# Quantum Boltzmann equation of composite fermions interacting with a gauge field

Yong Baek Kim,* Patrick A. Lee, and Xiao-Gang Wen
*Department of Physics, Massachusetts Institute of Technology, Cambridge, Massachusetts 02139*
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We derive the quantum Boltzmann equation (QBE) of composite fermions at/near the $\nu=\frac{1}{2}$ state using the nonequilibrium Green's-function technique. The lowest-order perturbative correction to the self-energy due to the strong gauge-field fluctuations suggests that there is no well-defined Landau quasiparticle. Therefore, we cannot assume the existence of the Landau quasiparticles *a priori* in the derivation of the QBE. Using an alternative formulation, we derive the QBE for the generalized Fermi-surface displacement which corresponds to the local variation of the chemical potential in momentum space. From this QBE, one can understand in a unified fashion the Fermi-liquid behaviors of the density-density and the current-current correlation functions at $\nu=\frac{1}{2}$ (in the long-wavelength and the low-frequency limits) and the singular behavior of the energy gap obtained from the finite-temperature activation behavior of the compressibility near $\nu=\frac{1}{2}$. Implications of these results for recent experiments are also discussed.

## I. INTRODUCTION

Since the discovery of the integer quantum Hall (IQH) and fractional quantum Hall (FQH) effects, the two-dimensional electron system in strong magnetic fields has often surprised us. Among recent developments, much attention has been given to the appearance of the interesting metallic state at the filling fraction $\nu=\frac{1}{2}$,[1] and the associated Shubnikov–de Haas oscillations of the longitudinal resistance around $\nu=\frac{1}{2}$.[2,3] The similarity between these phenomena near $\nu=\frac{1}{2}$ and those of electrons in weak magnetic fields was successfully explained by the composite fermion approach.[4] Using the fermionic Chern-Simons gauge theory of composite fermions,[5,6] Halperin, Lee, and Read (HLR) developed a theory that describes the metallic state at $\nu=\frac{1}{2}$.[6]

A composite fermion is obtained by attaching an even number $2n$ of flux quanta to an electron, and the transformation can be realized by introducing an appropriate Chern-Simons gauge field.[4–6] At the mean-field level, one takes into account only the average of the statistical magnetic field due to the attached magnetic flux. If the interaction between fermions is ignored, the system can be described as the free fermions in an effective magnetic field $\Delta B=B-B_{1/2n}$, where $B_{1/2n}=2nn_e hc/e$ is the averaged statistical magnetic field, and $n_e$ is the density of electrons. Therefore, in mean-field theory, the FQH states with $\nu=p/(2np+1)$ can be described as the IQH states of the composite fermions with $p$-filled Landau levels occupied in an effective magnetic field $\Delta B$.[4–6] In particular, $\Delta B=0$ at the filling fractions $\nu=1/2n$, so that the ground state of the system is the filled Fermi sea with a well-defined Fermi wave vector $k_F$.[6,7] As a result, the Shubnikov–de Haas oscillations near $\nu=\frac{1}{2}$ can be explained by the presence of a well-defined Fermi wave vector at $\nu=\frac{1}{2}$.[6] The mean-field energy gap of the system with $\nu=p/(2p+1)$ in the $p\to\infty$ limit is given by $E_g=e\Delta B/mc$, where $m$ is the mass of the composite fermions. Note that, in the large-$\omega_c$ limit, the finite $m$ is caused by the Coulomb interaction between the fermions. The effective mass $m$ should be chosen such that the Fermi energy $E_F$ is given by the Coulomb energy scale.

There are a number of experiments which show that there is a well-defined Fermi wave vector at $\nu=\frac{1}{2}$.[8–10] They observed the geometrical resonances between the semiclassical orbit of the composite fermions and another length scale artificially introduced to the system near $\nu=\frac{1}{2}$.

However, it is possible that fluctuations and two-particle interactions, which are ignored in the mean-field theory, are very important. Note that the density fluctuations correspond to the fluctuations of the statistical magnetic field. Therefore, the density fluctuations above the mean-field state induce the gauge-field fluctuations.[5,6] If the fermions are interacting via a two-particle interaction $v(\mathbf{q})=V_0/q^{2-\eta}(1\le\eta\le2)$, the effects of the gauge-field fluctuations can be modified. In fact, the gauge-field fluctuations become more singular as the interaction range becomes shorter (larger $\eta$). The reason is that the longer-range interaction (smaller $\eta$) suppresses more effectively the density fluctuations, and thus induces the less singular gauge-field fluctuations. Therefore, it is important to examine whether the mean-field Fermi-liquid state is stable against the gauge-field fluctuations, which also includes the effects of the two-particle interaction.

One way to study the stability of the mean-field Fermi-liquid state is to examine the low-energy behavior of the self-energy correction induced by the gauge-field fluctuations. It is found that the most singular contribution to the self-energy $\Sigma(\mathbf{k},\omega)$ comes from the transverse part of the gauge-field fluctuations.[6,11] The lowest-order perturbative correction to the self-energy (due to the transverse gauge field) has been calculated by several authors.[6,11] It turns out that $\mathrm{Re}\Sigma\sim\mathrm{Im}\Sigma\sim\omega^{2/(1+\eta)}$ for $1<\eta\le2$, and $\mathrm{Re}\Sigma\sim\omega\ln\omega$, $\mathrm{Im}\Sigma\sim\omega$ for $\eta=1$ (Coulomb

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interaction). Thus the Landau criterion for the quasiparticle is violated in the case of $1<\eta\leq 2$, and the case of $\eta=1$ shows marginal Fermi-liquid behavior. In both cases, the effective mass of the fermions diverges, as $m^*/m \propto |\xi_k|^{-(\eta-1)/(\eta+1)}$ for $1<\eta\leq 2$, and as $m^*/m \propto |\ln \xi_k|$ for $\eta=1$, where $\xi_k=(k^2/2m)-\mu$, and $\mu$ is the chemical potential.[6]

In a self-consistent treatment of the self-energy,[6] the energy gap of the system in the presence of a small effective magnetic field $\Delta B$ can be determined as $E_g \propto |\Delta B|^{(1+\eta)/2}$ for $1<\eta\leq 2$ and $E_g \propto |\Delta B|/|\ln \Delta B|$ for $\eta=1$. Therefore, the energy gap of the system vanishes faster than the mean-field prediction, or, equivalently, the effective mass diverges in a singular way as $\nu=\frac{1}{2}$ is approached. These results suggest that the effective Fermi velocity of the fermion $v_F^*$ goes to zero at $\nu=\frac{1}{2}$ even though the Fermi wave vector $k_F$ is finite and the quasiparticles have a very short lifetime $\tau \approx (T/\varepsilon_F)^{-2/(1+\eta)}(1/\varepsilon_F)$, where $T$ is the temperature and $\varepsilon_F$ is the Fermi energy. However, a recent magnetic focusing experiment[10] suggests that the fermion has a long lifetime or a long mean free path which seems inconsistent with the above picture.

Since the one-particle Green's function is not gauge invariant, the singular self-energy could be an artifact of the gauge choice. To address this question, we recently examined the lowest-order perturbative corrections to the gauge-invariant density-density and current-current correlation functions.[12] It was found that there are important cancellations between the self-energy corrections and the vertex corrections due to the Ward identity.[12,13] As a result, the density-density and current-current correlation functions show a Fermi-liquid behavior for all ratios of $\omega$ and $v_F q$.[12] In particular, the edge of the particle-hole continuum $\omega=v_F q$ is essentially not changed, which may suggest a finite effective mass. From the current-current correlation function, the transport scattering rate (due to the transverse part of the gauge field) is given by $1/\tau_{\rm tr} \propto \omega^{4/(1+\eta)} \ll \omega$ after the cancellation (the scattering rate would be much larger, $1/\tau_{\rm tr} \propto \omega^{2/(1+\eta)} \gg \omega$, had we ignored the vertex correction).[12] Therefore, the fermions have a long transport lifetime, which explains a long free path in the magnetic focusing experiment. From these results, one may suspect whether the divergent mass obtained from the self-energy has any physical meaning.

However, due to the absence of the underlying quasiparticle picture, we cannot simply conclude that fermions have a finite effective mass associated with the long lifetime which was obtained from the small-$q$ and -$\omega$ behaviors of the density-density and current-current correlation functions. In fact, it is found that $2k_F$ response functions show singular behaviors compared to the usual Fermi-liquid theory.[13] We also like to mention that recent experiments on the Shubnikov–de Haas oscillations[3] have observed some features which were interpreted as a sign of the divergent effective mass of the fermions as $\nu=\frac{1}{2}$ is approached. The experimentally determined effective mass diverges in a more singular way than any theoretical prediction. However, their determination of the effective mass is based on a theory for the noninteracting fermions, and also the disorder effect is very important near $\nu=\frac{1}{2}$ because the static fluctuations of the density due to the impurities induce an additional static random magnetic field. Since there is no satisfactory theory in the presence of disorder, it is difficult to compare the present theory and the experiments.

In order to answer the question about the effective mass, it is important to examine other gauge-invariant quantities which may potentially show a divergent effective mass. In a recent paper,[14] we calculated the lowest-order perturbative correction to the compressibility with a fixed $\Delta B$, which shows a thermally activated behavior when the chemical potential lies exactly at the middle of the successive effective Landau levels. It turns out that the corrections to the activation energy gap and the corresponding effective mass are singular, and consistent with previous self-consistent treatment of the self-energy.[6] Thus it is necessary to understand the apparently different behaviors of the density-density correlation function at $\nu=\frac{1}{2}$ and the activation energy gap determined from the compressibility near $\nu=\frac{1}{2}$.

One resolution of the problem was suggested by Stern and Halperin[15] within the usual Landau-Fermi-liquid theory framework. The idea is that both the effective mass and Landau-interaction function are singular in such a way that they cancel each other in the density-density correlation function. Recently, Stern and Halperin[15] put forward this idea and constructed a Fermi-liquid theory of the fermion-gauge system in the case of Coulomb interaction. Even though the use of the Landau-Fermi-liquid theory or equivalently the existence of well-defined quasiparticles can be *marginally* justified in the case of the Coulomb interaction, we feel that it is necessary to construct a more general framework which applies to the arbitrary two-particle interaction ($1<\eta\leq 2$ as well as $\eta=1$), and allows us to check the validity of the Fermi-liquid theory and to judge when the divergent mass shows up. In particular, it is worthwhile to provide a unified picture for understanding previous theoretical studies.[16–24]

In the usual Fermi-liquid theory, the quantum Boltzmann equation (QBE) of the quasiparticles provides useful information about the low-lying excitations of the system. Our objective is to construct a similar QBE which describes all the low-energy physics of the composite fermion system. One important difficulty we are facing here is that we cannot assume the existence of the quasiparticles *a priori* in the derivation of the QBE, even though the conventional derivation of the QBE of the Fermi-liquid theory relies on the existence of these quasiparticles. Following closely the work of Prange and Kadanoff[25] about the electron-phonon system, where there is also no well-defined quasiparticle at temperatures high compared with the Debye temperature, we concentrate on a generalized Fermi-surface displacement which, in our case, corresponds to the local variation of the chemical potential in momentum space. Due to the nonexistence of a well-defined quasiparticle, the usual distribution function $n_k$ in the momentum space cannot be described by a closed equation of motion. However, we will see below that the generalized Fermi surface displacement does satisfy a closed equation of motion. This equation of motion will be also called QBE.

We use the nonequilibrium Green's-function technique[26–28] to derive the QBE and calculate the generalized Landau-interaction function which has frequency dependence as well as the usual angular dependence due to the retarded nature of the gauge interaction. The QBE at $\nu=\frac{1}{2}$ consists of three parts. One is the contribution from the self-energy correction which gives the singular mass correction, the other one comes from the generalized Landau-interaction function, and finally it contains the collision integral. These quantities are calculated to the lowest order in the coupling to the gauge field.

By studying the dynamic properties of the collective modes using the QBE, we find that the smooth fluctuations of the Fermi surface (or the small angular momentum modes) show the usual Fermi-liquid behavior, while the rough fluctuations (or the large angular momentum modes) show the singular behavior determined by the singular self-energy correction. Here the angular momentum is the conjugate variable of the angle measured from a given direction in momentum space. There is a forward scattering cancellation between the singular self-energy correction and the singular (generalized) Landau-interaction function and a similar cancellation exists in the collision integral as far as the small angular momentum modes $l<l_c$ ($l_c \propto \Omega^{-1/(1+\eta)}$, where $\Omega$ is the small external frequency) are concerned. However, in the case of the large angular momentum modes $l>l_c$, the contribution from the Landau-interaction function becomes very small so that the self-energy correction dominates, and the collision integral also cannot be ignored in general. In this case the behaviors of the low-lying modes are very different from those in the Fermi liquids.

If we ignore the collision integral, it can be shown that the system has numerous collective modes between $\Omega \propto q^{(1+\eta)/2}(1<\eta\leq 2)$, $\Omega \propto q/|\ln q|(\eta=1)$, and $\Omega=v_F q$, while there is the particle-hole continuum below $\Omega \propto q^{(1+\eta)/2}(1<\eta\leq 2)$ and $\Omega \propto q/|\ln q|(\eta=1)$. The distinction between these two types of low-lying excitations is obscured by the existence of the collision integral.

From the above results, we see that the density-density and current-current correlation functions, being dominated by the small angular momentum modes $l<l_c$, show the usual Fermi-liquid behavior. On the other hand, the energy gap away from $\nu=\frac{1}{2}$ is determined by the behaviors of the large angular momentum modes $l>l_c$, so that the singular mass correction shows up in the energy gap of the system.

The outline of the paper is as follows. In Sec. II, we introduce the model and explain the way we construct the QBE without assuming the existence of the quasiparticles. In Sec. III, the QBE for the generalized distribution function is derived for $\Delta B=0$. In Sec. IV, we construct the QBE for the generalized Fermi-surface displacement for $\Delta B=0$. We also determine the generalized Landau-interaction function and discuss its consequences. In Sec. V, the QBE in the presence of a small $\Delta B$ is constructed, and the energy gap of the system is determined. In Sec. VI, we discuss the collective excitations of the system for the cases of $\Delta B=0$ and $\Delta B\neq 0$. We conclude the paper and discuss the implications of our results to experiments in Sec. VII. We concentrate on the zero-temperature case in the main text and provide the derivation of the QBE at finite temperatures in the Appendix, which requires some special treatments compared to the zero-temperature counterpart.

