\mathcal{H} \equiv K \sum_{I=1}^{N/b} \sum_{J=1}^{N/b} \sum_{i \in I} \sum_{j \in J} \frac{1}{r_{ij}^{\alpha}} S_i^I S_j^J \quad (i \neq j), \tag{2}$$

with $K \equiv \beta J$ ($\beta = 1/k_B T$; hereafter we take $k_B = 1$), a renormalized (block) Hamiltonian is determined by the following RG transformation:

$$e^{\mathcal{H}' + \mathcal{C}} = \mathrm{Tr}_{\{S_i^I\}} P\left(\{S_i^I\}, \{S_I'\}\right) e^{\mathcal{H}}, \tag{3}$$

where $\mathrm{Tr}_{\{S_i^I\}}$ denotes a sum over all the configurations of site spins S_i^I, $\mathcal{C}$ is a constant, and

$$P\left(\{S_i^I\}, \{S_I'\}\right) = \prod_I \frac{1}{2}\left[1 + S_I' \,\mathrm{sgn}\left(\sum_{i=1}^{b} S_i^I\right)\right] \tag{4}$$

is a weight function which characterizes the majority-rule recipes.[12]

The Hamiltonian $\mathcal{H}$ can be divided into two parts: $\mathcal{H} = \mathcal{H}_0 + \mathcal{V}$, where $\mathcal{H}_0 = \sum_I \mathcal{H}_0^I$ and $\mathcal{V} = \sum_{(I,J)} \mathcal{V}_{IJ}$; $\mathcal{H}_0^I$ includes only the interactions between spins *inside* the block I, whereas $\mathcal{V}_{IJ}$ includes the interactions between spins belonging to *different* blocks I and J. We then introduce the following expectation value:

$$\langle \mathcal{O} \rangle_0 \equiv \frac{1}{Z_0} \mathrm{Tr}_{\{S_i^I\}} P\left(\{S_i^I\}, \{S_I'\}\right) \exp\left[\mathcal{H}_0\left(\{S_i^I\}\right)\right] \mathcal{O}, \tag{5}$$

with

$$Z_0 \equiv \mathrm{Tr}_{\{S_i^I\}} P\left(\{S_i^I\}, \{S_I'\}\right) \exp\left[\mathcal{H}_0\left(\{S_i^I\}\right)\right]. \tag{6}$$

Then, Eq. (3) can be written as

$$e^{\mathcal{H}' + \mathcal{C}} = Z_0 \left\langle e^{\mathcal{V}} \right\rangle_0. \tag{7}$$

Using a cumulant expansion of $\left\langle e^{\mathcal{V}} \right\rangle_0$, a *first-order* approximation of $\mathcal{H}'$ can be obtained as

$$\mathcal{H}' = \langle \mathcal{V} \rangle_0 = \sum_{(I,J)} \langle \mathcal{V}_{IJ} \rangle_0. \tag{8}$$

Notice that this approach *retains the long-range character of the interactions*. This fact will allow us to catch, at least qualitatively, the correct behavior of the long-range phase transition.

Since the expectation value (5) is carried out with a block-independent probability distribution, we have

$$\langle \mathcal{V}_{IJ} \rangle_0 = K \sum_{i \in I} \sum_{j \in J} \frac{1}{r_{ij}^{\alpha}} \langle S_i^I \rangle_0 \langle S_j^J \rangle_0. \tag{9}$$

Let r_{IJ} be the distance between the center sites of the blocks I and J (see Fig. 1), measured *in units of the rescaling length* b. We have $r_{ij} = b\, r_{IJ} - (b-1)$, $b\, r_{IJ} - (b-2)$, $\ldots, b\, r_{IJ} - 1$, $b\, r_{IJ}$, $b\, r_{IJ} + 1$, $\ldots b\, r_{IJ} + (b-1)$. Then, for $r_{IJ} >> 1$ we can approximate

$$r_{ij} \approx b\, r_{IJ} \tag{10}$$

and consequently

$$\langle \mathcal{V}_{IJ} \rangle_0 \approx \frac{K}{b^{\alpha}\, r_{IJ}^{\alpha}} \sum_{i \in I} \sum_{j \in J} \langle S_i^I \rangle_0 \langle S_j^J \rangle_0.$$

It can be easily verified that $\left\langle S_i^I \right\rangle_0 = a_i(K)\, S_I'$, where $a_i(K)$ does not depend on the block I. Replacing into Eq. (8) we obtain

$$\mathcal{H}' = K_b'(K) \sum_{(I,J)} \frac{1}{r_{IJ}^{\alpha}} S_I' S_J', \tag{11}$$

where

$$K_b'(K) = \frac{K}{b^{\alpha}} \left[\sum_{i=1}^{b} a_i(K)\right]^2 \tag{12}$$

is our RG recurrence equation.

