Finite-size corrections to the correlation function of the spherical model at $d \geqslant 4$

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Finite-size effects in the correlation function $G(\mathbf{R},T;L) \equiv \langle \Phi(0) \cdot \Phi(\mathbf{R}) \rangle$ of a spherical-model ferromagnet, confined to geometry $L^{d-d'} \times \infty^{d'}$ ($d \geqslant 4$, $d' \leqslant 2$) and subjected to twisted boundary conditions, are analyzed. Focusing attention on the region of first-order phase transition ($T < T_c$), we examine the influence of the *twist* parameter $\boldsymbol{\tau}$ on the function $G(\mathbf{R},T;L)$ in different regimes of the distance parameter $\boldsymbol{\varepsilon}(=\mathbf{R}/L)$. Two complementary methodologies are employed: (i) dimensional regularization involving the limit $d \to 4$ as approached from the mean-field regime at short separations $\varepsilon \ll 1$, and (ii) ζ-function regularization at comparably larger separations $\varepsilon = O(1)$ using generators from more simplified classes of sums so as to properly handle the singular features of the correlation function at $d=4$. Results following from the two methodologies are found to be completely consistent with one another.

I. INTRODUCTION

In a recent paper[1] (hereafter referred to as paper I), we investigated the scaling behavior of the correlation function $G(\mathbf{R},T;L) \equiv \langle \Phi(0) \cdot \Phi(\mathbf{R}) \rangle$ of a finite-sized (FS) spherical model ferromagnet, confined to geometry $L^{d-d'} \times \infty^{d'}$ ($2 < d < 4$, $d' \leqslant 2$) and subjected to twisted boundary conditions (TBC's). These boundary conditions generalize the ones more commonly applied by introducing a continuously varying parameter $\boldsymbol{\tau}(=\mathbf{k}_0 L/2\pi$, $\mathbf{k}_0$ being the ground-state or infrared cutoff wave vector), with components τ_j ($j=1,\ldots,d^*$), of which the periodic boundary conditions (PBC's: $\tau_j = 0$) and the antiperiodic boundary condition (APBC's: $\tau_j = \frac{1}{2}$) are two extremes. Results obtained in paper I clearly revealed the role played by τ in restricting fluctuations and inducing a nonuniformity or *twist* in the order parameter field $\langle \Phi(\mathbf{r}) \rangle$ through a rather intricate analytical study of the universal scaling function for $G(\mathbf{R})$ in terms of the spin-spin separation $\mathbf{R}$, the temperature T, characteristic length L, dimension d, and geometry d'. However, due to the complexity of handling several parameters simultaneously, we kept that study confined to the hyperscaling regime $2 < d < 4$. Recently, one of us (S.A.) investigated the corresponding (local) susceptibility from the correlation function by way of the fluctuation-response theorem.[2]

In this paper we examine the same system for mean field regime $d > 4$ and derive explicit results valid at the upper critical dimension $d=4$ where one observes logarithmic dependence of $G(\mathbf{R})$ on R $(=|\mathbf{R}|)$ when $\tau > 0$. This is especially evident at temperatures below the bulk critical temperature ($T < T_c$) and for spin separations such that the scaled parameter ε $(=R/L)$ is much less than unity. To minimize repetition, we highlight only those results that differ substantially from the ones pertaining to the hyperscaling regime. In Sec. II we state various properties of the scaling function for $G(\mathbf{R})$ at $d > 4$, including an identity (representing a generalized reflection formula) which then allows for analytical continuation of the function from the region of second-order phase transition ($T \approx T_c$) into the region of first-order phase transition ($T < T_c$). In Sec. III we employ a dimensional regularization approach to get $G(\mathbf{R})$ at shorter distances ($\varepsilon \ll 1$) in the regime $4 < d < 6$ and then proceed to the limit $d \to 4$. In Sec. IV a ζ-function regularization procedure is employed at $d=4$ from the beginning at comparatively larger spin separations $[R = O(L)]$, from which the limit $R \ll L$ is extracted and in turn compared with the results of the previous section. We find the two procedures to be completely consistent with one another, suggesting, at least for the spherical model, that they are equivalent.