## II. MODEL AND QUANTUM BOLTZMANN EQUATION IN THE ABSENCE OF QUASIPARTICLES

Two-dimensional electrons interacting via a two-particle interaction can be transformed to composite fermions interacting via the same two-particle interaction, and also coupled to an appropriate Chern-Simons gauge field which appears due to the statistical magnetic flux quanta attached to each electron.[5,6] The model can be constructed as follows ($\hbar=e=c=1$):

$$Z=\int D\psi\, D\psi^* Da_\mu e^{i\int dt\, d^2r \mathcal{L}} , \tag{1}$$

where the Lagrangian density $\mathcal{L}$ is

$$\begin{aligned}\mathcal{L}=&\psi^*(\partial_0+ia_0-\mu)\psi-\frac{1}{2m}\psi^*(\partial_i-ia_i+iA_i)^2\psi\\&-\frac{i}{2\pi\tilde{\phi}}a_0\varepsilon^{ij}\partial_i a_j\\&+\tfrac{1}{2}\int d^2r'\psi^*(\mathbf{r})\psi(\mathbf{r})v(\mathbf{r}-\mathbf{r}')\psi^*(\mathbf{r}')\psi(\mathbf{r}') ,\end{aligned} \tag{2}$$

where $\psi$ represents the fermion field and $\tilde{\phi}$ is an even number $2n$ which is the number of flux quanta attached to an electron, and $v(\mathbf{r})\propto V_0/r^\eta$ is the Fourier transform of $v(\mathbf{q})=V_0/q^{2-\eta}$ $(1\leq\eta\leq 2)$ which represents the interaction between the fermions. $\mathbf{A}$ is the external vector potential ($B=\nabla\times\mathbf{A}$), and we choose the Coulomb gauge $\nabla\cdot\mathbf{a}=0$ for the Chern-Simons gauge field. Note that the integration over $a_0$ enforces the following constraint:

$$\nabla\times\mathbf{a}=2\pi\tilde{\phi}\psi^*(\mathbf{r})\psi(\mathbf{r}) , \tag{3}$$

which represents the fact that $\tilde{\phi}$ number of flux quanta are attached to each electron.

The saddle point of the action is given by the following conditions:

$$\nabla\times\langle\mathbf{a}\rangle=2\pi\tilde{\phi}n_e=B_{1/2n} \quad\text{and}\quad \langle a_0\rangle=0 . \tag{4}$$

Therefore, at the mean-field level, the fermions see an effective magnetic field ($\Delta\mathbf{A}=\mathbf{A}-\langle\mathbf{a}\rangle$)

$$\Delta B=\nabla\times\Delta\mathbf{A}=B-B_{1/2n} , \tag{5}$$

which becomes zero at the Landau-level filling factor $\nu=1/2n$. The IQH effect of the fermions may appear when the effective Landau-level filling factor $p=2\pi n_e/\Delta B$ becomes an integer. This implies that the real external magnetic field is given by $B=B_{1/2n}+\Delta B=2\pi n_e[(2np+1)/p]$, which corresponds to a FQH state of electrons with the filling factor $\nu=p/(2np+1)$.

The fluctuations of the Chern-Simons gauge field, $\delta a_\mu=a_\mu-\langle a_\mu\rangle$, can be incorporated as follows:

$$Z=\int D\psi D\psi^* D\delta a_\mu e^{i\int dt\, d^2r \mathcal{L}} , \tag{6}$$

where

$$\mathcal{L}=\psi^*(\partial_0+i\delta a_0-\mu)\psi-\frac{1}{2m}\psi^*(\partial_i-i\delta a_i+i\Delta A_i)^2\psi$$
$$-\frac{i}{2\pi\tilde{\phi}}\delta a_0 \varepsilon^{ij}\partial_i\delta a_j$$
$$+\frac{1}{2(2\pi\tilde{\phi})^2}\int d^2r'[\nabla\times\delta\mathbf{a}(\mathbf{r})]$$
$$\times v(\mathbf{r}-\mathbf{r}')[\nabla\times\delta\mathbf{a}(\mathbf{r}')] . \tag{7}$$

After integrating out the fermions and including gauge-field fluctuations within the random-phase approximation (RPA),[6] the effective action of the gauge field can be obtained as

$$S_{\rm eff}=\frac{1}{2}\int\frac{d^2q}{(2\pi)^2}\frac{d\omega}{2\pi}\delta a_\mu^*(\mathbf{q},\omega)D_{\mu\nu}^{-1}(\mathbf{q},\omega,\Delta B)\delta a_\nu(\mathbf{q},\omega) , \tag{8}$$

where $D_{\mu\nu}^{-1}(\mathbf{q},\omega,\Delta B)$ was calculated by several authors.[6,29,30] For our purpose, the $2\times 2$ matrix form for $D_{\mu\nu}^{-1}$ is sufficient so that $\mu,\nu=0$, 1, and 1 represents the direction that is perpendicular to $\mathbf{q}$. In particular, when $\Delta B=0$, the gauge-field propagator has the following form:[6]

$$D_{\mu\nu}^{-1}(\mathbf{q},\omega)=\begin{pmatrix} \frac{m}{2\pi} & -i\frac{q}{2\pi\tilde{\phi}} \\ i\frac{q}{2\pi\tilde{\phi}} & -i\gamma\frac{\omega}{q}+\tilde{\chi}(q)q^2 \end{pmatrix} , \tag{9}$$

where $\gamma=2n_e/k_F$ and $\tilde{\chi}(q)=1/24\pi m+v(q)/(2\pi\tilde{\phi})^2$. Since the most singular contribution to the self-energy correction comes from the transverse part of the gauge field,[6,11] we concentrate on the effect of the transverse gauge-field fluctuations. In the infrared limit, the transverse gauge-field propagator can be taken as[12,14]

$$D_{11}(\mathbf{q},\omega)=\frac{1}{-i\gamma\frac{\omega}{q}+\chi q^\eta} , \tag{10}$$

where $\chi=1/24\pi m+V_0/(2\pi\tilde{\phi})^2$ for $\eta=2$ and $\chi=V_0/(2\pi\tilde{\phi})^2$ for $1\le\eta<2$.

Before explaining the way we construct the QBE for a fermion-gauge-field system in which there is no well-defined Landau quasiparticle in general, we review the usual derivation of the QBE for a Fermi liquid with well-defined quasiparticles.[19,21] The QBE is nothing but the equation of motion of the fermion distribution function. Therefore, it can be derived from the equation of motion of the nonequilibrium one-particle Green's function. Following Kadanoff and Baym,[26] let us consider the following one-particle Green's function:

$$G^<(x_1,x_2)=i\langle\psi^\dagger(x_2)\psi(x_1)\rangle , \tag{11}$$

where $x_1=(\mathbf{r}_1,t_1)$ and $x_2=(\mathbf{r}_2,t_2)$. At nonequilibrium, $G^<(x_1,x_2)$ does not satisfy the translational invariance in space-time, so that it cannot be written as $G^<(x_1-x_2)$. By the following change of variables:

$$(\mathbf{r}_{\rm rel},t_{\rm rel})=x_1-x_2 \quad\text{and}\quad (\mathbf{r},t)=(x_1+x_2)/2 . \tag{12}$$

$G^<(x_1,x_2)$ can be written as

$$G^<(\mathbf{r}_{\rm rel},t_{\rm rel};\mathbf{r},t)=i\left\langle\psi^\dagger\left[\mathbf{r}-\frac{\mathbf{r}_{\rm rel}}{2},t-\frac{t_{\rm rel}}{2}\right]\right.$$
$$\left.\times\psi\left[\mathbf{r}+\frac{\mathbf{r}_{\rm rel}}{2},t+\frac{t_{\rm rel}}{2}\right]\right\rangle . \tag{13}$$

By the Fourier transformation for the relative coordinates $t_{\rm rel}$ and $\mathbf{r}_{\rm rel}$, we obtain $G^<(\mathbf{p},\omega;\mathbf{r},t)$. At equilibrium, $G^<$ can be written as[26–28]

$$G_0^<(\mathbf{p},\omega)=if_0(\omega)A(\mathbf{p},\omega) , \tag{14}$$

where $f_0(\omega)=1/(e^{\omega/T}+1)$ is the equilibrium Fermi distribution function and ($\Sigma^R$ is the retarded self-energy)

$$A(\mathbf{p},\omega)=\frac{-2\,{\rm Im}\Sigma^R(\mathbf{p},\omega)}{[\omega-\xi_p-{\rm Re}\Sigma^R(\mathbf{p},\omega)]^2+({\rm Im}\Sigma^R(\mathbf{p},\omega))^2} . \tag{15}$$

In the usual Fermi-liquid theory, ${\rm Im}\Sigma^R\ll\omega$, so that $A(\mathbf{p},\omega)$ is a peaked function of $\omega$ around $\omega=\xi_p+{\rm Re}\Sigma^R$. In this case, the equilibrium spectral function can be taken as[26–28]

$$A(\mathbf{p},\omega)=2\pi\delta[\omega-\xi_p-{\rm Re}\Sigma^R(\mathbf{p},\omega)] . \tag{16}$$

Using this property, if the system is not far away from equilibrium, one can construct a closed equation for the fermion distribution function $f(\mathbf{p},\mathbf{r},t)$,[26–28] which is the QBE. The linearized QBE of $\delta f(\mathbf{p},\mathbf{r},t)=f(\mathbf{p},\mathbf{r},t)-f_0(\mathbf{p})$, where $f_0(\mathbf{p})$ is the equilibrium distribution function, is the QBE of the quasiparticles in the Fermi-liquid theory. From this QBE, the equation of motion for the Fermi-surface deformation, which is defined as[26–28]

$$\nu(\theta,\mathbf{r},t)=\int d|\mathbf{p}|\delta f(\mathbf{p},\mathbf{r},t) , \tag{17}$$

can be also constructed.

In the case of the fermion-gauge-field system, as mentioned in Sec. I, ${\rm Im}\Sigma^R(\omega)$ is larger than $\omega$ ($1<\eta\le 2$) or comparable to $\omega$ ($\eta=1$); i.e., strictly speaking, there is no well-defined Landau quasiparticle from the viewpoint of perturbation theory. However, Stern and Halperin[15] showed that, within a self-consistent treatment, the Fermi-liquid theory can be barely applied to the case of Coulomb interaction in the sense that ${\rm Re}\Sigma^R$ is logarithmically larger than ${\rm Im}\Sigma^R$. Note that, in general, $A(\mathbf{p},\omega)$ at equilibrium is no longer a peaked function of $\omega$ in the fermion-gauge-field system. Because of this, $f(\mathbf{p},\mathbf{r},t)$ does not satisfy a closed equation of motion even near the equilibrium. However, if $\Sigma^R$ is only a function of $\omega$, $A(\mathbf{p},\omega)$ is still a well-peaked function of $\xi_p$ around $\xi_p=0$ for sufficiently small $\omega$.[25] This observation leads us to define the following generalized distribution function:[25]

$$f(\theta,\omega;\mathbf{r},t)=-i\int \frac{d\xi_p}{2\pi} G^<(\mathbf{p},\omega;\mathbf{r},t)\ , \tag{18}$$

where $\theta$ is the angle between $\mathbf{p}$ and a given direction. The linearized quantum Boltzmann equation for $\delta f(\theta,\omega;\mathbf{r},t)=f(\theta,\omega;\mathbf{r},t)-f_0(\omega)$ can be derived, which is analogous to the QBE of the quasiparticles in the usual Fermi-liquid theory. From this QBE, one can also construct the equation of motion for the generalized Fermi-surface displacement[25]

$$u(\theta,\mathbf{r},t)=\int \frac{d\omega}{2\pi}\delta f(\theta,\omega;\mathbf{r},t)\ , \tag{19}$$

which corresponds to the variation of the local chemical potential in the momentum space. This object can be still well defined even in the absence of a sharp Fermi surface. This is because one can always define a chemical potential in each angle $\theta$, which is the energy required to put an additional fermion in the direction labeled by $\theta$ in momentum space. In Sec. III, we derive the linearized QBE for the generalized distribution function $\delta f(\theta,\omega;\mathbf{r},t)$.

## III. QUANTUM BOLTZMANN EQUATION FOR GENERALIZED DISTRIBUTION FUNCTION

In the nonequilibrium Green's-function formulation, the following matrices of the Green's function and the self-energy satisfy Dyson's equation[28]

$$\tilde{G}=\begin{bmatrix} G_t & -G^< \\ G^> & -G_{\bar{t}} \end{bmatrix} \text{ and } \tilde{\Sigma}=\begin{bmatrix} \Sigma_t & -\Sigma^< \\ \Sigma^> & -\Sigma_{\bar{t}} \end{bmatrix}\ , \tag{20}$$

where

$$\begin{aligned} G^>(x_1,x_2)&=-i\langle \psi(x_1)\psi^\dagger(x_2)\rangle\ , \\ G^<(x_1,x_2)&=i\langle \psi^\dagger(x_2)\psi(x_1)\rangle\ , \\ G_t(x_1,x_2)&=\Theta(t_1-t_2)G^>(x_1,x_2) \\ &\quad+\Theta(t_2-t_1)G^<(x_1,x_2)\ , \\ G_{\bar{t}}(x_1,x_2)&=\Theta(t_2-t_1)G^>(x_1,x_2) \\ &\quad+\Theta(t_1-t_2)G^<(x_1,x_2)\ , \end{aligned} \tag{21}$$

and $\Sigma^>$, $\Sigma^<$, $\Sigma_t$, and $\Sigma_{\bar{t}}$ are the associated self-energies. $\Theta(t)=1$ for $t>0$ and zero for $t<0$. $G^R$ (retarded) and $G^A$ (advanced) Green's functions can be expressed in terms of $G_t$ (time ordered), $G_{\bar{t}}$ (antitime ordered), $G^<$ and $G^>$ as follows:

$$\begin{aligned} G^R&=G_t-G^<=G^>-G_{\bar{t}}\ , \\ G^A&=G_t-G^>=G^<-G_{\bar{t}}\ . \end{aligned} \tag{22}$$

Similarly, $\Sigma^R$ and $\Sigma^A$ are given by

$$\begin{aligned} \Sigma^R&=\Sigma_t-\Sigma^<=\Sigma^>-\Sigma_{\bar{t}}\ , \\ \Sigma^A&=\Sigma_t-\Sigma^>=\Sigma^<-\Sigma_{\bar{t}}\ . \end{aligned} \tag{23}$$

The matrix Green's function satisfies the following equations of motion:

$$\begin{aligned} \left[i\frac{\partial}{\partial t_1}-H_0(\mathbf{r}_1)\right]\tilde{G}(x_1,x_2) &=\delta(x_1-x_2)\tilde{I}+\int dx_3\tilde{\Sigma}(x_1,x_3)\tilde{G}(x_3,x_2)\ , \\ \left[-i\frac{\partial}{\partial t_2}-H_0(\mathbf{r}_2)\right]\tilde{G}(x_1,x_2) &=\delta(x_1-x_2)\tilde{I}+\int dx_3\tilde{G}(x_1,x_3)\tilde{\Sigma}(x_3,x_2)\ , \end{aligned} \tag{24}$$

where

$$H_0(\mathbf{r}_1)=-\frac{1}{2m}\left[\frac{\partial}{\partial \mathbf{r}_1}\right]^2-\mu$$

and

$$H_0(\mathbf{r}_2)=-\frac{1}{2m}\left[\frac{\partial}{\partial \mathbf{r}_2}\right]^2-\mu\ . \tag{25}$$

For our purpose, we need only the equation of motion for $G^<$,

$$\begin{aligned} \left[i\frac{\partial}{\partial t_1}-H_0(\mathbf{r}_1)\right]G^<(x_1,x_2) &=\int dx_3[\Sigma_t(x_1,x_3)G^<(x_3,x_2) \\ &\qquad-\Sigma^<(x_1,x_3)G_{\bar{t}}(x_3,x_2)]\ , \\ \left[-i\frac{\partial}{\partial t_2}-H_0(\mathbf{r}_2)\right]G^<(x_1,x_2) &=\int dx_3[G_t(x_1,x_3)\Sigma^<(x_3,x_2) \\ &\qquad-G^<(x_1,x_3)\Sigma_{\bar{t}}(x_3,x_2)]\ . \end{aligned} \tag{26}$$

