Theory of mesoscopic transport in disordered wires

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We present a nonperturbative microscopic approach to mesoscopic transport in disordered phase-coherent systems, based on the exact scaling ("evolution") equations for the transfer and scattering matrices and Landauer's formula. In this work, we apply the method to the quasi-one-dimensional limit and provide a systematic treatment of all the transport regimes present. In the *ballistic* regime, we consider the transition between the Sharvin and Drude conductances and recover many exact results of the classic transport theory. In the *weak localization* (diffusive) regime, the bulk of the probability distribution of the conductance of a wire is shown to be normal Gaussian, in agreement with the predictions based on the nonlinear σ model. The formalism developed provides a convenient way of calculating the quantum-interference effects not only in the conductance but in arbitrary linear statistics on the transmission eigenvalues as well. In the classic diffusive limit, the eigenvalue density is independent on the geometry of the conductor. In the *strong localization* regime, the distribution of the conductance is found to coincide with Abrikosov's solution (approximately log normal), describing a purely one-dimensional chain. The calculated value of the localization length is proportional to the cross section of the wire, in precise agreement with Efetov's result obtained with the use of supersymmetric techniques. We recover the theory of the coherent backscattering peak and calculate the enhancement factor. The angular structure of the transmission and reflection coefficients is calculated exactly and found to be nonisotropic, which is contrary to the standard isotropy assumption of the existing macroscopic models. The approach is shown to provide microscopical proof of the Dorokhov-Mello-Pereyra-Kumar equation for the distribution of transmission eigenvalues within the realistic model of a wire with the isotropic dispersion law.

I. INTRODUCTION

The question about electron conductivity of low-dimensional systems has a long history. It started with the work of Mott and Twose,[1] who predicted that the localization of all the electron states by a random potential at $T = 0$ (Anderson localization[2]) takes place in a purely one-dimensional (1D) metal even at an arbitrarily weak disorder. Another important step was done by Thouless,[3] who mentioned that electron localization should take place even in the case of a wire of a *finite* thickness. An analogous prediction and further generalization to films of finite thickness was made by Abrahams *et al.*[4]

The assertion of Ref. 3 was based on the scaling hypothesis. It did not matter whether a chain or a thick wire was considered. At the same time, generally speaking, these models are quite different in physics as well as in the *microscopical* methods used to deal with them.

The general assertion about the electron localization implies that the length of the wire L is much larger than the radius of a typical one-electron wave function, i.e., the localization length L_c. In the case of a purely 1D chain, L_c is of the order of the free mean path l, and for the only present regime — that of strong localization — there exist powerful yet ideologically simple microscopical approaches,[5–8] which enable one to calculate practically any kinetic characteristic of interest. In particular, it has been found that the average conductance G decays exponentially with the length of the chain, and that the distribution function of the conductance is approximately log normal,[9,10] i.e., a Gaussian distribution for $\ln G$ rather than for G itself. The physical reason behind those findings turned out to be that backward scattered waves and propagating ones remain coherent even after multiple acts of scattering, thus leading to the formation of *standing* waves, which can carry only an exponentially small current in the long length limit.

In the opposite case of a sufficiently thick wire, there are two well distinguishable regimes. Since the electron motion in the transverse direction is quantized, the number of channels available for scattering $N \sim k_F^2 S$ (with $\hbar k_F$ the Fermi momentum and S the cross section of the wire) is finite, yet may be very big. Only one of these channels — that corresponding to backward (thus coherent) scattering — actually counts towards localization, while scattering into the others is diffusive and rather maintains the population balance between the channels. As a result, the characteristic length needed for complete localization increases by a factor of N, as compared to a purely 1D chain, i.e., the localization length $L_c \sim Nl$. Since $L_c \sim lk_F^2 S$ considerably exceeds the mean free path l at a weak disorder with $lk_F \gg 1$ ($\hbar k_F$ is the Fermi momentum, S is the cross section of the wire), it is the diffusive (metallic) regime and not the localization one, which occurs in a relatively short wire with $l \ll L \ll L_c$, or, in terms of measurable quantities, with the residual conductance $G \gg e^2/h$ [$e^2/h \approx (25.8\ \mathrm{k}\Omega)^{-1}$]. In this regime (referred to in the given context as weak localization), the mean conductance $\langle G \rangle$ (angular brackets denote en-

semble averaging) obeys Ohm's law (known also as the Drude formula)

$$\langle G\rangle \approx \frac{e^2}{h}\frac{2}{3\pi}\frac{lk_F^2 S}{L} = \frac{S}{L}\sigma_0 \tag{1.1}$$

with small corrections, which can be calculated by means of the perturbation theory in the form of the impurity-averaged Green's-function technique (see, e.g., Ref. 11 and also Appendix D in Ref. 12). These weak localization (WL) corrections can be qualitatively understood as quantum effects arising due to the already mentioned coherent backscattering (in the momentum space) or, equally, due to the interference of time-reversed pairs of trajectories forming closed loops: this phenomenon gives rise to an enhanced probability of return of a diffusing electron to its starting point (compared to the value given by the usual kinetic equation) and thus decreases the electrical conductivity at low temperatures. (For a qualitative description of weak localization, see Refs. 13–15.)

A noteworthy effect predicted by this technique and widely observed experimentally is that for any small metal sample the WL correction appears both as a temperature-dependent decrease in the conductance and as a low-field negative magnetoresistance.[16,17] In the case of a wire, the correction at zero temperature is of order e^2/h; that is, it does not depend on the sample's length.

In the strong localization regime, in which the sample length L is much longer than the localization length L_c, the perturbation theory ceases to work. Much of our present understanding of this regime has come from the so-called *macroscopical* approach,[18–22,12] which has proven to be very successful in describing 1D and quasi-1D systems. At the same time, the method suffers from the rather unphysical "isotropy assumption" (known also as the *maximum-entropy criterion*), a generalization of the *random-phase* hypothesis[23] in 1D. To find the statistical distribution of the transfer matrix (see below), this model requires that each of the scattering channels be randomly mixed with all the others. Since the dimensionality can enter this scheme only through the number of channels N, the macroscopic approach appears not to allow the study of the two- or three-dimensional limits.[12] Even in the 1D and quasi-1D limits, this method requires independent calculations within the framework of a *microscopical* method as *a posteriori* proof of the validity of the assumptions made.

There are also solvable models of N weakly coupled chains.[24–27] However, the *microscopical* description in terms of separate chains corresponds to sufficiently thin (in comparison with the free mean path l) wires, and the number of the chains or channels cannot be large;[28] effectively, the isotropic assumption is again made use of here.

It seems that so far the only developed *microscopical* approach suitable for dealing with both the strong and weak localization regimes is the supermatrix σ model approach pioneered by Wegner,[29] Shäfer and Wegner,[30] and Efetov and co-workers.[31,32,28] By mapping the original problem of electron conductivity in a random potential to the thermodynamics of a supermatrix field, Efetov and Larkin[33] managed to calculate the density-density correlator at large distances and whereby the exact value of the localization length. More recently, Zirnbauer[34] obtained in the same vein the exact result for the mean conductance of a wire of an arbitrary length.

However, despite these remarkable achievements and other results obtained within the framework of the nonlinear σ model, how to find microscopically more general quantities, such as the distribution function of the conductance $P_L(G)$, remains in large part an open question. It is worth reminding that the general shift in the study of disordered systems from the mean values to fluctuations and, ultimately, toward the entire distribution functions has occurred when the so-called mesoscopic fluctuation phenomena were discovered in the conductance of small samples at low temperatures. In particular, time-independent reproducible aperiodic oscillations in the resistance as a function of magnetic field or Fermi energy[35,36] have been observed. From the theory of these effects (the Green's-function technique),[37–43] there follows the striking prediction that the variance of the conductance fluctuations is anomalously large and always of order $(e^2/h)^2$ when the sample is in the metallic regime (that is, when $L \ll L_c, L_{\rm in}$, with $L_{\rm in}$ the phase-coherence length set by inelastic processes). The presence of these "universal conductance fluctuations" (UCF) means that the conductance is not a self-averaged quantity, and that besides the mere calculation of the mean or typical value, it is necessary to study the entire distribution function of the conductance in an ensemble of samples having identical macroscopic characteristics. (This approach was named mesoscopic.)

This fundamental in the given context problem of calculating the conductance distribution function has been solved — but only in the metallic regime — by Al'tshuler, Kravtsov, and Lerner[44] (see also Ref. 45 where some corrections have been made) who used the σ model. They have shown that the body of the distribution is normal Gaussian; at the same time, the probability of large fluctuations is much higher than Gaussian: The tails of the G distribution function turn out to be logarithmically normal.

Much less is known about the distribution functions in the strong localization regime. As a rule, one has to rely on the insight gained by the 1D theories, semiphenomenological models, and on extrapolations from the metallic region. The already mentioned approximate log normality of the distribution function in the purely 1D case gives rise to gigantic fluctuations in the conductance,[9,46] so that the relative fluctuations are large compared to unity and do not decrease with $L \to \infty$ but increase. The same results have been obtained within the framework of the model of N weakly coupled chains[27] as well as by means of the macroscopical approach.[47]

The patently not fortuitous similarity between these results for $P_L(G)$ at $G \ll e^2/h$ and the dimensionality $d = 1$ on one hand, and the shape of the *tails* of the distribution obtained in Ref. 44 at $G \gg e^2/h$ and any dimensionality on the other hand, strongly suggests that the log-normal distribution (or, to be more precise, Abrikosov's solution[9]) is a universal distribution in

the localized regime. This would be quite natural since the conductance is expected to decay *exponentially* for any realization of the random potential, i.e., to be of *multiplicative* nature, and, therefore, $\ln G$ as an *additive* variable should eventually have a Gaussian distribution. Nonetheless, despite strong qualitative hints, there has so far been no microscopical derivation of the conduction distribution function $P_L(G)$ at $G \ll e^2/h$ even in the simplest case of a quasi-1D wire. As it comes to the 2D case, the situation becomes even more uncertain. The universally accepted notion of strong electron localization in quasi-2D films at arbitrarily weak disorder, which is one of the main conclusions of the scaling theory of localization,[4] is of fundamental importance and has become one of the cornerstones of the quantum Hall effect theory. Yet the only microscopical confirmation of this notion stems in fact from extrapolation of the expression for the conductance at $G \gg e^2/h$ (WL regime) to the region of small $G \ll e^2/h$ (strong localization regime) by means of the one-parameter scaling. At the same time, it has been shown in Ref. 44 that in the crossover region $G \lesssim e^2/h$, the distribution $P_L(G)$ at $d = 2$ *cannot* be specified by just one parameter. Thus, the one-parameter scaling, even if it is understood in its advanced form as the assumption of a one-parameter distribution function,[48] breaks down just in the region it ought to work if any conclusions concerning the "behind-the-wall" insulating regime are to be made.

As we see, the microscopic theory of strong localization in low-dimensional systems (except the purely 1D case) is far from being complete. On the other hand, the recent interest in mesoscopic effects in transport properties other than the conductance[49–52] (so-called linear statistics, such as the shot-noise power), as well as significant difficulties with the current-conserving treatment of WL in disordered[53] and ballistic[54] conductors, have clearly revealed the insufficiency of the conventional diagrammatic technique even in the domain of weak localization. This state of affair is in striking contrast to a practically complete physical picture of 1D transport. It is the straightforward and analytically tractable character of most of the 1D models that motivated us to seek a possible resolution to some of the outlined problems on the way of generalization of those 1D methods to realistic 3D systems. The physical reason that provides this possibility is the above mentioned one-dimensional character of the underlying phenomenon — coherent backward scattering in the momentum space. We present such a generalized approach below and apply it to the simplest case of a quasi-1D wire in which only potential scattering is present. The further development of the method, which is to include quasi-2D systems, is left for separate publications.[55]

II. OUTLINE AND PRINCIPAL RESULTS

Technically speaking, there are two different forms of the starting formula for the conductance often used in the literature. The first expresses conductance as an average over the sample volume[56,57] (the Kubo approach), and the second expresses conductance in terms of scattering coefficients[58,56] (the Landauer approach); as a matter of fact, the two are equivalent.[56,57] The former approach has been used in most studies of WL and UCF, for it leads to a simpler set of diagrams than does the Landauer approach. The latter is commonly used in studies of strong localization and will be used in the present paper.

Our starting point is standard for any theory based on the generalized two-probe Landauer formula,[58,56] which relates the conductance of the system to the transmission matrix of the disordered conductor $\hat{t}$,

$$g = 2\,\mathrm{Tr}[\hat{t}\,\hat{t}^{\dagger}] = 2\sum_{\mathbf{n},\mathbf{n}'} |t_{\mathbf{n},\mathbf{n}'}|^2, \tag{2.1}$$

where g is the dimensionless conductance measured in units of e^2/h (the factor of 2 accounts for spin degeneracy) and $t_{\mathbf{n},\mathbf{n}'}$ is the transmission amplitude between the propagating states $\mathbf{n}$, $\mathbf{n}'$ in the leads (assumed ideal). In Sec. III, we review how a disordered mesoscopic conductor of fixed transverse sizes and variable length L is described by a transfer matrix, which is the product of many transfer matrices determining the wave propagation through each cross section of the sample. The simple multiplicative nature of the transfer matrix suggests the existence of the simple linear scaling equations it must obey; they are found in Sec. IV. In Sec. IV, we also find the scaling equations for the components of the total scattering matrix (that is for the transmission and reflection matrices). All these equations may be thought of as "dynamical equations" of random motion (the length L acts as the time variable). Taken together with the initial conditions, they provide a "causal" description of the system, thus immediately allowing statistical averaging, which will be carried out in Sec. V. We shall see that the statistical description in terms of the moments of the conductance $\langle g^n \rangle$ requires the solution of an infinite set of coupled equations, which govern the "evolution" with length of the averages of various products of the transmission and reflection amplitudes. The key feature of this set is that the structure of the averages as a function of channel indices can be obtained without solving the whole system. Section VI is devoted to studying this structure, which is closely related to the problem of finding the exact ladder sum for a sample with boundaries in the theory of WL[59,60] and in the theory of backscattering effects in optics.[60–64] Although the exact *ladder sum* is customarily treated in the *diffuson* approximation (the terms are often used interchangeably in the literature), which makes the specification of boundary conditions uncertain and causes a breakdown of local and global current conservation (see Ref. 53 for a discussion), it turns out that an exact analytical solution does exist; moreover, it has been known (in a slightly different context) to the astrophysicists' community since the 1930s.[65] The dependence of the averages on channel indices (the channels are labeled by the corresponding transverse momenta) is shown to be nonisotropic, which is contrary to the standard isotropy assumption of the macroscopic approach.[18–22,12] As an immediate application of the derived formulas, we consider the transition from the bal-

listic regime (the Sharvin conductance) to the diffusive regime (Ohm's law) and calculate the so-called "injection depth." In that section we also recover the theory of the coherent backscattering peak and calculate the enhancement factor.

