Slave-particle quantization and sum rules in the t-J model

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In the framework of constrained systems, we give the classical Hamiltonian formulation of slave-particle models and their correct quantization. The electron-momentum distribution function in the t-J and Hubbard models is then studied in the framework of slave-particle approaches and within the decoupling scheme. We show that criticisms that have been addressed in this context coming from a violation of the sum rule for the physical electron are not valid. Due to the correct quantization rules for the slave particles, the sum rule for the physical electron is indeed obeyed, both exactly and within the decoupling scheme.

I. INTRODUCTION

Several strongly correlated fermionic systems such as liquid ^{3}He, heavy-fermion compounds, high-T_c superconductors, and Kondo systems are the subject of intense theoretical as well as experimental interest. The Hubbard model, originally introduced to describe correlation effects in narrow d-band materials, has been put forward as a possible key to the understanding of high-T_c oxide superconductivity.[1]

Describing electrons on a lattice with one orbital per site, the Hubbard Hamiltonian,

$$\hat{H}=\sum_{ij,\sigma}t_{ij}\hat{c}^\dagger_{i\sigma}\hat{c}_{j\sigma}+U\sum_i\hat{n}_{i\sigma}\hat{n}_{i(-\sigma)}\ , \tag{1}$$

includes a kinetic hopping term t_{ij} between different sites and an on-site Coulomb repulsion U between electrons of different spin. The operator $\hat{c}_{i\sigma}$ annihilates an electron of spin σ at site i and the occupation number operator is $\hat{n}_{i\sigma}=\hat{c}^\dagger_{i\sigma}\hat{c}_{i\sigma}$.

Of particular interest is the strong-coupling regime where an effective Hamiltonian of the Hubbard model with nearest-neighbor hopping is the t-J model Hamiltonian:[2]

$$\hat{H}=t\sum_{\langle ij\rangle,\sigma}(\hat{c}^\dagger_{i\sigma}\hat{c}_{j\sigma}+\text{H.c.})+J\sum_{\langle ij\rangle}(\hat{\mathbf{S}}_i\cdot\hat{\mathbf{S}}_j-\tfrac{1}{4}\hat{n}_i\hat{n}_j) \tag{2}$$

(with $J=4t^2/U$ and $\hat{\mathbf{S}}_i=\frac{1}{2}\sum_{\alpha\beta}\hat{c}^\dagger_{i\alpha}\sigma_{\alpha\beta}\hat{c}_{i\beta}$). The on-site Coulomb repulsion there is very large as compared with the electron-hopping energy, and therefore when there is less than half filling the system will avoid configuration with doubly occupied sites. One thus has the constraint

$$\sum_\sigma\hat{c}^\dagger_{i\sigma}\hat{c}_{i\sigma}\leq 1\ . \tag{3}$$

Apart from numerical approaches, a popular analytical approach to the t-J model is the slave-particle theory[3] where, for instance, in the slave-fermion representation the electron operator $\hat{c}^\dagger_{i\alpha}$ is written as $\hat{c}^\dagger_{i\alpha}=\hat{b}^\dagger_{i\alpha}\hat{f}_i$, with f the slave fermion and b_α a boson. Instead of the constraint of Eq. (3), which is difficult to handle, one considers the *a priori* more convenient slave-particle constraint avoiding double occupancy at site i:

$$\hat{f}^\dagger_i\hat{f}_i+\sum_\alpha\hat{b}^\dagger_{i\alpha}\hat{b}_{i\alpha}=1\ . \tag{4}$$

With the boson operator $\hat{b}^\dagger_{i\alpha}$ keeping track of the spin and the fermion operator $\hat{f}^\dagger_i$ keeping track of the charge, this formalism is well adapted to study the problem of the decoupling of spin and charge degrees of freedom in the large-U-limit Hubbard model. This decoupling, characteristic of Luttinger liquids, appears to occur in one dimension (1D),[4] the situation being still confused in 2D. At the mean-field level, spinons and holons, the elementary spin and charge excitations, may be separated in this formalism but are strongly coupled beyond the mean-field approximation.

Such slave-particle approaches are usually studied in a functional integral (over coherent states of Fermi and Bose fields) representation of the partition function, the slave-particle constraints being enforced by functional integration over Lagrange multipliers.

However, one may wonder what happens within a direct operator quantum approach of such slave-particle theories. Indeed, sum rules, coming from operator commutation relations, for spectral functions of the boson and fermion were used recently in Ref. 5 in a study of the electron-momentum distribution function in the t-J model in the framework of the slave-particle approach and within the decoupling scheme for the electron Green's function. It was claimed there that the sum rule for the physical electron was not obeyed within this framework and correspondingly that the electron Fermi surface (EFS) was not explained. However, it happens that the operator commutation relations used in Ref. 5 for the slave fermion and boson were the usual naive ones, i.e., the same as if the slave-particle constraint was not present. One then can question the use of such operator commutation relations in a slave-particle approach.

In this paper, we study the direct operator approach of such slave-particle theories. In Sec. II, we present at the classical level the consistent Hamiltonian formulation of models having a slave-particle constraint for their fields.

We show in Sec. III at the quantum level, for the slave-fermion and the slave-boson representations of the t-J model, the modifications in the sum rules for the slave particles coming from the fact that the correct canonical relations compatible with the constraints are not the naive ones. We present in Sec. IV a direct explicit operator quantization of the slave-particle approaches of the t-J model which confirms the results obtained in Sec. III. In Sec. V, we extend our analysis to a slave-boson representation which has been introduced[6] for the finite-U Hubbard model. Section VI summarizes our conclusions, and an Appendix presents calculations omitted for clarity in Sec. II.

II. HAMILTONIAN FORMULATION FOR SLAVE MODELS

Let us consider the general classical Lagrangian (written in real time) for n bosons b_α and m fermions f_σ (Grassmann variables) on a lattice:

$$L = i\sum_{i,\alpha} b_{i\alpha}^\dagger \partial_t b_{i\alpha} + i\sum_{i,\sigma} f_{i\sigma}^\dagger \partial_t f_{i\sigma} + X(b^\dagger, b, f^\dagger, f) + \sum_i \Lambda_i \left[\sum_\alpha b_{i\alpha}^\dagger b_{i\alpha} + \sum_\sigma f_{i\sigma}^\dagger f_{i\sigma} - 1 \right] , \quad (5)$$

bosons and fermions being submitted to the slave-particle constraint at each site i:

$$\Phi_i \equiv \sum_\alpha b_{i\alpha}^\dagger b_{i\alpha} + \sum_\sigma f_{i\sigma}^\dagger f_{i\sigma} - 1 = 0 . \quad (6)$$

One shows in the Appendix that, after a Dirac treatment,[7] one gets a Hamiltonian formalism with the nonzero brackets:

$$\{b_{i\alpha}, b_{j\beta}^\dagger\} = -i\delta_{ij}\delta_{\alpha\beta} , \quad \{f_{i\sigma}, f_{j\tau}^\dagger\} = -i\delta_{ij}\delta_{\sigma\tau} ,$$
$$\{\Lambda_i, \Pi_j\} = \delta_{ij} \quad (7)$$

but with the *first class*[7–9] constraints

$$\Phi_i \approx 0 \quad \text{and} \quad \Pi_i \approx 0 , \quad (8)$$

where, following Dirac, first class means here $\{\Phi_i, \Pi_i\} \approx 0$ and where the symbol ≈ 0 (weakly zero) means that one has to set the constraints only after computing all the brackets. We use graded Poisson-Dirac brackets[9] such that, for instance,

$$\{B, A\} = -(-1)^{ab}\{A, B\} ,$$
$$\{A, BC\} = \{A, B\}C + (-1)^{ab}B\{A, C\} , \quad (9)$$

where $a=0$ if A is a bosonic quantity and $a=1$ if A is a fermionic quantity.

The role of first class constraints is to generate infinitesimal contact transformations (that we shall call as usual[8] gauge transformations) in the Hamiltonian formalism that do not affect the physical state of the system. We have here for the nonzero brackets of the fundamental variables with the first class constraints:

$$\{b_{i\alpha}, \Phi_j\} = -ib_{i\alpha}\delta_{ij} , \quad \{b_{i\alpha}^\dagger, \Phi_j\} = +ib_{i\alpha}^\dagger\delta_{ij} , \quad (10)$$

$$\{f_{i\sigma}, \Phi_j\} = -if_{i\sigma}\delta_{ij} , \quad \{f_{i\sigma}^\dagger, \Phi_j\} = +if_{i\sigma}^\dagger\delta_{ij} , \quad (11)$$

$$\{\Lambda_i, \Pi_j\} = \delta_{ij} , \quad (12)$$

which means that the gauge transformations are $b \to e^{-i\theta}b$, $b^\dagger \to e^{i\theta}b^\dagger$, $f \to e^{-i\theta}f$, $f^\dagger \to e^{i\theta}f^\dagger$, $\Lambda \to \Lambda + a$, $\Pi \to \Pi$.

The standard strategy[8] is then to fix the gauge by choosing explicit forms for each gauge and imposing them as constraints not following from the Lagrangian. The choice of gauges should be made in such a way that the constraints Φ_i and Π_i will cease to be first class. It happens that a convenient choice here is the linear one,

$$\Phi_i' \equiv \sum_\alpha (G_{i\alpha}b_{i\alpha} + G_{i\alpha}^\dagger b_{i\alpha}^\dagger) + \sum_\sigma (H_{i\sigma}f_{i\sigma} - H_{i\sigma}^\dagger f_{i\sigma}^\dagger) + K_i \approx 0 , \quad (13)$$

$$\Lambda_i \approx 0 , \quad (14)$$

where the G's, the K's, and the (Grassmannian) H's are parameters.

We have then for the nonzero brackets of the fundamental variables with these gauge fixing constraints

$$\{b_{i\alpha}, \Phi_j'\} = -iG_{i\alpha}^\dagger\delta_{ij} , \quad \{b_{i\alpha}^\dagger, \Phi_j'\} = +iG_{i\alpha}\delta_{ij} , \quad (15)$$

$$\{f_{i\sigma}, \Phi_j'\} = -iH_{i\sigma}^\dagger\delta_{ij} , \quad \{f_{i\sigma}^\dagger, \Phi_j'\} = +iH_{i\sigma}\delta_{ij} , \quad (16)$$

$$\{\Pi_i, \Lambda_j\} = -\delta_{ij} \quad (17)$$

and we obtain the bracket of the slave-particle constraint with its gauge fixing constraint as

$$\{\Phi_i, \Phi_j'\} = i\left[\sum_\alpha (G_{i\alpha}b_{i\alpha} - G_{i\alpha}^\dagger b_{i\alpha}^\dagger) + \sum_\sigma (H_{i\sigma}f_{i\sigma} + H_{i\sigma}^\dagger f_{i\sigma}^\dagger) \right]\delta_{ij} \equiv iD_i\delta_{ij} , \quad (18)$$

which is not weakly zero.