Taking the difference of the two equations of Eq. (26), and using the relations

$$\begin{aligned} G_t&=\mathrm{Re}G^R+\tfrac{1}{2}(G^<+G^>)\ , \\ G_{\bar{t}}&=\tfrac{1}{2}(G^<+G^>)-\mathrm{Re}G^R\ , \end{aligned} \tag{27}$$

we obtain

$$\begin{aligned} &\left[i\frac{\partial}{\partial t_1}+i\frac{\partial}{\partial t_2}+\frac{1}{2m}\left[\frac{\partial}{\partial \mathbf{r}_1}\right]^2-\frac{1}{2m}\left[\frac{\partial}{\partial \mathbf{r}_2}\right]^2\right]G^<(x_1,x_2) \\ &=\int dx_3[\mathrm{Re}\Sigma^R(x_1,x_3)G^<(x_3,x_2)+\Sigma^<(x_1,x_3)\mathrm{Re}G^R(x_3,x_2)-\mathrm{Re}G^R(x_1,x_3)\Sigma^<(x_3,x_2)-G^<(x_1,x_3)\mathrm{Re}\Sigma^R(x_3,x_2) \\ &\quad+\tfrac{1}{2}\Sigma^>(x_1,x_3)G^<(x_3,x_2)-\tfrac{1}{2}\Sigma^<(x_1,x_3)G^>(x_3,x_2)-\tfrac{1}{2}G^>(x_1,x_3)\Sigma^<(x_3,x_2)+\tfrac{1}{2}G^<(x_1,x_3)\Sigma^>(x_3,x_2)]\ . \end{aligned} \tag{28}$$

Near equilibrium, one can linearize this equation assuming that $\delta\tilde{G}=\tilde{G}-\tilde{G}_0$ and $\delta\tilde{\Sigma}=\tilde{\Sigma}-\tilde{\Sigma}_0$ are small, where $\tilde{G}_0$ and $\tilde{\Sigma}_0$ are matrices of the equilibrium Green's function and the self-energy. The Fourier transform $\tilde{G}(p_1,p_2)$ $[p_1=(\mathbf{p}_1,\omega_1),p_2=(\mathbf{p}_2,\omega_2)]$ of $\tilde{G}(x_1,x_2)$ can be written in terms of the variables defined by

$$p=(\mathbf{p},\omega)=(p_1-p_2)/2 \quad \text{and} \quad q=(\mathbf{q},\Omega)=p_1+p_2 \ . \tag{29}$$

Using these variables, the Fourier-transformed linearized equation of $\delta G^<(p,q)$ can be written as

$$\begin{aligned}
&[\Omega-v_F|\mathbf{q}|\cos\theta_{\mathbf{pq}}]\delta G^<(p,q)-[\mathrm{Re}\Sigma_0^R(p+q/2)-\mathrm{Re}\Sigma_0^R(p-q/2)]\delta G^<(p,q)\\
&\quad+[G_0^<(p+q/2)-G_0^<(p-q/2)]\delta(\mathrm{Re}\Sigma^R(p,q))-[\Sigma_0^<(p+q/2)-\Sigma_0^<(p-q/2)]\delta(\mathrm{Re}G^R(p,q))\\
&\quad+[\mathrm{Re}G_0^R(p+q/2)-\mathrm{Re}G_0^R(p-q/2)]\delta\Sigma^<(p,q)\\
&\qquad=G_0^<(p)\delta\Sigma^>(p,q)+\Sigma_0^>(p)\delta G^<(p,q)-G_0^>(p)\delta\Sigma^<(p,q)-\Sigma_0^<(p)\delta G^>(p,q)\ ,
\end{aligned}\tag{30}$$

where $\theta_{\mathbf{pq}}$ is the angle between $\mathbf{p}$ and $\mathbf{q}$. In the presence of an external potential $U(q)$, one should add a term $U(q)[G_0^<(p+q/2)-G_0^<(p-q/2)]$ on the left-hand side of Eq. (30).

We next check that this expression is equivalent to the usual QBE for $\delta G^<(\mathbf{p},\omega;\mathbf{r},t)$, where $\mathbf{r}$ and $t$ are conjugate variables of $\mathbf{q}$ and $\Omega$. Note that

$$F(p+q/2)-F(p-q/2)\approx\mathbf{q}\cdot\frac{\partial F}{\partial\mathbf{p}}+\Omega\frac{\partial F}{\partial\omega}\ , \tag{31}$$

for small $|\mathbf{q}|$ and $\Omega$. From Eqs. (30) and (31), one can check that $\delta G^<(\mathbf{p},\omega;\mathbf{r},t)$, which is the Fourier transform of $\delta G^<(p,q)$, satisfies the following equation:

$$\begin{aligned}
&[\omega-p^2/2m,\delta G^<(\mathbf{p},\omega;\mathbf{r},t)]-[\mathrm{Re}\Sigma_0^R(\mathbf{p},\omega),G^<(\mathbf{p},\omega;\mathbf{r},t)]-[\delta(\mathrm{Re}\Sigma^R(\mathbf{p},\omega)),G_0^<(\mathbf{p},\omega)]\\
&\quad-[\Sigma_0^<(\mathbf{p},\omega),\delta(\mathrm{Re}G^R(\mathbf{p},\omega;\mathbf{r},t))]-[\delta\Sigma^<(\mathbf{p},\omega;\mathbf{r},t),\mathrm{Re}G_0^R(\mathbf{p},\omega)]\\
&\qquad=G_0^<(\mathbf{p},\omega)\delta\Sigma^>(\mathbf{p},\omega;\mathbf{r},t)+\Sigma_0^>(\mathbf{p},\omega)\delta G^<(\mathbf{p},\omega;\mathbf{r},t)-G_0^>(\mathbf{p},\omega)\delta\Sigma^<(\mathbf{p},\omega;\mathbf{r},t)-\Sigma_0^<(\mathbf{p},\omega)\delta G^>(\mathbf{p},\omega;\mathbf{r},t)\ ,
\end{aligned}\tag{32}$$

where $[X,Y]$ is the Poisson bracket

$$[X,Y]=\frac{\partial X}{\partial\omega}\frac{\partial Y}{\partial t}-\frac{\partial X}{\partial t}\frac{\partial Y}{\partial\omega}-\frac{\partial X}{\partial\mathbf{p}}\cdot\frac{\partial Y}{\partial\mathbf{r}}+\frac{\partial X}{\partial\mathbf{r}}\cdot\frac{\partial Y}{\partial\mathbf{p}}\ . \tag{33}$$

Note that this equation is just the linearized version of the usual QBE for $G^<(\mathbf{p},\omega;\mathbf{r},t)$ given by[25–28]

$$\begin{aligned}
&[\omega-p^2/2m-\mathrm{Re}\Sigma^R(\mathbf{p},\omega;\mathbf{r},t),G^<(\mathbf{p},\omega;\mathbf{r},t)]-[\Sigma^<(\mathbf{p},\omega;\mathbf{r},t),\mathrm{Re}G^R(\mathbf{p},\omega;\mathbf{r},t)]\\
&\qquad=\Sigma^>(\mathbf{p},\omega;\mathbf{r},t)G^<(\mathbf{p},\omega;\mathbf{r},t)-G^>(\mathbf{p},\omega;\mathbf{r},t)\Sigma^<(\mathbf{p},\omega;\mathbf{r},t)\ .
\end{aligned}\tag{34}$$

We directly deal with Eq. (30) in momentum space $(\mathbf{q},\Omega)$ rather than the long-time, long-wavelength expansion in real space $(\mathbf{r},t)$ given by Eq. (32). The nonequilibrium one-loop self-energy correction, which is given by the diagram in Fig. 1, can be written as[27,28]

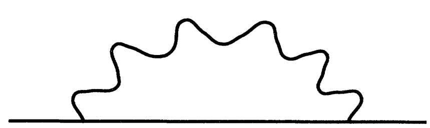

FIG. 1. The one-loop Feynman diagram for the self-energy of the fermions. Here the solid line represents the fermion propagator, and the wavy line denotes the RPA gauge-field propagator.

$$\begin{aligned}
\Sigma^<(\mathbf{p},\omega)&=\sum_{\mathbf{q}}\int_0^\infty\frac{d\nu}{\pi}\left|\frac{\mathbf{p}\times\hat{\mathbf{q}}}{m}\right|^2\mathrm{Im}D_{11}(\mathbf{q},\nu)\\
&\quad\times\{[n_0(\nu)+1]G^<(\mathbf{p}+\mathbf{q},\omega+\nu)\\
&\qquad+n_0(\nu)G^<(\mathbf{p}+\mathbf{q},\omega-\nu)\}\ ,\\
\Sigma^>(\mathbf{p},\omega)&=\sum_{\mathbf{q}}\int_0^\infty\frac{d\nu}{\pi}\left|\frac{\mathbf{p}\times\hat{\mathbf{q}}}{m}\right|^2\mathrm{Im}D_{11}(\mathbf{q},\nu)\\
&\quad\times\{n_0(\nu)G^>(\mathbf{p}+\mathbf{q},\omega+\nu)\\
&\qquad+[n_0(\nu)+1]G^>(\mathbf{p}+\mathbf{q},\omega-\nu)\}\ ,
\end{aligned}\tag{35}$$

where $n_0(\nu)=1/(e^{\nu/T}-1)$ is the equilibrium boson distribution function. The real part of the retarded self-energy is given by

$$\mathrm{Re}\Sigma^R(\mathbf{p},\omega;\mathbf{q},\Omega) = -\int \frac{d\omega'}{\pi} \mathcal{P} \frac{\mathrm{Im}\Sigma^R(\mathbf{p},\omega';\mathbf{q},\Omega)}{\omega-\omega'} = -\int \frac{d\omega'}{2\pi i} \mathcal{P} \frac{\Sigma^>(\mathbf{p},\omega';\mathbf{q},\Omega)-\Sigma^<(\mathbf{p},\omega';\mathbf{q},\Omega)}{\omega-\omega'} , \quad (36)$$

where $\mathcal{P}$ represents the principal value and $\mathrm{Im}\Sigma^R=(1/2i)(\Sigma^>-\Sigma^<)$ is used. The same relations hold for the Green's functions $G^R$,

$$\mathrm{Re}G^R(\mathbf{p},\omega;\mathbf{q},\Omega) = -\int \frac{d\omega'}{2\pi i} \mathcal{P} \frac{G^>(\mathbf{p},\omega';\mathbf{q},\Omega)-G^<(\mathbf{p},\omega';\mathbf{q},\Omega)}{\omega-\omega'} , \quad (37)$$

and $\mathrm{Im}G^R=(1/2i)(G^>-G^<)$.

At equilibrium, the Green's functions $G^<$ and $G^>$ can be written as[26–28]

$$G^<(\mathbf{p},\omega)=if_0(\omega)A(\mathbf{p},\omega) ,$$
$$G^>(\mathbf{p},\omega)=-i[1-f_0(\omega)]A(\mathbf{p},\omega) , \quad (38)$$

where $A(\mathbf{p},\omega)$ is given by Eq. (15). From these relations, the one-loop self-energy $\Sigma_0^R$ at equilibrium can be written as

$$\Sigma_0^R=\sum_{\mathbf{q}} \int_0^\infty \frac{d\nu}{\pi} \left| \frac{\mathbf{p}\times\hat{\mathbf{q}}}{m} \right|^2 \times \mathrm{Im}D_{11}(\mathbf{q},\nu) \left[ \frac{1+n_0(\nu)-f_0(\xi_{\mathbf{p}+\mathbf{q}})}{\omega+i\delta-\xi_{\mathbf{p}+\mathbf{q}}-\nu} + \frac{n_0(\nu)+f_0(\xi_{\mathbf{p}+\mathbf{q}})}{\omega+i\delta-\xi_{\mathbf{p}+\mathbf{q}}+\nu} \right] . \quad (39)$$

As emphasized in Sec. II, if the self-energy depends only on the frequency $\omega$, $A(\mathbf{p},\omega)$ at equilibrium is a peaked function of $\xi_p$. Therefore, as long as the system is not far away from the equilibrium, the generalized distribution function $f(\theta_{\mathbf{pq}},\omega;\mathbf{q},\Omega)$, which is given by the following relations, can be well defined at zero temperature:[25]

$$\int \frac{d\xi_p}{2\pi}[-iG^<(\mathbf{p},\omega;\mathbf{q},\Omega)]\equiv f(\theta_{\mathbf{pq}},\omega;\mathbf{q},\Omega) ,$$
$$\int \frac{d\xi_p}{2\pi}[iG^>(\mathbf{p},\omega;\mathbf{q},\Omega)]\equiv 1-f(\theta_{\mathbf{pq}},\omega;\mathbf{q},\Omega) , \quad (40)$$

where $\theta_{\mathbf{pq}}$ is the angle between $\mathbf{p}$ and $\mathbf{q}$.

The extension to the case of finite temperatures requires special care because, even at equilibrium, $\mathrm{Im}\Sigma_0^R(\mathbf{p},\omega)$ is known to be divergent,[11] so that $A(\mathbf{p},\omega)$, $G_0^<$, and $G_0^>$ at equilibrium are not well defined. Therefore, the nonequilibrium $G^<$ and $G^>$ are also not well defined near equilibrium. In order to resolve this problem, let us first separate the gauge-field fluctuations into two parts, i.e, $\mathbf{a}(\mathbf{q},\nu)\equiv\mathbf{a}_-(\mathbf{q},\nu)$ for $\nu<T$ and $\mathbf{a}(\mathbf{q},\nu)\equiv\mathbf{a}_+(\mathbf{q},\nu)$ for $\nu>T$, then examine the effects of $\mathbf{a}_+$ and $\mathbf{a}_-$ separately. The classical fluctuation $\mathbf{a}_-$ of the gauge field can be regarded as a vector potential which corresponds to a static but spatially varying magnetic field $\mathbf{b}_-=\nabla\times\mathbf{a}_-$. In order to remove the divergence in the self-energy, one can consider the one-particle Green's function $\tilde{G}_-\equiv\tilde{G}(\mathbf{P}_-,\omega;\mathbf{r},t)$ as a function of a variable $\mathbf{P}_-=\mathbf{p}-\mathbf{a}_-$. Since we effectively separate out $\mathbf{a}_-$ fluctuations, the self-energy, which appears in the equation of motion given by Eq. (24), should contain only $\mathbf{a}_+$ fluctuations and free of divergences. Therefore, $\delta G_-^<\equiv\delta G^<(\mathbf{P}_-,\omega;\mathbf{r},t)$ is well defined, and its equation of motion is given by the Fourier transform of Eq. (30) with the following replacement. In the first place, the variable $\mathbf{p}$ should be changed to a variable $\mathbf{P}_-=\mathbf{p}-\mathbf{a}_-$. Second, the self-energy $\tilde{\Sigma}$ should be changed to $\tilde{\Sigma}_+$, which now contains only $\mathbf{a}_+$ fluctuations. Finally, the equation of motion contains a term which depends on $\mathbf{b}_-$. We argue in the Appendix that ignoring this term does not affect the physical interpretations of the QBE, which will appear in Secs. IV, V, and VI. We provide the details of the analysis for the finite temperature case in the Appendix. From now on, we will adopt the notation that $G^<$ should be understood as $G_-^<$ for finite temperatures. For example, the generalized distribution function at finite temperatures is given by Eq. (40) with the replacement that $G^<,G^>\to G_-^<,G_-^>$. The same type of abuse of notation applies to the self-energy, where only $\mathbf{a}_+$ fluctuations should be included; i.e., the QBE is valid at finite $T$, provided that the lower cutoff $T$ is introduced for the frequency integrals.