It may be mentioned here that there are hardly any significant results for very large spin separations ($\varepsilon \gg 1$) beyond what is already reported in paper I, other than the effect the correlation length has on FS corrections to $G(\mathbf{R},T;L)$ in higher dimensions; see Ref. 3. This is because, for $R \gg L$, the correlation function behaves in a manner very similar to the one for a d'-dimensional "bulk" system and its dependence on d is no longer an explicit feature;[1–3] however, interesting symmetries involving universal amplitudes should arise if one were to study $G(\mathbf{R})$ for a singular FS system[4] ($d' > 2$) at very short ($\varepsilon \ll 1$) and very long distances ($\varepsilon \gg 1$). In related investigations, O'Connor and co-workers[5–7] have studied dimensional crossovers for FS Ising and O(n) model systems subjected to PBC's through an extensive use of the so-called "environmentally friendly" L-dependent renormalization group.

II. THE SCALING FUNCTION FOR $G(\mathbf{R},T;L)$ AT $d>4$

The correlation function of the FS spherical model subject to TBC's at $d > 2$ and $d' \leqslant 2$ is given by [see Eq. (27) of paper I]

$$G(\mathbf{R},T;L) \approx \frac{T\pi^{(d-4)/2}}{4J}\left(\frac{a}{L}\right)^{d-2} Q_{\tau}^{\varepsilon}\left(\frac{d-2}{2}\middle| d^*;y\right), \quad (1)$$

with the scaling function

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$$Q_{\tau}^{\varepsilon}(\nu|d^*;y)=\sum_{q_1=-\infty}^{\infty}\cdots\sum_{q_{d^*}=-\infty}^{\infty}\prod_{j=1}^{d^*}\cos(2\pi\tau_j q_j)\times(y/\pi^2 q^*)^{\nu}K_{\nu}(2yq^*),\qquad(2)$$

where

$$q^*=\sqrt{|\mathbf{q}+\boldsymbol{\varepsilon}_{\perp}|^2+\varepsilon_{\parallel}^2}>0,\quad d^*=d-d'.\qquad(3)$$

The sum over q_j appearing in the argument of the modified Bessel function $K_{\nu}(z)$ requires that $\varepsilon=\sqrt{\varepsilon_{\perp}^2+\varepsilon_{\parallel}^2}>0$, where $\boldsymbol{\varepsilon}_{\perp}=(R_1,\ldots,R_{d^*})/L$ has components between 0 and 1 while $\boldsymbol{\varepsilon}_{\parallel}=(R_{d^*+1},\ldots,R_d)/L$ is unrestricted. The *thermogeometric* parameter y is given in terms of the spherical field λ and the spin-spin interaction parameter J by the relation

$$y=(L/2a)\sqrt{\lambda/J-2d},\qquad(4)$$

a being the lattice constant. One can readily see that, in all dimensions greater than 2, standard bulk results are reproduced by the $(\mathbf{q}=\mathbf{0})$ term in (2); the FS effects, therefore, arise from the $(\mathbf{q}\neq\mathbf{0})$ terms of the sum.

In the region of first-order phase transition $(T<T_c)$, where $y^2\simeq-\pi^2\tau^2$, expression (2) is not in a form easily amenable to analysis. Application of the Poisson summation formula (PSF), however, leads to an alternate representation of (2) which is seen to obey a *multiparameter* reflection formula, viz.

$$Q_{\tau}^{\varepsilon}(\nu|d^*;y)=\pi^{-\nu}\sum_{n_1=-\infty}^{\infty}\cdots\sum_{n_{d^*}=-\infty}^{\infty}\prod_{j=1}^{d^*}\cos[2\pi\varepsilon_j(n_j+\tau_j)]\left(\frac{\varepsilon_{\parallel}}{n^*}\right)^{(1/2)d^*-\nu}K_{(1/2)d^*-\nu}(2\pi\varepsilon_{\parallel}n^*),\qquad(5)$$

where

$$n^*=\sqrt{|\mathbf{n}+\boldsymbol{\tau}|^2+y^2/\pi^2}>0.\qquad(6)$$

Equation (5) allows for analytic continuation to negative values of y^2, so long as $y^2>-\pi^2\tau^2$; this includes all of $T<T_c$ plus a part of the core region where $T\approx T_c$. As shown in paper I, the function $Q_{\tau}^{\varepsilon}(\nu|d^*;y)$, for $-\pi^2\tau^2<y^2<0$, may as well be written in terms of the quantity $v=\sqrt{-y^2}$, with the result

