Static crossover behavior in the neighborhood of a Lifshitz point

Ibraheem Nasser*
Physics Department, King Fahd University of Petroleum and Minerals, Dhahran 31261, Saudi Arabia

R. Folk
Institute of Theoretical Physics, University of Linz, Linz, Austria
(Received 12 June 1995)

We consider the phase transition from the para- to the ferro- phase for a system in which an m-fold Lifshitz point exists. Nearby the Lifshitz point, the critical behavior at finite distance from T_c is nonasymptotic, described by crossover phenomena, and strongly influenced by the Lifshitz point. The asymptotic behavior on the para-ferro segment of the critical line belongs to the universality class of an isotropic magnet with short-range interaction, with an upper critical dimension $d_c=4$. The crossover depends on the nonuniversal static parameters further away from the critical temperature $T_c(x)$, especially on the dispersion (wave-vector dependence) in the quadratic part of Landau-Ginzburg-Wilson Hamiltonian. The susceptibility and the specific heat have been calculated using the field-theoretical renormalization-group procedure. The effective exponents, which characterize the crossover behavior, have also been discussed.

I. INTRODUCTION

Crossover phenomena are common near phase transitions. They can occur for several reasons, and one of them, which is related to our work, is that irrelevant, but strong interactions, connected with some unstable fixed points, may dominate further away from the critical temperature T_c until the relevant interactions have driven the system to its asymptotic stable fixed point.

The renormalization-group method, incorporated with ϵ expansion,[1] has been applied extensively to study the crossover behavior for many different systems. The crossover map for magnetic fixed points for $d<4$ is given in the review article by Fisher.[2] Lawrie and Sarbach in their review article[3] studied the crossover behavior phenomena, which is a particular interest in the observation of tricritical systems. Recently, this method has been applied to study the crossover behavior in uniaxial dipolar ferromagnets.[4] Using another theoretical treatment, the uniaxial dipolar ferromagnets' crossover in d dimension was investigated by Stephens.[5] The crossover from isotropic to directed percolation has been reviewed in Ref. 6. Most of these crossover phenomena are connected with a change in the upper critical dimension of the system.

In this paper we study the crossover in a system that exhibits an m-fold Lifshitz multicritical point type, (see Hornreich, Luban, and Shtrikman[7]). As proposed in Ref. 7, a Lifshitz multicritical point divides a second-order phase transition (λ) line into two segments, (λ_1) and (λ_2), such that on only one of them (λ_1) is the critical order parameter, characterized by a fixed equilibrium wave vector, usually $q=0$. On the segment of λ_2 the wave vector increases continuously, and a phase transition from a paramagnetic to a helical phase takes place. In other words, an m-fold Lifshitz point is characterized by an instability, associated with the absence of the q^2 term in the Landau-Ginzburg-Wilson (LGW) Hamiltonian for all $i=1,2,\ldots,m\leq d$, where d is the dimensionality of the space.

Experimental evidence for a Lifshitz point in the magnet MnP material has been reviewed by Shapira.[8] The specific-heat exponent and its corresponding critical amplitude ratio for the Lifshitz point in MnP were measured experimentally by Bindilatti, Becerra, and Oliveira.[9] However, other realizations have been proposed, e.g., in structural phase transition NbO_2,[10] $RbCaF_2$,[11] (there even a tricritical Lifshitz point is expected) and in liquid crystals.[12] Recently, Vysochanskii and Slivka[13] analyzed the phase diagrams of the ferroelectric crystal $Sn_2P_2(Se_xS_{1-x})_6$ and a Lifshitz point was found. Selke, in his article,[14] reviewed most of the current theoretical and experimental work related to this subject.

The paper is arranged as follows: Section II is written to introduce our theoretical model using the generalized renormalization-group procedure, incorporating the minimal subtraction scheme formulated by Amit and Goldschmidt.[15] In Sec. III, we review the limits of the second-order critical and multicritical points. In Sec. IV, we calculate the renormalization constants and the flow of the coupling constants. In Sec. V, the susceptibility will be calculated and analyzed in terms of the effective exponent. Section VI is devoted to the calculation of the specific heat, which is followed by a summary of our calculations in Sec. VII.