In Eq. (35), one can change the variables such that $\mathbf{p}'=\mathbf{p}+\mathbf{q}$ and $\omega'=\omega+\nu$. The gauge-field propagator can be written in terms of these variables as $D_{11}(\mathbf{q},\nu)=D_{11}(\mathbf{p}'-\mathbf{p},\omega'-\omega)$, where $(\mathbf{p},\omega)$ and $(\mathbf{p}',\omega')$ represent the incoming and outgoing fermions. Assuming that $|\mathbf{p}|\approx|\mathbf{p}'|\approx k_F$ and using $|\mathbf{p}'-\mathbf{p}|\approx k_F|\theta_{\mathbf{p'q}}-\theta_{\mathbf{pq}}|$, we obtain $D_{11}(\mathbf{q},\nu)\approx D_{11}(k_F|\theta_{\mathbf{p'q}}-\theta_{\mathbf{pq}}|,\omega'-\omega)$. Using the above results and the fact that $G^<$ and $G^>$ are well-peaked functions of $\xi_p$ near the equilibrium, $\mathrm{Re}\Sigma^R$ can be written as

$$\mathrm{Re}\Sigma^R=N(0)\int \frac{d\theta_{\mathbf{p'q}}}{2\pi}\int d\omega' v_F^2 \mathrm{Re}D_{11}(k_F|\theta_{\mathbf{p'q}}-\theta_{\mathbf{pq}}|,\omega'-\omega) f(\theta_{\mathbf{p'q}},\omega';\mathbf{q},\Omega) , \quad (41)$$

where $N(0)=m/2\pi$ is the density of state. For simplicity, here we ignore the dependence of the gauge-field propagator $D_{11}$ on the fermion Green's function. In Sec. VII, we discuss the results of this approximation, and argue that the additional contributions coming from the dependence of the gauge-field propagator on the fermion Green's function do not change the consequences of the present analysis. Now $\delta(\mathrm{Re}\Sigma^R)$, which is the deviation from the equilibrium, can be written as

$$\delta(\mathrm{Re}\Sigma^R)=N(0)\int\frac{d\theta_{\mathrm{p'q}}}{2\pi}\int d\omega' v_F^2 \mathrm{Re}D_{11}(k_F|\theta_{\mathrm{p'q}}-\theta_{\mathrm{pq}}|,\omega'-\omega)\delta f(\theta_{\mathrm{p'q}},\omega';\mathbf{q},\Omega)\ . \tag{42}$$

We also assume that the nonequilibrium self-energy depends only on $\omega$, like that of the equilibrium case, which is plausible as long as the system is not far away from the equilibrium. In order to obtain the equation for $f(\theta_{\mathrm{pq}},\omega;\mathbf{q},\Omega)$, we perform $\int d\xi_p/2\pi$ integration on both sides of Eq. (30). Note that

$$\begin{aligned}\int\frac{d\xi_p}{2\pi}\mathrm{Re}G^R(\mathbf{p},\omega';\mathbf{q},\Omega)&=\int\frac{d\omega'}{2\pi}\mathcal{P}\frac{[1-f(\theta_{\mathrm{pq}},\omega';\mathbf{q},\Omega)]+f(\theta_{\mathrm{pq}},\omega';\mathbf{q},\Omega)}{\omega-\omega'}\\&=\int\frac{d\omega'}{2\pi}\mathcal{P}\frac{1}{\omega-\omega'}=0\ .\end{aligned}\tag{43}$$

Thus the fourth and fifth terms on the left-hand side of the QBE [given by Eq. (30)] vanish after $\int d\xi_p/2\pi$ integration. After this integration, using Eqs. (36), (40), and (42), the remaining parts of Eq. (30) can be written as $[\delta f(\theta_{\mathrm{pq}},\omega)\equiv\delta f(\theta_{\mathrm{pq}},\omega;\mathbf{q},\Omega)]$

$$\begin{aligned}&[\Omega-v_F q\cos\theta_{\mathrm{pq}}]\delta f(\theta_{\mathrm{pq}},\omega)\\&\quad-N(0)\int\frac{d\theta_{\mathrm{p'q}}}{2\pi}\int d\omega' v_F^2\mathrm{Re}D_{11}(k_F|\theta_{\mathrm{p'q}}-\theta_{\mathrm{pq}}|,\omega'-\omega)[f_0(\omega'+\Omega/2)-f_0(\omega'-\Omega/2)]\delta f(\theta_{\mathrm{pq}},\omega)\\&\quad+N(0)\int\frac{d\theta_{\mathrm{p'q}}}{2\pi}\int d\omega' v_F^2\mathrm{Re}D_{11}(k_F|\theta_{\mathrm{p'q}}-\theta_{\mathrm{pq}}|,\omega'-\omega)[f_0(\omega+\Omega/2)-f_0(\omega-\Omega/2)]\delta f(\theta_{\mathrm{p'q}},\omega')\\&\quad=N(0)\int d\theta_{\mathrm{p'q}}\int_0^\infty\frac{d\nu}{\pi}\int d\omega' v_F^2\mathrm{Im}D_{11}(k_F|\theta_{\mathrm{p'q}}-\theta_{\mathrm{pq}}|,\nu)\\&\qquad\times(\delta(\omega'-\omega+\nu)\{\delta f(\theta_{\mathrm{pq}},\omega)[1-f_0(\omega')+n_0(\nu)]-\delta f(\theta_{\mathrm{p'q}},\omega')[f_0(\omega)+n_0(\nu)]\}\\&\qquad-\delta(\omega'-\omega-\nu)\{\delta f(\theta_{\mathrm{p'q}},\omega')[1-f_0(\omega)+n_0(\nu)]-\delta f(\theta_{\mathrm{pq}},\omega)[f_0(\omega')+n_0(\nu)]\})\ .\end{aligned}\tag{44}$$

Some explanations of each term in Eq. (44) are in order. In the first place, as mentioned in Sec. II, Eq. (44) is the analog of the usual QBE for the quasiparticle distribution function $\delta f(\mathbf{p},\mathbf{q},\Omega)$; thus the structures of the QBE's in both cases are similar. The first term on the left-hand side of the equation corresponds to the free fermions. The second term on the left-hand side corresponds to the self-energy correction which renormalizes the mass of the fermions. The third term on the left-hand side can be regarded as the contribution from the generalized Landau-interaction function, which can be defined as

$$F(\theta_{\mathrm{p'q}}-\theta_{\mathrm{pq}},\omega'-\omega)=v_F^2\mathrm{Re}D_{11}(k_F|\theta_{\mathrm{p'q}}-\theta_{\mathrm{pq}}|,\omega'-\omega)\ . \tag{45}$$

Note that this generalized Landau-interaction function contains the frequency dependence as well as the usual angular dependence. This is due to the fact that the gauge interaction is retarded in time, and it is also one of the major differences between the fermion-gauge-field system and the usual Fermi liquid. The right-hand side of the equation is nothing but the collision integral $I_{\mathrm{collision}}$ and is given by the Fermi golden rule. Thus Eq. (44) can be written as

$$\begin{aligned}&[\Omega-v_F q\cos\theta_{\mathrm{pq}}]\delta f(\theta_{\mathrm{pq}},\omega)-[\mathrm{Re}\Sigma_0^R(\omega+\Omega/2)-\mathrm{Re}\Sigma_0^R(\omega-\Omega/2)]\delta f(\theta_{\mathrm{pq}},\omega)\\&\qquad+N(0)\int\frac{d\theta_{\mathrm{p'q}}}{2\pi}\int d\omega' F(\theta_{\mathrm{p'q}}-\theta_{\mathrm{pq}},\omega'-\omega)[f_0(\omega+\Omega/2)-f_0(\omega-\Omega/2)]\delta f(\theta_{\mathrm{p'q}},\omega')=I_{\mathrm{collision}}\ .\end{aligned}$$

After taking the integral $\int d\omega/2\pi$ on both sides of Eq. (44), it can be seen that one cannot write the QBE only in terms of $u(\theta_{\mathrm{pq}},\mathbf{q},\Omega)=\int(d\omega/2\pi)\delta f(\theta_{\mathrm{pq}},\omega;\mathbf{q},\Omega)$, which is the generalized Fermi-surface displacement. That is, the QBE becomes

$$\begin{aligned}&[\Omega-v_F q\cos\theta_{\mathrm{pq}}]u(\theta_{\mathrm{pq}},\mathbf{q},\Omega)\\&\quad-N(0)\int\frac{d\theta_{\mathrm{p'q}}}{2\pi}\int d\omega\int d\omega' v_F^2\mathrm{Re}D_{11}(k_F|\theta_{\mathrm{p'q}}-\theta_{\mathrm{pq}}|,\omega'-\omega)[f_0(\omega'+\Omega/2)-f_0(\omega'-\Omega/2)][\delta f(\theta_{\mathrm{pq}},\omega)-\delta f(\theta_{\mathrm{p'q}},\omega)]\\&\quad=N(0)\int d\theta_{\mathrm{p'q}}\int_0^\infty\frac{d\nu}{\pi}\int d\omega\int d\omega' v_F^2\mathrm{Im}D_{11}(k_F|\theta_{\mathrm{p'q}}-\theta_{\mathrm{pq}}|,\nu)\\&\qquad\times\{\delta(\omega'-\omega+\nu)[1-f_0(\omega')+n_0(\nu)]+\delta(\omega'-\omega-\nu)[f_0(\omega')+n_0(\nu)]\}\\&\qquad\times[\delta f(\theta_{\mathrm{pq}},\omega)-\delta f(\theta_{\mathrm{p'q}},\omega)]\ .\end{aligned}\tag{46}$$

In the presence of the external potential $U(\mathbf{q},\Omega)$, one should add an additional term $v_F q \cos\theta_{\mathbf{pq}} U(\mathbf{q},\Omega)$ on the left-hand side of Eq. (46), which requires a careful derivation. Note that the contributions from the self-energy and the generalized Landau-interaction function are combined on the left-hand side of the QBE. Even though the above equation is already useful, it is worthwhile to transform this equation to the more familiar one. In Sec. IV, we provide the approximate QBE for $u(\theta_{\mathbf{pq}},\mathbf{q},\Omega)$ which is more useful to understand the low-energy excitations of the system.

## IV. QUANTUM BOLTZMANN EQUATION FOR GENERALIZED FERMI SURFACE DISPLACEMENT

In order to transform the QBE given by Eq. (45) or (46) to a more familiar form, it is necessary to simplify the generalized Landau-interaction function $F(\theta,\omega) = v_F^2 \mathrm{Re} D_{11}(k_F|\theta|,\omega)$. Note that

$$\mathrm{Re}D_{11}(k_F|\theta|,\omega) = \frac{(\chi/\gamma^2)k_F^{2+\eta}|\theta|^{2+\eta}}{\omega^2 + (\chi/\gamma)^2 k_F^{2+2\eta}|\theta|^{2+2\eta}} . \tag{47}$$

It can be checked from Eq. (44) that $\delta f(\theta_{\mathbf{pq}},\omega;\mathbf{q},\Omega)$ is finite only when $|\omega| \lesssim \Omega$ at zero temperature. Therefore, the frequency $\omega$ in $\mathrm{Re}D_{11}(k_F|\theta|,\omega)$ is cut off by $\Omega$. In this case, one can introduce the $\Omega$-dependent cutoff $\theta_c \approx (1/k_F)(\gamma\Omega/\chi)^{1/(1+\eta)}$ in the angle variable, and approximate $F(\theta,\omega)$ by

$$F_{\mathrm{Landau}}(\theta) = \begin{cases} F(\theta,\omega=0) & \text{if } |\theta| > \theta_c \\ F(\theta=\theta_c,\omega=0) & \text{otherwise} , \end{cases} \tag{48}$$

where

$$F(\theta,\omega=0) = \frac{v_F^2}{\chi k_F^\eta} \frac{1}{|\theta|^\eta} . \tag{49}$$

Using this approximation and $f_0(\omega) = \Theta(-\omega)$ at zero temperature, the QBE given by Eq. (46) at zero temperature can be transformed into (the finite temperature case is discussed in the Appendix)

$$[\Omega - v_F q \cos\theta_{\mathbf{pq}}]u(\theta_{\mathbf{pq}},\mathbf{q},\Omega) + \Omega N(0)\int \frac{d\theta_{\mathbf{p'q}}}{2\pi} F_{\mathrm{Landau}}(\theta_{\mathbf{p'q}}-\theta_{\mathbf{pq}})[u(\theta_{\mathbf{pq}},\mathbf{q},\Omega) - u(\theta_{\mathbf{p'q}},\mathbf{q},\Omega)]$$
$$= N(0)\int d\theta_{\mathbf{p'q}} \int_0^\infty \frac{d\nu}{\pi} \int d\omega \int d\omega' v_F^2 \mathrm{Im}D_{11}(k_F|\theta_{\mathbf{p'q}}-\theta_{\mathbf{pq}}|,\nu)\{\delta(\omega'-\omega+\nu)[1-f_0(\omega')] + \delta(\omega'-\omega-\nu)f_0(\omega')\}$$
$$\times [\delta f(\theta_{\mathbf{pq}},\omega) - \delta f(\theta_{\mathbf{p'q}},\omega)] . \tag{50}$$

Note that $\Omega N(0)\int d\theta_{\mathbf{p'q}}/2\pi F_{\mathrm{Landau}}(\theta_{\mathbf{p'q}}-\theta_{\mathbf{pq}}) \propto \Omega^{2/(1+\eta)}$ $(1<\eta\le 2)$ or $\Omega \ln\Omega$ $(\eta=1)$ corresponds to the contribution from the real part of the retarded self-energy. On the other hand, $\Omega N(0)\int d\theta_{\mathbf{p'q}}/2\pi F_{\mathrm{Landau}}(\theta_{\mathbf{p'q}}-\theta_{\mathbf{pq}}) u(\theta_{\mathbf{p'q}},\mathbf{q},\Omega)$ represents the Landau-interaction part.

For smooth fluctuations of the generalized Fermi-surface displacement, $u(\theta,\mathbf{q},\Omega)$ is a slowly varying function of $\theta$, so that there is a forward-scattering cancellation between the self-energy part and the Landau-interaction part. Therefore, for smooth fluctuations, the singular behavior of the self-energy does not appear in the dynamics of the generalized Fermi-surface displacement. On the other hand, for rough fluctuations, $u(\theta,\mathbf{q},\Omega)$ is a rapidly varying function. In this case, the Landau-interaction part becomes very small, and the self-energy part dominates. Thus, for rough fluctuations, the dynamics of the generalized Fermi-surface displacement should show the singular behavior of the self-energy. From these results, one can expect that the smooth and rough fluctuations provide very different physical pictures of the elementary excitations of the system.