Let us note that the Hamiltonian (see the Appendix) is now

$$H_3 = -X(b^\dagger, b, f^\dagger, f) - \sum_i (x_i - \Lambda_i)\left[\sum_\alpha b_{i\alpha}^\dagger b_{i\alpha} + \sum_\sigma f_{i\sigma}^\dagger f_{i\sigma} - 1 \right] + \sum_i \Pi_i w_i , \quad (19)$$

where the first class constraints coefficients x and w (which are in H_3 independent of the fields) are determined by the requirement that the time derivative of the gauge fixing constraints is weakly zero

$$\{\Phi_i', H_3\} = i\sum_\alpha \left[G_{i\alpha}\frac{\partial X}{\partial b_{i\alpha}^\dagger} - G_{i\alpha}^\dagger \frac{\partial X}{\partial b_{i\alpha}} \right] + i\sum_\sigma \left[H_{i\sigma}\frac{\partial X}{\partial f_{i\sigma}^\dagger} - H_{i\sigma}^\dagger \frac{\partial X}{\partial f_{i\sigma}} \right] - i(x_i - \Lambda_i)D_i \approx 0 , \quad (20)$$

$$\{\Lambda_i, H_3\} = w_i \approx 0 . \quad (21)$$

Defining $\varphi_1 \equiv \Phi_i$, $\varphi_2 \equiv \Phi_i'$, $\varphi_3 \equiv \Pi_i$, $\varphi_4 \equiv \Lambda_i$, the matrix

$C_{ab}\equiv\{\varphi_a,\varphi_b\}$ is nonsingular and all the constraints are now *second class.*[7–9] Systematic use of the standard Dirac bracket[7] of two quantities A and B,

$$\{A,B\}_*\equiv\{A,B\}-\sum_{a,b}\{A,\varphi_a\}(C^{-1})_{ab}\{\varphi_b,B\}\ , \quad (22)$$

then allows one to set all these second class constraints strongly to zero because the Dirac bracket of anything with a second class constraint vanishes.

We thus obtain the correct classical nonzero canonical relations for slave-particle models, compatible with the constraints:

$$i\{b_{i\alpha},b_{j\beta}^\dagger\}_*=[\delta_{\alpha\beta}-(G_{i\beta}b_{i\alpha}-G_{i\alpha}^\dagger b_{i\beta}^\dagger)/D_i]\delta_{ij}\ , \quad (23)$$

$$i\{b_{i\alpha}^\dagger,b_{j\beta}^\dagger\}_*=[(G_{i\beta}b_{i\alpha}^\dagger-G_{i\alpha}b_{i\beta}^\dagger)/D_i]\delta_{ij}\ , \quad (24)$$

$$i\{b_{i\alpha},b_{j\beta}\}_*=[(G_{i\beta}^\dagger b_{i\alpha}-G_{i\alpha}^\dagger b_{i\beta})/D_i]\delta_{ij}\ , \quad (25)$$

$$i\{f_{i\sigma},f_{j\tau}^\dagger\}_*=[\delta_{\sigma\tau}+(H_{i\tau}f_{i\sigma}+H_{i\sigma}^\dagger f_{i\tau}^\dagger)/D_i]\delta_{ij}\ , \quad (26)$$

$$i\{f_{i\sigma}^\dagger,f_{j\tau}^\dagger\}_*=-[(H_{i\tau}f_{i\sigma}^\dagger+H_{i\sigma}f_{i\tau}^\dagger)/D_i]\delta_{ij}\ , \quad (27)$$

$$i\{f_{i\sigma},f_{j\tau}\}_*=-[(H_{i\tau}^\dagger f_{i\sigma}+H_{i\sigma}^\dagger f_{i\tau})/D_i]\delta_{ij}\ , \quad (28)$$

$$i\{f_{i\sigma},b_{j\alpha}^\dagger\}_*=-[(G_{i\alpha}f_{i\sigma}-H_{i\sigma}^\dagger b_{i\alpha}^\dagger)/D_i]\delta_{ij}\ , \quad (29)$$

$$i\{f_{i\sigma}^\dagger,b_{j\alpha}\}_*=-[(G_{i\alpha}^\dagger f_{i\sigma}^\dagger-H_{i\sigma}b_{i\alpha})/D_i]\delta_{ij}\ , \quad (30)$$

$$i\{f_{i\sigma}^\dagger,b_{j\alpha}^\dagger\}_*=[(G_{i\alpha}f_{i\sigma}^\dagger-H_{i\sigma}b_{i\alpha}^\dagger)/D_i]\delta_{ij}\ , \quad (31)$$

$$i\{f_{i\sigma},b_{j\alpha}\}_*=[(G_{i\alpha}^\dagger f_{i\sigma}-H_{i\sigma}^\dagger b_{i\alpha})/D_i]\delta_{ij}\ . \quad (32)$$

It is then clear, as we shall explicitly show below, that a correct operator quantization of such slave-particle models must be based on these new canonical relations, and not on the naive ones of Eq. (7) which are not compatible with the constraints.

Let us note that we have $\{\Pi_i,\Lambda_i\}_*=\{\Pi_i,\Pi_i\}_*=\{\Lambda_i,\Lambda_i\}_*=0$ indeed compatible with the constraints $\Pi_i=\Lambda_i=0$. On the other hand, the Hamiltonian is finally

$$H_4=-X(b^\dagger,b,f^\dagger,f) \quad (33)$$

and we have consistently, for example, $\{b_{i\alpha},H_3\}=\{b_{i\alpha},H_4\}_*$ since

$$\{b_{i\alpha},H_3\}=i\frac{\partial X}{\partial b_{i\alpha}^\dagger}-i(x_i-\Lambda_i)b_{i\alpha}\ , \quad (34)$$

$$\{b_{i\alpha},H_4\}_*=i\frac{\partial X}{\partial b_{i\alpha}^\dagger}-i\frac{b_{i\alpha}}{D_i}\left[\sum_\beta\left[G_{i\beta}\frac{\partial X}{\partial b_{i\beta}^\dagger}-G_{i\beta}^\dagger\frac{\partial X}{\partial b_{i\beta}}\right]+\sum_\sigma\left[H_{i\sigma}\frac{\partial X}{\partial f_{i\sigma}^\dagger}-H_{i\sigma}^\dagger\frac{\partial X}{\partial f_{i\sigma}}\right]\right]\ , \quad (35)$$

which are indeed equal using Eq. (20).

One may wonder at this stage about what happens for the classical version of the Sl(1|2) superalgebra obeyed in the t-J model[10] by the Hubbard[11] operators X_i^{ab}:

$$[X_i^{ab},X_j^{cd}]_\pm=(X_i^{ad}\delta_{bc}\pm X_i^{cb}\delta_{ad})\delta_{ij}\quad \text{with}\ \ X_i^{00}+\sum_\alpha X_i^{\alpha\alpha}=1\ , \quad (36)$$

which opens the way for a supersymmetric t-J model[10] and which is verified in the literature within a slave representation using the naive canonical relations. In fact, taking, for instance, in the t-J model the slave-fermion representation $c_{i\alpha}^\dagger=b_{i\alpha}^\dagger f_i$ of the electron field $c_{i\alpha}^\dagger$ which corresponds to $X_i^{\alpha 0}$, the Sl(1|2) superalgebra is obeyed using either the naive initial brackets of Eq. (7) or the Dirac brackets of Eqs. (23)–(32). For example, one has

$$i\{c_{i\alpha}^\dagger,c_{j\beta}\}=i\{c_{i\alpha}^\dagger,c_{j\beta}\}_*=[b_{i\alpha}^\dagger b_{i\beta}+f_i^\dagger f_i\delta_{\alpha\beta}]\delta_{ij}\ , \quad (37)$$

where $b_{i\alpha}^\dagger b_{i\beta}$ corresponds to $X_i^{\alpha\beta}$ and $f_i^\dagger f_i$ to X_i^{00}. The reason for this property is that the Dirac bracket of any two gauge invariant quantities is the same as their initial bracket. As all the generators of the Sl(1|2) superalbegra are gauge invariant, it is licit to use the initial brackets of the b's and f's to compute the Dirac brackets of these generators. One must, however, realize that this property is valid only for gauge invariant quantities. At the quantum level, a proper quantization should inherit the same property for (anti)commutators of gauge-invariant quantities, but not for products of these quantities. Let us also note that, using the slave-particle constraint, one has from Eq. (37)

$$\sum_\alpha i\{c_{i\alpha}^\dagger,c_{j\alpha}\}=\sum_\alpha i\{c_{i\alpha}^\dagger,c_{j\alpha}\}_*=[1+f_i^\dagger f_i]\delta_{ij}\ , \quad (38)$$

a relation that we shall recover below in a sum rule at the quantum level.

III. MODIFICATIONS OF THE NAIVE SUM RULES FOR THE SLAVE *t-J* MODELS

As mentioned in the Introduction, it has been stressed in the literature[5] that a study of the electron-momentum distribution function in the t-J model in the framework of the slave-particle approach and within the decoupling scheme would give rise to a violation of the sum rule of electron number. However, to obtain this result, sum rules for the slave particles using quantization of the naive relations of Eq. (7) were used. On the contrary we shall show in this section that starting from the correct canonical relations compatible with the constraints produces modifications in these sum rules for the slave particles in such a way that the sum rule of electron number is indeed obeyed. Though we shall present in the next section a direct explicit quantization which confirms this result, we think that it is important for the clarity of the exposition to see here the simple structure of these modifications in the sum rules.

A. The slave-fermion representation

Let us consider in the t-J model the slave-fermion representation $\hat{c}_{i\alpha}^\dagger=\hat{b}_{i\alpha}^\dagger\hat{f}_i$ of the electron operator $\hat{c}_{i\alpha}^\dagger$, with the slave-particle constraint avoiding double occupancy at site i:

$$\hat{f}_i^\dagger \hat{f}_i + \sum_\alpha \hat{b}_{i\alpha}^\dagger \hat{b}_{i\alpha} = 1 \ . \tag{39}$$

As in Ref. 5, we assume that there is no Bose condensation of spinons; i.e., the temperature of the system is 0^+. The hole-doping concentration δ is given by $\delta = \langle \hat{f}_i^\dagger \hat{f}_i \rangle$.

For discussing the electron-momentum distribution function, the Matsubara electron Green's function in imaginary time[12,13]

$$E_\alpha(r,\tau) = -\langle T_\tau(\hat{f}_i^\dagger(\tau)\hat{b}_{i\alpha}(\tau)\hat{b}_{j\alpha}^\dagger(0)\hat{f}_j(0))\rangle \quad (r = i - j) \ , \tag{40}$$

where $\langle \cdots \rangle$ means the thermodynamical average, was considered in Ref. 5 within the decoupling approximation written on Fourier transform:

$$E_\alpha(k,\omega_n) = \frac{1}{N}\sum_q \frac{1}{\beta}\sum_{\omega_m} F(q,\omega_m) B_\alpha(q+k,\omega_m+\omega_n) \ . \tag{41}$$

F is the Green's function for the slave-fermion f and B_α is the Green's function for the boson b_α. Introducing the Lehmann's spectral representations,

$$\begin{bmatrix} E_\alpha \\ F \\ B_\alpha \end{bmatrix}(k,\omega_n) = \int_{-\infty}^{+\infty} \frac{d\omega}{2\pi} \begin{bmatrix} A_{e\alpha} \\ A_f \\ A_{b\alpha} \end{bmatrix}(k,\omega) \frac{1}{i\omega_n - \omega} \ , \tag{42}$$

one easily obtains the electron spectral function as

$$A_{e\alpha}(k,\omega) = \frac{1}{N}\sum_q \int_{-\infty}^{+\infty} \frac{d\omega'}{2\pi} A_f(q,\omega') A_{b\alpha}(q+k,\omega+\omega') \times [n_F(\omega') + n_B(\omega+\omega')] \ , \tag{43}$$

where n_F and n_B are, respectively, the Fermi and Bose distribution functions.