II. THEORETICAL MODEL AND RENORMALIZATION

The LGW free-energy functional, describing the critical behavior of a system that exhibits an m-fold Lifshitz point, can be written in the form

$$H=-\frac{1}{2}\int_k (r_0+p^2+g_0q^2+\alpha_0q^4)\varphi_0(\mathbf{k})\varphi_0(-\mathbf{k}) -\frac{u_0}{4!}\int_{k_1}\int_{k_2}\int_{k_3}\int_{k_4}\varphi_0(\mathbf{k}_1)\varphi_0(\mathbf{k}_2)\varphi_0(\mathbf{k}_3)\varphi_0(\mathbf{k}_4) \times\delta\left[\sum_{i=1}^{4}\mathbf{k}_i\right]. \quad (1)$$

Here $\varphi_0(\mathbf{k})$ represents the order parameter, e.g., magnetization, polarization, etc. The d-dimensional wave vector $\mathbf{k}$ is decomposed into **q** and **p** components of dimension m and $(d-m)$, respectively. In Eq. (1) we used the notations $\int_k=\int d^dk/(2\pi)^d$, $r_0=(T-T_c)/T_c$ is the bare reduced temperature, g_0, and α_0 are the parameters of the dispersion. Near the Lifshitz point, when $\alpha_0 \gg g_0 \neq 0$, the α_0 term drops out asymptotically; however it may be of importance for the critical behavior in the experimental regime when $T\neq T_c$. The Gaussian propagator for the graphical expansion has the following form:

$$G_0(r_0,g_0,\alpha_0)=(r_0+p^2+g_0q^2+\alpha_0q^4)^{-1}. \qquad (2)$$

Proceeding as usual, in the minimal subtraction scheme adopted from Amit and Goldschmidt,[15] and used for dimensional crossover by Frey and Schwabl,[4] the dimension-dependent pole and singularities, which appear in the unrenormalized vertex functions $\Gamma_B^{(M,N)}$, could be absorbed into a renormalization factors Z's. After introducing the momentum scale μ, the renormalized parameter, coupling constant, and fields are defined by $r_0=Z_\varphi^{-1}Z_r r$, $u_0=\mu^\epsilon Z_\varphi^{-2}Z_u u K_{d-m,m}^{-1}$, $g_0=Z_\varphi^{-1}Z_g g$, $\alpha_0=Z_\varphi^{-1}Z_\alpha\alpha$, and $\varphi_0=Z_\varphi^{1/2}\varphi$, with the dimensions of $[r_0]=\mu^2$, $[\alpha_0]=\mu^{-2}$, $[g_0]=\mu^0$, $[\varphi_0]=\mu^{-[(d+2)/2]}$, $[u_0]=\mu^\epsilon$, and $\epsilon=4-d$. The factor $K_{d-m,m}=\Gamma[(d-m)/2]S_{d-m}S_m$, where $S_d=[2^{d-1}\pi^{d/2}\Gamma(d/2)]^{-1}$ has been introduced for convenience. In our calculations, the mass and coupling renormalization factors (Z_r and Z_u) are determined in one-loop order, and in this order we have $Z_\alpha=Z_g=Z_\varphi=1$.

Regarding the phase transition phenomena, the accessible quantities are for instance the susceptibility, or specific heat. In order to obtain these quantities using the field renormalization-group procedure, we have to calculate the renormalized vertex functions $\Gamma_R^{(2,0)}$ and $\Gamma_R^{(0,2)}$, by means of solving the renormalization-group equations in the form

$$\left(\mu\frac{\partial}{\partial\mu}+\beta_u(\Omega)\frac{\partial}{\partial u}+\zeta_r(\Omega)\left\{\begin{bmatrix}2\\0\end{bmatrix}+r\frac{\partial}{\partial r}\right\}+\zeta_g(\Omega)g\frac{\partial}{\partial g}+\zeta_\alpha(\Omega)\alpha\frac{\partial}{\partial\alpha}+\zeta_\varphi(\Omega)\right)\begin{bmatrix}\Gamma_R^{(0,2)}\\ \Gamma_R^{(2,0)}\end{bmatrix}(r,g,\alpha,u,\mu)=\begin{bmatrix}\mu^{-\epsilon}\hat{B}_{\varphi 2}(g,\alpha,\mu)\\0\end{bmatrix}. \qquad (3)$$