In Sec. VII, we proceed to finding the closed set of equations for the averages already summed up with respect to the channel indices. In the case of a quasi-1D geometry, the averages are readily reduced to various crossed moments of the traces $t_n = \mathrm{Tr}[(\hat{t}\hat{t}^\dagger)^n]$. The resulting infinite set of equations is equivalent to a single one-parameter Fokker-Planck (FP) equation, which governs the evolution with length of the joint distribution function $P_L(t_1, t_2, \dots,)$ for all the t_n. Interestingly, we shall see that this equation closely resembles the renormalization-group equation for the additional charges, which describe additional contributions to the cumulants $\langle g^n \rangle_c^{\mathrm{add}}$ in the nonlinear σ model.[44]

The introduced equations can be evaluated in the metallic regime $L \ll Nl$ and in the insulating regime $L \gg Nl$. These two opposite regimes are discussed separately in Sec. VIII and Sec. IX. We reproduce the known results in the weakly localized (metallic) limit; in particular, the WL correction to the conductance and to an arbitrary linear statistic, and UCF. The bulk of the conductance distribution function $P_L(G)$ is found to be Gaussian, in agreement with the results of Refs. 44 and 45. Proceeding to the limit of strong localization, we find that $P_L(G)$ is given in this case by Abrikosov's distribution[9] (approximately log normal) indeed, provided the 1D localization length l is replaced in the distribution by $L_c = (6\pi)^{-1} l k_F^2 L_y L_z$. That enables us to identify this parameter as the exact value of the quasi-1D localization length, in precise agreement with Efetov's result.[28]

We stress that the FP equation derived in Sec. VII is of an infinite order, yet all its variables t_n can be easily expressed in terms of just N eigenvalues $\mathcal{T}_i$ of the matrix $\hat{\mathcal{T}} = \hat{t}\hat{t}^\dagger$: $t_n = \sum_{i=1}^N \mathcal{T}_i^n$. Thus, the set of variables t_n is excessive, which means that any solution must contain a product of δ functions of some combinations of the t_n. The FP equation for the joint probability distribution of the eigenvalues $\mathcal{T}_n$ [or, to be more precise, of variables $\lambda_n = (1 - \mathcal{T}_n)/\mathcal{T}_n$], known also as the Dorokhov-Mello-Pereyra-Kumar (DMPK) equation, has been derived within the framework of the model of weakly coupled chains[26] and, independently, with the help of the macroscopical approach.[18] Lately, the DMPK equation has become a subject of a very extensive research.[12,47,51,66,67] In Sec. X, we show that the Fokker-Planck equation for the variables $t_n = \mathrm{Tr}[\hat{\mathcal{T}}^n]$ is equivalent to the DMPK equation, thus providing *microscopical* proof of the latter within the realistic model of a quasi-1D wire with the isotropic dispersion law (that is, a real 3D object with 1D *geometry*). Thus, despite the fact that the isotropy assumption, which is crucial for the derivation of the DMPK equation in Refs. 26 and 18, is shown to be invalid in general, the resulting equation still holds. In that section we also discuss some other connections with the existing theories.

III. SCATTERING AND TRANSFER MATRICES

Let us consider the propagation of an electron through a disordered conductor of transverse sizes $L_y \times L_z$ at $T = 0$; it is assumed that the disordered region of length L is placed between two perfect leads. Such a process is described by the Schrödinger equation

$$[-\nabla^2 + 2mU(\mathbf{r})]\Psi = k_F^2 \Psi, \tag{3.1}$$

where the impurity potential $U(\mathbf{r})$ is nonzero only when $x_1 \le x \le x_2$ $(x_2 - x_1 = L)$, which is the region occupied by the disordered layer.

We shall adopt the standard white-noise model for the statistics of the disordered potential (this corresponds to the Born approximation for scattering), assuming that $U(\mathbf{r})$ is a Gaussian random potential with a zero average and the correlator

$$\langle U(\mathbf{r})U(\mathbf{r}')\rangle = \frac{1}{2\pi\nu\tau}\delta(\mathbf{r}-\mathbf{r}') = \frac{\pi}{m^2 l}\delta(\mathbf{r}-\mathbf{r}'), \tag{3.2}$$

where ν is the density of states, $\tau = l/v_F$, and l is the mean free path.

In the absence of disorder, the transverse surfaces make the transverse momentum quantized; since the theory is not sensitive to the details of the boundary conditions, we choose them to be infinite hard walls for definiteness. Then, the eigenfunctions of an ordered sample are given by

$$e^{\pm i k_{\mathbf{n}} x}\phi_{\mathbf{n}}(y,z), \quad \mathbf{n} \equiv (n_y, n_z), \quad n_y, n_z = 1, 2, \dots, \tag{3.3}$$

where the transverse part of the wave function is

$$\phi_{\mathbf{n}}(\boldsymbol{\rho}) = \frac{2}{(L_y L_z)^{1/2}} \sin k_y y \sin k_z z, \tag{3.4}$$
$$k_y = \frac{\pi n_y}{L_y}, \quad k_z = \frac{\pi n_z}{L_z},$$

the longitudinal momentum $k_{\mathbf{n}}$ and the quantized transverse momentum $\mathbf{k}_{\perp\mathbf{n}}$ satisfying the relation

$$k_{\mathbf{n}}^2 + \mathbf{k}_{\perp\mathbf{n}}^2 = k_F^2. \tag{3.5}$$

The various $\mathbf{k}_{\perp\mathbf{n}} = (\pi n_y/L_y, \pi n_z/L_z)$, which satisfy Eq. (3.5) constrained by the condition that $k_{\mathbf{n}}$ be real, i.e., $\mathbf{k}_{\perp\mathbf{n}}^2 \le k_F^2$, define the $N = \frac{1}{4\pi} k_F^2 L_y L_z$ channels available for scattering. [Note that by definition the channels correspond to the *standing* waves (3.4), that is n_y, n_z are positive.] Since each channel can carry two waves propagating in opposite directions, the most general solution in the ordered regions is a linear combination of $2N$ unit-flux waves traveling to the right and to the left,

$$\Psi(\mathbf{r}) = \sum_{\mathbf{n}_1}\left[a_{1\mathbf{n}_1}\frac{e^{ik_{\mathbf{n}_1}(x-x_1)}}{k_{\mathbf{n}_1}^{1/2}} + b_{1\mathbf{n}_1}\frac{e^{-ik_{\mathbf{n}_1}(x-x_1)}}{k_{\mathbf{n}_1}^{1/2}}\right]\phi_{\mathbf{n}_1}(\boldsymbol{\rho}), \quad x \le x_1, \tag{3.6a}$$

$$\Psi(\mathbf{r}) = \sum_{\mathbf{n}_1}\left[a_{2\mathbf{n}_1}\frac{e^{ik_{\mathbf{n}_1}(x-x_2)}}{k_{\mathbf{n}_1}^{1/2}} + b_{2\mathbf{n}_1}\frac{e^{-ik_{\mathbf{n}_1}(x-x_2)}}{k_{\mathbf{n}_1}^{1/2}}\right]\phi_{\mathbf{n}_1}(\boldsymbol{\rho}), \quad x \ge x_2. \tag{3.6b}$$

One of the particularly important solutions describes an electron with the momentum $(k_{\mathbf{n}}, \mathbf{k}_{\perp\mathbf{n}})$ propagating from the left to the right; in the ordered leads it has the asymptotes

$$\Psi_{1\mathbf{n}}(\mathbf{r}) = \frac{e^{ik_{\mathbf{n}}(x-x_1)}}{k_{\mathbf{n}}^{1/2}}\phi_{\mathbf{n}}(\boldsymbol{\rho}) + \sum_{\mathbf{n}_1} r_{\mathbf{n}_1,\mathbf{n}} \frac{e^{-ik_{\mathbf{n}_1}(x-x_1)}}{k_{\mathbf{n}_1}^{1/2}}\phi_{\mathbf{n}_1}(\boldsymbol{\rho}), \quad x \le x_1, \tag{3.7a}$$

$$\Psi_{1\mathbf{n}}(\mathbf{r}) = \sum_{\mathbf{n}_1} t_{\mathbf{n}_1,\mathbf{n}} \frac{e^{ik_{\mathbf{n}_1}(x-x_2)}}{k_{\mathbf{n}_1}^{1/2}}\phi_{\mathbf{n}_1}(\boldsymbol{\rho}), \quad x \ge x_2, \tag{3.7b}$$

where $r_{\mathbf{n}',\mathbf{n}}$ and $t_{\mathbf{n}',\mathbf{n}}$ denote the reflected and transmitted amplitudes in channel $\mathbf{n}'$ when there is a unit-flux incident from the left in channel $\mathbf{n}$; one can also define amplitudes $r'_{\mathbf{n}',\mathbf{n}}$ and $t'_{\mathbf{n}',\mathbf{n}}$ with a similar meaning, except that the incident flux comes from the right. These elements form the $N \times N$ reflection and transmission matrices $\hat{r}$, $\hat{t}$ ($\hat{r}'$, $\hat{t}'$), respectively.

Now we are ready to introduce the scattering and transfer matrices. By definition, the $2N \times 2N$ scattering matrix $\check{S}$ relates the incoming flux to the outgoing flux,

$$\begin{pmatrix} \mathbf{b}_1 \\ \mathbf{a}_2 \end{pmatrix} = \check{S} \begin{pmatrix} \mathbf{a}_1 \\ \mathbf{b}_2 \end{pmatrix}, \tag{3.8}$$

where $\mathbf{a}_1$, $\mathbf{b}_1$, $\mathbf{a}_2$, and $\mathbf{b}_2$ are the N-component vectors formed by the coefficients of Eq. (3.6). [Note that due to the presence of the phase factors $\exp(\pm x_{1(2)})$ in Eq. (3.6), our definition of the $\check{S}$ matrix (and the $\check{M}$ below) is slightly different from a standard one.] Comparing the definitions (3.6)–(3.8), one can easily find that $\check{S}$ has the block structure

$$\check{S} = \begin{pmatrix} \hat{r} & \hat{t}' \\ \hat{t} & \hat{r}' \end{pmatrix}. \tag{3.9}$$

Current conservation implies that in Eq. (3.8)

$$|\mathbf{a}_1|^2 + |\mathbf{b}_2|^2 = |\mathbf{b}_1|^2 + |\mathbf{a}_2|^2, \tag{3.10}$$

which is equivalent to the unitarity of the $\check{S}$ matrix, $\check{S}\check{S}^\dagger = \check{S}^\dagger\check{S} = \check{1}$.

Time-reversal invariance imposes an additional constraint on the scattering matrix $\check{S}$. This being the case, if $\Psi(\mathbf{r})$ is a solution, then $\Psi'(\mathbf{r}) = \Psi^*(\mathbf{r})$ is also, with the coefficients given by $\mathbf{a}'_{1(2)} = \mathbf{b}^*_{1(2)}$ and $\mathbf{b}'_{1(2)} = \mathbf{a}^*_{1(2)}$. These coefficients must be related by the same $\check{S}$ matrix as that in Eq. (3.8), so that

$$\begin{pmatrix} \mathbf{a}_1 \\ \mathbf{b}_2 \end{pmatrix}^* = \check{S} \begin{pmatrix} \mathbf{b}_1 \\ \mathbf{a}_2 \end{pmatrix}^*. \tag{3.11}$$

Comparing with Eq. (3.8) and using the unitarity of the $\check{S}$ matrix, one can easily find the time-reversal invariance requirement $\check{S} = \check{S}^T$, that is, the $\check{S}$ matrix must be symmetric. From the definition (3.9) and the condition that $\check{S}$ be unitary and symmetric, we immediately obtain the relations for the transmission and reflection matrices

$$\hat{t}' = \hat{t}^T, \quad \hat{r} = \hat{r}^T, \tag{3.12a}$$

$$\hat{t}^\dagger\hat{t} + \hat{r}^\dagger\hat{r} = \hat{t}\hat{t}^\dagger + \hat{r}'\hat{r}'^\dagger = \hat{1}, \quad \hat{t}\hat{r}^* + \hat{r}'\hat{t}^* = \hat{0}. \tag{3.12b}$$

Although the $\check{S}$ matrix determines the conductance through Eq. (2.1), it is not immediately obvious how to derive a composition rule $\check{S}$ satisfies, which would be suitable for introducing a scaling approach. We are led now to consider the transfer matrix which contains the same physical information as $\check{S}$ does, but in a different form.

By definition the $2N \times 2N$ transfer matrix $\check{M}$ relates the flux amplitudes on the left-hand side of the disordered region to those on the right,

$$\begin{pmatrix} \mathbf{a}_2 \\ \mathbf{b}_2 \end{pmatrix} = \check{M} \begin{pmatrix} \mathbf{a}_1 \\ \mathbf{b}_1 \end{pmatrix}. \tag{3.13}$$

Similarly to $\check{S}$, we can write the transfer matrix $\check{M}$ in terms of four $N \times N$ blocks,

$$\check{M} = \begin{pmatrix} \hat{m}_{11} & \hat{m}_{12} \\ \hat{m}_{21} & \hat{m}_{22} \end{pmatrix}, \tag{3.14}$$

and from the definitions (3.8) and (3.13), one finds the relations

$$\begin{aligned} &\hat{m}_{11} = (\hat{t}^\dagger)^{-1}, \quad \hat{m}_{12} = \hat{r}'(\hat{t}')^{-1}, \\ &\hat{m}_{21} = -(\hat{t}')^{-1}\hat{r}, \quad \hat{m}_{22} = (\hat{t}')^{-1}. \end{aligned} \tag{3.15}$$

Note that as with the matrices $\hat{r}$, $\hat{t}$, the elements of the matrices $\hat{m}_{ij}$ can be found as coefficients in the asymptote of a particular solution of the Schrödinger equation (3.1), namely, the solution with a unit flux transmitted into a single channel. For instance,

$$\Phi_{1\mathbf{n}}(\mathbf{r}) = \frac{e^{-ik_{\mathbf{n}}(x-x_1)}}{k_{\mathbf{n}}^{1/2}}\phi_{\mathbf{n}}(\boldsymbol{\rho}), \quad x \le x_1, \tag{3.16a}$$

$$\Phi_{1\mathbf{n}}(\mathbf{r}) = \sum_{\mathbf{n}_1}\left[(\hat{m}_{12})_{\mathbf{n}_1,\mathbf{n}} \frac{e^{ik_{\mathbf{n}_1}(x-x_1)}}{k_{\mathbf{n}_1}^{1/2}} + (\hat{m}_{22})_{\mathbf{n}_1,\mathbf{n}} \frac{e^{-ik_{\mathbf{n}_1}(x-x_2)}}{k_{\mathbf{n}_1}^{1/2}}\right]\phi_{\mathbf{n}_1}(\boldsymbol{\rho}), \quad x \ge x_2. \tag{3.16b}$$

Just as with $\check{S}$, one can easily obtain the current conservation constraint on $\check{M}$,

$$\check{M}^\dagger\check{\Sigma}_z\check{M} = \check{\Sigma}_z, \quad \check{M}\check{\Sigma}_z\check{M}^\dagger = \check{\Sigma}_z, \tag{3.17}$$

and the time-reversal invariance constraint

$$\check{M}^* = \check{\Sigma}_x\check{M}\check{\Sigma}_x, \tag{3.18}$$

where $\check{\Sigma}_z$ and $\check{\Sigma}_x$ denote the $2N \times 2N$ Pauli block matrices

$$\check{\Sigma}_z = \begin{pmatrix} \hat{1} & \hat{0} \\ \hat{0} & -\hat{1} \end{pmatrix}, \qquad \check{\Sigma}_x = \begin{pmatrix} \hat{0} & \hat{1} \\ \hat{1} & \hat{0} \end{pmatrix}. \tag{3.19}$$

Note that Eqs. (3.17) are simply an alternative way of expressing the condition (3.10) rewritten in the form

$$|\mathbf{a}_1|^2 - |\mathbf{b}_1|^2 = |\mathbf{a}_2|^2 - |\mathbf{b}_2|^2. \tag{3.20}$$

The latter implies that unlike the unitary $\check{S}$ matrix, the transfer matrix $\check{M}$ is a $U(N,N)$ (pseudounitary) matrix.

