

Composite-fermion description of correlated electrons in quantum dots: Low-Zeeman-energy limit

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We study the applicability of composite-fermion theory to electrons in two-dimensional parabolically confined quantum dots in a strong perpendicular magnetic field in the limit of low Zeeman energy. The noninteracting composite fermion spectrum correctly specifies the primary features of this system. Additional features are relatively small, indicating that the residual interaction between the composite fermions is weak.

I. INTRODUCTION

Recent progress in microlithography¹ has made it possible to fabricate artificial semiconductor structures containing only a few electrons, called quantum dots. Electrons can be added to a quantum dot one by one, and its properties can be investigated by various techniques, e.g., tunneling,^{2,3} capacitance measurements,⁴ and optical spectroscopy.⁵ In particular, the chemical potential μ_N of the N -electron system can be measured as a function of various parameters, e.g., the confining potential or the external magnetic field B . The chemical potential is given by $\mu_N = E_N - E_{N-1}$, where E_N is the ground-state energy of the N -electron system, and consequently contains information about the ground-state energy of interacting electrons. As some parameter (say B) is changed, the ground states of the N and $N - 1$ electrons vary continuously until a level crossing occurs, at which time μ_N exhibits a cusp;⁶⁻¹² such cusps are observed in various experiments.^{2-5,13}

A completely noninteracting electron model is not sufficient for describing the quantum dot physics, and the repulsive Coulomb interaction must be taken into account at some level. In the simplest scenario, the energy to add an extra electron can be modeled in terms of a classical capacitance, which is a smooth function of the number of electrons in the dot.¹⁴ Superimposed over this smooth classical contribution are small fluctuations that originate from either the quantization of single-particle energy levels in the quantum dot, or correlations due to the Coulomb interaction, or a combination of the two. In the zeroth-order approximation, the corrections to the classical energy may be computed in terms of noninteracting electrons in the quantum dot.¹³ Then, whenever there is a level crossing in the noninteracting ground state, a cusp appears in μ_N . At high B , a level crossing occurs when an electron changes its Landau level (LL) index, or reverses its spin within the same LL. Klein *et al.*³ have investigated in detail the region where both spin species of only the lowest LL are occupied, and found that a Hartree-Fock theory^{6,11,15} provides a reasonably accurate quantitative account of the cusp positions; the cusps in this case originate when the electrons flip their

spin one by one, until they are fully polarized.

At still higher B , when the filling factor $\nu < 1$, and all electrons are fully polarized, an interplay of different LL's is not possible. A noninteracting electron model or a Hartree-Fock-type calculation will obtain the ground-state energy to be a smooth function of various parameters, and as a result will not produce any level crossings. However, detailed exact diagonalization studies on small systems show that level crossings do indeed occur in this regime.^{6,7,12,16,17} These originate exclusively from Coulomb correlations. Phase diagrams^{6,7} for the ground state as a function of the magnetic field and the strength of the confinement have been constructed using exact diagonalization studies. Such studies, however, are possible only for quantum dots with very few electrons.

The results of exact diagonalization studies of fully polarized electronic states have been interpreted in terms of the composite fermion theory.^{18,19} The motivation comes from the relevance of composite fermions to the phenomenon of the fractional quantum Hall effect (FQHE) (Ref. 20) that occurs when two-dimensional electron systems (2DES's) are exposed to very high magnetic fields. In this framework, the FQHE can be understood as the integer quantum Hall effect (IQHE) of composite fermions.²¹ The FQHE occurs as a result of incompressibility, i.e., cusps in the ground-state energy as a function of the magnetic field, which suggests that the physics of the cusps in quantum dots is related to the physics of the FQHE. It was shown that the cusps in the quantum dot states can be understood in terms of noninteracting composite fermions.^{18,19} Their origin is discussed briefly as follows. The electron system completely confined to the lowest LL maps on to composite fermions with several *quasi*-LL's occupied, and the interaction energy of electron is mapped into an effective cyclotron energy of the composite fermions. Level crossings in E_N then occur as a result of an interplay between various *quasi*-LL's of composite fermions, i.e., as the composite fermions change their *quasi*-LL index one by one. Thus the composite fermion (CF) theory effectively provides a single-particle description for the correlation effects, and gives a very simple intuitive picture for understanding the principal features of the strongly correlated quantum

dot ground states.