In Eq. (3), we have defined $\zeta_i(\Omega)=\mu(\partial/\partial\mu)\ln Z_i^{-1}|_0$, $i=\varphi,r,g,\alpha$ and $\beta_u=\mu(\partial/\partial\mu)u|_0$. $\Omega\equiv(g,\alpha\mu^2,u)$, and the symbol $|_0$ indicates that all derivatives are to be taken at fixed bare parameter r_0, g_0, α_0, and u_0. The inhomogeneity of Eq. (3), which is related to the additive renormalization of the specific heat, has the form

$$\hat{B}_{\varphi 2}(g,\alpha,\mu)=\mu^\epsilon Z_r^2\mu\frac{d}{d\mu}Z_r^{-2}[Z_r^2\Gamma_B^{(0,2)}(p=0)]_{\text{sing}}. \qquad (4)$$

Equation (3) could be solved using the method of characteristics[16] by means of $\mu(l)=\mu l$ and $\zeta_i(l)=\zeta_i(\Omega(l))$, which in the present case leads to the set of the equations

$$l\frac{d\mu(l)}{dl}\equiv\mu(l), \qquad (5)$$

$$l\frac{dr(l)}{dl}\equiv r(l)\zeta_r(l), \qquad (6)$$

$$l\frac{dg(l)}{dl}\equiv g(l)\zeta_g(l), \qquad (7)$$

$$l\frac{d\alpha(l)}{dl}\equiv\alpha(l)\zeta_\alpha(l), \qquad (8)$$

$$l\frac{du(l)}{dl}\equiv\beta_u(l), \qquad (9)$$

with the initial conditions $\mu(1)=\mu$, $r(1)=r$, $g(1)=g$, $\alpha(1)=\alpha$, and $u(1)=u$. The flow Eqs. (6), (7), and (8) are solved by

$$r(l)=re^{\int_1^l(d\rho/\rho)\zeta_r(\rho)}, \qquad (10)$$

$$g(l)=ge^{\int_1^l(d\rho/\rho)\zeta_g(\rho)}, \qquad (11)$$

and

$$\alpha(l)=\alpha e^{\int_1^l(d\rho/\rho)\zeta_\alpha(\rho)}. \qquad (12)$$

Via Eq. (10), a connection between the flow parameter l and the correlation length ξ or the temperature distance $T-T_c$ is made by the relation $r(l)/\mu^2l^2=1$.

The general solutions of the renormalization-group equation (3) read[16]

$$\Gamma_R^{(2,0)}(r,g,\alpha,u,\mu)=(\mu l)^2e^{\int_1^l(d\rho/\rho)\zeta_\varphi(\rho)}\hat{\Gamma}_R^{(2,0)}\times\left[\frac{r(l)}{\mu^2l^2},g,\alpha\mu^2l^2,u(l)\right], \qquad (13)$$

for the susceptibility, and

$$\Gamma_R^{(0,2)}(r,g,\alpha,u,\mu)=(\mu l)^{-\epsilon}e^{2\int_1^l(d\rho/\rho)\zeta_r(\rho)}\hat{\Gamma}_R^{(0,2)}\left(\frac{r(l)}{\mu^2l^2},u(l),g,\alpha\mu^2l^2\right)-(\mu l)^{-\epsilon}\int_1^l\frac{d\rho}{\rho}\hat{B}_{\varphi 2}(\rho)e^{\int_1^\rho(d\rho'/\rho')[2\zeta_r(\rho')-\epsilon]}, \qquad (14)$$

for the specific heat.

III. LIMITS OF SECOND-ORDER CRITICAL AND MULTICRITICAL POINT

Before we study the general case of crossover behavior, let us first review the behavior of our system in the limits of $g\equiv0$, or $\alpha\equiv0$. In order to do so, we have to work out the Lifshitz integrals in the form:

$$I_r(r,g,\alpha)=(2\pi)^{-d}\int d^{d-m}p\int d^mq(r+p^2+gq^2+\alpha q^4)^{-r}, \qquad (15)$$

where $r=1$ and 2, to calculate the vertex functions, and other related parameters.

A. Second-order critical behavior limit ($\alpha\equiv 0$ and $g\neq 0$)

The general solution of the integrals is given as follows:

$$I_1(r,g,\alpha=0)=\frac{1}{4}K_{d-m,m}\Gamma\left(\frac{m}{2}\right)\Gamma\left(\frac{\epsilon}{2}-1\right)g^{-m/2}r^{1-\epsilon/2},$$

$$I_2(r,g,\alpha=0)=\frac{1}{4}K_{d-m,m}\Gamma\left(\frac{m}{2}\right)\Gamma\left(\frac{\epsilon}{2}\right)g^{-m/2}r^{-\epsilon/2},$$