Using the expressions for the numbers of slave fermions, bosons, and electrons in state q,

$$\begin{aligned} n_f(q) &= \int_{-\infty}^{+\infty} \frac{d\omega}{2\pi} n_F(\omega) A_f(q,\omega) \ , \\ n_{b\alpha}(q) &= \int_{-\infty}^{+\infty} \frac{d\omega}{2\pi} n_B(\omega) A_{b\alpha}(q,\omega) \ , \\ n_e(q) &= \sum_\alpha \int_{-\infty}^{+\infty} \frac{d\omega}{2\pi} n_F(\omega) A_{e\alpha}(q,\omega) \ , \end{aligned} \tag{44}$$

the identity

$$n_F(\omega)[n_F(\omega') + n_B(\omega+\omega')] = n_B(\omega+\omega')[1 - n_F(\omega')] \ ,$$

and the sum rules

$$\begin{aligned} \int_{-\infty}^{+\infty} \frac{d\omega}{2\pi} A_f(q,\omega) &= \langle [\hat{f}_q, \hat{f}_q^\dagger]_+ \rangle \ , \\ \sum_\alpha \int_{-\infty}^{+\infty} \frac{d\omega}{2\pi} A_{b\alpha}(q,\omega) &= \left\langle \sum_\alpha [\hat{b}_{q\alpha}, \hat{b}_{q\alpha}^\dagger]_- \right\rangle \ , \end{aligned} \tag{45}$$

one obtains the sum rule for the electron spectral function

$$\begin{aligned} \sum_\alpha \int_{-\infty}^{+\infty} \frac{d\omega}{2\pi} A_{e\alpha}(k,\omega) &= \frac{1}{N}\sum_q n_f(q) \left\langle \sum_\alpha [\hat{b}_{(q+k)\alpha}, \hat{b}_{(q+k)\alpha}^\dagger]_- \right\rangle \\ &\quad + \frac{1}{N}\sum_{q,\alpha} n_{b\alpha}(q+k) \langle [\hat{f}_q, \hat{f}_q^\dagger]_+ \rangle \end{aligned} \tag{46}$$

and the expression for the number of electrons in state k

$$\begin{aligned} n_e(k) &= \frac{1}{N}\sum_{q,\alpha} n_{b\alpha}(q+k) \langle [\hat{f}_q, \hat{f}_q^\dagger]_+ \rangle \\ &\quad - \frac{1}{N}\sum_{q,\alpha} n_{b\alpha}(q+k) n_f(q) \ . \end{aligned} \tag{47}$$

Let us make the remark, not mentioned in Ref. 5, that the last equation can in fact be directly derived from the very definition of the decoupling approximation:

$$E_\alpha(r,\tau) = B_\alpha(r,\tau) F(-r,-\tau) \ , \tag{48}$$

where we used

$$\begin{aligned} \langle T_\tau(\hat{b}_{i\alpha}(\tau)\hat{b}_{j\alpha}^\dagger(0))\rangle &= -B_\alpha(r,\tau) \ , \\ \langle T_\tau(\hat{f}_i^\dagger(\tau)\hat{f}_j(0))\rangle &= F(-r,-\tau) \ . \end{aligned} \tag{49}$$

Then, with $\eta \to 0^+$,[12] one has

$$\begin{aligned} n_e &= \frac{1}{N}\sum_{i,\alpha} E_\alpha(0,-\eta) = \frac{1}{N}\sum_{i,\alpha} B_\alpha(0,-\eta) F(0,\eta) \\ &= \frac{1}{N}\sum_i \left[\sum_\alpha \langle \hat{b}_{i\alpha}^\dagger \hat{b}_{i\alpha} \rangle \right] \langle \hat{f}_i \hat{f}_i^\dagger \rangle \end{aligned} \tag{50}$$

using $B_\alpha(0,-\eta) = -\langle \hat{b}_{i\alpha}^\dagger \hat{b}_{i\alpha} \rangle$ and $F(0,\eta) = -\langle \hat{f}_i \hat{f}_i^\dagger \rangle$. This gives in Fourier transform the expression

$$\begin{aligned} n_e(k) &= \frac{1}{N}\sum_{q,\alpha} B_{(q+k)\alpha}(-\eta) F_q(\eta) \\ &= \frac{1}{N}\sum_{q,\alpha} \langle \hat{b}_{(q+k)\alpha}^\dagger \hat{b}_{(q+k)\alpha} \rangle \langle \hat{f}_q \hat{f}_q^\dagger \rangle \ , \end{aligned} \tag{51}$$

which, using the anticommutator of the fermion, can immediately be written in the form of Eq. (47).

The problem now is to evaluate the thermodynamical average of the commutator and anticommutator entering in these expressions. Using the quantization from the Dirac brackets

$$i\{A,B\}_* \to [\hat{A},\hat{B}]_\pm \ , \tag{52}$$

one obtains from our results Eqs. (23) and (26) of the preceding section the form of the commutator and anticommutator at equal times:

$$\begin{aligned} [\hat{b}_{i\alpha}, \hat{b}_{j\alpha}^\dagger]_- &= (1 - \hat{\Theta}_{i\alpha})\delta_{ij} \ , \\ [\hat{f}_i, \hat{f}_j^\dagger]_+ &= (1 + \hat{\Theta}_i)\delta_{ij} \ , \end{aligned} \tag{53}$$

where we have introduced the $\hat{\Theta}$ operators defined through the quantizations:

$$\begin{aligned} (G_{i\alpha} b_{i\alpha} - G_{i\alpha}^\dagger b_{i\alpha}^\dagger)/D_i &\equiv \Theta_{i\alpha} \to \hat{\Theta}_{i\alpha} \ , \\ (H_i f_i + H_i^\dagger f_i^\dagger)/D_i &\equiv \Theta_i \to \hat{\Theta}_i \ . \end{aligned} \tag{54}$$

Let us note that the definition of D_i from Eq. (18) gives by quantization the relation

$$\sum_\alpha \hat{\Theta}_{i\alpha}+\hat{\Theta}_i=1 \; . \tag{55}$$

With $\hat{\Theta}_n=\sum_p e^{ipn}\hat{\Xi}_p$ and $\hat{\Theta}_{n\alpha}=\sum_p e^{ipn}\hat{\Xi}_{p\alpha}$, it follows that

$$\langle [\hat{f}_q,\hat{f}_q^\dagger]_+\rangle=1+\langle\hat{\Xi}_0\rangle \; , \tag{56}$$

$$\left\langle \sum_\alpha [\hat{b}_{(q+k)\alpha},\hat{b}^\dagger_{(q+k)\alpha}]_-\right\rangle=2-\langle \sum_\alpha \hat{\Xi}_{0\alpha}\rangle =1+\langle\hat{\Xi}_0\rangle \; . \tag{57}$$

Since with the hole-doping concentration δ one has

$$\frac{1}{N}\sum_q n_f(q)=\delta \; , \quad \frac{1}{N}\sum_{q,\alpha} n_{b\alpha}(q)=1-\delta \; , \tag{58}$$

one obtains the following results:

$$\sum_\alpha \int_{-\infty}^{+\infty}\frac{d\omega}{2\pi}A_{e\alpha}(k,\omega)=1+\langle\hat{\Xi}_0\rangle \; , \tag{59}$$

$$n_e(k)=(1-\delta)(1+\langle\hat{\Xi}_0\rangle)-\frac{1}{N}\sum_{q,\alpha}n_{b\alpha}(q+k)n_f(q) \; , \tag{60}$$

$$\frac{1}{N}\sum_k n_e(k)=(1-\delta)(1+\langle\hat{\Xi}_0\rangle-\delta) \; . \tag{61}$$

One then sees clearly the modifications of the sum rules and of the results of Ref. 5 coming from the presence of the $\hat{\Theta}$'s in the correct canonical relations. The main point is that our results show that one indeed obtains the expected result that, if δ holes are introduced into the half-filled system, the total electron number per site would be $1-\delta$, instead of the $(1-\delta)^2$ found in Ref. 5.

In fact, from the quantization of Eq. (38) at the end of the preceding section, the sum rule for the electron spectral function must be

$$\sum_\alpha \int_{-\infty}^{+\infty}\frac{d\omega}{2\pi}A_{e\alpha}(q,\omega)=\left\langle \sum_\alpha [\hat{c}_{q\alpha},\hat{c}^\dagger_{q\alpha}]_+\right\rangle =1+\frac{1}{N}\sum_k\langle \hat{f}_k^\dagger\hat{f}_k\rangle=1+\delta \; , \tag{62}$$

being unchanged with respect to Ref. 5. Comparing with Eq. (59), this gives

$$\langle\hat{\Xi}_0\rangle=\delta \; . \tag{63}$$

From our results, the expected sum rule of the electron number is indeed recovered with $\langle\hat{\Xi}_0\rangle=\delta$, since

$$\frac{1}{N}\sum_k n_e(k)=(1-\delta)(1+\langle\hat{\Xi}_0\rangle-\delta)=1-\delta \; .$$

However, this does not guarantee the existence of an EFS within the decoupling scheme in the slave-fermion approach of the t-J model, and the arguments of Ref. 5 against an EFS still apply.

The electron operator anticommutator furthermore shows that one has not only the average equality $\langle\hat{\Theta}_i\rangle=\langle\hat{\Xi}_0\rangle=\delta$ but also the operator equality $\hat{\Theta}_i=\hat{f}_i^\dagger\hat{f}_i$: on one hand, quantization of Eq. (38) gives $\sum_\alpha[\hat{c}_{i\alpha},\hat{c}^\dagger_{i\alpha}]_+=1+\hat{f}_i^\dagger\hat{f}_i$, while, on the other hand, one gets by expressing each term of the anticommutator with the boson and fermion operators

$$\sum_\alpha[\hat{c}_{i\alpha},\hat{c}^\dagger_{i\alpha}]_+=1+\hat{f}_i^\dagger\hat{f}_i+\sum_\alpha b^\dagger_{i\alpha}(\hat{\Theta}_i-\hat{f}_i^\dagger\hat{f}_i)\hat{b}_{i\alpha} +\hat{f}_i^\dagger(\hat{\Theta}_i-\hat{f}_i^\dagger\hat{f}_i)\hat{f}_i \; , \tag{64}$$

effectively leading to a consistent quantization expressed by $\hat{\Theta}_i=\hat{f}_i^\dagger\hat{f}_i$.

B. The slave-boson representation

Our analysis of the t-J model in the slave-boson representation $\hat{c}^\dagger_{i\sigma}=\hat{f}^\dagger_{i\sigma}\hat{b}_i$ of the electron operator $\hat{c}^\dagger_{i\sigma}$, with the slave-particle constraint avoiding double occupancy at site i,

$$\hat{b}_i^\dagger\hat{b}_i+\sum_\sigma \hat{f}^\dagger_{i\sigma}\hat{f}_{i\sigma}=1 \; , \tag{65}$$

follows along the same lines. As in Ref. 5, we assume that there is no Bose condensation of holons. The hole-doping concentration δ is given by $\delta=\langle\hat{b}_i^\dagger\hat{b}_i\rangle$.