The study of Refs. 18 and 19 extended the applicability of the CF model to systems with nonuniform densities and edges. (Most earlier studies used the spherical geometry which has no edges.) Also, since the composite fermions are many-body objects, one may ask how many electrons are needed before the CF description becomes valid. (Of course, there is no composite fermion for a single electron.) In the case of fully polarized electrons, the CF description was found to be reasonably good even for as few as three electrons.¹⁹

This work investigates the applicability of the CF theory to quantum dots in the limit of vanishing Zeeman energy. This limit is relevant (in certain parameter range) because the Zeeman splitting is rather small in GaAs, roughly 1/60 of the cyclotron energy, since the band g factor ($g^* \sim 0.44$) of electrons and their band mass ($m^* \sim 0.067m_e$) are both very small. Experimental²² and theoretical²³ studies in FQHE have shown that even for moderately strong magnetic fields, the ground states are not always fully spin polarized (e.g., $\nu = \frac{2}{3}, \frac{2}{5}, \frac{2}{7}$, etc., have spin-singlet ground states). Similar physics can be expected in quantum dots. Certain features in single-electron capacitance spectroscopy⁴ and tunneling^{2,3} experiments in quantum dots have been interpreted in terms of spin-singlet states, and the effect of the spin degree of freedom has been investigated in several theoretical studies.^{6,7}

In not fully polarized states electrons may overlap spatially and hence one would expect the residual interaction between composite fermions to be of greater importance than for fully polarized composite fermions.²⁴ However, we find that the noninteracting CF model still identifies the relatively strong cusps. The additional weaker cusps are a signature of a residual interaction between composite fermions. While the true residual interaction between the composite fermions is not known, even a minimal “ δ -function” hard-core interaction gives a reasonable qualitative picture for the weaker structure. The low Zeeman-energy quasi-LL gap is found to be roughly an order of magnitude smaller than that of the fully polarized composite fermions. For example, in units of $\frac{e^2}{\epsilon a}$, where a is the effective magnetic length, the quasi-LL gap in the large Zeeman-energy limit¹⁹ was ~ 0.124 ($N = 5$), whereas in the small Zeeman energy limit the gap is ~ 0.019 ($N = 5$). Interestingly, we find that the CF description improves as the number of electrons confined in the dot increases.

Our main objective here is to investigate the validity of the CF model, for which we will compare its consequences with the results of exact diagonalization studies. Therefore, we restrict our study to small systems. It should however be emphasized that the details of the CF model may be worked out rather straightforwardly even for a large number of electrons. The noninteracting CF model should therefore prove useful in analyzing experimental results in the high-field regime where the Hartree-Fock approximation is not applicable, and exact diagonalization cannot be performed.

The plan of this paper is as follows. Section II gives the results of exact numerical diagonalization. Section

III compares the CF theory with numerical studies at zero Zeeman energy. In Sec. IV, we consider the electron ground-state phase diagrams as a function of the confinement strength and magnetic field, and also some typical addition spectra.

II. NUMERICAL CALCULATIONS

We will consider two-dimensional quantum dots with parabolic confinement $(1/2)m^*\omega_o|\mathbf{r}|^2$.²⁵ Further, we will assume sufficiently high B that the cyclotron energy $\hbar\omega_c$ is large compared to the confinement energy $\hbar\omega_o$, but the Zeeman energy remains sufficiently small (which will be set to zero throughout this work). The total angular momentum L commutes with the Coulomb interaction, which makes it possible to diagonalize the problem in various L subspaces separately. For zero Zeeman energy, the total spin S is also a good quantum number, so it is sufficient to work in the subspace of the lowest S_z , provided it is remembered that each state in this subspace represents a multiplet of $2S + 1$ degenerate states of the full Hilbert space. In the high B limit, the ground-state energy $E(L)$ in a given L subspace nicely separates into two parts, the confinement energy $E_c(L)$ and the interaction energy $V(L)$. The former is given by

$$E_c = \frac{\hbar\omega_c}{2} \left[\left(1 + 4\frac{\omega_o^2}{\omega_c^2} \right)^{\frac{1}{2}} - 1 \right] L, \quad (1)$$

and $V(L)$ is the same as the interaction energy of the ground state at L *without* any confinement, provided the magnetic length is replaced by an effective magnetic length a , given by $a = \sqrt{\frac{\hbar}{m^*}(\omega_c^2 + 4\omega_o^2)^{-\frac{1}{2}}}$. As a result, it is sufficient to calculate $V(L)$ for a system without any confinement.