The essential feature of the transfer matrix $\check{M}$, which follows directly from the definition (3.13), is the multiplicative nature of $\check{M}$: If two units with matrices $\check{M}_1$ and $\check{M}_2$ are connected in series, the transfer matrix $\check{M}$ of the combined system is simply the product $\check{M} = \check{M}_1\check{M}_2$. If one of the units is infinitesimally thin, this composition rule should be eventually transformed into a *linear* differential relation. We introduce such a scaling equation in the next section. On this basis, we derive the scaling equations for the $\check{S}$ matrix.

IV. SCALING EQUATIONS

Most of the equations in the previous section closely resemble the familiar equations describing a purely 1D system, with the understandable difference that $N \times N$ matrices substitute for 1D variables. To advance the analogy even further, we need a matrix analog of the 1D wave function. With this in mind we take the *set* of solutions of Eq. (3.1) with the asymptotes (3.7) and introduce the operator $\widehat{\Psi}_1(x)$ associated with this set in such a way that in the transverse momentum representation the corresponding matrix has the form

$$\Psi_{1\mathbf{n},\mathbf{n}'}(x) = \int d^2\boldsymbol{\rho}\, \phi_{\mathbf{n}}(\boldsymbol{\rho})\Psi_{1\mathbf{n}'}(x,\boldsymbol{\rho}). \tag{4.1}$$

It is to be noted that $\widehat{\Psi}_1(x)$ depends on the coordinate x as a *parameter*. This definition can be rewritten in an equivalent form

$$\widehat{\Psi}_1(x)\phi_{\mathbf{n}} = \Psi_{1\mathbf{n}}(x,\boldsymbol{\rho}), \tag{4.2}$$

which clearly reveals the physical meaning of the operator $\widehat{\Psi}_1(x)$. When acting on the transverse wave function of an *ordered* sample, it creates the corresponding (with the same transverse momentum in the leads) wave function of the *disordered* sample. In this sense, it resembles the creation operators in the representation of secondary quantization.

Now it is easy to see that Eqs. (3.7) are equivalent to the operator relation

$$\widehat{\Psi}_1(x) = \begin{cases} \widehat{k}_\parallel^{-1/2}\exp[-i\widehat{k}_\parallel(x-x_1)] \\ \quad +\widehat{k}_\parallel^{-1/2}\exp[-i\widehat{k}_\parallel(x-x_1)]\,\hat{r}, & x \le x_1, \\ \widehat{k}_\parallel^{-1/2}\exp[i\widehat{k}_\parallel(x-x_2)]\,\hat{t}, & x \ge x_2, \end{cases} \tag{4.3}$$

while the Schrödinger equation (3.1) takes the form

$$-\frac{d^2}{dx^2}\widehat{\Psi}_1 + 2m\widehat{U}\widehat{\Psi}_1 = \widehat{k}_\parallel^2\widehat{\Psi}_1. \tag{4.4}$$

Here $\widehat{U}(x)$ is the random potential operator given by

$$U_{\mathbf{n},\mathbf{n}'}(x) = \int d^2\boldsymbol{\rho}\, \phi_{\mathbf{n}}(\boldsymbol{\rho})U(x,\boldsymbol{\rho})\phi_{\mathbf{n}'}(\boldsymbol{\rho}), \tag{4.5}$$

and $\widehat{k}_\parallel$ is given by the diagonal matrix $k_{\parallel\mathbf{n},\mathbf{n}'} = k_{\mathbf{n}}\delta_{\mathbf{n},\mathbf{n}'}$.

The operator $\widehat{\Psi}_1$ describes scattering when there is a unit flux *incident* from the left. In the same manner we can introduce the operator $\widehat{\Phi}_1$ when there is a unit flux *transmitted* to the left. Then we find that the asymptotes of the corresponding set (3.16) are reduced to

$$\widehat{\Phi}_1(x) = \begin{cases} \widehat{k}_\parallel^{-1/2}\exp[-i\widehat{k}_\parallel(x-x_1)], & x \le x_1, \\ \widehat{k}_\parallel^{-1/2}\exp[i\widehat{k}_\parallel(x-x_2)]\,\widehat{m}_{12} \\ \quad +\widehat{k}_\parallel^{-1/2}\exp[-i\widehat{k}_\parallel(x-x_2)]\,\widehat{m}_{22}, & x \ge x_2, \end{cases} \tag{4.6}$$

whereas in the disordered region $\widehat{\Phi}_1$ satisfies the same Eq. (4.4).

It is important to note that being a differential equation with *boundary* conditions, Eq. (4.4) is not suitable for direct averaging over realizations of the random potential. Bearing in mind the statistical description of scattering, let us turn to finding the "dynamical" equations, i.e., differential equations with only *initial* conditions, that the $\check{M}$ and $\check{S}$ must satisfy. To this end we introduce the new variables $\widehat{u}_+$ and $\widehat{u}_-$,

$$\widehat{u}_\pm = \frac{1}{2}\widehat{k}_\parallel^{1/2}\left[\widehat{\Phi}_1 \pm i\widehat{k}_\parallel^{-1}\frac{d}{dx}\widehat{\Phi}_1\right]. \tag{4.7}$$

As Eq. (4.4) implies, these variables satisfy the following system of linear equations:

$$\frac{d}{dx}\widehat{u}_+ = -i\widehat{k}_\parallel\widehat{u}_+ + i\widehat{k}_\parallel^{-1/2}m\widehat{U}\widehat{k}_\parallel^{-1/2}[\widehat{u}_+ + \widehat{u}_-], \tag{4.8a}$$

$$\frac{d}{dx}\widehat{u}_- = i\widehat{k}_\parallel\widehat{u}_- - i\widehat{k}_\parallel^{-1/2}m\widehat{U}\widehat{k}_\parallel^{-1/2}[\widehat{u}_+ + \widehat{u}_-], \tag{4.8b}$$

while the boundary conditions (4.6) are reduced to

$$\widehat{u}_+(x_1) = \hat{1}, \quad \widehat{u}_-(x_1) = \hat{0}, \tag{4.9a}$$

$$\widehat{u}_+(x_2) = \widehat{m}_{22}, \quad \widehat{u}_-(x_2) = \widehat{m}_{12}. \tag{4.9b}$$

As we see, the operators $\widehat{u}_+$, $\widehat{u}_-$, as functions of x, satisfy the system (4.8) and the *boundary* conditions (4.9). On the other hand, the condition (4.9b) implies that the operators $\widehat{m}_{12}$, $\widehat{m}_{22}$, when considered as functions of the position of the right boundary x_2, satisfy the very same system (4.8) with the *initial* conditions (4.9a). Thus, we have arrived at the required "dynamical" equations for the components $\widehat{m}_{12}$, $\widehat{m}_{22}$ of the transfer matrix $\check{M}$. It is to be noted that this system constitutes a generalization of the equation of state employed (in a different notation) by Abrikosov and Ryzhkin[6] in their approach to the kinetics of one-dimensional disordered systems. One can derive the equation for the two remaining components along the same lines as above, or simply by making use of the time-reversal invariance constraint (3.18). The four equations obtained can be written in the block matrix form as

$$\frac{d}{dx_2}\check{M}(x_1,x_2) = \left[i\widehat{k}_\parallel\check{\Sigma}_z - i\widehat{k}_\parallel^{-1/2}m\widehat{U}(x_2)\widehat{k}_\parallel^{-1/2}\left(\check{\Sigma}_z + i\check{\Sigma}_x\right)\right]\check{M} \tag{4.10a}$$

(notation $\hat{a}\check{B}$ means that all four components of $\check{B}$ are multiplied by $\hat{a}$), with the initial condition $\check{M}(x_1,x_1) = \check{1}$. The similar dynamical equation with respect to x_1 reads

$$\frac{d}{dx_1}\check{M}(x_1,x_2) = -\check{M}\left[i\widehat{k}_\parallel\check{\Sigma}_z - i\widehat{k}_\parallel^{-1/2}m\widehat{U}(x_1)\widehat{k}_\parallel^{-1/2}\left(\check{\Sigma}_z + i\check{\Sigma}_x\right)\right]. \tag{4.10b}$$

We now go over to finding the dynamic equations for the components of the scattering matrix $\check{S}$. To this end, we employ the relations (3.15) rewritten in the form

$$\hat{r} = -\widehat{m}_{22}^{-1}\widehat{m}_{21}, \quad \hat{t}' = \widehat{m}_{22}^{-1}, \qquad \hat{t} = (\widehat{m}_{11}^\dagger)^{-1}, \quad \hat{r}' = \widehat{m}_{12}\widehat{m}_{22}^{-1}. \tag{4.11}$$

Differentiating equations (4.11) with respect to x_1 and x_2 and allowing for Eq. (4.10a), we find after some simple algebra that the resulting eight equations can be written in the block-matrix form as

$$\frac{d}{dx_1}\check{S}(x_1,x_2) = -i\widehat{k}_\parallel\check{\tau}_+\check{S} - i\check{S}\check{\tau}_+\widehat{k}_\parallel + i\left(\check{1}+\check{S}\right)\widehat{k}_\parallel^{-1/2} \times m\widehat{U}(x_1)\widehat{k}_\parallel^{-1/2}\check{\tau}_+\left(\check{1}+\check{S}\right), \tag{4.12a}$$

$$\frac{d}{dx_2}\check{S}(x_1,x_2) = i\widehat{k}_\parallel\check{\tau}_-\check{S} + i\check{S}\check{\tau}_-\widehat{k}_\parallel - i\left(\check{1}+\check{S}\right)\widehat{k}_\parallel^{-1/2} \times m\widehat{U}(x_2)\widehat{k}_\parallel^{-1/2}\check{\tau}_-\left(\check{1}+\check{S}\right), \tag{4.12b}$$

with the condition that $\check{S}(x_0,x_0) = \check{\Sigma}_x$ for any x_0. Here the following matrices have been used:

$$\check{\tau}_+ = \begin{pmatrix}\hat{1} & \hat{0}\\ \hat{0} & \hat{0}\end{pmatrix}, \qquad \check{\tau}_- = \begin{pmatrix}\hat{0} & \hat{0}\\ \hat{0} & \hat{1}\end{pmatrix}. \tag{4.13}$$

Each of the scaling equations (4.10) and (4.12) that we have derived determines a "dynamical" system (the coordinate acts as the time variable) influenced by random forces due to the potential $\widehat{U}(x)$. The "evolution" of the statistical distribution associated with such a system is governed by a Fokker-Planck equation, or alternatively can be described in terms of the moments of dynamical variables. We will discuss the properties of this statistical description in the next section.

V. STATISTICAL DESCRIPTION

Since the $\check{S}$ matrix is symmetric, not all the equations for its components are independent ones. In fact, the equations for $\hat{t}$ and $\hat{r}'$ in (4.12b) already constitute a closed system

$$\frac{d}{dx_2}\hat{r}'(x_1,x_2) = i\widehat{k}_\parallel\hat{r}' + i\hat{r}'\widehat{k}_\parallel - i\left(1+\hat{r}'\right)\widehat{k}_\parallel^{-1/2}m\widehat{U}(x_2)\widehat{k}_\parallel^{-1/2}\left(1+\hat{r}'\right), \tag{5.1}$$

$$\frac{d}{dx_2}\hat{t}(x_1,x_2) = i\left[\widehat{k}_\parallel - \left(1+\hat{r}'\right)\widehat{k}_\parallel^{-1/2}m\widehat{U}(x_2)\widehat{k}_\parallel^{-1/2}\right]\hat{t}, \tag{5.2}$$

which is sufficient for finding the quantities of interest. Nonetheless, having the other equations at hand is useful for various applications and sometimes leads to significantly simpler calculations (see below). Aside from Eqs. (5.1) and (5.2), we will use their counterparts from (4.12a),

$$\frac{d}{dx_1}\hat{r}'(x_1,x_2) = i\hat{t}\,\widehat{k}_\parallel^{-1/2}m\widehat{U}(x_1)\widehat{k}_\parallel^{-1/2}\,\hat{t}', \tag{5.3}$$

$$\frac{d}{dx_1}\hat{t}(x_1,x_2) = i\hat{t}\left[-\widehat{k}_\parallel + \widehat{k}_\parallel^{-1/2}m\widehat{U}(x_1)\widehat{k}_\parallel^{-1/2}\left(1+\hat{r}\right)\right]. \tag{5.4}$$

It is worth noting that as $L\to\infty$ and $\hat{t}\to 0$, the operators $\hat{r}$ and $\hat{r}'$ become unitary. It is obvious already from Eq. (3.12b), and follows as a matter of course from Eq. (5.1) since the latter conserves unitarity. This is why we have to deal with Eqs. (5.2) and (5.4), whereas Eq. (5.1) alone (otherwise entirely self-sufficient) is inconvenient for finding any quantities related to the conductance. One would need to extract a small quantity $\mathrm{Tr}[\hat{t}\hat{t}^\dagger] \ll N$ as a difference between $\mathrm{Tr}[\hat{r}'\hat{r}'^\dagger] \approx N$ and $\mathrm{Tr}[\hat{1}] = N$.