The Matsubara electron Green's function in imaginary time,

$$E_\sigma(r,\tau)=-\langle T_\tau(\hat{b}_i^\dagger(\tau)\hat{f}_{i\sigma}(\tau)\hat{f}^\dagger_{j\sigma}(0)\hat{b}_j(0))\rangle \; , \tag{66}$$

within the decoupling approximation written on Fourier transform reads

$$E_\sigma(k,\omega_n)=-\frac{1}{N}\sum_q\frac{1}{\beta}\sum_{\omega_m}B(q,\omega_m-\omega_n)F_\sigma(q+k,\omega_m) \tag{67}$$

and the electron spectral function is

$$A_{e\sigma}(k,\omega)=\frac{1}{N}\sum_q\int_{-\infty}^{+\infty}\frac{d\omega'}{2\pi}A_b(q,\omega'-\omega)A_{f\sigma}(q+k,\omega') \times[n_F(\omega')+n_B(\omega'-\omega)] \; . \tag{68}$$

Using the identity

$$n_F(\omega)[n_F(\omega')+n_B(\omega'-\omega)]=n_F(\omega')[1+n_B(\omega'-\omega)]$$

and the sum rules analogous to Eq.(45), one obtains the sum rule for the electron spectral function,

$$\sum_\sigma\int_{-\infty}^{+\infty}\frac{d\omega}{2\pi}A_{e\sigma}(k,\omega) =\frac{1}{N}\sum_q n_b(q)\left\langle\sum_\sigma[\hat{f}_{(q+k)\sigma},\hat{f}^\dagger_{(q+k)\sigma}]_+\right\rangle +\frac{1}{N}\sum_{q,\sigma}n_{f\sigma}(q+k)\langle[\hat{b}_q,\hat{b}_q^\dagger]_-\rangle \; , \tag{69}$$

and the expression for the number of electrons in state k,

$$n_e(k)=\frac{1}{N}\sum_{q,\sigma}n_{f\sigma}(q+k)\langle[\hat{b}_q,\hat{b}_q^\dagger]_-\rangle +\frac{1}{N}\sum_{q,\sigma}n_{f\sigma}(q+k)n_b(q) \; . \tag{70}$$

Again, the last equation can in fact be directly derived from the very definition of the decoupling approximation:

$$E_\sigma(r,\tau) = -F_\sigma(r,\tau)B(-r,-\tau) , \quad (71)$$

where we used

$$\langle T_\tau(\hat{f}_{i\sigma}(\tau)\hat{f}^\dagger_{j\sigma}(0))\rangle = -F_\sigma(r,\tau) ,$$
$$\langle T_\tau(\hat{b}^\dagger_i(\tau)\hat{b}_j(0))\rangle = -B(-r,-\tau) . \quad (72)$$

Then, with $\eta \to 0^+$, one has

$$n_e = \frac{1}{N}\sum_{i,\sigma} E_\sigma(0,-\eta) = -\frac{1}{N}\sum_{i,\sigma} F_\sigma(0,-\eta)B(0,\eta)$$
$$= \frac{1}{N}\sum_i \left[\sum_\sigma \langle \hat{f}^\dagger_{i\sigma}\hat{f}_{i\sigma}\rangle\right]\langle \hat{b}_i\hat{b}^\dagger_i\rangle \quad (73)$$

using $F_\alpha(0,-\eta) = +\langle \hat{f}^\dagger_{i\sigma}\hat{f}_{i\sigma}\rangle$ and $B(0,\eta) = -\langle \hat{b}_i\hat{b}^\dagger_i\rangle$. This gives in Fourier transform the expression,

$$n_e(k) = -\frac{1}{N}\sum_{q,\sigma} F_{(q+k)\sigma}(-\eta)B_q(\eta)$$
$$= \frac{1}{N}\sum_{q,\sigma}\langle \hat{f}^\dagger_{(q+k)\sigma}\hat{f}_{(q+k)\sigma}\rangle\langle \hat{b}_q\hat{b}^\dagger_q\rangle , \quad (74)$$

which, using the commutator of the boson, can immediately be written in the form of Eq. (70).

Now, with the quantizations

$$(G_i b_i - G^\dagger_i b^\dagger_i)/D_i \to \hat{\Theta}_i ,$$
$$(H_{i\sigma} f_{i\sigma} + H^\dagger_{i\sigma} f^\dagger_{i\sigma})/D_i \to \hat{\Theta}_{i\sigma} , \quad \sum_\sigma \hat{\Theta}_{i\sigma} + \hat{\Theta}_i = 1 \quad (75)$$

one obtains from our results of the preceding section the relations at equal times:

$$[\hat{b}_i,\hat{b}^\dagger_j]_- = (1-\hat{\Theta}_i)\delta_{ij} , \quad [\hat{f}_{i\sigma},\hat{f}^\dagger_{j\sigma}]_+ = (1+\hat{\Theta}_{i\sigma})\delta_{ij} \quad (76)$$

and with the same notations as above:

$$\langle [\hat{b}_q,\hat{b}^\dagger_q]_-\rangle = 1 - \langle \hat{\Xi}_0\rangle , \quad (77)$$

$$\left\langle \sum_\sigma [\hat{f}_{(q+k)\sigma},\hat{f}^\dagger_{(q+k)\sigma}]_+\right\rangle = 2 + \left\langle \sum_\alpha \hat{\Xi}_{0\alpha}\right\rangle = 3 - \langle \hat{\Xi}_0\rangle . \quad (78)$$

Inserting these results in Eqs. (69) and (70), one finally gets

$$\sum_\sigma \int_{-\infty}^{+\infty}\frac{d\omega}{2\pi}A_{e\sigma}(k,\omega) = 1 + 2\delta - \langle \hat{\Xi}_0\rangle , \quad (79)$$

$$n_e(k) = (1-\delta)(1-\langle \hat{\Xi}_0\rangle) + \frac{1}{N}\sum_{q,\sigma} n_{f\sigma}(q+k)n_b(q) , \quad (80)$$

$$\frac{1}{N}\sum_k n_e(k) = (1-\delta)(1-\langle \hat{\Xi}_0\rangle + \delta) . \quad (81)$$

Again, one sees clearly the modifications of the sum rules and of the results of Ref. 5 coming from the presence of the $\hat{\Theta}$'s in the correct canonical relations and that with

$$\langle \hat{\Xi}_0\rangle = \delta \quad (82)$$

one has the correct sum rules for the electron spectral function:

$$\sum_\sigma \int_{-\infty}^{+\infty}\frac{d\omega}{2\pi}A_{e\sigma}(q,\omega) = 1 + \delta \quad (83)$$

and for the electron number

$$\frac{1}{N}\sum_k n_e(k) = (1-\delta) \quad (84)$$

[instead of the $(1-\delta^2)$ found in Ref. 5]. However, this does not guarantee the existence of an EFS within the decoupling scheme in the slave-boson approach of the *t-J* model, and the arguments of Ref. 5 against an EFS still apply. Furthermore, the same arguments as above lead, using the electron operator anticommutator, to the operator equality $\hat{\Theta}_i = \hat{b}^\dagger_i\hat{b}_i$.

IV. EXPLICIT QUANTIZATION OF THE SLAVE-PARTICLE APPROACHES OF THE *t-J* MODEL

In the preceding section, it was shown that the expected sum rules for the electron spectral function and for the electron number are indeed found in the slave-particle approaches of the *t-J* model due to the fact that $\langle \hat{\Theta}_i\rangle = \langle \hat{\Xi}_0\rangle = \delta$, the operator $\hat{\Theta}_i$ being the new term which is present in the canonical relation of the slave particle when one quantizes the correct (Dirac) brackets compatible with the constraints.

We furthermore proved using the electron operator anticommutator that $\hat{\Theta}_i$ was the slave-particle number operator. We shall show in this section that a direct explicit operator quantization of the slave-particle approaches of the *t-J* model effectively confirms this result without invoking the electron operator.

A. The slave-fermion representation

Let us first take the slave-fermion case, where we obtain from the results of Sec. II

$$[\hat{f}_i,\hat{f}^\dagger_j]_+ = (1+\hat{\Theta}_i)\delta_{ij} , \quad [\hat{b}_{i\alpha},\hat{b}^\dagger_{j\alpha}]_- = (1-\hat{\Theta}_{i\alpha})\delta_{ij} , \quad (85)$$

where [cf. Eqs. (53) and (54)] the $\hat{\Theta}$'s correspond to the quantizations

$$(H_i f_i + H^\dagger_i f^\dagger_i)/D_i \to \hat{\Theta}_i , \quad (G_{i\alpha}b_{i\alpha} - G^\dagger_{i\alpha}b^\dagger_{i\alpha})/D_i \to \hat{\Theta}_{i\alpha} . \quad (86)$$

However, since $D_i \equiv \sum_\alpha (G_{i\alpha}b_{i\alpha} - G^\dagger_{i\alpha}b^\dagger_{i\alpha}) + (H_i f_i + H^\dagger_i f^\dagger_i)$, the explicit expressions of the $\hat{\Theta}$ operators in terms of the operators of the fermion and bosons are *a priori* not obvious. We shall now obtain these expressions through the structure of the Fock space.

From the slave-particle constraint,

$$\hat{f}^\dagger_i\hat{f}_i + \sum_\alpha \hat{b}^\dagger_{i\alpha}\hat{b}_{i\alpha} = 1 , \quad (87)$$

the operators $\hat{f}^\dagger_i\hat{f}_i$ and $\hat{b}^\dagger_{i\alpha}\hat{b}_{i\alpha}$ must be particle-number operators satisfying

$$(\hat{f}^\dagger_i\hat{f}_i)^2 = \hat{f}^\dagger_i\hat{f}_i \quad \text{and} \quad (\hat{b}^\dagger_{i\alpha}\hat{b}_{i\alpha})^2 = \hat{b}^\dagger_{i\alpha}\hat{b}_{i\alpha} . \quad (88)$$

Now it follows from Eq. (85) that

$$(\hat{f}_i^\dagger \hat{f}_i)^2 = \hat{f}_i^\dagger \hat{f}_i - \hat{f}_i^\dagger(\hat{f}_i^\dagger \hat{f}_i - \hat{\Theta}_i)\hat{f}_i \ ,$$
$$(\hat{b}_{i\alpha}^\dagger \hat{b}_{i\alpha})^2 = \hat{b}_{i\alpha}^\dagger \hat{b}_{i\alpha} + \hat{b}_{i\alpha}^\dagger(\hat{b}_{i\alpha}^\dagger \hat{b}_{i\alpha} - \hat{\Theta}_{i\alpha})\hat{b}_{i\alpha} \tag{89}$$

and a consistent quantization is thus

$$\hat{\Theta}_i = \hat{f}_i^\dagger \hat{f}_i \ , \quad \hat{\Theta}_{i\alpha} = \hat{b}_{i\alpha}^\dagger \hat{b}_{i\alpha} \ , \tag{90}$$

which from Eq. (85) leads to the relations

$$\hat{f}_i \hat{f}_i^\dagger = 1 \ , \quad \hat{b}_{i\alpha}\hat{b}_{i\alpha}^\dagger = 1 \ . \tag{91}$$

We will show below that the dimension of the Fock space at site i is in fact infinite (but without contradiction with the non-double-electron occupancy at site i). Thus Eqs. (91) are consistent with the fact that the particle-number operators are not unity. Let us remark that we have consistently, in accordance with Eqs. (55) and (87),

$$\hat{\Theta}_i + \sum_\alpha \hat{\Theta}_{i\alpha} = \hat{f}_i^\dagger \hat{f}_i + \sum_\alpha \hat{b}_{i\alpha}^\dagger \hat{b}_{i\alpha} = 1 \ . \tag{92}$$