The true ground state of the system is determined by the minimum of $E = E_c + V$. While it will depend on various parameters, such as B or ω_o , it is in general a state where the plot of $E(L)$ vs L has a downward

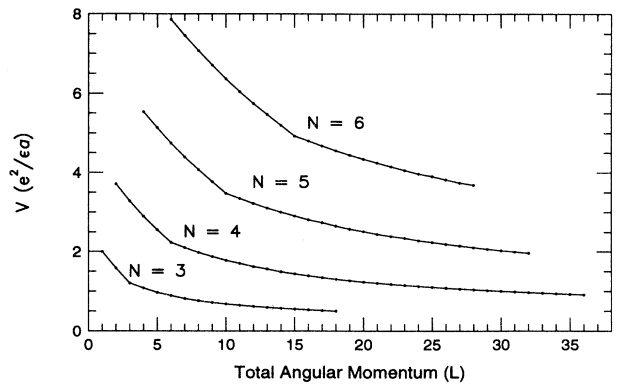


FIG. 1. The interaction energy of the ground state in the L subspace, $V(L)$, as a function of the total angular momentum L . N is the number of electrons.

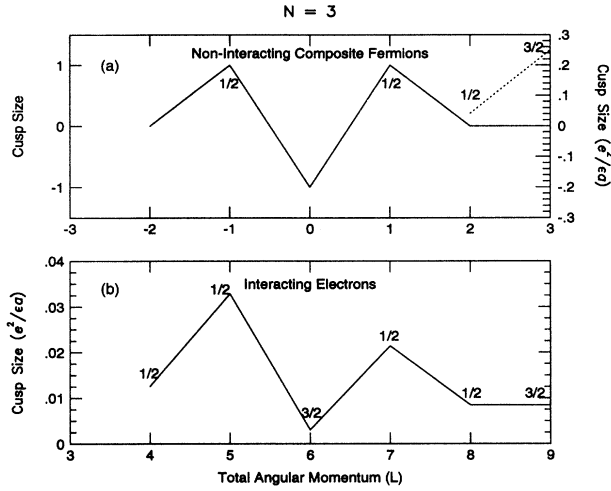


FIG. 2. The cusp size $\Delta(L)$, defined in the text, for non-interacting composite fermions (upper panel, solid line, left scale) and interacting electrons (lower panel, left scale). The dashed line in the upper panel shows the cusp size for interacting composite fermions (right scale). The Δ of noninteracting composite fermions is given in units of $\hbar\omega^{CF}$; the rest are in units of $\frac{e^2}{\epsilon a}$. The ground-state spins are shown on the plot itself. For noninteracting composite fermions, spins are not shown when there are many degenerate ground states; for interacting electrons (or interacting composite fermions), two spins are shown when the first excited state is almost degenerate with the ground state (the lower spin corresponds to the actual ground state).

cusp; states with upward cusps will never become ground states. It is convenient to define the size of a cusp at L as $\Delta(L) = E(L+1) + E(L-1) - 2E(L)$. Then, only states with positive Δ may become ground states with suitable choice of parameters. Since E_c is a linear function of L , $\Delta = V(L+1) + V(L-1) - 2V(L)$, and the

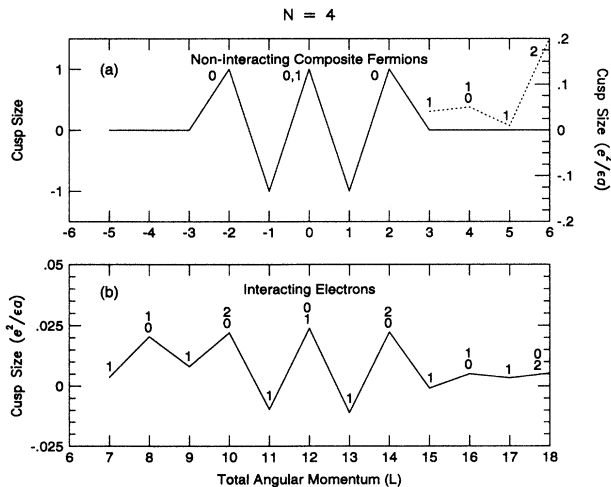


FIG. 3. Same as in Fig. 2 for $N = 4$.

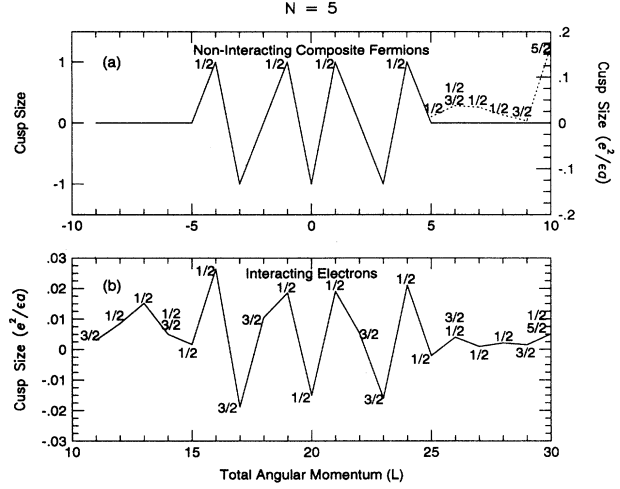


FIG. 4. Same as in Fig. 2 for $N = 5$.

possible ground states can be identified directly from the plot of $V(L)$.