with the pole $\epsilon=d_c-d$, and the upper critical dimension is $d_c=4$. The Z factors are calculated as usual[17] apart from a trivial scaling with the anisotropy factor g, as can be seen from the integrals. The renormalization constants for the mass (Z_r), coupling (Z_u), and the corresponding ζ_r and β_u equations are given by

$$Z_r(g,\alpha=0)=1+\frac{1}{4}\frac{u}{\epsilon}\Gamma\left(\frac{m}{2}\right)g^{-m/2},\tag{16}$$

$$Z_u(g,\alpha=0)=1+\frac{3}{4}\frac{u}{\epsilon}\Gamma\left(\frac{m}{2}\right)g^{-m/2},\tag{17}$$

$$\zeta_r(g,\alpha=0)=\frac{u}{4}\Gamma\left(\frac{m}{2}\right)g^{-m/2},\tag{18}$$

and

$$\beta_u(g,\alpha=0)=-\epsilon u+\frac{3}{4}\Gamma\left(\frac{m}{2}\right)u^2g^{-m/2}.\tag{19}$$

In Eq. (19), using $\tilde{u}_g=u/2\Gamma(m/2)g^{-m/2}$, one finds the usual flow equation in the form (independent of g)

$$\beta(\tilde{u}_g)=-\epsilon\tilde{u}_g+\frac{3}{2}\tilde{u}_g^2,\tag{20}$$

which gives the fixed points $\tilde{u}_g^*\equiv\{0,2/3\epsilon\}$, $\zeta_r=(1/3)\epsilon$ as $l\to 0$, and $\gamma=1+(1/2)\zeta_r=1.16$ for $d=3$. The additive renormalization constant is also calculated as

$$\hat{B}_{\varphi 2}(g,\alpha=0)=\frac{1}{4}K_{d-m,m}\Gamma\left(\frac{m}{2}\right)g^{-m/2}.\tag{21}$$

Note that, the factor of $\Gamma(m/2)$ which appears in our results, could be easily absorbed in the constant $K_{d-m,m}$. Also, the factor $g^{-m/2}$ in Eq. (19) comes from the overall scaling of the vertex function when one introduces $\tilde{u}_g$, i.e.,

$$\hat{\Gamma}_R^{(0,2)}(u,g)=g^{-m/2}\hat{\Gamma}_R^{(0,2)}(\tilde{u}_g,1).$$

B. Multicritical behavior limit ($g\equiv 0$ and $\alpha\neq 0$)

In this case, the general solution for the integrals are

$$I_1(r,g=0,\alpha)=\frac{1}{8}K_{d-m,m}\Gamma\left(\frac{m}{4}\right)\Gamma\left(\frac{\tilde{\epsilon}}{2}-1\right)\alpha^{-m/4}r^{1-\tilde{\epsilon}/2},$$

$$I_2(r,g=0,\alpha)=\frac{1}{8}K_{d-m,m}\Gamma\left(\frac{m}{4}\right)\Gamma\left(\frac{\tilde{\epsilon}}{2}\right)\alpha^{-m/4}r^{-\tilde{\epsilon}/2}$$

with $\tilde{\epsilon}=\epsilon+m/2$. The Z factors have poles in $\tilde{\epsilon}$ instead of ϵ, and read

$$Z_r(g=0,\alpha,u,\mu)=1+\frac{1}{8}\frac{u}{\tilde{\epsilon}}\Gamma\left(\frac{m}{4}\right)(\alpha\mu^2)^{-m/4},\tag{22}$$

$$Z_u(g=0,\alpha,\mu)=1+\frac{3}{8}\frac{u}{\tilde{\epsilon}}\Gamma\left(\frac{m}{4}\right)(\alpha\mu^2)^{-m/4}.\tag{23}$$

The corresponding ζ_r and β_u equations are

$$\zeta_r(g=0,\alpha\mu^2)=\frac{u}{8}\Gamma\left(\frac{m}{4}\right)(\alpha\mu^2)^{-m/4},\tag{24}$$

$$\beta_u(g=0,\alpha\mu^2)=-\epsilon u+\frac{3}{8}u^2\Gamma\left(\frac{m}{4}\right)(\alpha\mu^2)^{-m/4}.\tag{25}$$