On the other hand, the fundamental Dirac brackets compatible with the constraints given by Eqs. (23)–(32) lead to the following expressions:

$$i\{b_{i\alpha}^\dagger, f_i^\dagger f_i\}_* = b_{i\alpha}^\dagger \Theta_i \ , \tag{93}$$

$$i\{f_i^\dagger, b_{i\alpha}^\dagger b_{i\alpha}\}_* = f_i^\dagger \Theta_{i\alpha} \ , \tag{94}$$

$$i\{b_{i\alpha}^\dagger, b_{i\beta}^\dagger b_{i\beta}\}_* = -b_{i\beta}^\dagger \delta_{\alpha\beta} + b_{i\alpha}^\dagger \Theta_{i\beta} \ . \tag{95}$$

Quantizing these expressions with the same ordering of operators gives via Eqs. (90) and (91)

$$\hat{f}_i^\dagger \hat{f}_i \hat{b}_{i\alpha}^\dagger = \hat{b}_{i\alpha}^\dagger(\hat{f}_i^\dagger \hat{f}_i - \hat{\Theta}_i) = 0 \ \text{ and thus } \ \hat{f}_i \hat{b}_{i\alpha}^\dagger = 0 \ , \tag{96}$$

$$\hat{b}_{i\alpha}^\dagger \hat{b}_{i\alpha} \hat{f}_i^\dagger = \hat{f}_i^\dagger(\hat{b}_{i\alpha}^\dagger \hat{b}_{i\alpha} - \hat{\Theta}_{i\alpha}) = 0 \ \text{ and thus } \ \hat{b}_{i\alpha}\hat{f}_i^\dagger = 0 \ , \tag{97}$$

$$\hat{b}_{i\beta}^\dagger \hat{b}_{i\beta} \hat{b}_{i\alpha}^\dagger = \hat{b}_{i\alpha}^\dagger(\hat{b}_{i\beta}^\dagger \hat{b}_{i\beta} - \hat{\Theta}_{i\beta}) + \hat{b}_{i\beta}^\dagger \delta_{\alpha\beta}$$
$$\text{and thus } \ \hat{b}_{i\beta}\hat{b}_{i\alpha}^\dagger = \delta_{\alpha\beta} \ . \tag{98}$$

Using all our results, one easily verifies that the operators $\hat{f}_i^\dagger \hat{f}_i$ and $\hat{b}_{i\alpha}^\dagger \hat{b}_{i\alpha}$ are indeed particle-number operators: for example, $\hat{f}_i^\dagger \hat{f}_i$ acting on the fermion state $\hat{f}_i^\dagger|0\rangle$ at site i has eigenvalue 1 and acting on the two boson states $\hat{b}_{i\alpha}^\dagger|0\rangle$ at site i has eigenvalue 0.

One can check the consistency of our quantization: for example, quantizing the expression

$$i\{b_{i\alpha}^\dagger, b_{i\beta}b_{i\beta}^\dagger\}_* = -b_{i\beta}^\dagger \delta_{\alpha\beta} + \Theta_{i\beta} b_{i\alpha}^\dagger \tag{99}$$

with the same ordering of operators gives

$$[\hat{b}_{i\alpha}^\dagger, \hat{b}_{i\beta}\hat{b}_{i\beta}^\dagger]_- = [\hat{b}_{i\alpha}^\dagger, 1]_- = 0 \tag{100}$$

$$= -\hat{b}_{i\beta}^\dagger \delta_{\alpha\beta} + \hat{b}_{i\beta}^\dagger \hat{b}_{i\beta} \hat{b}_{i\alpha}^\dagger$$
$$= -\hat{b}_{i\beta}^\dagger \delta_{\alpha\beta} + \hat{b}_{i\beta}^\dagger \delta_{\alpha\beta} = 0 \ . \tag{101}$$

Through an explicit operator quantization of the slave-fermion approach of the t-J model we have thus obtained, at site i,

$$\hat{f}_i \hat{f}_i^\dagger = 1 \ , \quad \hat{b}_{i\alpha}\hat{b}_{i\beta}^\dagger = \delta_{\alpha\beta} \ , \quad \hat{f}_i \hat{b}_{i\alpha}^\dagger = 0 \ , \quad \hat{b}_{i\alpha}\hat{f}_i^\dagger = 0 \tag{102}$$

or in other words:

$$[\hat{f}_i, \hat{f}_j^\dagger]_+ = (1 + \hat{f}_i^\dagger \hat{f}_i)\delta_{ij} \ ,$$
$$[\hat{b}_{i\alpha}, \hat{b}_{j\beta}^\dagger]_- = (\delta_{\alpha\beta} - \hat{b}_{i\beta}^\dagger \hat{b}_{i\alpha})\delta_{ij} \ , \tag{103}$$

$$[\hat{f}_i, \hat{b}_{j\alpha}^\dagger]_- = -\hat{b}_{i\alpha}^\dagger \hat{f}_i \delta_{ij} \ ,$$
$$[\hat{b}_{i\alpha}, \hat{f}_j^\dagger]_- = -\hat{f}_i^\dagger \hat{b}_{i\alpha}\delta_{ij} \ . \tag{104}$$

Looking in the same way as above at the quantization of the classical canonical relations involving only creators or annihilators, we found either no informations or identities. Such relations might thus be only identities at the quantum level. The results of Eq. (102) will, however, be sufficient for the purposes of this paper.

In fact, as a consequence of this study we directly recover the expressions

$$\hat{\Theta}_i = \hat{f}_i^\dagger \hat{f}_i \ , \quad \langle \hat{\Theta}_i \rangle = \langle \hat{\Xi}_0 \rangle = \delta \ , \tag{105}$$

which through the analysis of Sec. III leads to the conclusion that the expected sum rules for the electron spectral function and for the electron number are indeed found in the slave-fermion approach of the t-J model within the decoupling approximation. Using Eq. (50) and our result $\hat{f}_i \hat{f}_i^\dagger = 1$, one can also see directly that

$$n_e = \frac{1}{N}\sum_i \left[\sum_\alpha \langle \hat{b}_{i\alpha}^\dagger \hat{b}_{i\alpha} \rangle \right] \langle \hat{f}_i \hat{f}_i^\dagger \rangle = (1-\delta) \ . \tag{106}$$

It is also important, with the expression of the electron operator,

$$\hat{c}_{i\alpha}^\dagger = \hat{b}_{i\alpha}^\dagger \hat{f}_i \tag{107}$$

and using our results for the quantization, to found the exact electron number operator

$$\hat{n}_i = \sum_\alpha \hat{c}_{i\alpha}^\dagger \hat{c}_{i\alpha} = \sum_\alpha \hat{b}_{i\alpha}^\dagger \hat{f}_i \hat{f}_i^\dagger \hat{b}_{i\alpha} = \sum_\alpha \hat{b}_{i\alpha}^\dagger \hat{b}_{i\alpha} = 1 - \hat{f}_i^\dagger \hat{f}_i \tag{108}$$

and to verify that

$$[\hat{c}_{i\alpha}^\dagger, \hat{c}_{i\beta}]_+ = \hat{b}_{i\alpha}^\dagger \hat{f}_i \hat{f}_i^\dagger \hat{b}_{i\beta} + \hat{f}_i^\dagger \hat{b}_{i\beta} \hat{b}_{i\alpha}^\dagger \hat{f}_i = \hat{b}_{i\alpha}^\dagger \hat{b}_{i\beta} + \hat{f}_i^\dagger \hat{f}_i \delta_{\alpha\beta} \ , \tag{109}$$

$$\sum_\alpha [\hat{c}_{i\alpha}^\dagger, \hat{c}_{i\alpha}]_+ = 1 + \hat{f}_i^\dagger \hat{f}_i \ , \tag{110}$$

effectively corresponding to the quantization of Eqs. (37) and (38).

We can now examine, as announced above, the structure of the Fock space at site i. Apart from the three states $|f_i\rangle = \hat{f}_i^\dagger|0\rangle$ and $|b_{i\alpha}\rangle = \hat{b}_{i\alpha}^\dagger|0\rangle$, we have an infinite number of states of the form either $\|f_i\rangle \equiv \hat{f}_i^\dagger(\hat{A}_i^\dagger)^n|0\rangle$ or $\|b_{i\alpha}\rangle \equiv \hat{b}_{i\alpha}^\dagger(\hat{A}_i^\dagger)^n|0\rangle$ where $(\hat{A}_i^\dagger)^n$ are products of $\hat{b}_{i\beta}^\dagger$ and $\hat{f}_i^\dagger$. The constraint of Eq. (87), counting only the last created particle, is satisfied for all these states. The particle content of these states can be found using a basis of new commuting operators; for example, one has $(\hat{b}_{i\alpha}^\dagger \hat{b}_{i\beta}^\dagger \hat{b}_{i\beta} \hat{b}_{i\alpha})\hat{b}_{i\alpha}^\dagger \hat{b}_{i\beta}^\dagger|0\rangle = \hat{b}_{i\alpha}^\dagger \hat{b}_{i\beta}^\dagger|0\rangle$. Nevertheless, the states $|\ \rangle$ and $\|\ \rangle$ share the same properties concerning the electron operators:

$$\hat{c}_{i\alpha}^\dagger|f_i\rangle = |b_{i\alpha}\rangle \ , \quad \hat{c}_{i\alpha}^\dagger|b_{i\beta}\rangle = 0 \ ,$$
$$\hat{c}_{i\alpha}^\dagger \hat{c}_{i\alpha}|f_i\rangle = 0 \ , \quad \hat{c}_{i\alpha}^\dagger \hat{c}_{i\alpha}|b_{i\beta}\rangle = \delta_{\alpha\beta}|b_{i\beta}\rangle \ ,$$

$$\hat{c}_{i\alpha}^{\dagger}\|f_i\rangle=\|b_{i\alpha}\rangle\ ,\quad \hat{c}_{i\alpha}^{\dagger}\|b_{i\beta}\rangle=0\ ,$$
$$\hat{c}_{i\alpha}^{\dagger}\hat{c}_{i\alpha}\|f_i\rangle=0\ ,\quad \hat{c}_{i\alpha}^{\dagger}\hat{c}_{i\alpha}\|b_{i\beta}\rangle=\delta_{\alpha\beta}\|b_{i\beta}\rangle\ .$$

Thus, there is always either no electron or one electron at site i, as is also expressed by the relation $\hat{c}_{i\alpha}^{\dagger}\hat{c}_{i\beta}^{\dagger}=\hat{b}_{i\alpha}^{\dagger}\hat{f}_i\hat{b}_{i\beta}^{\dagger}\hat{f}_i=0$ using Eq. (102). In spite of the infinite dimension of the Fock space at site i, the non-double-electron occupancy at site i is well satisfied, and furthermore it is equivalent for physical purposes to use only the sector of the three states $|f_i\rangle$ and $|b_{i\alpha}\rangle$.