The Coulomb Hamiltonian was diagonalized exactly for Hilbert spaces of sizes less than 3500, using standard numerical subroutines. For bigger Hilbert spaces, we used a modified Lanczos technique²⁶ to obtain a few of the low-energy states. The maximum size we have studied is approximately 45 000. The Lanczos scheme requires care in dealing with states that are almost degenerate. Exact diagonalization has been carried out in a number of earlier studies to which the reader is referred for details.^{6-10,12,16,17,27-29}

Figure 1 shows $V(L)$ as a function of L for 3-6 electrons. The structure on the curves is small compared to that seen in the analogous plot for spin-polarized electrons.¹⁹ In order to bring out the cusps more clearly, we plot in Figs. 2-5 $\Delta(L) = V(L+1) + V(L-1) - 2V(L)$

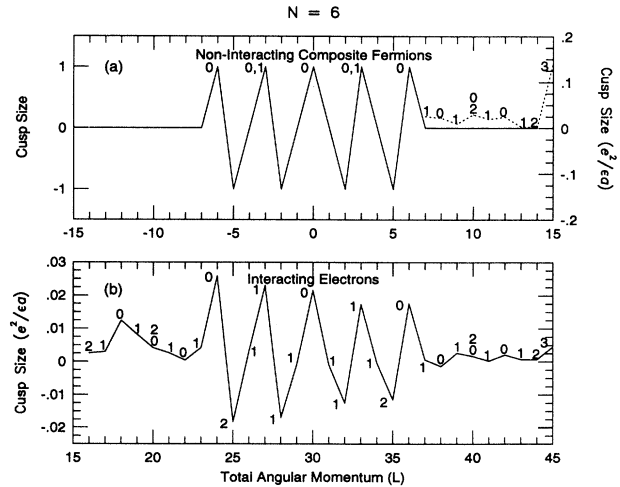


FIG. 5. Same as in Fig. 2 for $N = 6$.

as a function of L . As stated earlier, Δ is positive for downward cusps and negative for upward cusps.

III. COMPOSITE FERMION DESCRIPTION

Composite fermions are relevant when two-dimensional electrons are subjected to a strong magnetic field. The essential role of the Coulomb interaction is presumed to bind an even number of vortices of the many-particle wave function to each electron. The resultant electron plus vortex combination has the statistics of a fermion and is called a composite fermion:

electron + $2m$ vortices \rightarrow composite fermion . (2)

Microscopically, the formation of composite fermions implies that the (unnormalized) low-energy wave functions of interacting electrons with total angular momentum L have the form:

$$\Psi_L = \mathcal{P} \prod_{j < k} (z_j - z_k)^{2m} \Phi_{L^*} , \quad (3)$$

$$L = L^* + mN(N - 1) = L^* + 2mM , \quad (4)$$

$$M = \frac{N(N - 1)}{2} , \quad (5)$$

where Φ_{L^*} is the wave function of electrons with total angular momentum L^* , and \mathcal{P} projects the wave function on to the LLL of electrons, as appropriate for $B \rightarrow \infty$. The Jastrow factor $\prod_{j < k} (z_j - z_k)^{2m}$ binds $2m$ vortices to each electron of Φ_{L^*} to convert it into a composite fermion. Noninteracting composite fermions are obtained when Φ_{L^*} is taken to be the wave function of noninteracting electrons. The LL's of noninteracting electrons of Φ_{L^*} map into quasi-LL's of composite fermions, separated by a quasicyclotron energy gap, which is treated as a parameter of the theory. The system of interacting electrons at arbitrary L is mapped into a system of composite fermions in the range $-M \leq L^* \leq M$ with a suitable choice of m . M is the angular momentum of the $\nu = 1$ state of fully polarized electrons. Note that the above discussion is applicable in both the large and the small Zeeman-energy limits. In the former, the electrons in Φ_{L^*} are assumed to be fully polarized, whereas in the latter, which is the case in the present study, electrons in Φ_{L^*} can have either spin, with the Zeeman energy set to zero.