We now turn to the procedure of statistical averaging of Eqs. (5.2) and (5.3). As it follows from Eqs. (3.2), (3.4), and (4.5), the elements of the $N\times N$ matrix $\widehat{U}(x)$ are Gaussian random quantities with zero averages and δ-functional correlators

$$\langle U_{\mathbf{n}_1,\mathbf{n}_2}(x)U_{\mathbf{n}_3,\mathbf{n}_4}(x')\rangle = \frac{\pi\Delta(\mathbf{n}_1,\mathbf{n}_2,\mathbf{n}_3,\mathbf{n}_4)}{16m^2lL_yL_z} \times\delta(x-x'), \tag{5.5}$$

where

$$\Delta(\mathbf{n}_1,\mathbf{n}_2,\mathbf{n}_3,\mathbf{n}_4) = \sum_{i_{1,2,3,4}=\pm1} i_1\cdots i_4\,\delta_{0,\,i_1n_{1x}+\cdots+i_4n_{4x}} \times \sum_{j_{1,2,3,4}=\pm1} j_1\cdots j_4\,\delta_{0,\,j_1n_{1y}+\cdots+j_4n_{4y}}. \tag{5.6}$$

The "casual" character of the above scaling equations, when, say, $\hat{t}(x_1,x_2)$ and $\hat{r}'(x_1,x_2)$ are determined solely by the values of the random potential at points $x_1\le x\le x_2$, allows an immediate averaging over such $U_{\mathbf{n}_1,\mathbf{n}_2}(x)$. This can be easily explained by a simplified example of a dynamical system of the type

$$dX/dx = a(X) + v(x)b(X), \tag{5.7}$$

where $v(x)$ is white noise. We recall that averaging a product of Gaussian variables is equivalent to taking all possible pairings; thus, upon integrating Eq. (5.7) over x and taking advantage of the δ-functional correlator $\langle v(x)v(x')\rangle = 2C\delta(x-x')$, we easily find that $\langle v(x)X(x)\rangle = C\langle b(X)\rangle$. Obviously this procedure can be applied to any average of the type $\langle v(x)F[X(x)]\rangle$ with $F[X]$ a function or even a functional of $X(x)$. It is clear now that already at this point we could easily write down a system of exact scaling equations for various crossed moments of matrix elements $t_{\mathbf{n},\mathbf{n}'}$, $r'_{\mathbf{n},\mathbf{n}'}$, or obtain a FP

equation for the probability density $P_L(\check{S})$. Nonetheless, such a "Cartesian representation" fails to provide any readily tractable description, nor is it very instructive as it ignores the specific statistical features of Eqs. (5.2) and (5.3), which we are about to discuss here and in the next two sections.

First of all, under the common assumption $lk_F \gg 1$ (weak disorder) we suppose to hold from now on, those terms on the right-hand side of Eqs. (5.2) and (5.3), which contain the operator $\widehat{U}$, are small in comparison with the rest. As a result, the reflection and transmission amplitudes $\hat{r}'$, $\hat{t}$ turn out to be fast oscillating variables,

$$\hat{r}' = e^{i\hat{k}_\parallel x_2}\,\hat{r}'_0\,e^{i\hat{k}_\parallel x_2}, \quad \hat{t} = e^{i\hat{k}_\parallel x_2}\,\hat{t}_0\,e^{-i\hat{k}_\parallel x_1}, \tag{5.8}$$

with $\hat{r}'_0$, $\hat{t}_0$ varying slowly on the scale of k_F^{-1}. As a matter of fact, the measurable *kinetic* quantities are expressed in terms of *intensities*, so that we will be interested in quantities such as

$$t_{\mathbf{ab}}t^*_{\mathbf{cd}} \propto \exp[i(k_{\mathbf{a}} - k_{\mathbf{c}})x_2 - i(k_{\mathbf{b}} - k_{\mathbf{d}})x_1], \tag{5.9a}$$
$$r'_{\mathbf{ab}}r'^*_{\mathbf{cd}} \propto \exp[i(k_{\mathbf{a}} + k_{\mathbf{b}} - k_{\mathbf{c}} - k_{\mathbf{d}})x_2], \tag{5.9b}$$

and the like products, which have an equal number of matrix elements and their complex conjugates. (We need to allow a mismatch in the indices in order to get a closed system.) Unlike the exponents in the amplitudes (5.8), the oscillating coefficients in (5.9) may get cancelled out if the index mismatches are small; in the case of (5.9a), we have

$$\mathbf{a} \approx \mathbf{c}, \quad \mathbf{b} \approx \mathbf{d}. \tag{5.10}$$

Whereas in the case of (5.9b), there are two possibilities

$$\mathbf{a} \approx \mathbf{c},\ \mathbf{b} \approx \mathbf{d}; \quad \mathbf{a} \approx \mathbf{d},\ \mathbf{b} \approx \mathbf{c}. \tag{5.11}$$

We are thus led to apply the procedure of separating the variables into the "fast" variables with a range of the order of the electron wave length, and "slow" ones that describe the diffusive motion; all moments with an unequal number of matrix elements and their complex conjugates belong to the former type and are averaged out at the level of a statistical description. This procedure is well-known in the theory of 1D systems.[8] We start by finding the evolution equation for $\langle t_{\mathbf{ab}}t^*_{\mathbf{cd}}\rangle$. In fact, since all averaged quantities depend solely on $x_2 - x_1 = L$ (the translational invariance, broken in each realization of disordered potential, is restored by statistical averaging), we may use either of Eqs. (5.2) and (5.4), so that there are two different forms of the required evolution equation. From Eq. (5.2), upon omitting the fast variables we have

$$\frac{d}{dL}\langle t_{\mathbf{ab}}t^*_{\mathbf{cd}}\rangle = \frac{d}{dx_2}\langle t_{\mathbf{ab}}t^*_{\mathbf{cd}}\rangle$$
$$= i(K_{\mathbf{a}} - K^*_{\mathbf{c}})\langle t_{\mathbf{ab}}t^*_{\mathbf{cd}}\rangle + \sum_{\mathbf{n}_1,\mathbf{n}_2,\mathbf{n}_3,\mathbf{n}_4} \frac{\pi}{16lL_yL_z}\frac{\Delta(\mathbf{n}_1,\mathbf{n}_2,\mathbf{n}_3,\mathbf{n}_4)}{(k_{\mathbf{n}_1}k_{\mathbf{n}_2}k_{\mathbf{n}_3}k_{\mathbf{n}_4})^{1/2}}\langle(\delta_{\mathbf{an}_1}\delta_{\mathbf{cn}_3} + r'_{\mathbf{an}_1}r'^*_{\mathbf{cn}_3})t_{\mathbf{n}_2\mathbf{b}}t^*_{\mathbf{n}_4\mathbf{d}}\rangle, \tag{5.12}$$

where

$$K_{\mathbf{a}} = k_{\mathbf{a}} + i\frac{\pi}{lL_yL_z}\sum_{\mathbf{n}_1}\frac{1}{k_{\mathbf{a}}k_{\mathbf{n}_1}} = k_{\mathbf{a}} + \frac{i}{4\pi k_{\mathbf{a}}}\int_{k_\perp \le k_F}\frac{d^2\mathbf{k}_\perp}{k_\parallel} = k_{\mathbf{a}} + i\frac{k_F}{2k_{\mathbf{a}}l}. \tag{5.13}$$

Rewriting the last expression once more as $K^2_{\mathbf{a}} = 2m(\varepsilon_F + i/2\tau) - \mathbf{k}^2_{\perp\mathbf{a}}$ makes it clear that the renormalized momentum $K_{\mathbf{a}}$ corresponds to the familiar value of the self-energy $\varepsilon + i/2\tau$ in the averaged retarded Green's function.[68]

Similarly to Eq. (5.12), we have for the crossed moment $\langle r'_{\mathbf{ab}}r'^*_{\mathbf{cd}}\rangle$,

$$\frac{d}{dL}\langle r'_{\mathbf{ab}}r'^*_{\mathbf{cd}}\rangle = i(K_{\mathbf{a}} + K_{\mathbf{b}} - K^*_{\mathbf{c}} - K^*_{\mathbf{d}})\langle r'_{\mathbf{ab}}r'^*_{\mathbf{cd}}\rangle + \sum_{\mathbf{n}_1,\mathbf{n}_2,\mathbf{n}_3,\mathbf{n}_4}\frac{\pi}{16lL_yL_z}\frac{\Delta(\mathbf{n}_1,\mathbf{n}_2,\mathbf{n}_3,\mathbf{n}_4)}{(k_{\mathbf{n}_1}k_{\mathbf{n}_2}k_{\mathbf{n}_3}k_{\mathbf{n}_4})^{1/2}}$$
$$\times\langle(\delta_{\mathbf{an}_1} + r'_{\mathbf{an}_1})(\delta_{\mathbf{n}_2\mathbf{b}} + r'_{\mathbf{n}_2\mathbf{b}})(\delta_{\mathbf{cn}_3} + r'^*_{\mathbf{cn}_3})(\delta_{\mathbf{n}_4\mathbf{d}} + r'^*_{\mathbf{n}_4\mathbf{d}}) - \delta_{\mathbf{an}_1}\delta_{\mathbf{n}_2\mathbf{b}}r'_{\mathbf{n}_3\mathbf{n}_4}r'^*_{\mathbf{cd}} - \delta_{\mathbf{cn}_1}\delta_{\mathbf{n}_2\mathbf{d}}r'_{\mathbf{ab}}r'^*_{\mathbf{n}_3\mathbf{n}_4}\rangle. \tag{5.14}$$

The calculations we have performed so far are applicable to systems of any dimensionality in the regime $k_Fl \gg 1$. From now on we will restrict our analysis to a quasi-1D geometry $L \gg L_y \sim L_z$ (unless stated otherwise). From Eqs. (5.12) and (5.14), it follows that in this case any index mismatch in Eqs. (5.10) and (5.11) leads to an *additional* smallness in the corresponding moments with respect to $L_{y(z)}/L$ and, therefore, is not allowed. (This corresponds to the dominance of the "zero" mode in the Green's-function technique.) The general rule here is that in all equations there survive only those terms that upon substitution (5.8) contain no oscillating coefficients. Now Eq. (5.12) assumes the form

$$\frac{d}{dL}\langle|t_{\mathbf{ab}}|^2\rangle = -\frac{k_F}{k_{\mathbf{a}}l}\langle|t_{\mathbf{ab}}|^2\rangle + \frac{\pi}{lL_yL_z}\sum_{\mathbf{n}_1,\mathbf{n}_2}\frac{1}{k_{\mathbf{n}_1}k_{\mathbf{n}_2}}$$
$$\times\langle(\delta_{\mathbf{an}_1} + |r'_{\mathbf{an}_1}|^2)|t_{\mathbf{n}_2\mathbf{b}}|^2$$
$$+(1 - \delta_{\mathbf{n}_1\mathbf{n}_2})r'_{\mathbf{an}_1}r'^*_{\mathbf{an}_2}t_{\mathbf{n}_2\mathbf{b}}t^*_{\mathbf{n}_1\mathbf{b}}\rangle. \tag{5.15}$$

The multiplier $(1-\delta_{\mathbf{n}_1\mathbf{n}_2})$ has been used here to avoid double counting. When dealing with electronic systems, one is usually interested in the conductance, i.e., in the total transmission coefficient rather than the angle-resolved transmission coefficient $T_{\mathbf{ab}} = |t_{\mathbf{ab}}|^2$. We introduce the matrix $\widehat{\mathcal{T}} = \hat{t}\hat{t}^\dagger$ (not to be confused with the matrix $T_{\mathbf{ab}}$, which is a *direct* product of $\hat{t}$ and $\hat{t}^*$) and the total transmission coefficient (for *one* direction of spin) $T = \mathrm{Tr}[\widehat{\mathcal{T}}] = g/2$. The sum of (5.15) over $\mathbf{a}$, $\mathbf{b}$ yields

$$\frac{d}{dL}\langle T\rangle = \frac{d}{dL}\sum_{\mathbf{a}}\langle \mathcal{T}_{\mathbf{aa}}\rangle = \frac{\pi}{lL_yL_z}\sum_{\mathbf{n}_1,\mathbf{n}_2}\frac{1}{k_{\mathbf{n}_1}k_{\mathbf{n}_2}} \times\langle \mathcal{T}_{\mathbf{n}_1\mathbf{n}_1}\mathcal{T}_{\mathbf{n}_2,\mathbf{n}_2} + \mathcal{T}_{\mathbf{n}_1,\mathbf{n}_2}\mathcal{T}_{\mathbf{n}_2,\mathbf{n}_1}\rangle. \tag{5.16}$$

Although the reflection amplitudes have dropped out of Eq. (5.16), it is still not closed. We see that on the right-hand side of Eq. (5.16) there appear higher moments of individual elements of the matrix $\mathcal{T}$, so that their evolution equations are needed as well. One can see that those equations contain, in turn, even higher moments, the reflection amplitudes appearing explicitly at this time even in equations summed over the indices. Proceeding in this manner we would have arrived at a chain of evolution equations of ever growing complexity. A drastic simplification of this situation would be, however, possible if we knew the dependence of the moments under consideration on the channel indices. Then, upon employing the relation between the transmission and reflection matrices (3.12b), all the evolution equations could be expressed exclusively in terms of the traces $\mathrm{Tr}[\widehat{\mathcal{T}}^n]$ $(n = 1, 2, \ldots,)$, thus providing a reasonably simple description.