Let us finally insist on the fact that we have now a systematic direct algebraic procedure to find the expression in the slave-particle approach of any operator initially written in terms of the electron operators. For instance, since using our result $\hat{f}_i\hat{f}_i^{\dagger}=1$ one has

$$\hat{c}_{i\alpha}^{\dagger}\hat{c}_{i\beta}=\hat{b}_{i\alpha}^{\dagger}\hat{f}_i\hat{f}_i^{\dagger}\hat{b}_{i\beta}=\hat{b}_{i\alpha}^{\dagger}\hat{b}_{i\beta}\ , \tag{111}$$

we obtain directly the t-J model Hamiltonian of Eq. (2) in the slave-fermion representation:

$$\hat{H}=t\sum_{\langle ij\rangle,\alpha}(\hat{b}_{i\alpha}^{\dagger}\hat{f}_i\hat{f}_j^{\dagger}\hat{b}_{j\alpha}+\text{H.c.})$$
$$+J\sum_{\langle ij\rangle}\frac{1}{2}\left[\sum_{\alpha,\beta}\hat{b}_{i\alpha}^{\dagger}\hat{b}_{i\beta}\hat{b}_{j\beta}^{\dagger}\hat{b}_{j\alpha}-\hat{n}_i\hat{n}_j\right]$$
$$+\mu\sum_i\hat{f}_i^{\dagger}\hat{f}_i-\mu N, \tag{112}$$

with $\hat{n}_i$ given by Eq. (108), and where we have added the μ chemical potential term, N being the number of lattice sites.

B. The slave-boson representation

The explicit quantization in the slave-boson representation of the t-J model proceeds exactly in the same way, and we shall only give the following results expressed by, at site i,

$$\hat{b}_i\hat{b}_i^{\dagger}=1\ ,\quad \hat{f}_{i\sigma}\hat{f}_{i\tau}^{\dagger}=\delta_{\sigma\tau}\ ,\quad \hat{b}_i\hat{f}_{i\sigma}^{\dagger}=0\ ,\quad \hat{f}_{i\sigma}\hat{b}_i^{\dagger}=0 \tag{113}$$

or in other terms:

$$[\hat{b}_i,\hat{b}_j^{\dagger}]_-=(1-\hat{b}_i^{\dagger}\hat{b}_i)\delta_{ij}\ ,$$
$$[\hat{f}_{i\sigma},\hat{f}_{j\tau}^{\dagger}]_+=(\delta_{\sigma\tau}+\hat{f}_{i\tau}^{\dagger}\hat{f}_{i\sigma})\delta_{ij}\ , \tag{114}$$

$$[\hat{b}_i,\hat{f}_{j\sigma}^{\dagger}]_-=-\hat{f}_{i\sigma}^{\dagger}\hat{b}_i\delta_{ij}\ ,$$
$$[\hat{f}_{i\sigma},\hat{b}_j^{\dagger}]_-=-\hat{b}_i^{\dagger}\hat{f}_{i\sigma}\delta_{ij}\ . \tag{115}$$

We therefore directly recover in this case the expressions

$$\hat{\Theta}_i=\hat{b}_i^{\dagger}\hat{b}_i\ ,\quad \langle\hat{\Theta}_i\rangle=\langle\hat{\Xi}_0\rangle=\delta\ , \tag{116}$$

which through the analysis of Sec. III leads to the conclusion that the expected sum rules for the electron spectral function and for the electron number are indeed found also in the slave-boson approach of the t-J model within the decoupling approximation.

With the expression of the electron operator

$$\hat{c}_{i\sigma}^{\dagger}=\hat{f}_{i\sigma}^{\dagger}\hat{b}_i \tag{117}$$

and using our results for the quantization, let us also found the exact electron number operator:

$$\hat{n}_i=\sum_\sigma\hat{c}_{i\sigma}^{\dagger}\hat{c}_{i\sigma}=\sum_\sigma\hat{f}_{i\sigma}^{\dagger}\hat{b}_i\hat{b}_i^{\dagger}\hat{f}_{i\sigma}=\sum_\sigma\hat{f}_{i\sigma}^{\dagger}\hat{f}_{i\sigma}=1-\hat{b}_i^{\dagger}\hat{b}_i \tag{118}$$

to be compared with the ambiguous result using the (incorrect) naive quantization: since boson and fermion operators would commute in this naive quantization, $\sum_\sigma\hat{c}_{i\sigma}^{\dagger}\hat{c}_{i\sigma}$ could either be written $\sum_\sigma\hat{f}_{i\sigma}^{\dagger}\hat{f}_{i\sigma}\hat{b}_i\hat{b}_i^{\dagger}$, which using the slave-particle constraint and the naive commutator of the boson would give $1-(\hat{b}_i^{\dagger}\hat{b}_i)^2$, or be written $\sum_\sigma\hat{b}_i\hat{f}_{i\sigma}^{\dagger}\hat{f}_{i\sigma}\hat{b}_i^{\dagger}$, which in the same way would give $-(\hat{b}_i^{\dagger}\hat{b}_i)[1+\hat{b}_i^{\dagger}\hat{b}_i]$.

Finally, since using our result $\hat{b}_i\hat{b}_i^{\dagger}=1$ one has

$$\hat{c}_{i\sigma}^{\dagger}\hat{c}_{i\tau}=\hat{f}_{i\sigma}^{\dagger}\hat{b}_i\hat{b}_i^{\dagger}\hat{f}_{i\tau}=\hat{f}_{i\sigma}^{\dagger}\hat{f}_{i\tau}\ , \tag{119}$$

we obtain directly the t-J model Hamiltonian of Eq. (2) in the slave-boson representation:

$$\hat{H}=t\sum_{\langle ij\rangle,\sigma}(\hat{f}_{i\sigma}^{\dagger}\hat{b}_i\hat{b}_j^{\dagger}\hat{f}_{j\sigma}+\text{H.c.})$$
$$+J\sum_{\langle ij\rangle}\frac{1}{2}\left[\sum_{\sigma,\tau}\hat{f}_{i\sigma}^{\dagger}\hat{f}_{i\tau}\hat{f}_{j\tau}^{\dagger}\hat{f}_{j\sigma}-\hat{n}_i\hat{n}_j\right]+\mu\sum_i\hat{b}_i^{\dagger}\hat{b}_i-\mu N\ , \tag{120}$$

with $\hat{n}_i$ given by Eq. (118), and where we have added the μ chemical potential term, N being the number of lattice sites.

V. GENERALIZATION TO A SLAVE-BOSON APPROACH OF THE HUBBARD MODEL

A slave-boson approach was introduced in Ref. 6 for the Hubbard model, i.e., without neglecting the doubly occupied sites, and it was also claimed in Ref. 5 that the sum rule of the electron number was still violated there. Let us show explicitly that it is not the case if one uses our present quantization.

In Ref. 6 the electron operator was written in the finite-U Hubbard model as

$$\hat{c}_{i\sigma}^{\dagger}=\hat{f}_{i\sigma}^{\dagger}\hat{b}_{1i}+\sigma\hat{b}_{2i}^{\dagger}\hat{f}_{i(-\sigma)}\quad(\sigma=\pm1) \tag{121}$$

the bosons $\hat{b}_1^{\dagger}$ and $\hat{b}_2^{\dagger}$ describing, respectively, the empty and doubly occupied states. The slave-particle constraint here is

$$\hat{b}_{1i}^{\dagger}\hat{b}_{1i}+\hat{b}_{2i}^{\dagger}\hat{b}_{2i}+\sum_\sigma\hat{f}_{i\sigma}^{\dagger}\hat{f}_{i\sigma}=1\ . \tag{122}$$

The explicit quantization of this slave-boson representation of the Hubbard model proceeds in the same way as shown above for the slave-particle representation of the t-J model, and we obtain the results at site i:

$$\hat{b}_{\alpha i}\hat{b}_{\beta i}^{\dagger}=\delta_{\alpha\beta}\ ,\quad \hat{f}_{i\sigma}\hat{f}_{i\tau}^{\dagger}=\delta_{\sigma\tau}\ ,\quad \hat{b}_{\alpha i}\hat{f}_{i\sigma}^{\dagger}=0\ ,\quad \hat{f}_{i\sigma}\hat{b}_{\alpha i}^{\dagger}=0 \tag{123}$$

or in other terms:

$$[\hat{b}_{\alpha i},\hat{b}_{\beta j}^{\dagger}]_-=(\delta_{\alpha\beta}-\hat{b}_{\beta i}^{\dagger}\hat{b}_{\alpha i})\delta_{ij}\ ,$$
$$[\hat{f}_{i\sigma},\hat{f}_{j\tau}^{\dagger}]_+=(\delta_{\sigma\tau}+\hat{f}_{i\tau}^{\dagger}\hat{f}_{i\sigma})\delta_{ij}\ , \tag{124}$$

$$[\hat{b}_{\alpha i},\hat{f}^{\dagger}_{j\sigma}]_{-}=-\hat{f}^{\dagger}_{i\sigma}\hat{b}_{\alpha i}\delta_{ij}\ ,\qquad [\hat{f}_{i\sigma},\hat{b}^{\dagger}_{\alpha j}]_{-}=-\hat{b}^{\dagger}_{\alpha i}\hat{f}_{i\sigma}\delta_{ij}\ , \tag{125}$$

where α or $\beta=1,2$ and σ or $\tau=\pm1$.

The electron field being gauge invariant, one verifies as above that at the classical level one can use the initial brackets of the b's and f's to compute the Dirac brackets of the electron field:

$$i\{c^{\dagger}_{i\sigma},c_{i\tau}\}=i\{c^{\dagger}_{i\sigma},c_{i\tau}\}_{*}=\delta_{\sigma\tau}\ , \tag{126}$$

while at the quantum level our results give

$$\hat{c}^{\dagger}_{i\sigma}\hat{c}_{i\tau}=\hat{f}^{\dagger}_{i\sigma}\hat{f}_{i\tau}+\hat{b}^{\dagger}_{2i}\hat{b}_{2i}\delta_{\sigma\tau}\ , \tag{127}$$

$$\hat{c}_{i\tau}\hat{c}^{\dagger}_{i\sigma}=\sigma\tau\hat{f}^{\dagger}_{i(-\tau)}\hat{f}_{i(-\sigma)}+\hat{b}^{\dagger}_{1i}\hat{b}_{1i}\delta_{\sigma\tau}\ , \tag{128}$$

which effectively leads to

$$[\hat{c}_{i\tau},\hat{c}^{\dagger}_{i\sigma}]_{+}=\delta_{\sigma\tau}\ . \tag{129}$$

One has from Eq. (127) the expression of the exact electron number operator:

$$\sum_{\sigma}\hat{c}^{\dagger}_{i\sigma}\hat{c}_{i\sigma}=1-(\hat{b}^{\dagger}_{1i}\hat{b}_{1i}-\hat{b}^{\dagger}_{2i}\hat{b}_{2i})\ . \tag{130}$$

Let us define Δ as the average number of empty sites, d the average number of doubly occupied sites, and δ the hole doping concentration:

$$\Delta=\langle\hat{b}^{\dagger}_{1i}\hat{b}_{1i}\rangle\ ,\quad d=\langle\hat{b}^{\dagger}_{2i}\hat{b}_{2i}\rangle\ ,\quad \delta=\Delta-d\ . \tag{131}$$

The total electron number per site is then correctly obtained from Eq. (130) as

$$n_e=1-\delta\ . \tag{132}$$

Let us again emphasize that the expression of the Hamiltonian in the slave-particle approach is obtained, using our results, through a direct algebraic procedure. Equations (127) and (123) lead to

$$\hat{n}_{i+}\hat{n}_{i-}=(\hat{f}^{\dagger}_{i+}\hat{f}_{i+}+\hat{b}^{\dagger}_{2i}\hat{b}_{2i})(\hat{f}^{\dagger}_{i-}\hat{f}_{i-}+\hat{b}^{\dagger}_{2i}\hat{b}_{2i})=\hat{b}^{\dagger}_{2i}\hat{b}_{2i} \tag{133}$$

and we obtain directly the Hubbard Hamiltonian of Eq. (1) with nearest-neighbor hopping in the slave-boson representation of Eq. (121):

$$\begin{aligned}\hat{H}=t&\sum_{\langle ij\rangle,\sigma}\hat{f}^{\dagger}_{i\sigma}(\hat{b}_{1i}\hat{b}^{\dagger}_{1j}-\hat{b}_{2i}\hat{b}^{\dagger}_{2j})\hat{f}_{j\sigma}\\&+t\sum_{\langle ij\rangle}[\hat{f}^{\dagger}_{i+}\hat{f}^{\dagger}_{j-}(\hat{b}_{1i}\hat{b}_{2j}+\hat{b}_{1j}\hat{b}_{2i})+\text{H.c.}]\\&+U\sum_{i}\hat{b}^{\dagger}_{2i}\hat{b}_{2i}+\mu\sum_{i}(\hat{b}^{\dagger}_{1i}\hat{b}_{1i}-\hat{b}^{\dagger}_{2i}\hat{b}_{2i})-\mu N\ ,\end{aligned} \tag{134}$$

where we have added the μ chemical potential term, N being the number of lattice sites.