A. Noninteracting composite fermions

According to the CF theory, the interacting electron system at L is equivalent to a weakly interacting CF system at $L^* = L - 2mM$. In Fig. 6 we also plot the kinetic energy of composite fermions as a function of L^* . This is the same as that of noninteracting electrons at L^* , but with the cyclotron energy of electrons replaced by an ef-

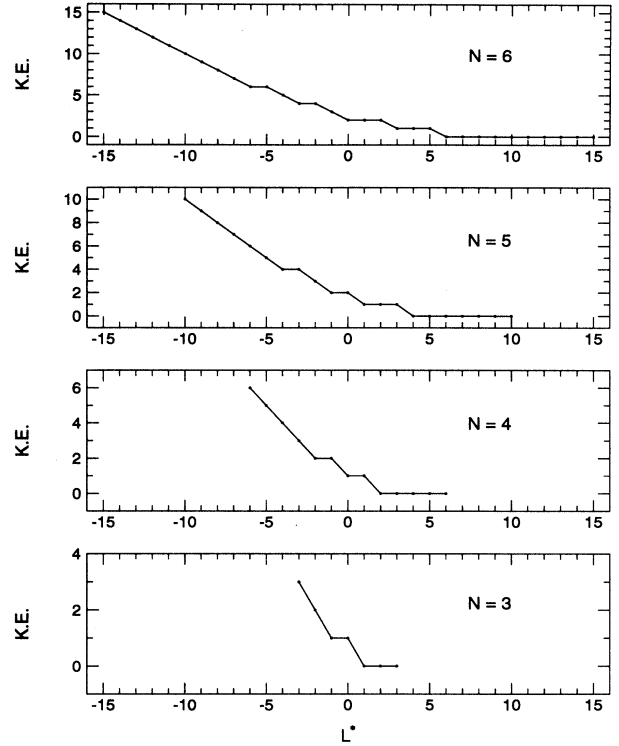


FIG. 6. The kinetic-energy (KE) spectrum of noninteracting electrons ($N=3,4,5,6$) with spin, in the range $-M \leq L^* \leq M$. The energies are given in units of $\hbar\omega^{\text{CF}}$.

fective cyclotron energy of composite fermions, denoted by $\hbar\omega^{\text{CF}}$, which is to be determined empirically by comparison with the interaction energy curve.

It is worthwhile to consider in some detail why cusps appear in Fig. 6. Consider $N = 4$. For noninteracting electrons, there are no cusps for $L \geq 2$; at $L = 2$ the occupied single-particle states in the lowest LL are $0 \uparrow$, $0 \downarrow$, $1 \uparrow$, and $1 \downarrow$, where $j \downarrow$ denotes the single-particle state with angular momentum j and spin down. The exact diagonalization calculation for interacting electrons, however, shows cusps. Let us go to the noninteracting CF basis. The total angular momentum of composite fermions is $L^* = L - 12$. All composite fermions can be accommodated in the lowest quasi-LL for $L^* \geq 2$, which corresponds to $L \geq 14$. As L^* is reduced, one composite fermion must be pushed into the second quasi-LL. The lowest L^* with only one composite fermion in the second quasi-LL is $L^* = 0$, where the ground state contains three composite fermions in the lowest quasi-LL, in angular momentum states $0, 0$, and 1 , and one in the second quasi-LL, in the angular momentum state -1 .³⁰ For decreasing L^* further, the CF kinetic energy must be raised further. This would result in a cusp in the interacting electron system at $L = 12$. Similar analysis produces the curves of Fig. 6.

The cusp sizes are plotted in Figs. 2–5. The prominent cusps in the interaction energy curve are well reproduced by the noninteracting CF theory. The effec-

tive cyclotron energy, in units of $\frac{e^2}{\epsilon_0 a}$, is determined to be $\sim 0.012(N=3)$, $0.017(N=4)$, $0.019(N=5)$, and $0.017(N=6)$. It is interesting that the description of interacting electrons in terms of composite fermions becomes better for larger N . For example, the negative CF cusps for $N=5$ and $N=6$ correspond to negative cusps in the interacting energy spectrum, but to positive cusps for $N=3$ and 4 (even though the overall shape is obtained correctly). This is not surprising, since composite fermions are inherently many-body objects, and may not be appropriate for systems with very few electrons. For zero Zeeman energy, the composite fermion description is only qualitatively valid for quantum dots with fewer than five electrons.