Let us discuss the approximation (10). We have seen that $r_{ij} = b[r_{IJ} + x]$, where $x = 0, \pm(1 - 1/b), \pm(1 - 2/b), \ldots, \pm 1/b$. Hence, it is expected to obtain good results for high values of α, where $[r_{IJ} + x]^{\alpha} \approx r_{IJ}^{\alpha}$. Moreover, the approximation will be systematically improved for increasingly high values of b. The greatest error in (10) occurs when $r_{IJ} = 1$ and $x = -(1 - 1/b)$, where $r_{ij}^{\alpha} = 1$ is approximated by $r_{ij}^{\alpha} \approx b^{\alpha}$. This case, however, corresponds to the interaction between first-neighbor sites belonging to adjacent blocks; such short-range interactions have no effect on the critical behavior of the one-dimensional system, at least as far as the critical exponents are concerned.

Notice that the majority rule is applied exactly inside each block; i.e., expression (6) gives the exact partition function of a system of noninteracting blocks of length b (except for an irrelevant factor of 2). Hence, approximation (8) corresponds to an exact averaging of the first-order cumulant of the interaction term between blocks. Since the n^{th} cumulant is of order $1/b^{n\alpha}$, approximation (8) can be seen as the leading order in a series expansion of Eq. (3) in powers of $1/b^{\alpha}$. Therefore, it is expected that the results will be systematically improved for increasingly high values of b.

We now analyze the recurrence equation (12) and its fixed points $K^* = K_b'(K^*)$. The qualitative behavior of $K_b'(K)$ can be appreciated in Fig. 2, where it is depicted for the particular case $b = 5$ and typical values of α. It shows two trivial fixed points: $K = 0$ ($T = \infty$) and $K = \infty$ ($T = 0$). For low temperatures $K \gg 1$ we have $a_i(K) \sim 1\ \forall\ i$; from Eq. (12) we obtain the asymptotic behavior $K_b'(K) \sim b^{2-\alpha} K$. For low values of α the slope of $K_b'(K)$ at $K = 0$ is greater than one and it does not present a (nontrivial) fixed point for finite values of K. In this case the fixed point $K = 0$ is repulsive and therefore $T_c = \infty$. For intermediate values of α, $K_b'(K)$ possesses a nontrivial fixed point at finite $K = K_c(\alpha)$. For $\alpha > 2$

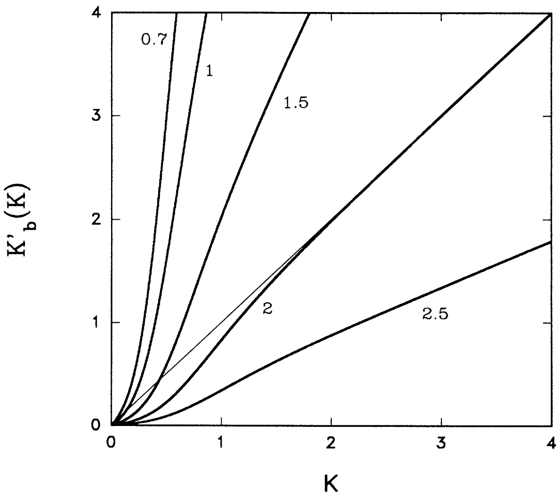

FIG. 2. Recurrence equation $K_b'(K)$ for $b = 5$. The numbers beneath the curves correspond to the values of α.

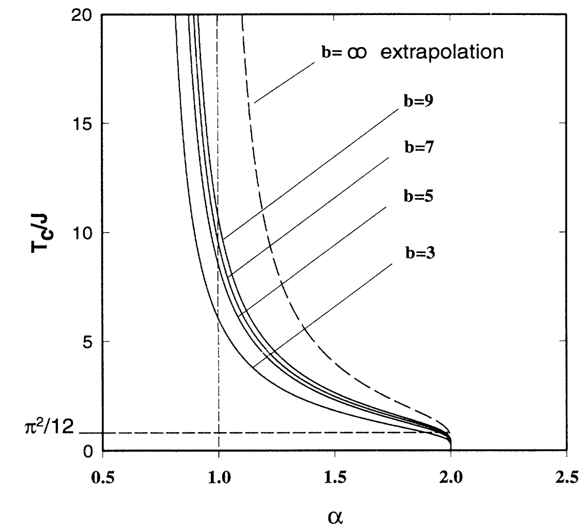

FIG. 3. Critical temperature $T_c(\alpha)/J$ vs α for different values of the rescaling length b. The dashed line corresponds to an extrapolation of the numerical results for $b \to \infty$ (see Fig. 4).

there is again no fixed point at finite K. In this case, however, the fixed point $K = 0$ is attractive and therefore $T_c = 0$ for all values of b, recovering the exact result. Summarizing, there exists some value $\alpha_1(b)$ such that (i) $T_c = \infty$ for $\alpha < \alpha_1(b)$; (ii) there is a phase transition at finite temperature $T_c(\alpha)$ for $\alpha_1(b) < \alpha < 2$; (iii) $T_c = 0$ for $\alpha \geq 2$.