$$Q_{\tau}^{\varepsilon}(\nu|d^*;y)=\frac{\pi}{2\sin(\pi\nu)}\sum_{q_1=-\infty}^{\infty}\cdots\sum_{q_{d^*}=-\infty}^{\infty}\prod_{j=1}^{d^*}\cos(2\pi\tau_j q_j)\left(\frac{v}{\pi^2 q^*}\right)^{\nu}J_{-\nu}(2vq^*),\quad \nu\neq0,\pm1,\pm2,\ldots,\qquad(7)$$

where $J_{-\nu}(x)$ is the ordinary Bessel function of *negative* order. We should point out that the scaling function for $G(\mathbf{R})$, in the form (7), fails to apply at dimensions $d=4$, 6, 8, ...; however, it is valid in the interval $4<d<6$ and, as will be shown below, the limit $d\to4$ can be taken successfully. In this regime of dimensionality, the spherical constraint that determines the parameter y takes the form[3,8]

$$\frac{1}{T}-\frac{1}{T_c}\approx\frac{1}{2J}\left(\frac{a}{L}\right)^{d-2}\left[-\frac{(2y)^2}{v_3}+\frac{\pi^{(d-4)/2}}{2}Q_{\tau}\left(\frac{d-2}{2}\middle|d^*;y\right)\right],\qquad(8)$$

where[9,10]

$$v_3=w^{-1}(a/L)^{d-4},\qquad(9a)$$

$$w=\tfrac{1}{4}\int_0^{\infty}[e^{-x}I_0(x)]^d x\,dx,\qquad(9b)$$

$I_0(x)$ being the other modified Bessel function. The variable v_3 is a dangerously irrelevant variable[11] that dramatically affects the scaling behavior of thermodynamic and dynamical observables for a near-critical bulk or FS system above its upper critical dimension. The function $Q_{\boldsymbol{\tau}}$ is given by Eq. (6) of Ref. 3 and has been studied in detail in Ref. 8. It relates intimately to the scaling function $Q_{\boldsymbol{\tau}}^{\varepsilon}$ for G in the limit $\varepsilon\to0$, viz.

$$\lim_{\varepsilon\to0}Q_{\tau}^{\varepsilon}(\nu|d^*;y)=\frac{\Gamma(\nu)}{2(\pi^2\varepsilon^2)^{\nu}}+Q_{\tau}(\nu|d^*;y),\qquad(10)$$

which follows readily from Eq. (2).

III. FINITE-SIZE EFFECTS IN THE REGIME $R\ll L$

For $\varepsilon\ll1$, we employ Eqs. (1) and (7) for $G(\mathbf{R})$ and take the appropriate limit in ε; at the same time, we make use of Eqs. (8)–(10) and obtain precisely the results embodied in Eqs. (18) and (19) of paper I *plus* an additional term

$$-(2T/J)\,w(va/L)^2,\qquad(11)$$

where $v^2=-y^2$ which, for a broad range of temperatures below T_c, is practically equal to $\pi^2\tau^2$. The resulting FS effects in the correlation function $G(\mathbf{R})$ for $4<d<6$ and $d'\leqslant2$, up to order ε^2, are then given by

$$G-G_B\simeq -\frac{2}{d^*}(v\varepsilon_\perp)^2\left(1-\frac{T}{T_c}\right)-\frac{2T}{J}w\left(\frac{va}{L}\right)^2\left\{1-\frac{2v^2\varepsilon_\perp^2}{d^*}\right\}+\frac{T}{8\pi^{d/2}J}\left(\frac{a}{L}\right)^{d-2}v^2\left\{\Gamma\left(\frac{d-4}{2}\right)\varepsilon^{-(d-4)}+\frac{v^2}{2}\Gamma\left(\frac{d-6}{2}\right)\varepsilon^{6-d}\right\}$$
$$+\frac{T}{4Jd^*}\pi^{d/2}\left(\frac{a}{L}\right)^{d-2}Q_\tau\left(\frac{d}{2}\middle|d^*;v\right)[d'\varepsilon_\perp^2-d^*\varepsilon_\parallel^2], \quad (12)$$

where G_B is the (isotropic) bulk correlation function

$$G_B\approx\left(1-\frac{T}{T_c}\right)+\frac{T}{8\pi^{d/2}J}\Gamma\left(\frac{d-2}{2}\right)\left(\frac{a}{R}\right)^{d-2} \quad (R\gg a, T<T_c). \quad (13)$$