The decoupling approximation for the Matsubara electron Green's function in imaginary time is here expressed by

$$\begin{aligned}E_{\sigma}(r,\tau)=&-F_{\sigma}(r,\tau)B_1(-r,-\tau)\\&+F_{-\sigma}(-r,-\tau)B_2(r,\tau)\ .\end{aligned} \tag{135}$$

On one hand, one obtains the sum rule for the electron spectral function within the decoupling approximation:

$$\begin{aligned}\sum_{\sigma}\int_{-\infty}^{+\infty}\frac{d\omega}{2\pi}A_{e\sigma}(k,\omega)=&\frac{1}{N}\sum_{q}n_{b_1}(q)\left\langle\sum_{\sigma}[\hat{f}_{(q+k)\sigma},\hat{f}^{\dagger}_{(q+k)\sigma}]_{+}\right\rangle+\frac{1}{N}\sum_{q,\sigma}n_{f\sigma}(q+k)\langle[\hat{b}_{1q},\hat{b}^{\dagger}_{1q}]_{-}\rangle\\&+\frac{1}{N}\sum_{q,\sigma}n_{f\sigma}(q)\langle[\hat{b}_{2(q+k)},\hat{b}^{\dagger}_{2(q+k)}]_{-}\rangle+\frac{1}{N}\sum_{q}n_{b_2}(q+k)\left\langle\sum_{\sigma}[\hat{f}_{q\sigma},\hat{f}^{\dagger}_{q\sigma}]_{+}\right\rangle\ ,\end{aligned} \tag{136}$$

which using our quantization gives

$$\sum_{\sigma}\int_{-\infty}^{+\infty}\frac{d\omega}{2\pi}A_{e\sigma}(k,\omega)=\Delta(3-\Delta-d)+(1-\Delta-d)(1-d)+(1-\Delta-d)(1-\Delta)+d(3-\Delta-d)=2 \tag{137}$$

as it should be from Eq. (129).

On the other hand, Eq. (135) gives the expression for the number of electrons in state k within the decoupling approximation:

$$\begin{aligned}n_e(k)=&\frac{1}{N}\sum_{q,\sigma}[\langle\hat{f}^{\dagger}_{(q+k)\sigma}\hat{f}_{(q+k)\sigma}\rangle\langle\hat{b}_{1q}\hat{b}^{\dagger}_{1q}\rangle+\langle\hat{b}^{\dagger}_{2(q+k)}\hat{b}_{2(q+k)}\rangle\langle\hat{f}_{q\sigma}\hat{f}^{\dagger}_{q\sigma}\rangle]\\=&\frac{1}{N}\sum_{q,\sigma}n_{f\sigma}(q+k)\langle[\hat{b}_{1q},\hat{b}^{\dagger}_{1q}]_{-}\rangle+\frac{1}{N}\sum_{q,\sigma}n_{f\sigma}(q+k)n_{b_1}(q)\\&+\frac{1}{N}\sum_{q,\sigma}n_{b_2}(q+k)\langle[\hat{f}_{q\sigma},\hat{f}^{\dagger}_{q\sigma}]_{+}\rangle-\frac{1}{N}\sum_{q,\sigma}n_{b_2}(q+k)n_{f\sigma}(q)\ .\end{aligned} \tag{138}$$

Using our results for the quantization of the b_1 boson and the f fermion, one obtains

$$\begin{aligned}n_e(k)=&(1-\Delta)^2-d^2+2d+\frac{1}{N}\sum_{q,\sigma}n_{f\sigma}(q+k)n_{b_1}(q)\\&-\frac{1}{N}\sum_{q,\sigma}n_{b_2}(q+k)n_{f\sigma}(q)\end{aligned} \tag{139}$$

from which we indeed get the expected sum rule of the electron number:

$$\frac{1}{N}\sum_{k}n_e(k)=(1-\Delta+d)=1-\delta\ . \tag{140}$$

This last result also directly follows from Eq. (135):

$$n_e=\frac{1}{N}\sum_i\langle\hat{b}_{1i}\hat{b}_{1i}^\dagger\rangle\left[\sum_\sigma\langle\hat{f}_{i\sigma}^\dagger\hat{f}_{i\sigma}\rangle\right]+\frac{1}{N}\sum_i\left[\sum_\sigma\langle\hat{f}_{i\sigma}\hat{f}_{i\sigma}^\dagger\rangle\right]\langle\hat{b}_{2i}^\dagger\hat{b}_{2i}\rangle$$
$$=1(1-\Delta-d)+2d=1-\delta \quad (141)$$

using the results of our quantization, instead of

$$(1+\Delta)(1-\Delta-d)+[2-(1-\Delta-d)]d=(1-\Delta^2+d^2)$$

using the (incorrect) naive quantization of Refs. 5 and 6.

VI. CONCLUSIONS

In this paper, we have first presented at the classical level the consistent Hamiltonian formulation of models having a slave-particle constraint for their fields. Due to this constraint, the naive canonical relations are replaced by modified canonical relations which are compatible with the constraint. This is achieved through the use of Dirac brackets, after fixing the gauge generated by the slave-particle constraint.

We have then shown at the quantum level, for the slave-fermion and the slave-boson representations of the *t-J* model and for a slave-boson representation of the Hubbard model, that a consistent quantization of these modified canonical relations changes the naive sum rules for the slave particles. These naive sum rules used in Ref. 5 were there shown to lead to difficulties in these slave-particle approaches for the *t-J* and Hubbard models, coming from the fact that the sum rule of the electron number was violated within a decoupling approximation. On the contrary, we find that, using our quantization and modified sum rules for the slave particles, the sum rule of the electron number is in fact well obeyed. On the other hand, we obtain a systematic direct algebraic procedure to find the exact expression in a slave-particle approach of any operator, e.g., the Hamiltonian, initially written in terms of the electron operators.

We thus show that one has to be careful about the canonical relations when using a direct quantum operator approach for slave-particle theories. Of course we are not concerned for these theories neither with the functional integral approach nor with the Abrikosov[14] method used in some slave-particle theories.[15]

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APPENDIX

In this appendix, we show how the brackets of Eq. (7) and the first class constraints of Eq. (8) are obtained and we present the expression of the corresponding Hamiltonian.

Let us consider the classical Lagrangian (written in real time) for n bosons b_α and m fermions f_σ (Grassmann variables) on a lattice:

$$L=i\tfrac{1}{2}(\lambda+1)\sum_{i,\alpha}b_{i\alpha}^\dagger\partial_t b_{i\alpha}+i\tfrac{1}{2}(\lambda-1)\sum_{i,\alpha}\partial_t b_{i\alpha}^\dagger b_{i\alpha}+i\tfrac{1}{2}(\lambda'+1)\sum_{i,\sigma}f_{i\sigma}^\dagger\partial_t f_{i\sigma}+i\tfrac{1}{2}(\lambda'-1)\sum_{i,\sigma}\partial_t f_{i\sigma}^\dagger f_{i\sigma}+X(b^\dagger,b,f^\dagger,f)+\sum_i\Lambda_i\left[\sum_\alpha b_{i\alpha}^\dagger b_{i\alpha}+\sum_\sigma f_{i\sigma}^\dagger f_{i\sigma}-1\right], \quad (A1)$$

the Lagrangians of Eq. (5) being obtained for $\lambda=\lambda'=1$. The canonical momenta are then

$$\bar{b}_{i\alpha}=i\tfrac{1}{2}(\lambda+1)b_{i\alpha}^\dagger,\quad \bar{b}_{i\alpha}^\dagger=i\tfrac{1}{2}(\lambda-1)b_{i\alpha},$$
$$\bar{f}_{i\sigma}=-i\tfrac{1}{2}(\lambda'+1)f_{i\sigma}^\dagger,\quad \bar{f}_{i\sigma}^\dagger=i\tfrac{1}{2}(\lambda'-1)f_{i\sigma},\quad \Pi_i=0. \quad (A2)$$

The canonical graded [see Eq. (9)] Poisson brackets, the nonzero ones being

$$\{b_{i\alpha},\bar{b}_{j\beta}\}=\{b_{i\alpha}^\dagger,\bar{b}_{j\beta}^\dagger\}=\delta_{ij}\delta_{\alpha\beta},$$
$$\{f_{i\sigma},\bar{f}_{j\tau}\}=\{f_{i\sigma}^\dagger,\bar{f}_{j\tau}^\dagger\}=-\delta_{ij}\delta_{\sigma\tau},\quad\{\Lambda_i,\Pi_j\}=\delta_{ij}, \quad (A3)$$

are, however, not compatible with the expressions of the canonical momenta. Following the general procedure of Dirac,[7] one has to consider here as Hamiltonian

$$H_1=-X(b^\dagger,b,f^\dagger,f)-\sum_i\Lambda_i\left[\sum_\alpha b_{i\alpha}^\dagger b_{i\alpha}+\sum_\sigma f_{i\sigma}^\dagger f_{i\sigma}-1\right]+\sum_{i,\alpha}(B_{i\alpha}u_{i\alpha}+u_{i\alpha}^\dagger B_{i\alpha}^\dagger)+\sum_{i,\sigma}(F_{i\sigma}v_{i\sigma}+v_{i\sigma}^\dagger F_{i\sigma}^\dagger)+\sum_i\Pi_i w_i, \quad (A4)$$

where $u,u^\dagger,v,v^\dagger,w$ are unknown (at this stage independent of the fields and of the canonical momenta) coefficients and where Π_i and

$$B_{i\alpha}\equiv\bar{b}_{i\alpha}-i\tfrac{1}{2}(\lambda+1)b_{i\alpha}^\dagger,\quad B_{i\alpha}^\dagger\equiv\bar{b}_{i\alpha}^\dagger-i\tfrac{1}{2}(\lambda-1)b_{i\alpha},$$
$$F_{i\sigma}\equiv\bar{f}_{i\sigma}+i\tfrac{1}{2}(\lambda'+1)f_{i\sigma}^\dagger,\quad F_{i\sigma}^\dagger\equiv\bar{f}_{i\sigma}^\dagger-i\tfrac{1}{2}(\lambda'-1)f_{i\sigma} \quad (A5)$$

are *primary constraints* which are weakly zero, meaning that one has to set the constraints only after computing all the brackets. In order to have a consistent system, we require the time derivatives of these primary constraints to be weakly zero (≈ 0), which gives

$$\{B_{i\alpha},H_1\}=\frac{\partial X}{\partial b_{i\alpha}}+\Lambda_i b_{i\alpha}^\dagger-iu_{i\alpha}^\dagger\approx 0,$$
$$\{B_{i\alpha}^\dagger,H_1\}=\frac{\partial X}{\partial b_{i\alpha}^\dagger}+\Lambda_i b_{i\alpha}+iu_{i\alpha}\approx 0, \quad (A6)$$

$$\{F_{i\sigma},H_1\}=\frac{\partial X}{\partial f_{i\sigma}}-\Lambda_i f_{i\sigma}^\dagger+iv_{i\sigma}^\dagger\approx 0,$$
$$\{F_{i\sigma}^\dagger,H_1\}=\frac{\partial X}{\partial f_{i\sigma}^\dagger}+\Lambda_i f_{i\sigma}-iv_{i\sigma}\approx 0, \quad (A7)$$

$$\{\Pi_i,H_1\}=\sum_\alpha b_{i\alpha}^\dagger b_{i\alpha}+\sum_\sigma f_{i\sigma}^\dagger f_{i\sigma}-1\equiv\Phi_i\approx 0. \quad (A8)$$