Equation (25) suggests the introduction of the effective fourth-order coupling $\tilde{u}_\alpha=u/4\Gamma(m/4)(\alpha\mu^2)^{-m/4}$, which is marginal at the upper critical dimension $d_c=4+m/2$, to give the flow equation (independent of α)

$$\beta(\tilde{u}_\alpha)=-\tilde{\epsilon}\tilde{u}_\alpha+\tfrac{3}{2}\tilde{u}_\alpha^2,\tag{26}$$

with the fixed points $\tilde{u}_\alpha^*\equiv\{0,2/3\tilde{\epsilon}\}$, $\zeta_r=(1/3)\tilde{\epsilon}$ as $l\to 0$, and $\gamma=1.25$ for $d=3$. In fact, it turns out from considering the loop graphs that this effective coupling appears in every order. The additive renormalization constant is also calculated as

$$\hat{B}_{\varphi 2}(g=0,\alpha\mu^2)=\frac{1}{8}K_{d-m,m}\Gamma\left(\frac{m}{4}\right)(\alpha\mu^2)^{-m/4}.\tag{27}$$

With the above expressions (22)–(27) we recover the one-loop results for the exponents of Hornreich, Luban, and Shtrikman,[7] if the effective coupling $\tilde{u}_\alpha$ is introduced in all vertex functions.

IV. FLOW EQUATION FOR THE EFFECTIVE COUPLING

As already mentioned above, at $g\neq 0$ the upper critical dimension is $d_c=4$, and in the other limit at $\alpha\neq 0$, the upper critical dimension is $d_c=4+m/2$. In the former case, as the perturbation theory suggests, the effective expansion coefficient is no longer the four-point coupling coefficient u, but a suitably chosen effective coupling $\tilde{u}$. In the following, the form of $\tilde{u}$ containing both limits will be derived by considering the four-point vertex function.

In case of renormalization of the coupling constant and the flow diagram in one-loop order, the bare vertex function $\Gamma_B^{(4)}$ is given by

$$\Gamma_B^{(4)}(r_0,g_0,\alpha_0,u_0)=u_0-\tfrac{3}{2}u_0^2I_2(r_0,g_0,\alpha_0).\tag{28}$$

Using standard techniques,[18] the divergent part of the integral I_2 is found to be

$$[I_2(r=0,g,\alpha,\mu)]_{\text{sing}}=K_{d-m,m}c_1\frac{(\alpha\mu^2)^{-m/4}}{\tilde{\epsilon}}\times\left[c_2+\left(\frac{\tilde{g}}{\mu}\right)^{2\sigma}\right]^{-\tilde{\epsilon}/2\sigma},\tag{29}$$

with $\tilde{g}=g/\sqrt{\alpha}$. The constants c_1, c_2, and σ will be fixed using the matching procedure[4] of the new β_u equation with the others in the limited cases. To do so, we have to calculate the renormalized four-point vertex function.

Since there is no field renormalization in one-loop order, hence the renormalized four-point vertex function is given by

$$\Gamma_R^{(4)}(r,g,\alpha,u,\mu)=\mu^{\epsilon}\left[Z_u u K_{d-m,m}^{-1} - \frac{3}{2} Z_u^2 u^2 K_{d-m,m}^{-2} I_2\left(\frac{Z_r r}{\mu^2},\frac{\tilde g}{\mu},\alpha\mu^2\right)\right]. \tag{30}$$

Therefore the coupling renormalization constant (Z_u), and the corresponding β_u function for the four-point coupling constant are

$$Z_u(g,\alpha,\mu)=1+\frac{3}{2}uc_1\frac{(\alpha\mu^2)^{-m/4}}{\tilde\epsilon}\left[c_2+\left(\frac{\tilde g}{\mu}\right)^{2\sigma}\right]^{-\tilde\epsilon/2\sigma}, \tag{31}$$

$$\beta_u(g,\alpha,\mu)=-\epsilon u+\frac{3}{2}c_1u^2(\alpha\mu^2)^{-m/4}\left[1+\frac{1}{c_2}\left(\frac{\tilde g}{\mu}\right)^{2\sigma}\right]^{-1}. \tag{32}$$

Matching the β_u equations (19), (25), and (32), one can find $\sigma=m/4$, $c_1=(1/4)\Gamma(m/4)$, and $c_2=2\Gamma(m/2)/\Gamma(m/4)$.