The borderline value $\alpha_1(b)$ is determined by the condition $dK_b'/dK \mid_{K=0}= 1$. This equation can be easily solved by noting that $a_i(0) = \gamma(b)\ \forall\ i$. Hence, from Eq. (12) we obtain $dK_b'/dK \mid_{K=0}= b^{2-\alpha}\gamma(b)^2$ and therefore

$$\alpha_1(b) = 2\left[1 + \frac{\ln\,\gamma(b)}{\ln\,b}\right].$$

After some algebra we find

$$\gamma(b) = \frac{(b-1)!}{2^{b-1}\,\left(\frac{b-1}{2}\,!\right)^2}.$$

Using Stirling's asymptotic expansion for $b \gg 1$ we find $\gamma(b) \sim 2/\sqrt{\pi}\ b^{-1/2}$ and we recover the exact result $\alpha_1(b) \to 1$ in the limit $b \to \infty$. Using Eq. (12) we computed $T_c(\alpha)$ numerically for several values of b. The corresponding results are shown in Fig. 3.

For $\alpha \to 2^-$ we see that $K_c \to \infty$. In such limit the recurrence equation (12) presents the following asymptotic behavior:

$$K_b'(K) \sim \frac{K}{b^\alpha}\left[b - 4\,e^{-B(b)\,K}\right]^2,$$

where $B(b) = 2\,\sum_{n=1}^{b-1} 1/n^2$; $B(b)\,K$ is the energy difference between the ground state and the first excited state of $\mathcal{H}_0^I$, in the limit $\alpha \to 2$. The asymptotic behavior of $T_c(\alpha)$ for $\alpha \to 2^-$ is then given by the Cauchy function

$$2 - \alpha \sim A(b)\,e^{-B(b)\,J/T_c}, \tag{13}$$

with $A(b) = 8/(b\,\ln\,b)$. In the limit $b \to \infty$ we have $B \to \pi^2/3$ and $A \to 0$. The coefficient $A(b)$ determines the region near $\alpha = 2$ in which the asymptotic regime (13) holds; therefore, for $b \to \infty$ such behavior disappears, suggesting a nonuniform convergence to a finite value $T_c(\alpha = 2) \neq 0$, consistent with the exact result.

For $\alpha \to \alpha_1^+(b)$ we see that $K_c \to 0$. Then, expanding (12) around $K = 0$ we obtain $T_c/J \sim C(b)/[\alpha - \alpha_1(b)]$, where

$$C(b) = \frac{2}{b\,\ln\,b}\sum_{n=1}^{b-1}\frac{b-n}{n^{\alpha_1(b)}}.$$

In the limit $b \to \infty$ we have $C(b) \to 2$. Since $\alpha_1(b) \to 1$, we can estimate the exact asymptotic behavior:

$$T_c/J \sim \frac{2}{\alpha - 1} \tag{14}$$

for $\alpha \to 1$. This result reproduces the asymptotic behavior derived from a closed-form approximation in Ref. 13: $T_c/J \sim 2\zeta(\alpha) \sim 2/(\alpha - 1)$, where $\zeta(\alpha)$ is the Riemann zeta function.[14] Now, we can use the above-derived asymptotic behavior to extrapolate the full curve $T_c(\alpha)$ vs α for $b \to \infty$. First, we define the rescaled variables $x \equiv (2-\alpha)/[2-\alpha_1(b)]$ and $y \equiv T_c/J\ [2-\alpha_1(b)]/C(b)$, so that $y(x) \sim 1/(1-x)$ for $x \to 1\ \forall\ b$. In Fig. 4 we plotted $y(x)$ vs x for different values of b. All the curves fall into a single one for $b > 5$ (solid line in Fig. 4). Transforming back such a curve into the (T_c, α) variables we obtain a good estimate of the exact critical temperature for $\alpha \in (1, 2)$, except in the neighborhood of $\alpha = 2$, where our calculation yields $T_c(2) = 0$ for all *finite* values of b. The extrapolated curve is depicted in Fig. 3 as a dashed line. The critical temperature at $\alpha = 2$ can be estimated as follows. First, we make a linear extrapolation of the inflection point of the Cauchy function (13) into the $\alpha = 2$ axis, for finite b. Then, we take the $b \to \infty$ limit, obtaining the value $T_c = B(\infty)/4 = \pi^2/12 \approx 0.823$ (see Fig. 3). This value is within the uncertainty of the Anderson and Yuval estimate $T_c/J = 0.79 \pm 0.05$.[11]