Equations (A6) and (A7) determine the $u,u^\dagger,v,v^\dagger$ and Eq. (A8) gives the slave-particle constraint which appears as a *secondary constraint.* Repeating the process, we require the time derivative of this secondary constraint to be weakly zero, which gives

$$\{\Phi_i,H_1\}=\sum_\alpha(b^\dagger_{i\alpha}u_{i\alpha}+u^\dagger_{i\alpha}b_{i\alpha})+\sum_\sigma(-f^\dagger_{i\sigma}v_{i\sigma}+v^\dagger_{i\sigma}f_{i\sigma})\approx 0\ . \tag{A9}$$

Using the expressions of the $u,u^\dagger,v,v^\dagger$ given by Eqs. (A6) and (A7), Eq. (A9) reads

$$\sum_\alpha\left[b^\dagger_{i\alpha}\frac{\partial X}{\partial b^\dagger_{i\alpha}}-\frac{\partial X}{\partial b_{i\alpha}}b_{i\alpha}\right]+\sum_\sigma\left[f^\dagger_{i\sigma}\frac{\partial X}{\partial f^\dagger_{i\sigma}}+\frac{\partial X}{\partial f_{i\sigma}}f_{i\sigma}\right]=0\ , \tag{A10}$$

which is identically satisfied if X is a function of $b^\dagger b$ and $f^\dagger f$.

One recalls that a quantity is called *first class* if its bracket with each of the (primary and secondary) constraints is weakly zero, and *second class* if at least one of these brackets is not weakly zero. Π_i is then first class, while our other constraints are *a priori* second class since

$$\{B_{i\alpha},B^\dagger_{j\beta}\}=-i\delta_{ij}\delta_{\alpha\beta}\ ,\quad \{F_{i\sigma},F^\dagger_{j\tau}\}=-i\delta_{ij}\delta_{\sigma\tau}\ , \tag{A11}$$

$$\{B_{i\alpha},\Phi_j\}=-b^\dagger_{i\alpha}\delta_{ij}\ ,\quad \{B^\dagger_{i\alpha},\Phi_j\}=-b_{i\alpha}\delta_{ij}\ , \tag{A12}$$

$$\{F_{i\sigma},\Phi_j\}=+f^\dagger_{i\sigma}\delta_{ij}\ ,\quad \{F^\dagger_{i\sigma},\Phi_j\}=-f_{i\sigma}\delta_{ij}\ . \tag{A13}$$

However, the following linear combination of these second class constraints

$$\Psi_i\equiv\Phi_i+i\sum_\alpha(-B_{i\alpha}b_{i\alpha}+b^\dagger_{i\alpha}B^\dagger_{i\alpha})+i\sum_\sigma(F_{i\sigma}f_{i\sigma}+f^\dagger_{i\sigma}F^\dagger_{i\sigma}) \tag{A14}$$

$$=i\sum_\alpha(-\bar b_{i\alpha}b_{i\alpha}+\bar b^\dagger_{i\alpha}b^\dagger_{i\alpha})+i\sum_\sigma(\bar f_{i\sigma}f_{i\sigma}-\bar f^\dagger_{i\sigma}f^\dagger_{i\sigma})-1 \tag{A15}$$

(which is weakly equal to Φ_i) can be verified to be first class.

Thus, Π_i and Ψ_i are first class, and $B_{i\alpha}$, $B^\dagger_{i\alpha}$, $F_{i\sigma}$, and $F^\dagger_{i\sigma}$ (which are such that no linear combination of them is first class) are second class. Defining $\varphi_1\equiv B_{i\alpha}$, $\varphi_2\equiv B^\dagger_{i\alpha}$, $\varphi_3\equiv F_{i\sigma}$, $\varphi_4\equiv F^\dagger_{i\sigma}$, the matrix $C_{ab}\equiv\{\varphi_a,\varphi_b\}$ is nonsingular. Systematic use of the standard Dirac bracket[7] of two quantities A and B,

$$\{A,B\}_D\equiv\{A,B\}-\sum_{a,b}\{A,\varphi_a\}(C^{-1})_{ab}\{\varphi_b,B\}\ , \tag{A16}$$

then allows to set all these second class constraints strongly to zero because the Dirac bracket of anything with a second class constraint vanishes.

One then finds the following results:

$$\{\bar b^\dagger_{i\alpha},b^\dagger_{j\beta}\}_D=\tfrac{1}{2}(\lambda-1)\delta_{ij}\delta_{\alpha\beta}\ ,\quad \{\bar b_{i\alpha},b_{j\beta}\}_D=-\tfrac{1}{2}(\lambda+1)\delta_{ij}\delta_{\alpha\beta}\ , \tag{A17}$$

$$\{\bar b^\dagger_{i\alpha},\bar b_{j\beta}\}_D=i\tfrac{1}{4}(\lambda^2-1)\delta_{ij}\delta_{\alpha\beta}\ ,\quad \{b_{i\alpha},b^\dagger_{j\beta}\}_D=-i\delta_{ij}\delta_{\alpha\beta}\ , \tag{A18}$$

$$\{\bar f^\dagger_{i\sigma},f^\dagger_{j\tau}\}_D=\tfrac{1}{2}(\lambda'-1)\delta_{ij}\delta_{\sigma\tau}\ ,\quad \{\bar f_{i\sigma},f_{j\tau}\}_D=-\tfrac{1}{2}(\lambda'+1)\delta_{ij}\delta_{\sigma\tau}\ , \tag{A19}$$

$$\{\bar f^\dagger_{i\sigma},\bar f_{j\tau}\}_D=-i\tfrac{1}{4}(\lambda'^2-1)\delta_{ij}\delta_{\sigma\tau}\ ,\quad \{f_{i\sigma},f^\dagger_{j\tau}\}_D=-i\delta_{ij}\delta_{\sigma\tau}\ , \tag{A20}$$

$$\{\Lambda_i,\Pi_j\}_D=\delta_{ij}\ . \tag{A21}$$

Since we can now set the second class constraints strongly to zero, i.e., use the expressions of $\bar b_{i\alpha},\bar b^\dagger_{i\alpha},\bar f_{i\sigma},\bar f^\dagger_{i\sigma}$ given by Eqs. (A2), a choice of independent nonzero canonical relations is

$$\{b_{i\alpha},b^\dagger_{j\beta}\}_D=-i\delta_{ij}\delta_{\alpha\beta}\ ,\quad \{f_{i\sigma},f^\dagger_{j\tau}\}_D=-i\delta_{ij}\delta_{\sigma\tau}\ ,\quad \{\Lambda_i,\Pi_j\}_D=\delta_{ij}\ , \tag{A22}$$

which are those of Eq. (7) of Sec. II where we have dropped for convenience the subscript D. Let us note that the parameters λ and λ' of the Lagrangian (A1) no longer appear, and can thus, as usual, be taken as unity in the Lagrangian.

At this stage, the Hamiltonian is

$$H_2=-X(b^\dagger,b,f^\dagger,f)-\sum_i\Lambda_i\left[\sum_\alpha b^\dagger_{i\alpha}b_{i\alpha}+\sum_\sigma f^\dagger_{i\sigma}f_{i\sigma}-1\right]+\sum_i\Pi_i w_i \tag{A23}$$

and one now has to deal, as shown in Sec. II, with the first class constraints:

$$\Phi_i\equiv\sum_\alpha b^\dagger_{i\alpha}b_{i\alpha}+\sum_\sigma f^\dagger_{i\sigma}f_{i\sigma}-1\approx 0\ ,\quad \Pi_i\approx 0\ . \tag{A24}$$

Let us note that we have consistently, for example, $\{b_{i\alpha},H_1\}=\{b_{i\alpha},H_2\}_D$ since

$$\{b_{i\alpha},H_1\}=u_{i\alpha}\ , \tag{A25}$$

$$\{b_{i\alpha},H_2\}_D=i\frac{\partial X}{\partial b^\dagger_{i\alpha}}+i\Lambda_i b_{i\alpha}\ , \tag{A26}$$

which are indeed equal using Eq. (A6).

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[1]P. W. Anderson, Science **235**, 1196 (1987).

[2]F. C. Zhang and T. M. Rice, Phys. Rev. B **37**, 3759 (1988).

[3]See, e.g., the review, L. Yu, in *Recent Progress in Many-Body Theories,* edited by T. L. Ainsworth *et al.* (Plenum, New York, 1992), Vol. 3, p. 157.

[4]M. Ogata and H. Shiba, Phys. Rev. B **41**, 2326 (1990); H. Shiba and M. Ogata, Int. J. Mod. Phys. B **5**, 31 (1991).

[5]Shiping Feng, J. B. Wu, Z. B. Su, and L. Yu, Phys. Rev. B **47**, 15 192 (1993).

[6]Z. Zou and P. W. Anderson, Phys. Rev. B **37**, 627 (1988).

[7]P. A. M. Dirac, Can. J. Math. **2**, 129 (1950); *Lectures on Quantum Mechanics* (Belfer Graduate School of Science, Yeshiva

University, New York, 1964).
[8]A. Hanson, T. Regge, and C. Teitelboim, *Constrained Hamiltonian Systems* (Academia Nazionale dei Lincei, Rome, 1976).
[9]K. Sundermeyer, *Constrained Dynamics,* Springer Lectures Notes in Physics Vol. 169 (Springer-Verlag, New York, 1982).
[10]P. B. Wiegmann, Phys. Rev. Lett. **60**, 821 (1988); D. Förster, *ibid.* **63**, 2140 (1989); S. Sarkar, J. Phys. A **24**, 1137 (1991).
[11]J. Hubbard, Proc. R. Soc. London Ser. A **285**, 542 (1965).
[12]A. A. Abrikosov, L. P. Gorkov, and I. E. Dzyaloshinski, *Methods of Quantum Field Theory in Statistical Physics* (Dover, New York, 1975).
[13]G. D. Mahan, *Many-Particle Physics,* 2nd ed. (Plenum, New York, 1990).
[14]A. A. Abrikosov, Physics **2**, 5 (1965).
[15]S. E. Barnes, J. Phys. F **6**, 1375 (1976); **7**, 2637 (1977); P. Coleman, Phys. Rev. B **28**, 5255 (1983); **29**, 3035 (1984).