The first terms on the right-hand sides of (12) and (13) combine to give the "initial twist" to the long-range order in the system whose magnitude is measured by the square of the spontaneous magnetization, i.e., $M_0^2(T)=\lim_{\mathbf{r}'\to\mathbf{r}}\langle\Phi(\mathbf{r})\rangle\cdot\langle\Phi(\mathbf{r}')\rangle$; see also Eq. (22). For PBC's ($\boldsymbol{\tau}=\mathbf{0}$) there would be no such inhomogeneity in the system. The last term in (12), as in the case $2<d<4$, is regular in v and approaches a definite value as $v\to\pi\tau$. The middle terms, on the other hand, are peculiar to the interval $4<d<6$.

We now proceed to examine the limiting case $d\to 4_+$ in (12) which evokes important connections to various (renormalized) field theory approaches.[5–7,12,13] To begin with, we examine the d dependence of the quantity w appearing in Eqs. (9). Splitting the integration over x into a $\int_0^\Lambda$ term and a $\int_\Lambda^\infty$ term, with $\Lambda\gg 1$, we get

$$w\approx\tfrac{1}{4}\int_0^\Lambda[e^{-x}I_0(x)]^d x\,dx+\tfrac{1}{4}\int_\Lambda^\infty(2\pi x)^{-d/2}x\,dx, \quad (14)$$

where the Bessel function $I_0(x)$, in the second term has been replaced by its asymptotic form for $x\gg 1$. The first term in (14), for large values of Λ, gives asymptotically[9,14]

$$\tfrac{1}{4}\int_0^\Lambda[e^{-x}I_0(x)]^4 x\,dx\approx\frac{1}{(4\pi)^2}\left[\ln\left(\frac{\Lambda}{2}\right)-C_4-\psi(2)\right], \quad (15)$$

where $\psi(2)$ is the digamma function[15] and $C_4=-4.7920\ldots$. The limiting form for (14), as $d\to 4_+$, may then be written as

$$\lim_{d\to 4_+} w=\frac{1}{8\pi^2(d-4)}-\frac{1}{(4\pi)^2}\{C_4+\psi(2)+\ln(4\pi)\}. \quad (16)$$

Employing Eq. (16) and similar limiting forms for the functions $\Gamma(\mu)$ appearing in Eq. (12), we get in the limit $d\to 4$

$$G-G_B\simeq -\frac{2}{d^*}(v\varepsilon_\perp)^2\left(1-\frac{T}{T_c}\right)$$
$$+\frac{T}{4J}\left(\frac{va}{\pi L}\right)^2\frac{(v\varepsilon_\perp)^2}{d^*}\left[\ln\left(\frac{L}{2\pi a}\right)^2+|C_4|-\psi(3)\right]$$
$$+\frac{T}{8J}\left(\frac{va}{\pi L}\right)^2 v^2\varepsilon^2[\ln(\pi\varepsilon)-\tfrac{1}{2}\psi(2)]$$
$$+\frac{T}{4Jd^*}\left(\frac{\pi a}{L}\right)^2 R_\tau(2|d^*;v)[d'\varepsilon_\perp^2-d^*\varepsilon_\parallel^2], \quad (17)$$

where we have also used the limiting formula[8]

$$\lim_{\nu\to l}Q_\tau(\nu|d^*;v)=\frac{1}{2l!}\frac{1}{l-\nu}\left(\frac{v}{\pi}\right)^{2l}+R_\tau(l|d^*;v)$$
$$(l=0,1,2,\ldots). \quad (18)$$