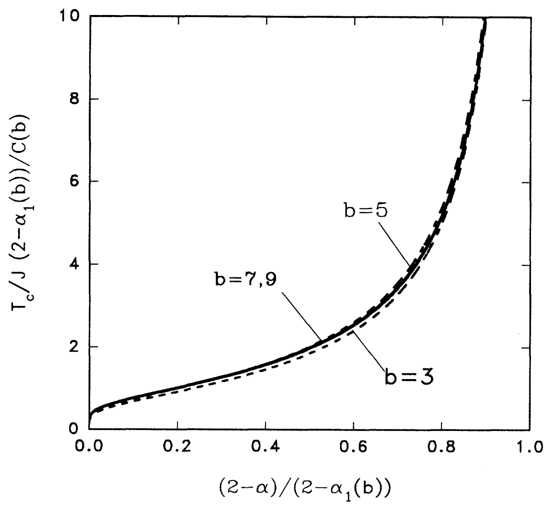

FIG. 4. Rescaled critical temperature. All curves with $b > 5$ fall into the solid line.

Finally, we calculate the correlation-length critical exponent

$$\nu(\alpha) = \frac{\ln b}{\ln \left(dK_b'/dK\ |_{K_c}\right)}.$$

The numerical results are depicted in Fig. 5 for several values of b; the results are compared with the exact one $\nu_{\rm ex} = 1/(\alpha - 1)$, for $1 < \alpha < 1.5$ (Ref. 6), and the asymptotic result from Kosterlitz $\nu_K \sim [2(2-\alpha)]^{-1/2}$ for $\alpha \to 2$.[9] For finite b we find $\nu \sim 1/[\alpha - \alpha_1(b)]$. Therefore, for $b \to \infty$ we recover the exact result in the $\alpha \sim 1$ region. Rescaling ν using the last asymptotic behaviour we obtain an extrapolated curve $\nu(\alpha)$ in the $b \to \infty$ limit (see Fig. 5). Such a curve presents a minimum for $\alpha \approx 1.64$ and reproduces the expected behaviors for $\alpha \sim 1$ and $\alpha \sim 2$, but it presents a little departure from the exact result for $\alpha \sim 1.5$.

The approach adopted here gives an estimate of the critical properties of the model as a function of α, based on an extrapolation of a systematic series of RG calculations. The analytic asymptotic results, obtained in the $b \to \infty$ limit, give confidence in the validity of the method, by recovering several known results. Therefore, we believe the obtained critical temperature approximates with high precision the exact one. In particular, we conjecture that $T_c(\alpha = 2) = \pi^2/12$ reproduces the exact value.

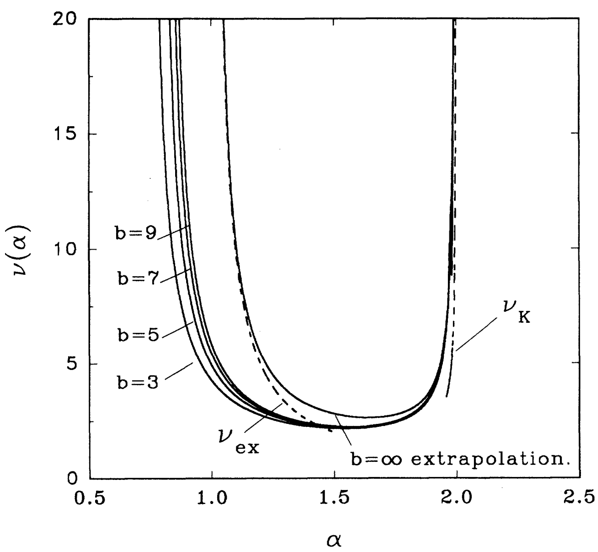

FIG. 5. Correlation-length critical exponent $\nu(\alpha)$ vs α; $\nu_{\rm ex}$ refers to the exact result for $1 < \alpha < 1.5$ and ν_K to the Kosterlitz's (Ref. 9) asymptotic result.

This work can be extended to higher-dimensional spin models with long-range interactions, where less rigorous information than in the one-dimensional case is known; it can also be used to treat more complex interactions like the RKKY one. Some works along these lines are in progress and will be published elsewhere.

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[14] Actually, the authors in Ref. 13 do not calculate the asymptotic behavior explicitly for the one-dimensional case; it can be derived with little effort from Eq. (20) in the abovementioned reference.