We are led now to consider the index dependence, i.e., an angular structure of the individual transmission and reflection coefficients $T_{\mathbf{ab}} = |t_{\mathbf{ab}}|^2$, $R'_{\mathbf{ab}} = |r'_{\mathbf{ab}}|^2$. Although being "nonobservable" quantities in the physics of disordered conductors, they are directly measured in light-scattering experiments.[60,63]

VI. ANGLE-RESOLVED SCATTERING

In order to find the angular structure of $T_{\mathbf{ab}}$ we go back to Eq. (5.15). In the cases of physical interest the number N of channels is very large, so that we can take advantage of an additional small parameter N^{-1}. One easily finds that the first two terms on the right-hand side of Eq. (5.15) are of the order of $l^{-1}N^{-2}\mathrm{Tr}[\widehat{\mathcal{T}}]$, while the third one is $\sim l^{-1}N^{-3}\mathrm{Tr}[\widehat{\mathcal{T}}^2]$ and can be omitted in the leading approximation in N^{-1}. (In the ballistic limit $L \lesssim l$, we have $\mathrm{Tr}[\widehat{\mathcal{T}}^n] \approx N$ and thus $\mathrm{Tr}[\widehat{\mathcal{T}}] \sim \mathrm{Tr}[\widehat{\mathcal{T}}^2]$; we shall see that the latter relation actually holds for *any* L.) Noticing that the conductance $g \propto L^{-1}$ (Ohm's law) in the diffusive limit $l \ll L \ll L_c$, whereas $g \propto \exp[-L/L_c]$ in the dielectric limit $L \gg L_c$, we estimate the derivative on the left-hand side as $\min^{-1}\{L, L_c\}N^{-2}\mathrm{Tr}[\widehat{\mathcal{T}}]$, so that it can be neglected as well if $L \gg l$. Finally, the second term on the right-hand side is decoupled in the main approximation in N^{-1} to yield an *algebraic* relation

$$0 = -\frac{k_F}{k_{\mathbf{a}}l}\langle |t_{\mathbf{ab}}|^2\rangle + \frac{\pi}{lL_yL_z} \times\sum_{\mathbf{n}_1,\mathbf{n}_2}\frac{1}{k_{\mathbf{n}_1}k_{\mathbf{n}_2}}\langle \delta_{\mathbf{an}_1} + |r'_{\mathbf{an}_1}|^2\rangle\langle |t_{\mathbf{n}_2\mathbf{b}}|^2\rangle. \tag{6.1}$$

Introducing functions

$$\tau_L(\mathbf{a}) = k_{\mathbf{a}}\sum_{\mathbf{n}_1}\frac{1}{k_{\mathbf{n}_1}}\langle |t_{\mathbf{n}_1\mathbf{a}}|^2\rangle, \tag{6.2a}$$

$$\rho_L(\mathbf{a}) = 1 + k_{\mathbf{a}}\sum_{\mathbf{n}_1}\frac{1}{k_{\mathbf{n}_1}}\langle |r'_{\mathbf{an}_1}|^2\rangle, \tag{6.2b}$$

we rewrite Eq. (6.1) in the form

$$\langle T_{\mathbf{ab}}\rangle = \frac{\pi}{k_FL_yL_z}\rho_L(\mathbf{a})\frac{\tau_L(\mathbf{b})}{k_{\mathbf{b}}}. \tag{6.3}$$

It is obvious that $\tau_L(\mathbf{a}) \equiv \tau_L(\mu_{\mathbf{a}})$, $\rho_L(\mathbf{a}) \equiv \rho_L(\mu_{\mathbf{a}})$, where $\mu_{\mathbf{a}} = k_{\mathbf{a}}/k_F = \cos\theta_{\mathbf{a}}$ with $\theta_{\mathbf{a}}$ being the incidence angle. The current conservation condition (3.12b) implies the identity

$$\int_0^1 d\mu\,[\tau_L(\mu) + \rho_L(\mu)] = 2. \tag{6.4}$$

In the limit $L \gg l$, the function ρ_L is approaching its limiting value $\rho(\mathbf{a}) \sim 1$, while $\tau_L(\mathbf{a}) \ll 1$. Since the *averaged* $T_{\mathbf{ab}}$ must be symmetric in $\mathbf{a}$, $\mathbf{b}$, from Eq. (6.3), we find that in the main approximation the angular dependence of $\langle T_{\mathbf{ab}}\rangle$ is given by the function ρ as follows:

$$\langle T_{\mathbf{ab}}\rangle = \frac{\rho(\mathbf{a})\rho(\mathbf{b})}{\left[\sum_{\mathbf{n}_1}\rho(\mathbf{n}_1)\right]^2}\langle T\rangle = \frac{1}{16D}\rho(\mu_{\mathbf{a}})\rho(\mu_{\mathbf{b}})\frac{\langle T\rangle}{N^2}, \tag{6.5}$$

where we have denoted $D = [\frac{1}{2}\int_0^1 \mu\rho(\mu)d\mu]^2$.

Next we calculate the function $\rho(\mu)$. To this end we note that in the limit of a quasi-1D geometry, Eq. (5.14) is reduced to

$$\frac{d}{dL}\langle |r'_{\mathbf{ab}}|^2\rangle = -\frac{k_F}{l}\left(\frac{1}{k_{\mathbf{a}}} + \frac{1}{k_{\mathbf{b}}}\right)\langle |r'_{\mathbf{ab}}|^2\rangle - \frac{\pi}{lL_yL_z}\frac{2}{k_{\mathbf{a}}k_{\mathbf{b}}}\langle |r'_{\mathbf{ab}}|^2\rangle + \frac{\pi}{lL_yL_z}\sum_{\mathbf{n}_1,\mathbf{n}_2}\frac{1}{k_{\mathbf{n}_1}k_{\mathbf{n}_2}} \times\langle (\delta_{\mathbf{an}_1} + |r'_{\mathbf{an}_1}|^2)(\delta_{\mathbf{bn}_2} + |r'_{\mathbf{bn}_2}|^2) + 2\delta_{\mathbf{ab}}\delta_{\mathbf{an}_2}|r'_{\mathbf{an}_1}|^2 + (1-\delta_{\mathbf{n}_1\mathbf{n}_2})r'_{\mathbf{an}_1}r'^*_{\mathbf{n}_1b}r'_{\mathbf{an}_2}r'^*_{\mathbf{n}_2b}\rangle. \tag{6.6}$$

[The obviously negligible second term on the right-hand side has been retained so as to make Eq. (6.6) fully compatible with the unitarity of the matrix $\hat{r}'$ at $L \to \infty$.] On making use of the same approximation as above (the leading order in l/L and N^{-1}), Eq. (6.6) yields in the general case $\mathbf{a} \neq \mathbf{b}$

$$\langle R'_{\mathbf{ab}}\rangle = \langle R_{\mathbf{ab}}\rangle = \frac{1}{4N}\frac{\rho(\mu_{\mathbf{a}})\rho(\mu_{\mathbf{b}})}{\mu_{\mathbf{a}} + \mu_{\mathbf{b}}}. \tag{6.7}$$

Combining Eq. (6.7) with the definition (6.2b), we obtain

$$\rho(\mu) = 1 + \frac{1}{2}\mu\rho(\mu)\int_0^1 \frac{\rho(\mu')}{\mu+\mu'}\,d\mu'. \tag{6.8}$$

The nonlinear integral equation, Eq. (6.8), is in fact one of the basic equations of the classical radiative transfer theory and as such is well-known in the astrophysical literature.[65] Its solution, the Chandrasekhar H function describes the angular distribution of the emergent radiations in semi-infinite atmospheres; it is long known to have an explicit analytical expression as follows:

$$H(\mu) = \exp\left\{-\frac{\mu}{\pi}\int_0^\infty \frac{dt}{1+\mu^2t^2}\ln\left[1-\frac{\arctan t}{t}\right]\right\}. \tag{6.9}$$

It is to be noted that Eq. (6.8) is equivalent to the Schwarzschild-Milne equation,[65] which in turn is equivalent to the ladder approximation for the Bethe-Salpeter equation for electron and light propagation through a disordered medium.[59–63] In mesoscopic physics, the *exact* ladder/crossed ladder sums are customarily treated in the diffuson/cooperon approximation, which is adequate for evaluating the conductivity in the Kubo approach. However, as a result of a breakdown of local current conservation, which ultimately leads to a global violation of unitarity of the scattering matrix, the diffuson/cooperon approximation is not adequate to calculate the average transmission and reflection coefficients in the Landauer approach (as has lately been discussed in Ref. 53). The exact ladder/crossed ladder sums can be found through solving the Milne equation by means of a rather complicated method of Wiener and Hopf.[69] The solution (6.9), in particular, has been originally derived by this very method. In the Appendix we provide an elementary derivation of the formula (6.9); we have not found such a compact exposition in the literature.

We shall see that as long as one is only interested in the total transmission coefficient at $L \gg l$, there is no need to know the explicit solution of Eq. (6.8). What is needed is the mean value of the function $\rho(\mu)$, which immediately follows from Eq. (6.4),

$$\int_0^1 \rho(\mu)\,d\mu = 2, \tag{6.10}$$

and its first moment, or more precisely, the diffusion constant D, which can be found as follows. Using identity (6.10), we rewrite Eq. (6.8) in the form

$$\int_0^1 \frac{\rho(\mu)\rho(\mu')}{\mu+\mu'}\mu'\,d\mu' = 2. \tag{6.11}$$

Multiplying Eq. (6.11) by μ^2 and integrating over the range of μ, we obtain

$$\begin{aligned}\frac{2}{3} &= \int_0^1\int_0^1 \frac{\mu^2\mu'}{\mu+\mu'}\rho(\mu)\rho(\mu')\,d\mu\,d\mu' \\ &= \frac{1}{2}\int_0^1\int_0^1 \frac{\mu^2\mu'+\mu\mu'^2}{\mu+\mu'}\rho(\mu)\rho(\mu')\,d\mu\,d\mu' \\ &= \frac{1}{2}\left[\int_0^1 \mu\rho(\mu)\,d\mu\right]^2 = 2D,\end{aligned} \tag{6.12}$$

that is $D = 1/3$.

A. Crossover between the Sharvin and Drude conductance

Expressions (6.5) and (6.7) give the angular structure of $T_{\mathbf{ab}}$ and $R_{\mathbf{ab}}$ in the main approximation at any $L \gg l$. In the metallic limit $L \ll Nl$, these formulas can be easily improved. Now we may keep the derivative on the left-hand side of Eq. (6.1), which gives a differential equation instead of Eq. (6.3):

$$\frac{d}{ds}\langle T_{\mathbf{ab}}\rangle = -\frac{1}{\mu_{\mathbf{a}}}\langle T_{\mathbf{ab}}\rangle + \frac{1}{4N}\frac{1}{\mu_{\mathbf{a}}\mu_{\mathbf{b}}}\rho_s(\mu_{\mathbf{a}})\tau_s(\mu_{\mathbf{b}}), \tag{6.13}$$

where we have denoted $s = L/l$. Recalling once more the definition (6.2a), we obtain the equation for τ_s as follows:

$$\frac{d}{ds}\tau_s(\mu) = -\frac{1}{\mu}\tau_s(\mu) + \frac{1}{2}\rho_s(\mu)\int_0^1 d\mu'\frac{\tau_s(\mu')}{\mu'}. \tag{6.14}$$

Employing Eq. (5.3) we obtain, *in addition* to Eq. (6.6), the following relation:

$$\frac{d}{ds}\langle R'_{\mathbf{ab}}\rangle = \frac{1}{4N}\frac{1}{\mu_{\mathbf{a}}\mu_{\mathbf{b}}}\tau_s(\mu_{\mathbf{a}})\tau_s(\mu_{\mathbf{b}}), \tag{6.15}$$

which yields for ρ_s

$$\frac{d}{ds}\rho_s(\mu) = \frac{1}{2}\tau_s(\mu)\int_0^1 d\mu'\frac{\tau_s(\mu')}{\mu'}. \tag{6.16}$$

Equations (6.13)–(6.16) describe the transport regime in which only incoherent scattering is present, so that WL corrections are neglected and all measurable quantities are self-averaged. Although the equations have been formally derived for a quasi-1D geometry, one can easily see that in fact they are valid for a general 3D geometry: Those terms in Eqs. (5.12) and (5.14), which have a mismatch in indices (as described in Sec. V), are obviously absent at $s = 0$ and contibute only to WL corrections when $0 < s \ll N$.

In the regime $s \ll 1$ ($L \ll l$) of *ballistic* transport, the transmission coefficient $T_{\mathbf{ab}} \approx T_{\mathbf{ab}}(s=0) = \delta_{\mathbf{ab}}$, so that the total transmission $T = N$ and hence the conductance is given by the familiar Sharvin's formula[70]

$$G = \frac{2e^2}{h}N = \frac{e^2}{h}\frac{k_F^2 L_y L_z}{2\pi}. \tag{6.17}$$

In the opposite regime $s \gg 1$ ($L \gg l$) of diffusive transport, when $\tau_s(\mu) \to 0$ and $\rho_s(\mu) \to \rho(\mu)$, the system (6.13)–(6.16) can be easily solved. At the leading order in s^{-1}, we obtain

$$\tau_s(\mu) = \frac{\mu\rho(\mu)}{s+2s_0}, \tag{6.18a}$$

$$\rho_s(\mu) = \rho(\mu)\left(1 - \frac{\mu}{s+2s_0}\right), \tag{6.18b}$$

$$\langle T_{\mathbf{ab}}\rangle = \frac{1}{4N}\frac{\rho(\mu_{\mathbf{a}})\rho(\mu_{\mathbf{b}})}{s+2s_0}, \tag{6.19a}$$

$$\langle R_{\mathbf{ab}}\rangle = \frac{1}{4N}\rho(\mu_{\mathbf{a}})\rho(\mu_{\mathbf{b}})\left(\frac{1}{\mu_{\mathbf{a}}+\mu_{\mathbf{b}}} - \frac{1}{s+2s_0}\right), \tag{6.19b}$$

where $s_0 \sim 1$ is a yet undefined constant. Calculating the total transmission, we obtain

$$\langle T \rangle = D\frac{4N}{s+2s_0} = \frac{lk_F^2 L_y L_z}{3\pi(L+2l_0)} \tag{6.20}$$

(here $l_0 = s_0 l$), which gives the Drude conductance (Ohm's law) (1.1). It is to be mentioned that results similar to Eqs. (6.19) and (6.20) have been recently obtained from Milne's equation by Nieuwenhuizen and Luck.[71]

Surprisingly enough, the approximation (6.18) and (6.19) satisfies Eqs. (6.13)–(6.16) *exactly* at any s_0, although it does not satisfy, of course, the initial conditions $\tau_{s=0}(\mu) = 1$, $\rho_{s=0}(\mu) = 1$. This means that once the correct value of s_0 is found, the corrections to (6.13)–(6.16) [and hence to Eq. (6.20)] would be exponentially small [$\propto \exp(-s)$]. To find s_0 we employ the identity, which follows from Eqs. (6.14) and (6.16):

$$\int_0^1 d\mu\, \mu[\rho_s(\mu) - \tau_s(\mu)] = s\int_0^1 d\mu\, \tau_s(\mu). \tag{6.21}$$

(It can be checked by differentiating with respect to s.) Substituting Eqs. (6.18) into Eq. (6.21), we get

$$s_0 = \frac{\int_0^1 d\mu\, \mu^2 \rho(\mu)}{\int_0^1 d\mu\, \mu\rho(\mu)} = 0.7104. \tag{6.22}$$

We recover a well-known numerical value[65] of the "injection depth." This physical interpretation becomes evident when expressing the resistance of the sample as the sum of the Drude resistance G_D^{-1} [G_D is given by Eq. (1.1)] and two extra interface resistances,

$$G^{-1} = G_D^{-1} + 2G_I^{-1}, \tag{6.23}$$

where

$$G_I^{-1} = \frac{3\pi h}{2e^2 lk_F^2 L_y L_z} l_0 \tag{6.24}$$

has the appearance of the Drude resistance of a surface layer of thickness l_0.