One can make this observation more concrete by looking at the QBE in angular momentum $l$ (which is the conjugate variable of $\theta$) space. By the Fourier expansion

$$u(\theta,\mathbf{q},\Omega) = \sum_l e^{il\theta} u_l(\mathbf{q},\Omega) \quad \text{and} \quad \delta f(\theta,\omega;\mathbf{q},\Omega) = \sum_l e^{il\theta} \delta f_l(\omega;\mathbf{q},\Omega) , \tag{51}$$

one can obtain

$$\Omega u_l(\mathbf{q},\Omega) - \frac{v_F q}{2}[u_{l+1}(\mathbf{q},\Omega) + u_{l-1}(\mathbf{q},\Omega)] + \Omega N(0)\int \frac{d\theta}{2\pi} F_{\mathrm{Landau}}(\theta)[1-\cos(l\theta)]u_l(\mathbf{q},\Omega)$$
$$= N(0)\int d\theta \int_0^\infty \frac{d\nu}{\pi} \int d\omega \int d\omega' v_F^2 \mathrm{Im}D_{11}(k_F|\theta|,\nu)[1-\cos(l\theta)]$$
$$\times \{\delta(\omega'-\omega+\nu)[1-f_0(\omega')] + \delta(\omega'-\omega-\nu)f_0(\omega')\}\delta f_l(\omega;\mathbf{q},\Omega) . \tag{52}$$

Note that, in the $1-\cos(l\theta)$ factor inside the integral on the left-hand side of the QBE given by Eq. (52), 1 comes from the self-energy part and $\cos(l\theta)$ comes from the Landau-interaction part. For $l<l_c\approx 1/\theta_c \propto \Omega^{-1/(1+\eta)}$, $1-\cos(l\theta)\approx l^2\theta^2/2$ and the additional $\theta^2$ dependence makes the angle integral less singular because typical $\theta$ is of the order of $\Omega^{1/(1+\eta)}$. Due to this cancellation for the small-angle (forward) scattering, the correction from the self-energy part and the Landau-interaction part becomes of the order of $\Omega^{4/(1+\eta)}$, so that it does not cause any singular correction. Note that a similar type of cancellation occurs in the collision integral. Therefore, for the small angular momentum modes $l<l_c$, the system behaves like the usual Fermi liquid. For $l>l_c$, the $\cos(l\theta)$ factor becomes highly oscillating as a function of $\theta$, so that the Landau-interaction part becomes very small. As a result, the self-energy part dominates and the dispersion relation for the dynamics of the generalized Fermi-surface displacement is changed from $\Omega=v_F q$ to $\Omega\propto q^{(1+\eta)/2}$ $(1<\eta\le 2)$ or $\Omega\propto q/|\ln q|$ $(\eta=1)$. Also, a similar thing happens in the collision integral, i.e., the $\cos(l\theta)$ factor does not contribute and the remaining contribution shows the singular behavior of the imaginary part of the self-energy, so that the collision integral cannot be ignored for $1<\eta\le 2$ and can be *marginally* ignored for $\eta=1$.

Using the above results, one can understand the density-density and current-current correlation functions, which show no anomalous behavior in the long-wavelength and low-frequency limits.[12,13] From the QBE, one can evaluate these correlation functions by taking the angular average of the density or the current disturbance due to the external potential, and calculating the linear response. As a result, in these correlation functions the small angular momentum modes are dominating, so that the results do not show any singular behavior. From these results, one can also expect that two different behaviors of the small $(l<l_c)$ and large $(l>l_c)$ angular momentum modes may show up even in the presence of the finite effective magnetic field $\delta B$, and that the large angular momentum modes may be responsible for the singular energy gap of the system,[6,14,15] which is the subject of Sec. V.

## V. QUANTUM BOLTZMANN EQUATION IN THE PRESENCE OF EFFECTIVE MAGNETIC FIELD AND ENERGY GAP

We follow Hänsch and Mahan[31] to derive the QBE in the presence of the finite effective magnetic field $\Delta B$. The only difference between the case of $\Delta B\neq 0$ and that of $\Delta B=0$ is that the external vector potential $\Delta\,\mathbf{A}=-\frac{1}{2}\mathbf{r}\times\Delta\mathbf{B}$ enters into the kinetic energy in the equation of motion of the one-particle Green's function.[31] The same procedure used in the case of $\Delta B=0$ can be employed to derive the QBE from the equation of motion of the one-particle Green's function. The resulting equation can be transformed to a convenient form by a change of variables given by

$$\mathbf{P}=\mathbf{p}-\Delta\,\mathbf{A}=\mathbf{p}+\tfrac{1}{2}\mathbf{r}\times\Delta\mathbf{B}\ , \tag{53}$$

so that one can construct the QBE for $G^<(\mathbf{P},\omega;\mathbf{q},\Omega)$, which is now a function of $\mathbf{P}$.[31] As a result, the change we have to make for the case of $\Delta B\neq 0$ [compared to the case of $\Delta B=0$ given by Eq. (30)] is that all momentum variables should be changed from $\mathbf{p}$ to $\mathbf{P}$, and the following additional terms should be added to Eq. (30):[31]

$$\frac{\mathbf{P}}{m}\cdot\Delta\mathbf{B}\times\frac{\partial}{\partial\mathbf{P}}\delta G^<(\mathbf{P},\omega;\mathbf{q},\Omega)+\frac{\partial}{\partial\mathbf{P}}\delta(\mathrm{Re}\Sigma^R(\mathbf{P},\omega;\mathbf{q},\Omega))\cdot\Delta\mathbf{B}\times\frac{\partial}{\partial\mathbf{P}}G_0^<(\mathbf{P},\omega)$$
$$-\Delta\mathbf{B}\cdot\frac{\partial}{\partial\mathbf{P}}\delta\Sigma^<(\mathbf{P},\omega;\mathbf{q},\Omega)\times\frac{\partial}{\partial\mathbf{P}}(\mathrm{Re}G_0^R(\mathbf{P},\omega))-\Delta\mathbf{B}\cdot\frac{\partial}{\partial\mathbf{P}}\Sigma_0^<(\mathbf{P},\omega)\times\frac{\partial}{\partial\mathbf{P}}\delta(\mathrm{Re}G^R(\mathbf{P},\omega;\mathbf{q},\Omega))\ . \tag{54}$$

Since the self-energy does not depend on the momentum $\mathbf{P}$ in the fermion-gauge-field system, the only term which contributes to the QBE is

$$\frac{\mathbf{P}}{m}\cdot\Delta\mathbf{B}\times\frac{\partial}{\partial\mathbf{P}}\delta G^<(\mathbf{P},\omega;\mathbf{q},\Omega)\ . \tag{55}$$

In principle, the self-energy and Green's function in the QBE also depend on the effective magnetic field $\Delta B$. In the semiclassical approximation for very small $\Delta B$, we ignore this type of $\Delta B$ dependence and, instead of that, we introduce a low-energy cutoff $E_g$ in the frequency integrals, which is the energy gap of the system. Then, after the integration $\int d\xi_{\mathbf{P}}/2\pi$, the equation becomes that of Eq. (30) with a low-energy cutoff $E_g$, and it also contains an additional term given by

$$\Delta\omega_c\frac{\partial}{\partial\theta_{\mathbf{Pq}}}\delta f(\theta_{\mathbf{Pq}},\omega;\mathbf{q},\Omega)\ , \tag{56}$$

where $\Delta\omega_c=\Delta B/m$. After $\int d\omega/2\pi$, the QBE for a generalized Fermi-surface displacement can be written as

$$[\Omega-v_F q\cos\theta_{\mathbf{Pq}}]u(\theta_{\mathbf{Pq}},\mathbf{q},\Omega)-i\Delta\omega_c\frac{\partial}{\partial\theta_{\mathbf{Pq}}}u(\theta_{\mathbf{Pq}},\mathbf{q},\Omega)+\Omega N(0)\int\frac{d\theta_{\mathbf{P'q}}}{2\pi}F_{\mathrm{Landau}}(\theta_{\mathbf{P'q}}-\theta_{\mathbf{Pq}})[u(\theta_{\mathbf{Pq}},\mathbf{q},\Omega)-u(\theta_{\mathbf{P'q}},\mathbf{q},\Omega)]$$
$$=N(0)\int d\theta_{\mathbf{P'q}}\int_0^\infty\frac{d\nu}{\pi}\int d\omega\int d\omega' v_F^2\mathrm{Im}D_{11}(k_F|\theta_{\mathbf{P'q}}-\theta_{\mathbf{Pq}}|,\nu)\{\delta(\omega'-\omega+\nu)[1-f_0(\omega')]+\delta(\omega'-\omega-\nu)f_0(\omega')\}$$
$$\times[\delta f(\theta_{\mathbf{Pq}},\omega)-\delta f(\theta_{\mathbf{P',q}},\omega)]\ , \tag{57}$$

where a low-energy cutoff $E_g$ is introduced in the frequency integrals. In particular, the angle cutoff $\theta_c$ in $F_{\rm Landau}(\theta)$ should be changed from $\theta_c \approx (1/k_F)(\gamma\Omega/\chi)^{1/(1+\eta)}$ ($\Delta B = 0$) to $\theta_c \approx (1/k_F)(\gamma E_g/\chi)^{1/(1+\eta)}$ ($\Delta B \neq 0$) in the low-frequency $\Omega$ limit.

Now similar interpretations can be made for the case of $\Delta B = 0$. For the smooth fluctuations ($l < l_c \approx 1/\theta_c$), there is a cancellation between the self-energy and the Landau-interaction parts. As a result, we have a term which is of the order of $\Omega E_g^{(3-\eta)/(1+\eta)}$, which can be ignored compared to $\Omega$ because $E_g$ is very small near $\nu = \frac{1}{2}$ or $\Delta B = 0$. Also, a similar thing happens in the collision integral. Therefore, the QBE for the smooth fluctuations can be written as

$$[\Omega - v_F q \cos\theta_{\rm Pq}]u(\theta_{\rm Pq},\mathbf{q},\Omega) - i\Delta\omega_c \frac{\partial}{\partial\theta_{\rm Pq}} u(\theta_{\rm Pq},\mathbf{q},\Omega) \approx 0 \ . \qquad (58)$$

On the other hand, for the rough fluctuations ($l > l_c$), the self-energy part dominates, and we have a contribution which is of the order $\Omega E_g^{-(\eta-1)/(\eta+1)}$ ($1 < \eta \leq 2$) or $\Omega|\ln E_g|$ ($\eta = 1$). Ignoring the $\Omega$ term compared to $\Omega E_g^{-(\eta-1)/(\eta+1)}$ ($1 < \eta \leq 2$) or $\Omega|\ln E_g|$ ($\eta = 1$) and multiplying the factor $E_g^{(\eta-1)/(\eta+1)}$ ($1 < \eta \leq 2$) or $1/|\ln E_g|$ ($\eta = 1$) on both sides of the equation, we obtain

$$[\Omega - v_F^* q \cos\theta_{\rm Pq}]\, u(\theta_{\rm Pq},\mathbf{q},\Omega) - i\Delta\omega_c^* \frac{\partial}{\partial\theta_{\rm Pq}} V(\theta_{\rm pq},\mathbf{q},\Omega) = \text{collision integral} \ , \qquad (59)$$

where $v_F^* = k_F/m^*$, $\Delta\omega_c^* = \Delta B/m^*$, and $m^*/m \propto E_g^{-(\eta-1)/(\eta+1)}$ ($1 < \eta \leq 2$) or $|\ln E_g|$ ($\eta = 1$).

Let us consider two different types of wave packets created along the Fermi surface. Note that the revolution of these wave packets is governed by two different frequencies $\Delta\omega_c$ and $\Delta\omega_c^*$. The frequency of the revolution of the broad wave packet [see Fig. 2(a)] is given by $\Delta\omega_c$, because it consists mainly of small angular momentum modes. On the other hand, if we ignore the collision integral in the QBE, the frequency of the revolution of the narrow wave packet [see Fig. 2(b)] is given by $\Delta\omega_c^*$ because it mainly contains the large angular momentum modes. The energy gap of the system can be obtained by quantizing the motion of revolution and taking the smallest quantized frequency as the energy gap of the system. Therefore, the energy gap of the system is given by $E_g = \Delta\omega_c^* \propto \Delta B E_g^{(\eta-1)/(\eta+1)}$ ($1 < \eta \leq 2$) or $\Delta B/|\ln E_g|$ ($\eta = 1$). Solving this self-consistent equation for $E_g$, we obtain

$$E_g \propto \begin{cases} |\Delta B|^{(1+\eta)/2} & \text{if } 1 < \eta \leq 2 \\ \dfrac{|\Delta B|}{|\ln\Delta B|} & \text{if } \eta = 1 \ . \end{cases} \qquad (60)$$

This result is the same as the self-consistent treatment of HLR (Ref. 6) and also the perturbative evaluation of the activation energy gap in the finite-temperature compressibility.[14] We see that the divergent effective mass shows up in the energy gap $E_g$. More detailed discussions of the low-lying excitations described by the QBE can be found in Sec. VI.

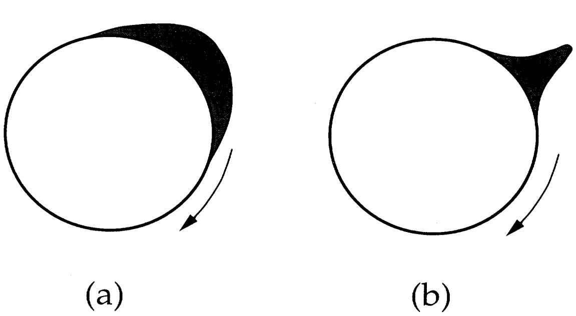


FIG. 2. A broad wave packet (a) and a narrow wave packet (b) (given by the shaded region) created in the momentum space. The circle is the schematic representation of the Fermi surface, which is actually not so well defined, and the arrow represents the direction of motion of the wave packet.

## VI. COLLECTIVE EXCITATIONS

Let us first study the collective excitations of the system with $\Delta B = 0$ by looking at the QBE given by Eq. (52). We ignore the collision integral for the time being, and discuss its influence below. In the absence of the collision integral, Eq. (52) can be considered as the Schrödinger equation of an equivalent tight-binding model in the angular momentum space. It is convenient to rewrite Eq. (52) as

$$\Omega v_l = \frac{v_F q}{2}\left[\frac{v_{l+1}}{\sqrt{g(l)g(l+1)}} + \frac{v_{l-1}}{\sqrt{g(l)g(l-1)}}\right] , \qquad v_l = \sqrt{g(l)}u_l \ , \qquad (61)$$

where

$$g(l,\Omega) = 1 + N(0)\int \frac{d\theta}{2\pi} F_{\rm Landau}(\theta)[1 - \cos(l\theta)] \ . \qquad (62)$$

Equation (61) describes a particle hopping in a one-dimensional (1D) lattice with a spatial-dependent hopping amplitude $t_l \approx v_F q/2g(l)$. Note that $g(l)$ is of the order one for $l < l_c$ and becomes much larger, $g(l) \propto \Omega^{-(\eta-1)/(\eta+1)}$, when $l > l_c$. Due to this type of spatial-dependent hopping amplitude, the eigenspectrum of Eq. (61) consists of two parts. That is, there is a continuous spectrum near the center of the band and a discrete spectrum in the tail of the band. The discrete spectrum appears above and below the continuous spectrum (see Fig. 3). The boundary between these two different spectra is determined from $\Omega = 2t_{l\to\infty} \propto v_F q \Omega^{(\eta-1)/(\eta+1)}$, which self-consistently generates a singular dispersion relation $\Omega(\theta) \propto q^{(1+\eta)/2}$ ($1 < \eta \leq 2$) or $\Omega(\theta) \propto q/|\ln q|$ ($\eta = 1$). Also, the tail of the band ends at $\Omega(\theta) = 2t_1 \sim v_F q$.