B. Residual interaction between composite fermions

There are additional cusps in the $V(L)$ curve, of relatively small sizes, that cannot be explained within the noninteracting CF model. In particular, this model predicts an absence of cusps in the region corresponding to $M' \leq |L^*| \leq M$, where M' is the smallest angular momentum possible within the lowest LL, but there are several cusps in $V(L)$ in the corresponding region $M' + 2mM \leq L \leq (2m+1)M$. The additional cusps can be interpreted as originating from a residual interaction between the composite fermions. These cusps are roughly an order of magnitude weaker than the primary cusps, showing that the residual interaction between the composite fermions is weak compared to their effective cyclotron energy.

The residual interaction between the composite fermions may be incorporated by taking the composite fermions at L^* to be interacting. We have considered two models in an attempt to mimic the residual interaction, one in which the composite fermions at L^* interact via the Coulomb interaction, and the other in which they

interact via a hard-core interaction. For simplicity, we have considered only the range $M' \leq L^* \leq M$, where all composite fermions are in the lowest quasi-LL. Figures 2–5 show the Coulomb interaction energy cusp-size [panel (a), broken lines]. It is clear that it captures the qualitative physics of the weaker cusps. The same is true of a hard-core δ function interaction. This suggests that the dominant part of the residual interaction is the contact interaction, explaining why the residual interaction is less important for fully polarized electrons.

C. Low-energy spectrum

Besides the shape of the $V(L)$ curve, the CF scheme also sheds light on the low-energy spectrum of states of interacting electrons. Figure 7(a) shows the low-energy spectrum of a five-electron system in the range $24 \leq L \leq 30$. The spin quantum number is shown in the figure for a few of the low-energy states. Figure 7(b) shows the spectrum of composite fermions in the range $4 \leq L^* \leq 10$, interacting via the Coulomb interaction. The low-energy spectrum of Fig. 7(b) matches nicely that of Fig. 7(a). At several values of L , there are two nearly degenerate ground states in Fig. 7(a). This is also seen in Fig. 7(b), although the ordering of the two states is sometimes reversed. Note that it is crucial to consider *interacting* composite fermions in order to explain the low-energy spectrum—for noninteracting composite fermions, all states at a given L^* in Fig. 7(b) would be degenerate.

The ground state at $L = (2m+1)M$ is fully spin polarized, known to be well described by the Laughlin wave function.²⁸ We find that there is another state very close to it in energy. We do not fully understand the origin of this state at the moment. It is believed that the Laughlin state is nondegenerate in the thermodynamic limit even for zero Zeeman energy, except for the spin multiplicity.

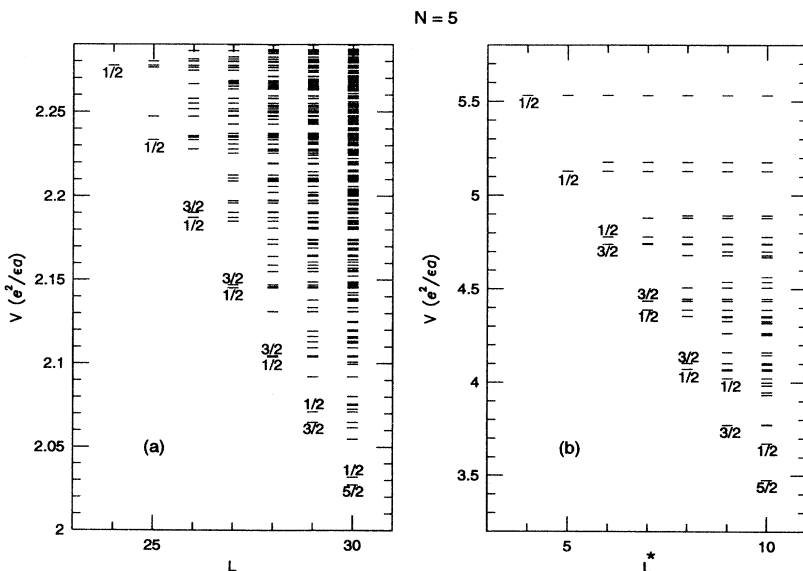


FIG. 7. (a) The low-energy spectrum of the five-electron system with Coulomb interaction. (b) The low-energy spectrum of a five-fermion system with Coulomb interaction. The energies are in units of $\frac{e^2}{\epsilon_0 a}$.

In our calculations, we find that the energy difference between the two states increases with N (it is 0.0027, 0.0046, and 0.0047, in units of $\frac{e^2}{\epsilon_0 a}$ for $N = 4, 5,$ and $6,$ respectively), suggesting that the other state may not be relevant in the thermodynamic limit. We also note here parenthetically that the ground-state quantum numbers in our calculations are sometimes in disagreement with those of Ref. 6.