Recently the transition between the Sharvin and Drude conductances has been considered by de Jong[72] through a semiclassical approach and investigated experimentally by Tarucha *et al.*[73] The proposed in Ref. 72 *interpolation* formula $G^{-1} = G_D^{-1} + G_S^{-1}$, which approximates the resistance of the sample by the sum of the Drude and Sharvin resistances, fits very well the experimental data. We note that this formula differs insignificantly from Eq. (6.23), although the latter has a somewhat different status. As we discussed above, Eq. (6.23) provides the best approximation possible (provided WL corrections are neglected, of course) for $L \gg l$. One should keep in mind that experimental studies of the ballistic transport are conducted on the two-dimensional electron gas. In this case the function ρ in the preceding formulas should be replaced by the solution of a two-dimensional Chandrasekhar's equation. It is easy to show that the required equation is obtained from Eq. (6.8) if the differential $d\mu$ is replaced by $2d\mu/\pi(1-\mu^2)^{1/2}$ (see also the Appendix).

B. Backscattering peak

In Sec. VI A, we considered incoherent scattering. The formulas derived above allow us to consider already here the simplest coherent effect — the backscattering enhancement. This phenomenon is clearly observed in optics as a peak in reflected intensity approaching a factor close to 2 in height in the backward direction.[74–76] To calculate the enhancement exactly, we recast Eq. (6.6) in the same approximation which led to Eq. (6.7), but this time the case $\mathbf{a} = \mathbf{b}$ is included into consideration:

$$\langle R'_{\mathbf{ab}} \rangle = \langle R_{\mathbf{ab}} \rangle = \frac{1}{4N}\frac{(1+\delta_{\mathbf{ab}})\rho(\mu_{\mathbf{a}})\rho(\mu_{\mathbf{b}}) - \delta_{\mathbf{ab}}}{\mu_{\mathbf{a}} + \mu_{\mathbf{b}}}. \tag{6.25}$$

Equation (6.25) means that backward scattering to the same channel is enhanced by a factor of

$$\eta = 2 - \rho^{-2}(\mu), \tag{6.26}$$

as compared with the scattering to the nearby channels. This is consistent with the well-known interpretation mentioned in the Introduction that every path is in phase with its time-reversed counterpart, which gives rise to a factor of 2 in the backward direction. In fact, the reflected intensity is enhanced by a factor that is slightly below 2, because the single-scattering events are physically identical to their own time-reversed transforms and therefore have to be discarded. The maximal value of the enhancement factor is achieved for normal incidence ($\mu = 1$) and given by $\eta_{\max} = 1.88$.

Finally, it is worth mentioning that the peak in reflected intensity at $\mathbf{a} = \mathbf{b}$ is accompanied by a slight suppression of the background [due to the second and the last terms at $\mathbf{a} \neq \mathbf{b}$ on the right of Eq. (6.6)], so that the unitarity is conserved: $\sum_{\mathbf{b}} R_{\mathbf{ab}} = 1$, $\sum_{\mathbf{a},\mathbf{b}} R_{\mathbf{ab}} = N$ at $L \to \infty$.

VII. FOKKER-PLANCK EQUATION

We are now ready to come back to finding the evolution equations initiated in Sec. V. As we have seen in the previous section, the evolution equation (5.15) for the averaged transmission coefficient has a "two-level" construction: on the level of *individual coefficients* $T_{\mathbf{ab}}$, it reduces in the main approximation to an algebraic equation (6.1), which allows us to find the dependence on the indices (angular structure) without solving the original differential equation; upon summing over the indices $\mathbf{a}$, $\mathbf{b}$, i.e., on the level of *matrix traces*, those very terms, which dominated on the previous level and determined the angular structure of the individual coefficients, cancel each other. This feature proves to be quite general for the averaged equations generated by the system (5.1) and (5.2) and is the key to obtaining a complete statistical description. Reasoning in the same manner as in Sec. VI, it is easy to show that the dependences on the indices $\mathbf{n}_1$, $\mathbf{n}_2$ on the right-hand side of Eq. (5.16) are given by the same Chandrasekhar function as the dependence on $\mathbf{a}$ [see Eq. (6.5)] on the left-hand side is. Recalling the

identities Eqs. (6.10) and (6.12), we immediately obtain

$$\frac{d}{dL}\langle \mathrm{Tr}\widehat{\mathcal{T}}\rangle = -\frac{3\pi}{lk_F^2 S}\left[\langle \mathrm{Tr}^2\widehat{\mathcal{T}}\rangle + \langle \mathrm{Tr}\widehat{\mathcal{T}}^2\rangle\right]. \tag{7.1}$$

Equation (7.1) is not closed and we need to find the evolution equations for the new quantities that have appeared on the right-hand side. Denoting

$$t_n = \mathrm{Tr}\widehat{\mathcal{T}}^n, \tag{7.2}$$

$$L_c = \frac{1}{6\pi} lk_F^2 L_y L_z = \frac{2}{3} lN, \tag{7.3}$$

and proceeding in the manner outlined above, we find consecutively

$$\begin{aligned}
\frac{d\langle t_1\rangle}{dL} &= -\frac{1}{2L_c}\left[\langle t_1^2\rangle + \langle t_2\rangle\right], \\
\frac{d\langle t_1^2\rangle}{dL} &= -\frac{1}{2L_c}\left[2\langle t_1^3\rangle + 2\langle t_1 t_2\rangle + 4\langle t_3\rangle - 4\langle t_2\rangle\right], \\
\frac{d\langle t_2\rangle}{dL} &= -\frac{1}{2L_c}\left[4\langle t_1 t_2\rangle + 4\langle t_3\rangle - 2\langle t_1^2\rangle - 2\langle t_2\rangle\right],
\end{aligned} \tag{7.4}$$

and so on. To get a closed set of equations, we need to construct the evolution equation for the general average:

$$\langle t_1^{s_1} t_2^{s_2}\cdots t_n^{s_n}\rangle = |\mathbf{s}\rangle, \tag{7.5}$$

where $\mathbf{s} = (s_1, \ldots, s_n)$ is a set of natural numbers. The manner in which this equation is constructed admits the following illustrative representation (compare with a similar illustration in Ref. 45). We depict the trace $\mathrm{Tr}\widehat{\mathcal{T}}^k = \sum_{\mathbf{n}_i} \mathcal{T}_{\mathbf{n}_1\mathbf{n}_2}\mathcal{T}_{\mathbf{n}_2\mathbf{n}_3}\cdots\mathcal{T}_{\mathbf{n}_k\mathbf{n}_1}$ in the form of a ring of $2k$ alternating black and white balls. (The balls correspond to the first and second indices of $\widehat{\mathcal{T}}$.) We set an assembly of such rings in correspondence with the cross moment (7.5). The right side of the required equation is constructed then by a sum of all possible operations of four types: (1) separation of one ring into two; (2) joining together two rings into one; (1+), (2+) operations (1), (2) with simultaneous insertion of a pair of balls. The resulting system of equations can be represented in an operator form as follows:

$$2L_c\frac{d}{dL}|\mathbf{s}\rangle = \widehat{\mathcal{L}}\,|\mathbf{s}\rangle, \tag{7.6}$$

where

$$\begin{aligned}
\widehat{\mathcal{L}} = &\sum_{k=1}^{\infty}[k(k-1)a_k^\dagger a_k - k^2 a_{k+1}^\dagger a_k] \\
&+ \sum_{k,m=1}^{\infty}\{a_k^\dagger a_m^\dagger[(k+m)a_{k+m} - (k+m-1)a_{k+m-1}] \\
&+ 2km[a_{k+m}^\dagger - a_{k+m+1}^\dagger]a_k a_m\},
\end{aligned} \tag{7.7}$$

and the "creation" and "annihilation" operators are

$$\begin{aligned}
a_k^\dagger|s_1, \ldots, s_k, \ldots,\rangle &= |s_1, \ldots, s_k + 1, \ldots,\rangle, \\
a_k|s_1, \ldots, s_k, \ldots,\rangle &= s_k\,|s_1, \ldots, s_k - 1, \ldots,\rangle.
\end{aligned} \tag{7.8}$$

The system of Eq. (7.6) is equivalent to a single (albeit of infinite order) FP equation for the characteristic function $F_L(v_1, v_2, \ldots,) = \langle \exp(iv_1 t_1 + iv_2 t_2 + \cdots)\rangle$:

$$2L_c\frac{\partial F_L}{\partial L} = \widehat{\mathcal{D}}F_L, \tag{7.9}$$

where

$$\begin{aligned}
\widehat{\mathcal{D}}(v_1, v_2, \ldots,) &= \widehat{\mathcal{L}}^\dagger(a_k^\dagger \to iv_k,\ a_k \to \partial/i\partial v_k) \\
&= \sum_{k,m=1}^{\infty}\left\{2km\,i^2 v_k v_m\left[\frac{\partial}{i\partial v_{k+m}} - \frac{\partial}{i\partial v_{k+m+1}}\right] + [(k+m)iv_{k+m} - (k+m-1)iv_{k+m-1}]\frac{\partial^2}{i^2\partial v_k\partial v_m}\right\} \\
&\quad + \sum_{k=1}^{\infty} iv_k\left[k(k-1)\frac{\partial}{i\partial v_k} - k^2\frac{\partial}{i\partial v_{k+1}}\right].
\end{aligned} \tag{7.10}$$

The cross moments $|\mathbf{s}\rangle$ are defined by the identity

$$\langle t_1^{s_1}\cdots t_n^{s_n}\rangle = \frac{\partial^{s_1+\cdots+s_n}}{\partial(iv_1)^{s_1}\cdots\partial(iv_n)^{s_n}}F_L(\mathbf{v})|_{\mathbf{v}=0}. \tag{7.11}$$

Taking Fourier transforms with respect to the v variables, we go over from Eq. (7.9) to the FP equation for the joint distribution function $P_L(t_1, t_2, \ldots,)$:

$$\begin{aligned}
2L_c\frac{\partial P_L}{\partial L} &= \widehat{D}P_L, \\
\widehat{D}(t_1, t_2, \ldots,) &= \widehat{\mathcal{L}}^\dagger(a_k^\dagger \to -\partial/\partial t_k,\ a_k \to t_k).
\end{aligned} \tag{7.12}$$

In a more explicit form, the expression for $\widehat{D}$ reads

$$\begin{aligned}
\widehat{D} = &\sum_{k,m=1}^{\infty}\left\{2km\frac{\partial^2}{\partial t_k\partial t_m}[t_{k+m} - t_{k+m+1}]\right. \\
&\left. - \left[(k+m)\frac{\partial}{\partial t_{k+m}} - (k+m-1)\frac{\partial}{\partial t_{k+m-1}}\right]t_k t_m\right\} \\
&- \sum_{k=1}^{\infty}\frac{\partial}{\partial t_k}[k(k-1)t_k - k^2 t_{k+1}].
\end{aligned} \tag{7.13}$$

Note that some of the equations of the hierarchy (7.4) have been obtained earlier by Mello and Stone[12] within the framework of the macroscopical approach. They were proceeding in the direction opposite to ours: from the DMPK equation (a FP equation for the probability density of the eigenvalues of $\widehat{\mathcal{T}}$) to individual equations for

the traces t_n. Now we have the complete set of equations. We shall yet return to the question about the equivalence of Eq. (7.12) and the DMPK equation.

In the diffusive regime the t variables must be self-averaged, i.e., deterministic in the first-order approximation, so that all the cumulants of the distribution function (higher than the first-order moments) become nonzero only in higher-order approximations. Thus, it should be of advantage, at least for the study of the diffusive regime, if we rewrite Eq. (7.12) in terms of the cumulants. For this purpose we introduce the generating function $\Theta_L(\mathbf{v}) = \ln F_L(\mathbf{v})$. The cumulants of the t's can be expressed as

$$\langle t_1^{s_1} \cdots t_n^{s_n} \rangle_c = \frac{\partial^{s_1+\cdots+s_n}}{\partial(iv_1)^{s_1} \cdots \partial(iv_n)^{s_n}} \Theta_L(\mathbf{v})|_{\mathbf{v}=0}. \quad (7.14)$$

Substituting $F_L = \exp \Theta_L$ into Eq. (7.9), we obtain

$$2L_c \frac{\partial \Theta_L}{\partial L} = \widehat{\mathcal{D}} \Theta_L + \sum_{k,m=1}^{\infty} [(k+m) i v_{k+m} - (k+m-1) i v_{k+m-1}] \frac{\partial \Theta_L}{i \partial v_k} \frac{\partial \Theta_L}{i \partial v_m}. \quad (7.15)$$

It is interesting to note that Eqs. (7.6) closely resemble the renormalization-group equation for the additional charges, which describe additional contributions to the cumulants $\langle g^n \rangle_c^{\text{add}}$ in the expanded nonlinear σ model.[44] Indeed, if we omit the second term in each square bracket in Eq. (7.7), the remaining operator will practically coincide with the operator given by Eq. (70) of Ref. 44. The implications of the indicated correspondence are beyond the scope of the present work. We only note that the "extra" terms in our Eq. (7.7) appear due to the global current conservation (3.12b) (they would be missing if the reflection matrix were unitary) and might conceivably be beyond the accuracy of the model considered in Ref. 44.

VIII. METALLIC REGIME

Equations (7.6), (7.12), and (7.15) hold for any $L \gg l$. Let us evaluate them first in the metallic regime $l \ll L \ll Nl$. Mello and Stone[12] have devised a procedure of solving the system (7.6), in which the inverse number of channels N^{-1} plays the role of an expansion parameter. This procedure makes substantial use of the ballistic initial conditions

$$t_n(0) = N, \quad n = 1, 2, \ldots . \quad (8.1)$$

Indeed, one can see that the derivation of the WL correction in Ref. 12 is dependent on the assumption that Eqs. (7.6) are to match these initial conditions *exactly* [being a constant the correction otherwise drops out of the differential equations (7.6)]. On the other hand, as we have seen in Sec. VI A, Eqs. (7.6) *cannot* be expected to work in the ballistic regime and therefore they are not bound to satisfy the exact ballistic initial conditions. Due to this circumstance we shall not impose any initial conditions at $L = 0$ but instead we require that the total transmission coefficient $T = t_1$ be given in the main approximation by the Drude formula (6.20), which we rewrite here in the form

$$t_1 = \frac{2L_c}{L + 2l_0}. \quad (8.2)$$

As we have discussed in Sec. VI A, Eq. (8.2) describes incoherent scattering with exponential accuracy.