One can map this energy spectrum to the diagram for the excitations in the usual $\Omega - q$ plane, which is given by

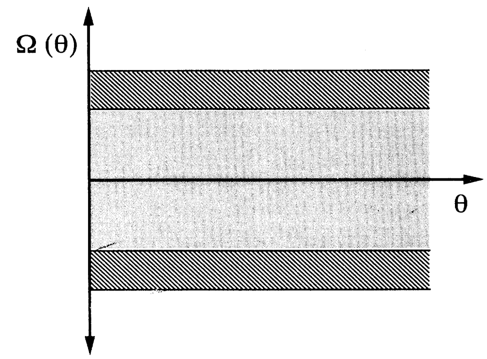


FIG. 3. The energy band $\Omega(\theta)$ of the tight-binding model given by Eq. (61) as a function of $\theta$. The shaded region around the center of the band corresponds to the continuum states, and the hatched region in the tails of the band corresponds to the bound states.

Fig. 4. Note that the continuum states ($l > l_c$) can be mapped to the particle-hole continuum which exist below $\Omega \propto q^{(1+\eta)/2}$ $(1<\eta\le 2)$ or $\Omega \propto q/|\ln q|$ $(\eta=1)$. On the other hand, the bound states (the discrete spectrum) ($l<l_c$) can be mapped to the collective modes which exist between $\Omega \propto q^{(1+\eta)/2}$ $(1<\eta\le 2)$, $\Omega \propto q/|\ln q|$, $(\eta=1)$ and $\Omega \sim v_F q$. However, the distinction between these two different elementary excitations is obscured by the presence of the collision integral, which provides the lifetime for the excitations. In particular, since $g(l,\Omega)$ does not provide a sharp boundary between $l>l_c$ and $l<l_c$, one expects a crossover from particle-hole excitations to the collective modes even in the absence of the collision integral.

Now let us consider the case of $\Delta B \neq 0$ (i.e., away from

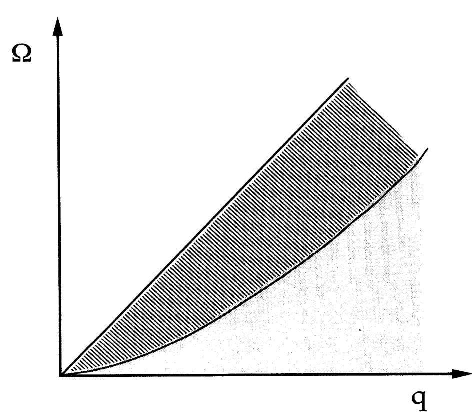


FIG. 4. The elementary excitations in $\Omega - q$ space in the absence of the collision integral. The shaded region corresponds to the particle-hole continuum, and the hatched region corresponds to the collective modes. The boundary is given by the singular dispersion relation $\Omega \propto q^{(1+\eta)/2}$ for $1<\eta\le 2$ and $\Omega \propto q/|\ln q|$ for $\eta=1$.

$\nu=\frac{1}{2}$ state). In this case, Eq. (61) becomes [see also Eq. (57)]

$$\Omega v_l = \frac{l\Delta\omega_c}{g(l)} v_l + \frac{v_F q}{2}\left[\frac{v_{l+1}}{\sqrt{g(l)g(l+1)}} + \frac{v_{l-1}}{\sqrt{g(l)g(l-1)}}\right] . \quad (63)$$

When $g(l)=1$, one can write the solution of Eq. (63) [or Eq. (57)] as

$$u(\theta_{\mathbf{Pq}},\mathbf{q},t) \propto e^{in\theta_{\mathbf{Pq}} - i\Omega t} e^{-i(v_F q/\Delta\omega_c)\sin\theta_{\mathbf{Pq}}} ,$$

with $\Omega = n\Delta\omega_c$. Thus we recover the well-known spectrum of degenerate Landau levels for free fermions.

When $g(l)\neq$const, it is difficult to calculate the spectrum of Eq. (63). However, using $g(l)=g(-l)$, we can show that the spectrum of Eq. (63) is symmetric about $\Omega=0$, and $\Omega=0$ is always an eigenvalue of Eq. (63). Also, for nonzero $\Delta\omega_c$, the spectrum is always discrete.

Note that, for small $q \ll l_c\Delta\omega_c/v_F$, $u(\theta_{\mathbf{Pq}},\mathbf{q},t)$ corresponds to a smooth fluctuation of the Fermi surface. For large $q \gg l_c\Delta\omega_c/v_F$, even the smooth parts of $u(\theta_{\mathbf{Pq}},\mathbf{q},t)$, around $\theta_{\mathbf{Pq}}=\pm\pi/2$, corresponds to a rough fluctuation; hence the whole function $u(\theta_{\mathbf{Pq}},\mathbf{q},t)$ corresponds to a rough fluctuation. Thus we expect that the small-$q$ and large-$q$ modes have very different dynamics. The small-$q$ modes should be controlled by the finite effective mass, and the large-$q$ modes by the divergent mass.

To understand the behavior of the modes in more detail, in the following we present a semiclassical calculation. The main result that we obtain is Eq. (75). The dispersion of the lowest-lying mode (for $q > \Delta\omega_c/v_F$) has a scaling form $\omega_{\text{cyc}}(q) \propto (\Delta\omega_c)^{(1+\eta)/2} f(q/q_c)$ with $f(\infty)=$const and $f(x \ll 1) \propto x^{1-\eta}$. The crossover momentum $q_c \propto \sqrt{\Delta\omega_c}$.

When $qv_F \ll \Delta\omega_c$, the spectrum can be calculated exactly and is given by

$$\Omega = \frac{l\Delta\omega_c}{g(l)} . \quad (64)$$

To obtain the spectrum for $qv_F > \Delta\omega_c$, we will use a semiclassical approach. Note that $(\theta_{\mathbf{Pq}}, l)$ is a canonical coordinate and momentum pair. The classical Hamiltonian that corresponds to the quantum system Eq. (63) can be found to be

$$H(\theta_{\mathbf{Pq}}, l) = \frac{l\Delta\omega_c}{g(l)} + \frac{v_F q}{g(l)}\cos(\theta_{\mathbf{Pq}}) . \quad (65)$$

Assuming $g(l)$ is a slowly varying function of $l$, one arrives at the following simple classical equations of motion:

$$\dot{\theta}_{\mathbf{Pq}} = \frac{\Delta\omega_c}{g(l)} , \quad \dot{l} = \frac{v_F q}{g(l)}\sin(\theta_{\mathbf{Pq}}) . \quad (66)$$

From this equation, one can easily show that

$$l = -\frac{v_F q}{\Delta\omega_c}\cos(\theta_{\mathbf{Pq}}) + l_0 , \quad (67)$$

where $l_0$ is a constant. Note that Eq. (67) with $l_0=0$ is an exact solution for the classical system Eq. (65), which describes a motion with zero energy. Now the first equation in Eq. (66) can be simplified as

$$\dot{\theta}_{\mathbf{Pq}}=\frac{\Delta\omega_c}{g\left[-\frac{v_F q}{\Delta\omega_c}\cos(\theta_{\mathbf{Pq}})+l_0\right]}, \tag{68}$$

which describes a periodic motion. The angular frequency of the periodic motion is given by

$$\omega=\frac{2\pi\Delta\omega_c}{\int_0^{2\pi} g\left[-\frac{v_F q}{\Delta\omega_c}\cos(\theta_{\mathbf{Pq}})+l_0\right]d\theta_{\mathbf{Pq}}}. \tag{69}$$

The above classical frequency $\omega$ has a quantum interpretation. It is the gap between neighboring energy levels, of which the energy is close to the classical energy associated with the classical motion described by Eq. (67). In particular, the cyclotron frequency $\omega_{\text{cyc}}$ is given by the gap between the $\Omega=0$ level and the first $\Omega>0$ level. Therefore

$$\omega_{\text{cyc}}=\frac{2\pi\Delta\omega_c}{\int_0^{2\pi} g\left[-\frac{v_F q}{\Delta\omega_c}\cos(\theta_{\mathbf{Pq}})+1\right]d\theta_{\mathbf{Pq}}}. \tag{70}$$

Here we have chosen $l_0=1$ (instead of $l_0=0$), so that Eq. (70) reproduces the exact result Eq. (64) when $q=0$. Note that $g(l)$ also depends on frequency $\Omega$ and we should set $\Omega=\omega_{\text{cyc}}$ in the function $g(l)$. Thus the cyclotron frequency should be self-consistently determined from Eq. (70).

We would like to remark that when $q \gg \Delta\omega_c/v_F$, the classical frequency in Eq. (69) is a smooth function of $l_0$, and hence a smooth function of the energy. This means that the gap between the neighboring energy levels is also a smooth function of the energy of the levels. The validity of the semiclassical approach requires that the gap between neighboring energy levels be almost a constant in the neighborhood of interested energies. Thus the above behavior of the classical frequency indicates that the semiclassical approach is at least self-consistent.

To analyze the behavior of $\omega_{\text{cyc}}$, we first make an approximation for Eq. (70) as

$$\omega_{\text{cyc}}=\frac{\Delta\omega_c}{g\left[\lambda\frac{v_F q}{\Delta\omega_c}+1\right]}, \tag{71}$$

where $\lambda$ is a nonzero constant between 0 and 1. We see that $\omega_{\text{cyc}}(q)$ has a sharp dependence on $q$ around $q\sim\Delta\omega_c/v_F$. The smaller the $\Delta\omega_c$, the sharper the $q$ dependence. This sharp dependence is not due to the singular gauge interaction, but is merely a consequence of the fact that $g(1)\neq g(2)\neq\cdots$.

As $q$ increases, $g[\lambda(v_F q/\Delta\omega_c)+1]$ becomes larger and larger, thus we expect that $\omega_{\text{cyc}}(q)$ decreases. When $q$ exceeds a crossover value $q_c$, $g[\lambda(v_F q/\Delta\omega_c)+1]$ saturates at a very large value, and $\omega_{\text{cyc}}(q)$ is drastically reduced. This phenomenon is a result of the singular gauge interaction. The crossover momentum $q_c$ is determined from

$$\frac{v_F q_c}{\Delta\omega_c}=l_c=k_F\left[\frac{\chi}{\gamma\omega_{\text{cyc}}(q\to\infty)}\right]^{1/(1+\eta)},$$

and

$$\omega_{\text{cyc}}(q\to\infty)=\frac{\Delta\omega_c}{C(\eta)[\omega_{\text{cyc}}(q\to\infty)]^{(1-\eta)/(1+\eta)}} \quad \text{for } 1<\eta\le 2\,, \tag{72}$$

$$\omega_{\text{cyc}}(q\to\infty)=\frac{\Delta\omega_c}{C(\eta=1)|\ln\omega_{\text{cyc}}(q\to\infty)|} \quad \text{for } \eta=1\,,$$

where

$$C(\eta)=\frac{v_F\cos\left[\frac{\pi}{2}\left[\frac{\eta-1}{\eta+1}\right]\right]}{2\pi(1+\eta)\sin\left[\frac{2\pi}{1+\eta}\right]\gamma^{(\eta-1)/(\eta+1)}\chi^{2/(1+\eta)}}$$

for $1<\eta\le 2$ and $C(\eta=1)=v_F/2\pi^2\chi$ for $\eta=1$. We find

$$\begin{aligned} &q_c=B(\eta)\sqrt{\Delta\omega_c} \quad \text{for } 1<\eta\le 2\,,\\ &q_c=B(\eta=1)\sqrt{\Delta\omega_c|\ln\Delta\omega_c|} \quad \text{for } \eta=1\,, \end{aligned} \tag{73}$$

where $B(\eta)=m(\chi/\gamma)^{1/(1+\eta)}\sqrt{C(\eta)}$. When $q\gg q_c$, the cyclotron frequency saturates at the following values:

$$\begin{aligned} &\omega_{\text{cyc}}(q\to\infty)=(\Delta\omega_c/C(\eta))^{(1+\eta)/2} \quad \text{for } 1<\eta\le 2\,,\\ &\omega_{\text{cyc}}(q\to\infty)=\frac{\Delta\omega_c/C(\eta=1)}{|\ln[\Delta\omega_c/C(\eta=1)]|} \quad \text{for } \eta=1\,. \end{aligned} \tag{74}$$

When $v_F q/\Delta\omega_c\gg 1$, the cyclotron frequency is expected to have the following scaling form:

$$\begin{aligned} &\omega_{\text{cyc}}(q)\propto(\Delta\omega_c)^{(1+\eta)/2}f(q/q_c)\,,\\ &f(\infty)=\text{const.} \quad \text{and } f(x\ll 1)\propto x^{1-\eta}\,, \end{aligned} \tag{75}$$

where $f(\infty)$ is determined from $\omega_{\text{cyc}}(q\to\infty)\propto(\Delta\omega_c)^{(1+\eta)/2}$, and $f(x\ll 1)$ can be obtained from the condition that $\omega_{\text{cyc}}(q)=\Delta\omega_c$ for $q\sim\Delta\omega_c/v_F$. Note that the divergence of $f(x)$ for small $x$ should be cut off when $x\sim\Delta\omega_c/v_F q_c$. As a result, the cyclotron spectrum of the system looks like the one given by Fig. 5.

The smaller gap for $q>q_c$ corresponds to a divergent effective mass $m^*\propto(\Delta\omega_c)^{(1-\eta)/(1+\eta)}$, while the larger gap near $q=0$ can be viewed as a cyclotron frequency derived from a finite effective mass. The thermal activation gap measured through the longitudinal conductance is given by the smaller gap at large wave vectors $q>q_c$. However, the cyclotron frequency measured through the cyclotron resonance for the uniform electric field should be given by the larger gap.