In view of Eqs. (16) and (18), the constraint equation (8) for $d=4$ assumes the form

$$\frac{1}{T}-\frac{1}{T_c}\approx\frac{1}{4J}\left(\frac{a}{L}\right)^2\left\{R_\tau(1|d^*;v)+\frac{1}{2}\frac{v^2}{\pi^2}\left[|C_4|-\psi(2)+\ln\left(\frac{L}{2\pi a}\right)^2\right]\right\}, \quad (19)$$

in perfect agreement with the special case $\boldsymbol{\tau}=\mathbf{0}$.[14]

IV. FINITE-SIZE EFFECTS FOR $R=O(L)$

For a detailed study of the problem at hand, we use representation (5) for the scaling function and follow a procedure which is closely related to the ζ-function regularization of Dowker, Critchley, and Hawking; see, for instance, Elizalde.[16,17] Elizalde[17] and Kirsten[18] have also analyzed a class of generalized multidimensional ζ functions based on the hierarchy of Riemann, Hurwitz, and Epstein and, although our physical motivation differs, the mathematical problem considered here is very similar to theirs. Essentially, we generalize the results of Elizalde and Kirsten[16–19] by including the m-vector *twist* (or phase modulation) parameter $\boldsymbol{\tau}$ together with the *shift* parameter $\boldsymbol{\varepsilon}_\perp$ in Eqs. (2) and (5), which readily reproduce some of the known solutions arising in the limit $\varepsilon_\parallel\to 0$, $\varepsilon_\perp\to 0$, and $\tau\to 0$. It appears that the parameter $\boldsymbol{\tau}$ in our problem corresponds to the toroidal components of the (Abelian) gauge potential in the problem of topological mass generation by a partially compactified spacetime.[19]

It has been shown in paper I that the scaling function $Q^{\varepsilon}_{\tau}(\nu|d^*;y)$ satisfies the recurrence relation

$$\frac{\partial}{\partial \varepsilon_{\parallel}^2} Q^{\varepsilon}_{\tau}(\nu|d^*;y) = -\pi^2 Q^{\varepsilon}_{\tau}(\nu+1|d^*;y), \tag{20}$$

which leads to the power series

$$Q^{\varepsilon}_{\tau}(\nu|d^*;y) = \sum_{l=0}^{\infty} Q^{\varepsilon_\perp}_{\tau}(\nu+l|d^*;y)\,(-\pi^2\varepsilon_{\parallel}^2)^l/l!\,. \tag{21}$$

The coefficients appearing in this expansion are, in fact, phase-modulated ζ functions of the Epstein-Hurwitz type[17] whose asymptotic behavior, as $y^2 \to -\pi^2\tau^2$, is given in Eqs. (B2) of paper I; this, in turn, introduces certain y-independent functions, $L^{\varepsilon_\perp}_{\tau}(\mu|d^*)$ and $N^{\varepsilon_\perp}_{\tau}(\mu|d^*)$, which are also defined in paper I. Through an extensive use of these asymptotic formulas, Eqs. (1) and (19) combine to yield the result

$$G(\mathbf{R},T;L) \approx \left\{ \left(1-\frac{T}{T_c}\right) - \frac{T}{8J}\left(\frac{a\tau}{L}\right)^2 \left[\ln\left(\frac{L}{2\pi a}\right)^2 + |C_4| - \psi(2) \right] \right\} \prod_{j=1}^{d^*} \cos(2\pi\varepsilon_j\tau_j)$$
$$+ \frac{T}{4J}\left(\frac{a}{L}\right)^2 \left\{ L^{\varepsilon_\perp}_{\tau}(1|d^*) - L_{\tau}(1|d^*) \prod_{j=1}^{d^*} \cos(2\pi\varepsilon_j\tau_j) \right\} + \frac{T}{4J}\left(\frac{a}{L}\right)^2 \sum_{l=1}^{\infty} N^{\varepsilon_\perp}_{\tau}(l+1|d^*) \frac{(-\pi^2\varepsilon_{\parallel}^2)^l}{l!}, \tag{22}$$

valid for $\varepsilon = O(1)$ in the region $T < T_c$ at $d=4$. Note that the first term in (22) implies a nonuniformity or *twist* of the order parameter field $\langle \Phi(\mathbf{r}) \rangle$ in the finite directions of the system.