D. Nature of the ground states

It was noted in Ref. 19 that the ground states of interacting electrons are “compact” states of composite fermions. In order to define compact states, and their relationship to ground states, let us first consider the system of noninteracting fermions in the range $-M \leq L^* \leq M$. Let us fix the number of electrons in the k th LL to be N_k ($\sum N_k = N$). Then, the state with the smallest total angular momentum is called compact, denoted by $[N_0, N_1, \dots, N_K]$, where K is the index of the highest occupied LL. This state has the property that the N_k electrons in the k th LL occupy the innermost angular momentum states (hence the name compact). It is clear that all of the positive cusp states of noninteracting fermions are compact, since otherwise the total angular momentum can always be decreased without increasing the kinetic energy. (Of course, every compact state is not necessarily associated with a positive cusp.)

We have already shown that the principal cusps of interacting electrons can be explained in terms of noninteracting composite fermions. These states are therefore compact states of composite fermions. Their wave functions, denoted by $[N_0, N_1, \dots, N_K]_{\text{CF}}$, are obtained according to Eq. (3), i.e., by multiplying the wave functions of compact electron states by the Jastrow factor, and then projecting on to the lowest LL. These wave functions have been tested successfully for the case of fully polarized electrons.^{19,31}

Note that the L^* spectrum from 0 to M is related by reflection symmetry to the other half (0 to $-M$). The horizontal lines of one side map into the tilted lines of the other side and vice-versa, and a cusp at L^* also implies a cusp at $-L^*$. Similar approximate mirror symmetry is also seen here in Figs. 2–5 about $L = 2$ mM in the

interacting electron ground-states. In particular, the spin quantum numbers and the cusp sizes are (approximately) reflected about $L = 2$ mM.

In the disk geometry, all cusps may not be associated with thermodynamic FQHE states, as indicated in Ref. 19. This is related to the fact that the IQHE state at $\nu^* = n$ cannot, in general, be identified precisely. For fully spin-polarized electrons, the only exception is the $\nu^* = 1$ state, which allows an identification of the Laughlin states at $\nu = \frac{1}{(2m+1)}$; the other cusps are not associated with any thermodynamic FQHE states. In the present case, with low Zeeman energy, the IQHE state at $\nu^* = 2$ is the only state that can be identified unambiguously: for an even number of electrons, it occurs at $L^* = \frac{N}{2}(\frac{N}{2} - 1)$. Consequently, the cusp at $L = 2M + \frac{N}{2}(\frac{N}{2} - 1)$ can be associated with the $\frac{2}{5}$ spin-singlet FQHE state, and the cusp at $L = 2M - \frac{N}{2}(\frac{N}{2} - 1)$ with the $\frac{2}{3}$ spin-singlet state. These are the last and the first cusps in the CF kinetic-energy curves. Other cusps do not correspond to FQHE states.

IV. CONCLUSION

This study analyzes interacting lowest-LL electrons in quantum dots in the small Zeeman-energy limit in the framework of composite fermions, and shows that the prominent features are understood in terms of noninteracting composite fermions. For more detailed information, it is necessary to incorporate the residual interaction between composite fermions. While the true interaction between composite fermions is necessarily quite complex, we find that any repulsive two-body interaction provides a reasonable qualitative picture.

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¹ M.A. Reed, J.N. Randall, R.J. Aggarwal, R.J. Matyi, T.M. Moore, and A.E. Wetsel, Phys. Rev. Lett. **60**, 535 (1988).

² V.J. Goldman, B. Su, and J.E. Cunningham, in *Nanostructures and Mesoscopic Systems*, edited by W.P. Kirk and M.A. Reed (Academic, New York, 1991), p. 173; B. Su, V.J. Goldman, and J.E. Cunningham, Science **255**, 313 (1992); Phys. Rev. B **46**, 7644 (1992).

³ O. Klein, C. de C. Chamon, D. Tang, D.M. Abusch-Magder, U. Meirav, X.G. Wen, M.A. Kastner, and S.J. Wind, Phys. Rev. Lett. **74**, 785 (1995).

⁴ R.C. Ashoori, H.L. Stormer, J.S. Weiner, L.N. Pfeiffer, K.N. Baldwin, and K.W. West, Phys. Rev. Lett. **71**, 613 (1993); Surf. Sci. **35**, 558 (1994).

⁵ B. Meurer, D. Heitman, and K. Ploog, Phys. Rev. Lett. **68**, 1371 (1992).

⁶ S.-R. Eric Yang, A.H. MacDonald, and M.D. Johnson, Phys. Rev. Lett. **71**, 3194 (1993).