The above discussion of the cyclotron frequency is for

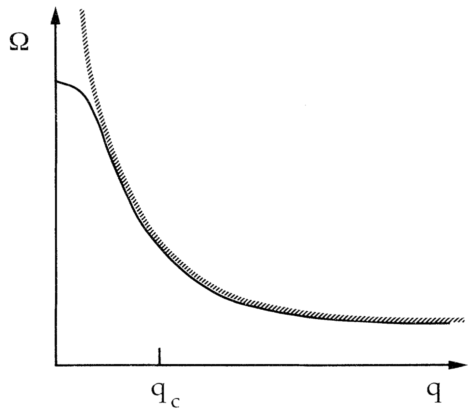


FIG. 5. The lowest excitation spectrum of the composite fermion system in the presence of the finite effective magnetic field $\Delta B$ as a function of the wave vector $q$ (solid line). The dashed line is the scaling curve described in the text. For $q \gg q_c$, the excitation gap becomes smaller and is proportional to $|\Delta B|^{(1+\eta)/2}$ for $1<\eta\leq 2$ and $|\Delta B|/|\ln\Delta B|$ for $\eta=1$. $q_c \propto \sqrt{|\Delta B|}$ for $1<\eta\leq 2$ and $q_c \propto \sqrt{|\Delta B||\ln\Delta B|}$ for $\eta=1$.

the toy model, where only the transverse gauge-field fluctuations are included. One may wonder whether the same picture also applies to the real $\nu=\frac{1}{2}$ state. In the real $\nu=\frac{1}{2}$ state, the lowest-lying plasma modes correspond to the intra-Landau-level excitations, for which the energy is much less than the inter-Landau-level gap $\omega_c$. In the $q\to 0$ limit, such modes decouple from the center-of-mass motion. This means that the $u_{\pm 1}$ components (which correspond to the dipolar distortions of the Fermi surface) of the eigenmodes must disappear in the $q\to 0$ limit as far as the lowest-lying modes (intra-Landau-level modes) are concerned. The mode that contains $u_{\pm 1}$ components should have a large inter-Landau-level gap in the $q\to 0$ limit in order to satisfy Kohn's theorem. Examining our solution for the eigenmodes in the $q\to 0$ limit, we find that the lowest eigenmodes are given by $u_l \propto \delta_{\pm 1,l}$. Therefore, according to the above consideration, we cannot identify the lowest-lying modes in the toy model with the lowest-lying intra-Landau-level plasma modes in the real model. However, this problem can be fixed following the procedure introduced in Ref. 30. That is, we may introduce an additional nondivergent Landau-Fermi-liquid parameter $\Delta F_1$ which modifies only the value of $g(\pm 1)$. We may fine tune the value of $\Delta F_1$ such that the $l=\pm 1$ modes in Eq. (64) will have the large inter-Landau-level gap $\Omega=\Delta\omega_c/g(\pm 1)=\omega_c$. In this case the $l=\pm 2$ modes become the lowest-lying modes in the $q\to 0$ limit. Such modes correspond to the quadrapolar distortions of the Fermi surface, and decouple from the center-of-mass motion. The above correction only affects the energy of the lowest-lying modes for the small momenta, $q<\Delta\omega_c/v_F$. With this type of correction, our results for the toy model essentially applies to the $\nu=\frac{1}{2}$ state. The only change is that the lowest-lying modes at small momenta, $q \ll \Delta\omega_c/v_F$, is given by the $l=\pm 2$ modes instead of the $l=\pm 1$ modes. This is because, as $q$ decreases below a value of order $\Delta\omega_c/v_F$, the $l=\pm 1$ modes start to have a higher energy than that of the $l=\pm 2$ modes, and the lowest-lying modes cross over to the $l=\pm 2$ modes.

In the absence of the singular gauge interaction, according to the picture developed in Ref. 30, one expects that the intra-Landau-level plasma mode near $\nu=\frac{1}{2}$ has a gap $2\Delta\omega_c$ for $q<\Delta\omega_c/v_F$. The gap is expected to be reduced by the factor 2 when $q>\Delta\omega_c/v_F$. In the presence of the singular gauge interaction, we find that the plasma mode has a gap of order $2\Delta\omega_c$ (since $g(\pm 2)\neq 1$) for $q<\Delta\omega_c/v_F$. However, the gap for the large momenta can be much less than $\Delta\omega_c$. Observing this drastic gap reduction will confirm the presence of the singular gauge interaction.

In the above discussion, we have ignored the effects of the collision term. The role of the collision integral is simply to provide the lifetime effects on the collective excitations. However, due to the energy conservation, only the collective modes with energy greater than $2\omega_{\rm cyc}(q_{\rm min})$ will have a finite lifetime. Here $\omega_{\rm cyc}(q_{\rm min})$ is the minimum energy gap of the lowest-lying plasma mode, and $q_{\rm min}$ is the momentum where the energy takes the minimum value. For large $q$, the modes above $2\omega_{\rm cyc}(q_{\rm min})$ may have a short lifetime such that the modes are not well defined.

## VII. SUMMARY, CONCLUSION, AND IMPLICATIONS TO EXPERIMENTS

In this section, we summarize the results and provide a unified picture of the composite fermions interacting with a gauge field. In this paper, we construct a general framework, which is the QBE of the system, to understand the previously known theoretical[6,12–16] and experimental[1–3,8–10] results. Since there is no well-defined Landau quasiparticle, we cannot use the usual formulation of the QBE, so we used an alternative formulation which was used by Prange and Kadanoff[25] for the electron-phonon problem. We used the nonequilibrium Green's-function technique[25–28] to derive the QBE of the generalized distribution function $\delta f(\theta_{\mathbf{pq}},\omega;\mathbf{q},\Omega)$ for $\Delta B=0$, and $\delta f(\theta_{\mathbf{Pq}},\omega;\mathbf{q},\Omega)$ ($\mathbf{P}=\mathbf{p}-\Delta\mathbf{A}$) for $\Delta B\neq 0$. From this equation, we also derived the QBE for the generalized Fermi-surface displacement $u(\theta_{\mathbf{pq}},\mathbf{q},\Omega)$ ($\Delta B=0$) or $u(\theta_{\mathbf{Pq}},\mathbf{q},\Omega)$ ($\Delta B\neq 0$) which corresponds to the local variation of the chemical potential in momentum space.

For $\Delta B=0$, the QBE consists of three parts: the self-energy part, the generalized Landau-interaction part, and the collision integral. The Landau-interaction function $F_{\rm Landau}(\theta)$ can be taken as $F_{\rm Landau}(\theta)\propto 1/|\theta|^\eta$ for $\theta>\theta_c\propto\Omega^{1/(1+\eta)}$ and $1/|\theta_c|^\eta$ for $\theta<\theta_c$. For the smooth fluctuations of the generalized Fermi-surface displacement ($l<l_c\approx 1/\theta_c\propto\Omega^{-[1/(1+\eta)]}$), where $l$ (the angular momentum in momentum space) is the conjugate variable of the angle $\theta$, there is a small-angle-(forward)-scattering cancellation between the self-energy part and the Landau-interaction part. Both the self-energy part and the Landau-interaction part are of the order $\Omega^{2/(1+\eta)}$

$(1<\eta\leq 2)$ or $\Omega\ln\Omega$ $(\eta=1)$. After cancellation, the combination of these contributions becomes of the order $\Omega^{4/(1+\eta)}$. There is also a similar cancellation in the collision integral, so that the transport scattering rate becomes of the order $\Omega^{4/(1+\eta)}$. As a result, the smooth fluctuations do not show the anomalous behavior expected from the singular self-energy correction. On the other hand, for rough fluctuations $(l>l_c)$, the Landau-interaction part becomes very small, and the self-energy part, which is proportional to $\Omega^{2/(1+\eta)}$, dominates. Also, the collision integral becomes of the order $\Omega^{2/(1+\eta)}$. Therefore, the rough fluctuations show an anomalous behavior of the self-energy correction, and suggest that the effective mass shows a divergent behavior $m^*\propto\Omega^{-(\eta-1)/(\eta+1)}$ for $1<\eta\leq 2$ and $m^*\propto|\ln\Omega|$ for $\eta=1$.

From these results, one can understand the density-density and current-current correlation functions calculated in the perturbation theory,[12,13] which show no anomalous behavior in the long-wavelength and low-frequency limits. Using the QBE, one can evaluate these correlation functions by taking the angular average of the density or current disturbance due to the external potential and calculating the linear response. Thus, in these correlation functions, the small angular momentum modes are dominating, so that the results do not show any singular behavior. Note that the cancellation which exists in the collision integral implies that the transport lifetime is sufficiently long to explain the long mean free path of the composite fermions in the recent magnetic focusing experiment.[10] For the $2k_F$ response functions, there is no corresponding cancellation between the self-energy part and the Landau-interaction part, so that it shows a singular behavior.[13]

The QBE in the presence of the small effective magnetic field $\Delta B$ was used to understand the energy gap $E_g$ of the system. As in the case of $\Delta B=0$, there can be two different behaviors of the generalized Fermi surface displacement. For the smooth fluctuations $(l<l_c\propto E_g^{-1/(1+\eta)})$, the frequency of revolution of the wave packet is given by $\Delta\omega_c=\Delta B/m$; i.e., there is no anomalous behavior after the cancellation between the self-energy and Landau-interaction parts. For rough fluctuations, the self-energy part dominates, and the frequency of revolution of the wave packet is renormalized as $\Delta\omega_c^*\propto\Delta\omega_c E_g^{-(\eta-1)/(\eta+1)}$. The energy gap can be obtained by quantizing the motion of the wave packet and taking the lowest quantized frequency which is nothing but $\Delta\omega_c^*$. Solving the self-consistent equation $E_g=\Delta\omega_c^*$, we obtain $E_g\propto|\Delta B|^{(1+\eta)/2}$ for $1<\eta\leq 2$ and $E_g\propto|\Delta B|/|\ln\Delta B|$ for $\eta=1$. These are consistent with the previous results.[6,14,15]

The excitations of the system were studied from the QBE of the generalized Fermi-surface displacement. For $\Delta B=0$, in the absence of the collision integral, there are two types of excitations which can be described most easily in the $\Omega-q$ plane. There are particle-hole excitations which exist below an edge $\Omega\propto q^{(1+\eta)/2}$ $(1<\eta\leq 2)$ or $\Omega\propto q/|\ln q|$ $(\eta=1)$. There are also collective modes which exist between $\Omega\propto q^{(1+\eta)/2}$ $(1<\eta\leq 2)$, $\Omega\propto q/|\ln q|$ $(\eta=1)$, and $\Omega\sim v_F q$. However, the distinction between these two different elementary excitations is obscured by the presence of the collision integral, which provides the lifetime of the excitations. In the case of $\Delta B\neq 0$, the QBE in the presence of the finite $\Delta B$ is again used to understand the low-lying plasma spectrum of the system as a function of $q$. For $q<q_c$, where $q_c\propto\sqrt{|\Delta B|}$ for $1<\eta\leq 2$ and $q_c\propto\sqrt{|\Delta B|\ln|\Delta B|}$ for $\eta=1$, the plasma mode corresponds to a smooth fluctuation of the Fermi surface, and the excitation gap is given by $\Delta\omega_c\sim\Delta B/m$. On the other hand, for $q>q_c$, the plasma mode corresponds to a rough fluctuation of the Fermi surface. As a consequence, the excitation gap becomes much smaller and proportional to $|\Delta B|^{(1+\eta)/2}$ for $1<\eta\leq 2$ and $|\Delta B|/|\ln\Delta B|$ for $\eta=1$. Thus the lowest excitation spectrum of the system looks like the one given by Fig. 5, which is consistent with the previous numerical calculations.[30]

We would like to remark that, in our derivation of the linearized QBE [Eq. (50)], we have ignored the dependence of the gauge-field propagator $D_{11}$ on the fermion Green's function [i.e., we have set $\delta D_{11}=(\partial D_{11}/\partial G)\delta G=0$] [see Eq. (42)]. However, in our model the gauge-field propagator is calculated through the RPA, and depends on the fermion Green's function. Therefore, the QBE contains additional terms that come from $\delta D_{11}$. In the following, we would like to argue that these additional terms do not affect the results we obtained in previous sections. Let us first look at the smooth modes of the generalized Fermi-surface fluctuation. One can argue that the additional corrections are still small, and the smooth modes retain a linear dispersion. To see this, let us imagine calculating the density-density or current-current correlation functions from the QBE. The QBE in Eq. (50) will generate two-particle correlation functions corresponding to diagrams in Figs. 1(a)–1(e) in Ref. 12. The additional terms coming from $\delta D_{11}$ will generate diagrams of Figs. 1(f) and 1(g) in Ref. 12. The cancellation between the self-energy and the vertex diagrams in Figs. 1(a)–1(e) for small $(\omega,\mathbf{q})$ is directly related to the cancellation of the self-energy and the Fermi-liquid-function terms in the QBE for smooth modes. As discussed in Ref. 12, a similar cancellation also happens for the diagrams in Figs. 1(f) and 1(g), and as a consequence the two-particle correlation functions have a Fermi-liquid form for small $(\omega,\mathbf{q})$. Therefore, contributions from $\delta D_{11}$ will not modify the dynamics of the smooth modes qualitatively. For the rough modes, we believe that there is again no cancellation, and the Fermi-liquid picture breaks down. The reason is that, as shown in Ref. 12, the diagrams of Figs. 1(f) and 1(g) have the same singular frequency dependence as those of self-energy contributions before the cancellation, which occurs only for small $(\omega,\mathbf{q})$.

Applying the picture developed in this paper for the $\nu=\frac{1}{2}$ metallic state to the magnetic focusing experiment of Ref. 10, we find that the observed oscillations should not be interpreted as effects due to the focusing of the quasiparticles. This is because the inelastic mean free path $L_q=v_F^*\tau$ and the lifetime $\tau\sim 1/T$ of the quasiparticle is quite short. Here $v_F^*$ is the renormalized Fermi velocity of the quasiparticle. For the Coulomb interaction, we find

$$L_q \sim \frac{\sqrt{4\pi n}}{mT\ln(E_F/T)} .$$

Here $n$ is the density of the electron, $T$ the temperature, $m$ the bare mass of the composite fermion, and $E_F = k_F^2/2m = 2\pi n/m$. Taking $n = 10^{11}$ cm$^{-2}$, and $m$ to be the electron mass in the vacuum (see Refs. 2, 3, and 10), we have

$$L_q \sim 0.26 \frac{100 \text{ mK}}{T} \ \mu\text{m} .$$

At $T = 35$ mK, $L_q \sim 0.7$ $\mu$m, which is much less than the length of the semicircular path, 6 $\mu$m, which connects the two slits. Therefore, the oscillations observed in Ref. 10 cannot be explained by the focusing of the quasiparticles, which have a divergent effective mass and a short lifetime.

There is another way to explain the observed oscillations in Ref. 10. We can inject a net current into one slit, which causes a dipolar distortion of the local Fermi surface near the slit. The current and the associated dipolar distortion propagate in space according to the QBE, and are bent by the effective magnetic field $\Delta B$. This causes the oscillation in the current received by the other slit. According to this picture, the oscillations observed in Ref. 10 are caused by the smooth fluctuations of the Fermi surface whose dynamics is identical to those of a Fermi liquid with a *finite* effective mass. Thus the oscillations in the magnetic focusing experiments behave as if they are caused by quasiparticles with a finite effective mass and a long lifetime. The relaxation time for the current distribution is given by $\tau_j \sim E_F/T^2 \ln(E_F/T)$ for the Coulomb interaction. This leads to a diffusion length (caused by the gauge fluctuations) $L_j = v_F \tau_j$, where $v_F$ is the bare Fermi velocity of the composite fermions. We find

$$L_j \sim 14 \left[ \frac{100 \text{ mK}}{T} \right]^2 \mu\text{m} .$$

The real diffusion length should be shorter than the above value due to other possible scattering mechanisms. Thus we expect that the crossover temperature, above which the oscillations disappear, should be lower than 150 mK. In the experiment,[10] no oscillations were observed above 100 mK. Another important consequence of our picture is that, if a time-of-flight measurement can be performed by pulsing the incoming current, the time is given by the bare velocity $v_F$ and *not* the quasiparticle velocity $v_F^*$.