A. Transmission eigenvalue density

We begin with the evaluation of the transmission eigenvalue density, i.e., the density of the eigenvalues $\mathcal{T}_n$ of the matrix $\widehat{\mathcal{T}}$,

$$p(\mathcal{T}) \equiv \left\langle \sum_n \delta(\mathcal{T} - \mathcal{T}_n) \right\rangle = \frac{1}{\pi} \operatorname{Im} \operatorname{Tr} \left(\frac{1}{\mathcal{T}\hat{1} - \widehat{\mathcal{T}} - i0} \right). \quad (8.3)$$

It is seen from Eq. (8.3) that the conductance can be expressed in terms of $p(\mathcal{T})$ as

$$\langle g \rangle = 2\langle T \rangle = 2 \left\langle \sum_{n=1}^{N} \mathcal{T}_n \right\rangle = 2 \int_0^1 d\mathcal{T} \mathcal{T} p(\mathcal{T}). \quad (8.4)$$

In fact, knowledge of $p(\mathcal{T})$ enables one to compute any *linear statistic* on the transmission eigenvalues

$$A = \sum_n a(\mathcal{T}_n). \quad (8.5)$$

Aside the conductance (with $a = 2\mathcal{T}$), examples of linear statistics include the shot-noise power[49] [with $a = \mathcal{T}(1-\mathcal{T})$] and the conductance of a normal-superconductor interface[50] [with $a = 2\mathcal{T}^2(2-\mathcal{T})^{-2}$].

In order to find $p(\mathcal{T})$, we compute the averages $\langle t_n \rangle$. From Eq. (7.15), we have

$$2L_c \frac{d\langle t_n \rangle}{dL} = n(n-1)\langle t_n \rangle - n^2 \langle t_{n+1} \rangle + n \sum_{k=1}^{n-1} \langle t_k t_{n-k} \rangle - n \sum_{k=1}^{n} \langle t_k t_{n-k+1} \rangle + n \sum_{k=1}^{n-1} \langle t_k t_{n-k} \rangle_c - n \sum_{k=1}^{n} \langle t_k t_{n-k+1} \rangle_c. \quad (8.6)$$

In the main approximation, all fluctuations of t_n's can be neglected, so that one deals with practically certain characteristics of propagation. This is the limit of a classical diffusive transport. As a result, all the correlators in Eq. (8.6) vanish. In view of Eq. (8.1), it is natural to assume that $\langle t_n \rangle \sim \langle t_1 \rangle \gg 1$ for all n, so that the first two terms on the right-hand side of Eq. (8.6) can be neglected at the leading order in L_c/L. The resulting equations can be consecutively solved. In view of Eq. (8.2), the solution assumes the form

$$\langle t_n\rangle = \alpha_n \frac{2L_c}{L+2l_0} \tag{8.7}$$

(so that the assumption $t_n \sim t_1$ is confirmed) with the constants α_n satisfying the recursive equation

$$\alpha_n = n\sum_{k=1}^{n}\alpha_k\alpha_{n-k+1} - n\sum_{k=1}^{n-1}\alpha_k\alpha_{n-k}, \quad \alpha_1 = 1. \tag{8.8}$$

Equation (8.8) can be rewritten in terms of the generating function $f(x) = \sum_{n=1}^{\infty}\alpha_n x^{n-1}$ as

$$f(x) = [(x-x^2)f^2(x)]'. \tag{8.9}$$

Differential equation (8.9) is in fact *linear* in $f(x)$, which immediately allows us to obtain a linear recursive relation $(2n-1)\alpha_n = (2n-2)\alpha_{n-1}$, and hence

$$\alpha_n = \frac{(2n-2)!!}{(2n-1)!!}. \tag{8.10}$$

On the other hand, solving Eq. (8.9) we obtain

$$f(x) = \frac{\arcsin x^{1/2}}{(x-x^2)^{1/2}}. \tag{8.11}$$

Setting $x = \sin(\phi/2)$ and recalling Eqs. (7.2) and (8.7), we get

$$Q(\phi) = \left\langle \mathrm{Tr}\left(\frac{\hat{\mathcal{T}}}{\hat{1}-\sin^2(\phi/2)\hat{\mathcal{T}}}\right)\right\rangle = \langle T\rangle\frac{\phi}{\sin\phi}. \tag{8.12}$$

When continued to the complex ϕ plane, this relation lets us find the transmission eigenvalues density (8.3). From Eq. (8.12), one finds

$$p(\cosh^{-2}z) = \frac{\cosh^4 z}{2\pi i}\{Q[2i(z+i0)+\pi] - Q[2i(z-i0)-\pi]\}, \tag{8.13}$$

which yields, for the eigenvalue density in the classical diffusive limit

$$p_D(\mathcal{T}) = \frac{1}{2}\langle T\rangle\frac{1}{\mathcal{T}\sqrt{1-\mathcal{T}}}\,\theta(\mathcal{T}-\mathcal{T}_0), \tag{8.14}$$

where $\mathcal{T}_0 \approx 4\exp(-3L/2l)$ is a cutoff at small $\mathcal{T}$ such that $\int_0^1 p(\mathcal{T})\,d\mathcal{T} = N$. We stress that just as in the case of the Drude formula (8.2), the results (8.7)–(8.12) and consequently the distribution (8.14) are independent in the classical diffusive limit on the geometry of the conductor (see Sec. VI A). This proves that the classical value of any linear statistic calculated with the distribution (8.14),

$$A_D = \int_0^1 p_D(\mathcal{T})a(\mathcal{T})\,d\mathcal{T} = \int_0^\infty a(\cosh^{-2}z)\,dz, \tag{8.15}$$

is geometry independent.

The result (8.14) was first obtained by Mello and Pichard,[22] by integration of the DMPK equation — the method which is formally valid only for a quasi-1D geometry. Nazarov[52] recently gave a quite different from ours microscopic derivation of Eqs. (8.12) and (8.14), which made use of a beautiful analogy with the nonequilibrium superconducting state problem.

B. Weak localization and UCF

The experience with Eq. (8.6) suggests that L_c/L can be employed as an expansion parameter. We will thus seek the solution to the foregoing equations as a series in descending powers of the dimensionless Drude conductance $g_D/2 = 2L_c/(L+2l_0)$. The key simplification of the procedure is to evaluate the cumulants of t_n's given by Eq. (7.15) as opposite to the moments. The following two simple rules are instrumental in reducing the original differential evolution equations to purely algebraic ones. (i) The expansions of $\langle t_n\rangle_c = \langle t_n\rangle$ start with the first power of g_D,

$$\langle t_n\rangle = \alpha_n\frac{2L_c}{L+2l_0} + \beta_n + \cdots, \tag{8.16}$$

whereas the expansions of all other cumulants (which equal to zero at $L=0$) start with zero or negative powers. (ii) Since the expansions contain no logarithmic terms [otherwise the solution would not be compatible with the Drude formula (8.2)], the first power of g_D is to vanish on the right-hand side of the evolution equations.

As an example, we evaluate the equation for $\langle t_1\rangle$:

$$2L_c\frac{d}{dL}\langle t_1\rangle = -\langle t_1\rangle^2 - \langle t_2\rangle - \langle t_1^2\rangle_c. \tag{8.17}$$

Equating the total coefficient of g_D in Eq. (8.17) to zero, we obtain

$$2\alpha_1\beta_1 + \alpha_2 = 0. \tag{8.18}$$

Recalling Eq. (8.10), we find $\beta_1 = -1/3$, a well-known dimensionless value of the WL correction to the conductance of a conductor at zero temperature in the limit of a quasi-1D geometry.[12] The negative sign of the correction is consistent with the known interpretation mentioned in the Introduction that interference effects enhance the probability of return for a diffusing electron, thus decreasing the conductivity.

Next we proceed to the equation for the variance $\langle t_1^2\rangle_c$. From Eq. (7.15), we have

$$2L_c\frac{d\langle t_1^2\rangle_c}{dL} = -2\langle t_1^3\rangle_c - 2\langle t_1t_2\rangle_c - 4\langle t_3\rangle + 4\langle t_2\rangle - 4\langle t_1^2\rangle_c\langle t_1\rangle. \tag{8.19}$$

Equating again the total coefficient of g_D [it originates from the last three terms on the right of Eq. (8.19)] to zero, we obtain

$$\langle t_1^2\rangle_c = \frac{\alpha_2-\alpha_3}{\alpha_1} = \frac{2}{15}, \tag{8.20}$$

a well-known value of UCF for a quasi-1D conductor.[38,12]

These results can be readily generalized to include an arbitrary linear statistic (8.5). Thus, to calculate a WL correction to the classical value (8.15), we return to Eq. (8.6) and, just as with Eq. (8.17), we obtain the following recursive relation for the constants β_n:

$$0 = n(n-1)\alpha_n - n^2\alpha_{n+1} + 2n\sum_{k=1}^{n-1}\alpha_k\beta_{n-k} + 2n\sum_{k=1}^{n}\alpha_k\beta_{n-k+1}, \tag{8.21}$$

which reduces to

$$n\alpha_{n+1} + 2\sum_{k=1}^{n}\alpha_k\beta_{n-k+1} = 0. \tag{8.22}$$

Reasoning along the same lines as in deriving Eqs. (8.9), (8.11), and (8.12), we introduce the generating function $g(x) = \sum_{n=1}^{\infty}\beta_n x^{n-1}$ by which means Eq. (8.22) is reduced to

$$f'(x) + 2f(x)g(x) = 0, \tag{8.23}$$

with $f(x)$ defined after Eq. (8.8). From Eq. (8.11), we now have

$$g(x) = \frac{1-2x}{4(x-x^2)} - \frac{1}{4(x-x^2)^{1/2}\arcsin x^{1/2}}, \tag{8.24}$$

so that the correction to the function Q in Eq. (8.12) is given by

$$\delta Q(\phi) = \frac{\cos\phi}{\sin^2\phi} - \frac{1}{\phi\sin\phi}. \tag{8.25}$$

Recalling Eq. (8.13) we find, for the WL correction δA to the average $\langle A\rangle = A_D + \delta A$ of an arbitrary linear statistic (8.5)

$$\delta A = -\left(\frac{1}{4}a(1) + \int_0^{\infty} dz\,\frac{a(\cosh^{-2}z)}{4z^2+\pi^2}\right). \tag{8.26}$$

This result obtained here by evaluating the cumulant equations is in precise agreement with the formula derived by Beenakker[51] from the DMPK equation. We stress that unlike the classical value A_D, the WL correction is geometry dependent, so that formula (8.26) is valid only in the quasi-1D limit.

The line of reasoning developed here allows one to calculate higher-order cumulants as well. In the general case, we obtain the following expression

$$\langle t_1^{s_1}t_2^{s_2}\cdots t_n^{s_n}\rangle_c \sim g_D^{2-(s_1+\cdots+s_n)}. \tag{8.27}$$

It can be seen from (8.27) that in the metallic regime $g_D \gg 1$, the higher cumulants $\langle t_1^n\rangle$ $(n>2)$ are small. This means the distribution of the conductance is close to normal Gaussian, in agreement with the result of Al'tshuler, Kravtsov, and Lerner.[44] As an illustrative example, we consider the equation for the third-order cumulant $\langle t_1^3\rangle$

$$2L_c\frac{d\langle t_1^3\rangle_c}{dL} = -6\langle t_1^3\rangle_c\langle t_1\rangle - \langle t_1^2\rangle_c^2 - 3\langle t_1^4\rangle_c - 3\langle t_1^2t_2\rangle_c - 12\langle t_1t_3\rangle_c + 12\langle t_1t_2\rangle_c. \tag{8.28}$$

We require (as we have done before) that the first power of g_D vanish from the expansion on the right of Eq. (8.28). The only source of such a contribution could be the first term on the right, which necessarily means that $\langle t_1^3\rangle \sim g_D^{-1}$, in accordance with (8.27). The conclusion that

$$\langle t_1^n\rangle \sim g_D^{2-n} \tag{8.29}$$

is drawn simply by evaluation of the leading contribution in Eq. (7.15), which comes from the second term on the right. As we already mentioned, the first term (operator $\mathcal{D}$) yields contributions corresponding to the additional charges in the expanded σ model, which have been shown[44] to make expression (8.29) invalid for very high moments $n \gtrsim k_F^2l^2 \gg 1$ and lead to the appearance of log-normal tails in the conductance distribution. This property requires further investigation.

IX. INSULATING REGIME

We turn now to the strong localization regime defined by the condition $L \gg L_c \sim NL$ or, equivalently, $G \ll e^2/h$. In this regime, the transmission eigenvalues are exponentially small, so that, ordering the $\mathcal{T}_n$'s from large to small, we have $1 \gg \mathcal{T}_1 \gg \mathcal{T}_2 \gg \cdots \gg \mathcal{T}_N$. The traces t_n are then dominated by $\mathcal{T}_1$, i.e., by the biggest of the $\mathcal{T}_n$'s:

$$t_n = \mathrm{Tr}\widehat{\mathcal{T}}^n = \mathcal{T}_1^n. \tag{9.1}$$

In particular, we have $T \equiv t_1 = \mathcal{T}_1$. Due to this reason, it is natural to propose the following "ansatz" for t_n:

$$t_n = T^n. \tag{9.2}$$

Substituting (9.2) into the system (7.6), we find that all equations of the same order $s_1+s_2+\cdots = n$ are reduced to a single equation for $\langle T^n\rangle$:

$$L_c\frac{d}{dL}\langle T^n\rangle = -[n^2\langle T^{n+1}\rangle - n(n-1)\langle T^n\rangle]. \tag{9.3}$$

Going over to the equation for the distribution function, we obtain

$$L_c\frac{\partial P_L(T)}{\partial L} = \frac{\partial}{\partial T}\left[(1-T)\frac{\partial}{\partial T}T^2P_L\right]. \tag{9.4}$$

In order to identify Eq. (9.4) we note that it is equivalent to the following equation for the distribution of the dimensional resistance $r = T^{-1}$:

$$L_c\frac{\partial\tilde{P}_L(r)}{\partial L} = \frac{\partial}{\partial r}\left[(r^2-r)\frac{\partial}{\partial r}\tilde{P}_L\right]. \tag{9.5}$$

[The two distributions are related through $P_L(T) = \tilde{P}(r)r^2$.] This is Abrikosov's scaling equation[9] for a 1D chain. We are led to conclude that in the insulating regime the conductance distribution for a wire coincides

with that for a 1D chain, provided the 1D localization length l is replaced by L_c. Thus, we can identify the parameter $L_c = \frac{1}{6\pi} l k_F^2 L_y L_z$ as the exact value of the quasi-1D localization length. This is precisely the value found in Ref. 33 by calculating the density-density correlator with the use of supersymmetric techniques.

The above derivation can be modified so as to get Eq. (9.4) directly from Eq. (7.12). For this purpose, the ansatz (9.2) is to be rewritten in terms of the joint distribution function $P(t_1, t_2, \dots,)$:

$$P_L(t_1, t_2, \dots,) = \int_0^1 dT\, \delta(t_1 - T)\delta(t_2 - T^2) \cdots P_L(T). \tag{9.6}$$

Substituting (9.6) into Eq. (7.12), one can see that the latter is satisfied identically if the distribution $P_L(T)$ obeys "one-dimensional" Eq. (9.4).