Our next task is to compare Eq. (22) to (17) by taking the limit $\varepsilon \ll 1$ and letting $\tau_j = \tau/\sqrt{d^*}$ for $j=1,\ldots,d^*$, retaining only the leading terms of the phase factor. For the third term in (22), we may write [for comparison, see Eq. (34) of paper I]

$$X^{\varepsilon_\perp}_{\tau} \equiv L^{\varepsilon_\perp}_{\tau}(1|d^*) - L_{\tau}(1|d^*) \prod_{j=1}^{d^*} \cos(2\pi\varepsilon_j\tau_j) = \frac{1}{2\pi^{d^*/2}} \sum_{\substack{r=0\\(r\neq 1)}}^{\infty} \frac{\tau^{2r}}{r!} \left\{ D^{\varepsilon_\perp}_{\tau}(1-r|d^*) - D_{\tau}(1-r|d^*) \prod_{j=1}^{d^*} \cos(2\pi\varepsilon_j\tau_j) \right\}$$
$$+ \frac{\tau^2}{2\pi^{d^*/2}} \left\{ D^{\varepsilon_\perp}_{\tau}(0|d^*) - \overline{D}_{\tau}(d^*) \prod_{j=1}^{d^*} \cos(2\pi\varepsilon_j\tau_j) \right\}, \tag{23}$$

where the quantities $D_{\varepsilon}(\mu|d^*)$ and $\overline{D}_{\tau}(d^*)$ are defined and analyzed in detail in Appendix A of Ref. 8; these quantities are related to $D^{\varepsilon_\perp}_{\tau}(\mu|d^*)$ via the limit $\varepsilon_\perp \to 0$, as shown in Eq. (B7) of paper I. For $\mu = 0, -1, -2, \ldots,$ the singular behavior of these phase-modulated *multidimensional* Hurwitz ζ functions, as $\varepsilon_\perp \to 0$, becomes logarithmic and we find, to $O(\varepsilon_\perp^2)$,

$$D^{\varepsilon_\perp}_{\tau}(-l|d^*) \approx \begin{cases} -\pi^{d^*/2}[\ln(\pi^2\varepsilon_\perp^2)+\gamma] + \overline{D}_{\tau}(d^*) + \left(\frac{2}{d^*}-1\right)\pi^2 D_{\tau}(1|d^*)\varepsilon_\perp^2 & (l=0) \quad \text{(24a)} \\ \pi^{d^*/2}[\ln(\pi^2\varepsilon_\perp^2)+\psi(2)]\pi^2\varepsilon_\perp^2 + D_{\tau}(-1|d^*) - \left[\overline{D}_{\tau}(d^*) + \frac{2\pi^{d^*/2}}{d^*}\right]\pi^2\varepsilon_\perp^2 & (l=1) \quad \text{(24b)} \\ \pi^{d^*/2}\frac{(-1)^{l+1}}{l!}(\pi^2\varepsilon_\perp^2)^l \ln(\pi^2\varepsilon_\perp^2) + D_{\tau}(-l|d^*) - \left[1+\frac{2(l-1)}{d^*}\right]\pi^2 D_{\tau}(1-l|d^*)\varepsilon_\perp^2 & (l=2,3,\ldots). \quad \text{(24c)} \end{cases}$$

Substituting these results into (23), we get, to leading orders in $\varepsilon_\perp$,

$$X^{\varepsilon_\perp}_{\tau} \approx \frac{1}{2\pi^2\varepsilon_\perp^2} - \ln(\pi\varepsilon_\perp)\frac{\tau}{\pi\varepsilon_\perp} J_1(2\pi\tau\varepsilon_\perp) + \frac{\tau^2}{2}\psi(1) + \frac{d'}{d^*}\pi^2 N_{\tau}(2|d^*)\varepsilon_\perp^2 - \left(\psi(2)+\frac{2}{d^*}\right)\left(\frac{\pi\varepsilon_\perp\tau^2}{2}\right)^2. \tag{25}$$

Substituting (25) into (22), with $\varepsilon_{\parallel}=0$ and $\varepsilon_\perp \ll 1$, yields FS effects for $G(\mathbf{R})$ that are identical with the ones given by Eq. (17).