⁷ M. Wagner, U. Merkt, and A.V. Chaplik, Phys. Rev. B **45**, 1951 (1992).

⁸ P. Hawrylak, Phys. Rev. Lett. **71**, 3347 (1993).

⁹ J.J. Palacios *et al.*, Phys. Rev. B **50**, 5760 (1994).

¹⁰ D. Pfannkuche, R.R. Gerhardt, P.A. Maksym, and V. Gudmundsson, Physica B **189**, 6 (1993).

¹¹ C. de C. Chamon and X.G. Wen, Phys. Rev. B **49**, 8227 (1994).

¹² W. Lai, K. Lu, Z. Su, and L. Yu, Solid State Commun. **52**,

- 339 (1994).
- ¹³ P.L. McEuen *et al.*, Phys. Rev. Lett. **66**, 1926 (1991); Physica B **189**, 70 (1993).
- ¹⁴ D.V. Averin and K.K. Likharev, in *Mesoscopic Phenomenon in Solids*, edited by B.L. Altshuler *et al.* (Elsevier, Amsterdam, 1991), and references therein.
- ¹⁵ D. Pfannkuche, V. Gudmundsson, and P.A. Maksym, Phys. Rev. B **47**, 2244 (1993).
- ¹⁶ S.M. Girvin and T. Jach, Phys. Rev. B **29**, 5617 (1984).
- ¹⁷ B.L. Johnson and G. Kirczenow, Phys. Rev. B **47**, 10563 (1993).
- ¹⁸ C.W.J. Beenaker and B. Rejaei, Physica B **189**, 147 (1993).
- ¹⁹ J.K. Jain and T. Kawamura, Europhys. Lett. **29**, 321 (1995).
- ²⁰ D.C. Tsui, H.L. Stormer, and A.C. Gossard, Phys. Rev. Lett. **48**, 1559 (1982).
- ²¹ J.K. Jain, Phys. Rev. Lett. **63**, 199 (1989); Phys. Rev. B **41**, 7653 (1990); Adv. Phys. **41**, 105 (1992).
- ²² R.L. Willet *et al.*, Phys. Rev. Lett. **59**, 1766 (1987); J.P. Eisenstein *et al.*, *ibid.* **61**, 997 (1988); R.G. Clark *et al.*, *ibid.* **62**, 1536 (1989); J.P. Eisenstein, *ibid.* **62**, 1540 (1989).
- ²³ B.I. Halperin, Helv. Phys. Acta **56**, 75 (1983); F.C. Zhang, and Tapash Chakraborty, Phys. Rev. B **30**, 7320 (1984); F.D.M. Haldane and E.H. Rezayi, Phys. Rev. Lett. **60**, 956 (1988); X.C. Xie and F.C. Zhang, Mod. Phys. Lett. B **5**, 471 (1991); T. Chakraborty and P. Pietilainen, *The Fractional Quantum Hall Effect* (Springer-Verlag, Berlin, 1988); X.G. Wu, G. Dev, and J.K. Jain, Phys. Rev. Lett. **71**, 153 (1993).
- ²⁴ X.G. Wu and J.K. Jain, Phys. Rev. B **49**, 7515 (1994).
- ²⁵ V. Fock, Z. Phys. **47**, 446 (1928); C.G. Darwin, Proc. Cambridge Philos. Soc. **27**, 86 (1930); G.W. Bryant, Phys. Rev. Lett. **59**, 1140 (1987).
- ²⁶ E.R. Gagliano, E. Dagotto, A. Moreo, and F.C. Alcaraz, Phys. Rev. B **34**, 1677 (1986).
- ²⁷ M. Stone, H.W. Wyld, and R.L. Schultz, Phys. Rev. B **45**, 14156 (1992).
- ²⁸ R.B. Laughlin, in *The Quantum Hall Effect*, edited by R.E. Prange and S.M. Girvin (Springer-Verlag, New York, 1990), and references therein.
- ²⁹ P.A. Maksym and T. Chakraborty, Phys. Rev. Lett. **65**, 108 (1990); Phys. Rev. B **45**, 1947 (1992). For recent reviews, see T. Chakraborty, Commun. Condens. Matter Phys. **16**, 35 (1992); M.A. Kastner, Rev. Mod. Phys. **64**, 849 (1992), and references therein.
- ³⁰ We remind the reader that in the n th LL ($n = 0, 1, \dots$) the single-particle angular momentum can take values $\ell = -n, -n + 1, \dots$.
- ³¹ G. Dev and J.K. Jain, Phys. Rev. B **45**, 1223 (1992).