Equation (9.5) was extensively studied in Ref. 9. The solution is given by

$$\tilde{P}_L(r) = \frac{\exp(-L/4L_c)}{\sqrt{2\pi}(L/L_c)^{3/2}} \int_{\operatorname{arccosh}(2r-1)}^{\infty} \times dx\, \frac{x \exp(-x^2 L_c/4L)}{\sqrt{\cosh x - 2r + 1}}, \tag{9.7}$$

and the mean value of the conductance decays exponentially with the sample length

$$\langle g \rangle = 2 \int_1^{\infty} \frac{dr}{r} \tilde{P}(r) \approx \frac{\pi^{5/2}}{2} \left(\frac{L_c}{L}\right)^{3/2} \exp\left(-\frac{L}{4L_c}\right). \tag{9.8}$$

The distributions $\tilde{P}_L(r)$ and $P_L(g)$ are asymptotically log-normal, which can be best seen directly from Eqs. (9.4) and (9.5). Thus, using the condition $T \ll 1$, one finds that $\ln g$ has a normal distribution, with mean $\langle \ln g \rangle = -L/L_c$ and variance $\operatorname{Var} \ln g = 2L/L_c$. Note that one has to use Eq. (9.7) rather than the log-normal distribution in order to find the preexponential factor in Eq. (9.8).

A final comment is in order. In the strong localization regime, the *average* value of the conductance $\langle g \rangle \propto \exp(-L/4L_c)$ is exponentially larger than the *typical* value $\exp\langle \ln g \rangle \propto \exp(-L/L_c)$, which by definition corresponds to the maximum of the distribution function; as a result, the conductance strongly fluctuates over the ensemble of macroscopically identical samples (the relative fluctuation $\delta g/\langle g \rangle \gg 1$). The reason for this is that due to the stretched form of the distribution, the mean value is dominated by exponentially *rare* yet almost transparent configurations of the random potential. In the metallic regime, the relatively small ensemble fluctuations of the conductance are known to manifest themselves in the form of "grass" on the Ohmic pedestal of a particular sample,[42] and it is natural to expect even more pronounced effects in the localized regime. Indeed, the above-noted difference between the mean and typical values of the conductance has been recently predicted[46] to lead to gigantic mesoscopic fluctuations around the superlinear on average I-V characteristic.

X. DISCUSSION

In this work we have introduced exact scaling equations that govern the evolution of the transfer and scattering matrices as the sample length increases. After ensemble averaging, these equations result in a rather complete statistical description of mesoscopic transport in quasi-1D conductors. It is appropriate to discuss here some results of the method presented, which can be of particular concern when comparing to the other existing techniques. Thus, we calculate exactly the angular structure of individual $T_{\mathbf{n},\mathbf{n}'}$, $R_{\mathbf{n},\mathbf{n}'}$, which, as can be shown, is tantamount (through Fourier transform) to finding the exact ladder sum $L(\mathbf{r}, \mathbf{r}')$. In so doing we manage to avoid the difficulties associated with the breakdown of current conservation in the diffuson/cooperon approximation. Because reflection is dominated by short trajectories of a few mean free paths in length, whereas the standard diffuson approximation is adequate only in the limit of long wavelengths, one needs an accurate expression for $L(\mathbf{r}, \mathbf{r}')$ whenever both the transmission and reflection coefficients are being evaluated in a single current-conserving approximation. This problem has been recently discussed in detail by Hastings, Stone, and Baranger[53] in connection with the question about inequivalence of coherent backscattering and WL, which is believed to be of some importance for the semiclassical theory of ballistic chaotic conductors. It is worth noting that although the physical reasons behind the WL correction and the backscattering peak are quite similar, these two effects are *not* complimentary in a disordered medium. As indicated in the last remark of Sec. VI B, the backscattering peak does *not* lead to any correction in the total reflection coefficient R. It is easy to understand if one remembers that the backscattering peak is present even for a semi-infinite disordered medium, when $R_\infty \equiv N$. In the metallic regime, the first-order correction (negative) to R_∞ appears due to the finite Drude conductance and only the second-order correction (positive) appears due to WL. This feature is not explicit in the results of Ref. 53.

Let us discuss in more detail the connection between our formalism and that of the macroscopic approach. In many cases we arrive microscopically at the same results as those derived from the DMPK scaling equation,

$$\tfrac{1}{2} l N \frac{\partial \mathcal{P}_L(\boldsymbol{\lambda})}{\partial L} = \sum_{i=1}^{N} \frac{\partial}{\partial \lambda_i} \left[\lambda_i (1 + \lambda_i) J(\boldsymbol{\lambda}) \frac{\partial}{\partial \lambda_i} \frac{\mathcal{P}_L(\boldsymbol{\lambda})}{J(\boldsymbol{\lambda})} \right], \tag{10.1}$$

which describes the evolution of the probability density $\mathcal{P}_L(\lambda_1, \dots, \lambda_N)$ for the variables $\lambda_n = (1 - \mathcal{T}_n)/\mathcal{T}_n$. In Eq. (10.1), $J(\boldsymbol{\lambda}) = \prod_{i<j} |\lambda_i - \lambda_j|$; the ballistic initial condition is $\mathcal{P}_0(\boldsymbol{\lambda}) = \delta(\boldsymbol{\lambda})$. Although in our view the method of cumulants of Sec. VIII is much simpler to work with

when computing observable quantities, it would be of significant interest to show that the FP equation (7.12) is equivalent to the DMPK equation. The ansatz (9.6) used in Sec. VIII to find the distribution of the maximum transmission eigenvalue, suggests a simple way to do that. In the general case, instead of (9.2) one has $t_n = \sum_{i=1}^{N} \mathcal{T}_i^n$, and it is natural to use the following ansatz:

$$P_L(t_1, t_2, \dots,) = \int d^N\boldsymbol{\lambda} \prod_{n=1}^{\infty} \times\delta\left(t_n - \sum_{i=1}^{N} \frac{1}{(1+\lambda_i)^n}\right) \mathcal{P}(\boldsymbol{\lambda}). \quad (10.2)$$

A direct check shows that our ansatz (10.2) identically satisfies Eq. (7.12) for any N, provided $\mathcal{P}_L$ obeys Eq. (10.1) with the numerical coefficient $\frac{2}{3}$ instead of $\frac{1}{2}$ on the left. Thus, we have provided the first microscopical proof of the DMPK equation for the realistic model of a wire with the isotropic dispersion law.

It is to be noted that the known derivations of the DMPK equation within the framework of the macroscopic approach rely on the isotropy assumption that the flux incident on any scattering channel becomes equally distributed among all outgoing channels. This concept of equivalent channels ignores the finite time scale for transverse diffusion and obviously must be dropped in order to describe correctly the 2D and 3D behavior. In fact, we have shown that as long as the channel velocities $k_{\mathbf{a}}/m$ in different channels differ from each other, the individual transmission coefficients have a nontrivial angular structure, given by Eq. (6.5). This makes it impossible to treat the channels in a completely symmetrical fashion even in the case of a wire geometry. Nonetheless, our derivation of the FP equation (7.12) shows that, in a *macroscopic* sense, the isotropy assumption could be eventually replaced by a weaker requirement that the angular structure of individual elements of the $\check{S}$ matrix be length independent. Indeed, if we were to ignore the results of Sec. VI and merely assumed that

$$\sum_{\mathbf{b}} \langle T_{\mathbf{ab}}\rangle / \langle T\rangle = \sum_{\mathbf{b}} \langle R_{\mathbf{ab}}\rangle / \langle R\rangle = \rho(\mathbf{a}), \quad (10.3)$$

with unspecified $\rho(\mathbf{a})$ being independent on L, we would have arrived at the same equations (7.4). The only thing that would become missing on the way is the correct value of the dimensionless diffusion constant D, which, in macroscopical models, is absorbed by the mean free path in any case. Notice, however, that any generalization to 2D and 3D geometries still requires knowledge of the angular structure in question. We believe that this concept should be of value when discussing the minimum set of assumptions needed to generate the known quantum-interference effects without referring to the microscopic Hamiltonian. Let us also note that the angular structure of $\langle T_{\mathbf{ab}}\rangle$ undergoes a drastic change in the ballistic regime: from the δ-functional dependence $\langle T_{\mathbf{ab}}\rangle \propto \delta_{\mathbf{ab}}$ at $L = 0$ to a smooth one, given at $L \gtrsim l$ by the Chandrasekhar function. Whether the DMPK equation can yield accurate results in the ballistic regime remains, therefore, an open question.

It was remarked in the text that the master equation (7.6) bears a close resemblance to the renormalization-group equations of the σ model in Ref. 44. This is especially striking if one remembers that the equations of Ref. 44 were derived in the *two-dimensional* metallic limit. The common underlying physics of the two models provides a basis for the conjecture that Eq. (7.6) can be directly generalized to the quasi-2D limit. This problem is left for future examination.

In conclusion, we have presented a new microscopic approach to the problem of mesoscopic transport in phase-coherent disordered systems that enables one to treat in a systematic way a wide range of physical phenomena in quasi-1D conductors including ballistic regime, weak localization effects, and strong localization.

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APPENDIX: CHANDRASEKHAR FUNCTION

We provide here an elementary solution of Eq. (6.8). To avoid dealing with certain peculiarities of Eq. (6.8), we will in fact consider the general equation of Chandrasekhar's type:

$$Z(\mu) = 1 + \frac{1}{2}\mu Z(\mu) \int_0^1 d\mu'\, K(\mu') \frac{Z(\mu')}{\mu + \mu'}, \quad (A1)$$

where the function $K(\mu)$ is assumed to be real, nonnegative in the interval $0 \le \mu \le 1$, and satisfy the condition $\int_0^1 K(\mu)\, d\mu \le 1$. The H function will be obtained by setting $K = 1$ in the final result. Note that provided the proper choice of $K(\mu)$, Eq. (A1) gives the angular distribution of $T_{\mathbf{ab}}$ for the case of 2D wire ($L_z \sim k_F^{-1}$) as well as the angular structure of $\langle t_{\mathbf{ab}} t^*_{\mathbf{cd}}\rangle$ in general 3D case.

Multiplying Eq. (A1) by $K(\mu)/(\mu - \mu'')$ and integrating over the range of μ, we find

$$\int_0^1 \frac{K(\mu)Z(\mu)}{\mu - \mu''}\, d\mu = \int_0^1 \frac{K(\mu)}{\mu - \mu''}\, d\mu + \frac{1}{2}\int_0^1 \frac{K(\mu)Z(\mu)}{\mu - \mu''}\, d\mu \int_0^1 \mu'' \frac{K(\mu')Z(\mu')}{\mu' + \mu''}\, d\mu' + \frac{1}{2}\int_0^1 \frac{K(\mu')Z(\mu')}{\mu' + \mu''}\, d\mu' \int_0^1 \mu' \frac{K(\mu)Z(\mu)}{\mu + \mu'}\, d\mu, \quad (A2)$$

or, repeatedly applying Eq. (A1),

$$\left[1+\mu''^2\int_0^1\frac{K(\mu)}{\mu^2-\mu''^2}\,d\mu\right]Z(\mu'')$$

$$=1+\frac{1}{2}\mu''\int_0^1 d\mu\,K(\mu)\frac{Z(\mu)}{\mu-\mu''}. \quad \text{(A3)}$$

We have obtained a linear integral equation, effectively Milne's equation. We are obviously interested only in solutions that are bounded in the interval $0\le\mu\le 1$. Under this condition, Eq. (A1) defines $Z(\mu)$ in the complex μ plane. Combining Eqs. (A1) and (A3), we obtain an algebraic equation

$$\left[1+\mu^2\int_0^1\frac{K(\mu')}{\mu'^2-\mu^2}\,d\mu'\right]Z(-\mu)Z(\mu)=1. \quad \text{(A4)}$$

We assume that the coefficient on the left-hand side of Eq. (A4) has no zeroes in the μ plane. (In physically relevant cases it does not; otherwise there would be *two* bounded in the interval $0\le\mu\le 1$ solutions.) Then, from Eqs. (A1) and (A4), the function $Z(\mu)$ is analytic and has no zeroes for $\mathrm{Re}\mu>-1$. From Eq. (A1), it also follows that there exists $\lim_{\mu\to\infty}Z(\mu)=Z_\infty$, and we assume for simplicity that Z_∞ is finite; the final result will not depend on this assumption. (In fact, as follows from Eq. (A4), $Z_\infty=[1-\int_0^1 K(\mu)\,d\mu]^{-1/2}$, so that $Z_\infty=\infty$ only when $K(\mu)\equiv 1$.) Next comes the crucial step. We rewrite Eqs. (A4) as follows:

$$\ln\{Z(\mu)/Z_\infty\}+\ln\{Z(-\mu)/Z_\infty\}$$

$$=-\ln\left\{\left[1+\mu^2\int_0^1\frac{K(\mu')}{\mu'^2-\mu^2}\,d\mu'\right]Z_\infty^2\right\}. \quad \text{(A5)}$$

We see that the right-hand side of Eq. (A5) is analytic in the strip $\mathrm{Re}\mu<1$ and goes to zero as $\mu\to\infty$. At the same time, the first term on the left-hand side is analytic for $\mathrm{Re}\mu>-1$ and the other for $\mathrm{Re}\mu<1$; both go to zero at infinity. These analytical properties already enable us to obtain a representation of $Z(\mu)$ as a complex integral. For this purpose we multiply Eq. (A5) by $1/(\mu-\mu_0)$ and integrate with respect to μ along the imaginary axis. Provided $0<\mathrm{Re}\mu_0<1$, by taking the residue we have

$$\ln\{Z(\mu_0)/Z_\infty\}=\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\frac{d\mu}{\mu-\mu_0}\ln\left\{\left[1+\mu^2\times\int_0^1\frac{K(\mu')}{\mu'^2-\mu^2}\,d\mu'\right]Z_\infty^2\right\}. \quad \text{(A6)}$$

Since $\ln\{Z(-\mu)/Z_\infty\}$ has no poles for $\mathrm{Re}\mu<0$, the corresponding integral has vanished. From Eq. (A6), we arrive at the final result

$$Z(\mu)=\exp\left\{-\frac{\mu}{\pi}\int_0^\infty\frac{d\xi}{\mu^2+\xi^2}\times\ln\left[1-\int_0^1 d\mu'\,\frac{\xi^2K(\mu')}{\mu'^2+\xi^2}\right]\right\}. \quad \text{(A7)}$$

Upon setting $K=1$, Eq. (A7) is reduced to Eq. (6.9).

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