Finally, we remark on the surface acoustic wave experiment. The condition for the resonance between the cyclotron radius and sound wave length is given by $\omega_{\text{cyc}} \gg \omega_s$, where $\omega_{\text{cyc}}$ is the cyclotron frequency and $\omega_s$ is the sound wave frequency. This is because we can regard the sound wave as a standing wave only when $\omega_{\text{cyc}} \gg \omega_s$. Let us imagine that we are changing $\omega_s$ such that $\omega_s \approx \Delta\omega_c^*$. If we use the quasiparticle picture to explain the above resonance, then the cyclotron frequency $\omega_{\text{cyc}}$ is determined by the divergent effective mass, and $\omega_{\text{cyc}}$ should be comparable to $\Delta\omega_c^*$. Therefore, there should not be any resonance because $\omega_{\text{cyc}} \approx \omega_s$ in this case. However, in reality, the resonance is governed by the smooth fluctuation of the Fermi surface, so that $\omega_{\text{cyc}} \approx \Delta\omega_c$ is a cyclotron frequency determined by the finite bare mass of the composite fermion. As a result, one should still see the resonance because $\omega_{\text{cyc}} \gg \omega_s \approx \Delta\omega_c^*$. Therefore, one can expect that there should be still resonance effects even when the phonon energy exceeds the energy gap determined from the Shubnikov–de Haas oscillations. The bottom line is that the cyclotron frequency measured in acoustic wave experiments can be much larger than the energy gap measured in transport experiments. In a recent experiment by Willet, West, and Pfeiffer,[32] resonance was observed when $\omega_s$ is larger than the energy gap of the system determined by the large effective mass obtained from the Shubnikov–de Haas oscillations.[3] The authors claimed that this is an apparent contradiction between the surface acoustic wave experiment and the Shubnikov–de Haas oscillations. We would like to point out that the cyclotron frequency (for small $q$) is determined by the bare mass (in a crude estimation,[6] the bare mass is about $\frac{1}{3}$ of the electron mass in vacuum). On the other hand, the mass obtained from Shubnikov–de Haas oscillations or from the activation gap in transport measurements is in principle a different mass, which in practice turns out to be of order of the electron mass in vacuum even away from $\nu = \frac{1}{2}$. Even though we do not quantitatively understand the mass difference, there is in principle no contradiction. The surface acoustic experiment is in fact an excellent way of measuring the bare mass.

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## APPENDIX

In this appendix, we consider the QBE at finite temperatures. Recall that $\text{Im}\Sigma^R(\mathbf{p},\omega)$ at equilibrium diverges at finite temperatures, which has no cutoff.[11] In this case, it is clear from Eq. (15) that $G_0^<(\mathbf{p},\omega) = if_0(\omega)A(\mathbf{p},\omega)$ is not well defined. Thus it is also difficult to define $G^<(\mathbf{p},\omega;\mathbf{r},t)$ for the nonequilibrium case. Since the divergent contribution to the self-energy comes from the gauge-field fluctuations with $\nu < T$, where $\nu$ is the energy transfer by the gauge field,[11] it is worthwhile to separate the gauge-field fluctuations into two parts, i.e., $\mathbf{a}(\mathbf{q},\nu) \equiv \mathbf{a}_-(\mathbf{q},\nu)$ for $\nu < T$ and $\mathbf{a}(\mathbf{q},\nu) \equiv \mathbf{a}_+(\mathbf{q},\nu)$ for $\nu > T$, and examine the effects of $\mathbf{a}_+$ and $\mathbf{a}_-$ separately.

The classical fluctuation $\mathbf{a}_-$ of the gauge field can be regarded as a vector potential which corresponds to static but spatially varying magnetic field $\mathbf{b}_- = \nabla \times \mathbf{a}_-$. For a given random magnetic field $\mathbf{b}_-(\mathbf{r})$, and in a fixed gauge, the fluctuation of the gauge potential $\mathbf{a}_-$ can be very large. The gauge potential can have huge differences from one point to another, as long as the two points are well separated. We know that locally the center of the Fermi surface is at the momentum $\mathbf{p}-\mathbf{a}_-(\mathbf{r})$ around the point $\mathbf{r}$ in space. The huge fluctuation of $\mathbf{a}_-$ indicates that the local Fermi surfaces at different points in space may appear in very different regions in the momentum space. This is the reason why the one-particle Green's function in the *momentum space* is not well defined. This also suggests that the fermion distribution in the momentum space, $f(\mathbf{p},\omega)$, may be ill defined. Note that the local Fermi surface can be determined in terms of the velocity of the fermions [i.e., the states with $(m/2)\mathbf{v}^2 = (1/2m)(\mathbf{p}-\mathbf{a}_-)^2 < E_F$ are filled)], and the velocity is a gauge-invariant physical quantity. This suggests that it is more reasonable to study the fermion distribution in the physical *velocity space*. The above discussion leads us to consider the one-particle Green's function $\tilde{G}(\mathbf{P}_-,\omega;\mathbf{r},t)$ as a function of a variable $\mathbf{P}_- = m\mathbf{v} = \mathbf{p}-\mathbf{a}_-$. Note that this transformation is reminiscent of the procedure we used in the case of the finite effective magnetic field (see Sec. V). We may follow the similar line of deviation to obtain the QBE in the random magnetic field. Since we effectively separate out $\mathbf{a}_-$ fluctuations, the self-energy, which appears in the equation of motion given by Eq. (24), should contain only $\mathbf{a}_+$ fluctuations. Therefore, the equation of motion for $\delta G^<(\mathbf{P}_-,\omega;\mathbf{r},t)$ is given by the Fourier transform of Eq. (30) with the following replacement. In the first place, the variable $\mathbf{p}$ should be changed to a variable $\mathbf{P}_- = \mathbf{p}-\mathbf{a}_-$. Second, the self-energy $\tilde{\Sigma}$ should be changed to $\tilde{\Sigma}_+$ which now contains only $\mathbf{a}_+$ fluctuations. Finally, as we can see from the case of the finite effective magnetic field in Sec. V, the following term should be added:

$$\frac{\mathbf{P}_-}{m}\cdot\mathbf{b}_-(\mathbf{r})\times\frac{\partial}{\partial\mathbf{P}_-}\delta G^<(\mathbf{P}_-,\omega;\mathbf{r},t) \ . \tag{A1}$$

Note that the equation of motion contains the term which depends on $\mathbf{b}_-$, but does not contain the terms which depend on $\mathbf{a}_-$ in an explicit way. Since we removed the source of the divergence (non-gauge-invariance with respect to $\mathbf{a}_-$), the Green's function $\tilde{G}(\mathbf{P}_-,\omega;\mathbf{r},t)$ or the corresponding self-energy is now finite for finite $T$ or $\omega$.

Now one can perform the integration $\int d\xi_{\mathbf{P}_-}/2\pi$ of $\delta G^<(\mathbf{P}_-,\omega;\mathbf{r},t)$ safely to define

$$\begin{aligned}\int\frac{d\xi_{\mathbf{P}_-}}{2\pi}[-iG^<(\mathbf{P}_-,\omega;\mathbf{r},t)] &\equiv f(\theta,\omega;\mathbf{r},t) \ ,\\ \int\frac{d\xi_{\mathbf{P}_-}}{2\pi}[iG^>(\mathbf{P}_-,\omega;\mathbf{r},t)] &\equiv 1-f(\theta,\omega;\mathbf{r},t) \ ,\end{aligned} \tag{A2}$$

where $\theta$ is the angle between $\mathbf{P}_-$ and a given direction. For a while, let us ignore the contribution from the term that depends on $\mathbf{b}_-(\mathbf{r})$ in the equation of motion for $\delta f(\theta,\omega;\mathbf{r},t)$, which is given by

$$\frac{b_-(\mathbf{r})}{m}\frac{\partial}{\partial\theta}\delta f(\theta,\omega;\mathbf{r},t) \ . \tag{A3}$$

In the absence of this term, the equation of motion of the generalized distribution function $\delta f(\theta,\omega;\mathbf{q},\Omega)$ is given by Eq. (44), with the constraint that the lower cutoff $T$ should be introduced in the frequency integrals, which is due to the fact that only $\mathbf{a}_+$ fluctuations should be included. Using the same procedure we used in Sec. IV, we can construct the equation of motion for the generalized Fermi-surface displacement (in the velocity space) $u(\theta,\mathbf{q},\Omega) = \int(d\omega/2\pi)\delta f(\theta,\omega;\mathbf{q},\omega)$. The corresponding equation is given by Eq. (50) with the change that $\theta_c$ in the definition of the Landau-interaction function $F_{\mathrm{Landau}}(\theta)$ is now given by $\theta_c = (1/k_F)[\gamma\max(\Omega,T)/\chi]^{1/(1+\eta)}$. Therefore, the same arguments for the small and large angular momentum modes can be used to discuss the physical consequences of the QBE, and the change is that the crossover angular momentum is now given by $l_c \approx 1/\theta_c \approx k_F[\gamma\max(\Omega,T)/\chi]^{-1(1+\eta)}$.

Now let us discuss the effect of the term which depends on $b_-(\mathbf{r})$. After integration $\int d\omega/2\pi$ of the QBE for the generalized distribution function $\delta f(\theta,\omega;\mathbf{r},t)$, this term has the following form in the QBE for $u(\theta,\mathbf{r},t)$:

$$\frac{b_-(\mathbf{r})}{m}\frac{\partial}{\partial\theta}u(\theta,\mathbf{r},t) \ . \tag{A4}$$

This term provides the scattering mechanism due to $\mathbf{a}_-$ fluctuations, and generates a dispersion of the angle $\theta$. The transport scattering rate $1/\tau_-$ which is due to $\mathbf{a}_-$ fluctuations can be estimated as follows. In order to examine $b_-$ fluctuations, let us first consider

$$\begin{aligned}\langle b_-(\mathbf{q})b_-(-\mathbf{q})\rangle &= \int_0^T\frac{d\omega}{2\pi}[n(\omega)+1]q^2\mathrm{Im}D_{11}(q,\omega)\\ &\approx \int_0^T\frac{d\omega}{2\pi}\frac{T}{\omega}q^2\frac{q\omega/\gamma}{\omega^2+(\chi q^{1+\eta}/\gamma)^2}\\ &\approx q^3/\gamma \quad \text{for } q \lesssim q_0 \ ,\end{aligned} \tag{A5}$$

where $q_0 = (\gamma T/\chi)^{1/(1+\eta)}$. Therefore, the typical length scale of $b_-(\mathbf{r})$ fluctuations is given by $l_0 = 1/q_0$. The typical value of $b_-(\mathbf{r})$ over the length scale $l_0$ can be estimated from $\langle b_-(\mathbf{r})b_-(\mathbf{r}')\rangle \approx 1/(\gamma l_0^5)$ for $|\mathbf{r}-\mathbf{r}'| \lesssim l_0$, so that typically $b_- \approx 1/\sqrt{\gamma l_0^5}$. The dispersion of the angle $\Delta\theta$ after the fermion travels over the length $l_0$ can be estimated as $\Delta\theta = (b_-/m)\Delta t \approx 1/(\sqrt{\gamma l_0^5}m)(l_0/v_F) \approx 1/(k_F l_0)^{3/2}$. Let $l_M = n l_0$ be the mean free path which is defined by the length scale after which the total dispersion of the angle becomes of order 1. The number $n$ can be estimated by requiring that the total dispersion of the angle $\sqrt{n}\,\Delta\theta \approx \sqrt{n}/(k_F l_0)^{3/2}$ becomes of order 1, so that $n \approx (k_F l_0)^3$. Thus $l_M \approx k_F^3 l_0^4$. From $l_M = v_F\tau_-$, the scattering rate due to $\mathbf{a}_-$ fluctuations can be estimated as $1/\tau_- \propto T^{4/(1+\eta)}$.

Note that $1/\tau_- \propto T^{4/(1+\eta)}$ is the same order as that of the scattering rate due to $\mathbf{a}_+$ fluctuations in the case of the small angular momentum modes ($l < l_c$). For $l < l_c$, the contribution from the imaginary part of the self-energy $\mathrm{Im}\Sigma^R \propto T^{2/(1+\eta)}$ is canceled by the contribution from the Landau-interaction function, so that the resulting scattering rate is proportional to $T^{4/(1+\eta)}$. In the other limit of large angular momentum modes ($l > l_c$), $1/\tau_-$ can be completely ignored. This is because the self-energy contribution dominates. Since $1/\tau_- < T$, and it is at most the same order as the scattering rate due to $\mathbf{a}_+$ fluctuations even in the case of the small angular momentum modes, ignoring this contribution does not affect the general consequences of the QBE, which are discussed in Secs. IV, V, and VI.

Therefore, the QBE for the generalized distribution function at finite temperatures is essentially given by Eq. (44) with the lower cutoff $T$ of the frequency integral in the expression of the contributions from the self-energy and the Landau-interaction function. As a result, the form of the QBE is the same as that of the zero-temperature case, and the only difference is that the crossover angle $\theta_c$ and the crossover angular momentum $l_c$ are now given by $\theta_c \approx (1/k_F)(\gamma \max(\Omega, T)/\chi)^{1/(1+\eta)}$ and $l_c \approx 1/\theta_c \propto [\max(\Omega, T)]^{-1/(1+\eta)}$, respectively.

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*Present address: AT&T Bell Laboratories, Murray Hill, NJ 07974.


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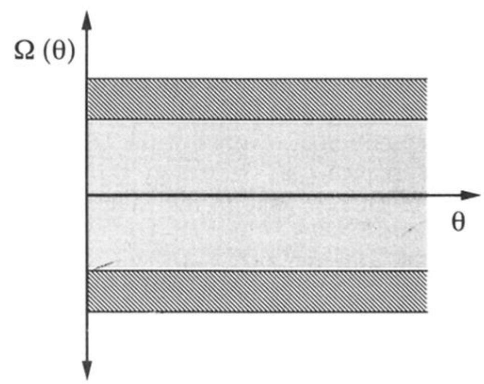


FIG. 3. The energy band $\Omega(\theta)$ of the tight-binding model given by Eq. (61) as a function of $\theta$. The shaded region around the center of the band corresponds to the continuum states, and the hatched region in the tails of the band corresponds to the bound states.

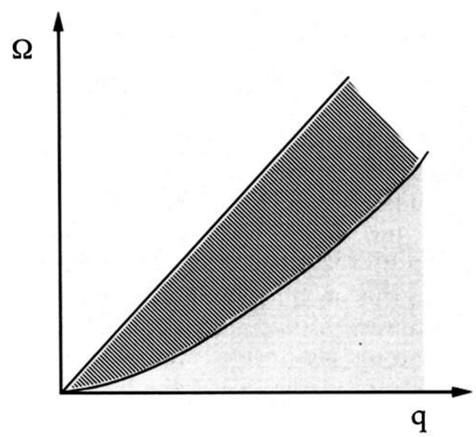


FIG. 4. The elementary excitations in $\Omega - q$ space in the absence of the collision integral. The shaded region corresponds to the particle-hole continuum, and the hatched region corresponds to the collective modes. The boundary is given by the singular dispersion relation $\Omega \propto q^{(1+\eta)/2}$ for $1<\eta\leq 2$ and $\Omega \propto q/|\ln q|$ for $\eta=1$.