Consider now the case with $\varepsilon_{\parallel} > 0$ to which the last term in (22) is relevant. To calculate this term for $\varepsilon \ll 1$, one has to evaluate the quantities $N^{\varepsilon_\perp}_{\tau}(l|d^*)$ for small $\varepsilon_\perp$; one gets

$$N^{\varepsilon_\perp}_{\tau}(l|d^*) = \frac{1}{2\pi^{d^*/2}} \sum_{r=0}^{\infty} \frac{\tau^{2r}}{r!} D^{\varepsilon_\perp}_{\tau}(l-r|d^*)$$
$$\approx \frac{\tau^{2l}}{2} \sum_{r=0}^{l-1} \frac{\Gamma(l-r)}{r!} (\pi\tau\varepsilon_\perp)^{-2(l-r)} - \frac{\tau^{2l}}{2} \ln(\pi^2\varepsilon_\perp^2) \frac{J_l(2\pi\tau\varepsilon_\perp)}{(\pi\tau\varepsilon_\perp)^l} + \frac{\tau^{2l}}{2}\frac{1}{l!}\psi(1) + N_{\tau}(l|d^*), \tag{26}$$

where the first two terms on the right-hand side of (26) represent all the singular features that arise in the limit $\varepsilon_\perp \to 0$. Equations (25) and (26) now give, for $(\varepsilon_\perp, \varepsilon_{\parallel}) \ll 1$,

$$X_{\tau}^{\varepsilon_\perp}+\sum_{l=1}^{\infty} N_{\tau}^{\varepsilon_\perp}(l+1|d^*)\frac{(-\pi^2\varepsilon_\parallel^2)^l}{l!}\approx\frac{1}{2\pi^2\varepsilon^2}-\ln(\pi\varepsilon)\frac{\tau}{\pi\varepsilon}J_1(2\pi\tau\varepsilon)+\frac{\tau^2}{2}\psi(1)-\frac{\tau^4}{4}\pi^2\psi(1)\varepsilon_\parallel^2-\left[\psi(2)+\frac{2}{d^*}\right]\left(\frac{\pi\varepsilon_\perp\tau^2}{2}\right)^2$$
$$+\pi^2\left[\left(\frac{d'}{d^*}\right)\varepsilon_\perp^2-\varepsilon_\parallel^2\right]N_\tau(2|d^*)\quad(\varepsilon=\sqrt{\varepsilon_\perp^2+\varepsilon_\parallel^2}\ll 1). \tag{27}$$

Putting this result back into (22), with $\varepsilon\ll 1$, reproduces Eq. (17) precisely; however, Eq. (27) now gives (confluent) singularities to *all* order in ε, not just to $O(\varepsilon^2)$. In the derivation that led to Eq. (17), this feature, which implies a well-defined limit when $\tau\to 0$, was not transparent for the higher-order terms in ε^2.

Finally, we consider a fully finite system ($\Omega=L^d$). For $\tau>0$, the last term in each of Eqs. (12), (17), and (27) vanishes since both d' and $\varepsilon_\parallel$ vanish; however, if $\tau=0$, then all other terms in $G-G_B$ vanish, except for the last term which in this case reduces to

$$G-G_B\approx(T/4Jd)(a/L)^{d-2}\varepsilon^2\quad(\tau=0), \tag{28}$$

in perfect agreement with the corresponding result reported previously.[20] It is remarkable how much information gets lost in the limit $\tau\to 0$; in fact, the leading FS corrections to $G(\mathbf{R})$ for a system confined to geometry $\Omega=L^d$ with $\tau>0$(TBC's) are mutually exclusive from the ones pertaining to systems under PBC's ($\tau=0$).

In conclusion, we have shown that, at least for the spherical model, two well-known procedures, viz. the ζ function regularization and the dimensional regularization, are completely equivalent in describing the spin-spin correlation function of a finite-sized ferromagnetic system under twisted boundary conditions at or near the upper critical dimension. In the light of recent work,[5–7,16] we expect that this equivalence is maintained for more realistic "insoluble" O(n) model systems of finite extent under general boundary conditions, surface interactions, etc. in the region of both first-order and second-order phase transitions.

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