Using the effective coupling $\tilde u=c_1u(\alpha\mu^2)^{-m/4}[1+1/c_2(\tilde g/\mu)^{m/2}]^{-1}$, we obtain

$$\beta_{\tilde u}(\tilde g,\tilde u)=-\tilde\epsilon_1\tilde u+\tfrac{3}{2}\tilde u^2, \tag{33}$$

with the "crossover" dimension $\tilde\epsilon_1(\mu)=\epsilon+m/2[1+1/c_2(\tilde g/\mu)^{m/2}]^{-1}$. Then, the flow Eq. (9) reads

$$l\frac{d\tilde u}{dl}=-\tilde\epsilon_1(l)\tilde u(l)+\frac{3}{2}\tilde u^2(l), \tag{34}$$

with

$$\tilde\epsilon_1(l)=\epsilon+\frac{m}{2}\left[1+\frac{1}{c_2}\left(\frac{\tilde g}{\mu l}\right)^{m/2}\right]^{-1}=\tilde\epsilon$$

if $g=0$, and $\tilde\epsilon_1(l)=\epsilon$ if $\alpha=0$. The fixed points in Eq. (34) are $\tilde u^*=0$, and $\tilde u^*=\lim_{l\to 0}(2/3)\tilde\epsilon_1(l)=(2/3)\epsilon$.

To study the behavior of the flow equation, one has to solve Eq. (34) numerically. $\tilde u(l)$ has been plotted in Fig. 1 for $m=\mu=\alpha=1$ as a function of the flow parameter l, and for different values of $\tilde g(l=1)$ as indicated in the graph. One can see from Fig. 1 the pronounced effect of small values of $\tilde g$ on the shape of $\tilde u(l)$. At small values of $\tilde g$ and in the interval ($10^{-1}\geqslant l\geqslant 10^{-3}$), $\tilde u(l)$ starts to increase to the Lifshitz fixed point value of one loop ($\simeq 1$). At small l value ($l<10^{-4}$), $\tilde u(l)$ starts to go back to the usual fixed-point value ($\simeq 0.66$).

V. TEMPERATURE DEPENDENCE OF THE SUSCEPTIBILITY

In the case of the mass renormalization and susceptibility, the same divergent part of the one-loop integral $I_1(r,g,\alpha)$ appears, therefore, the renormalization constant for the mass is

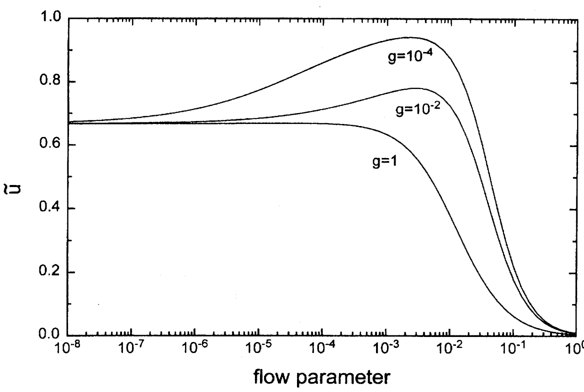

FIG. 1. Effective coupling $\tilde u(l)$ as a function of the flow parameter l with $m=\alpha=\mu=1$, and for different values of g. The initial value $\tilde u(1)=0.01$ has been taken.

$$Z_r(g,\alpha,u,\mu)=1+\frac{1}{2}uc_1\frac{(\alpha\mu^2)^{-m/4}}{\tilde\epsilon}\left[c_2+\left(\frac{\tilde g}{\mu}\right)^{m/2}\right]^{-2\tilde\epsilon/m}, \tag{35}$$

and the corresponding function ζ_r is given by

$$\zeta_r[\tilde u(l)]=\frac{1}{2}u(l)c_1(\alpha l^2)^{-m/4}\left[1+\frac{1}{c_2}\left(\frac{\tilde g}{\mu}\right)^{m/2}\right]^{-1}=\frac{\tilde u(l)}{2}. \tag{36}$$

The renormalized two-point vertex function is expressed as

$$\Gamma_R^{(2,0)}(r,g,\alpha,u,\mu)=\mu^2\left[\frac{Z_r r}{\mu^2}+\frac{1}{2}uK_{d-m,m}^{-1}J\left(\frac{r}{\mu^2},\frac{\tilde g}{\mu},\alpha\mu^2\right)\right], \tag{37}$$

where $J(r/\mu^2,\tilde g/\mu,\alpha\mu^2)=I_1(r/\mu^2,\tilde g/\mu,\alpha\mu^2)-I_1(0,\tilde g/\mu,\alpha\mu^2)$. The parameter integral $J(r/\mu^2,\tilde g/\mu,\alpha\mu^2)$ will be calculated in Appendix A.

Using standard techniques[18] for the evaluation of the integral $I_1(r/\mu^2,\tilde g/\mu,\alpha\mu^2)$ in the dimensional regularization scheme, one has

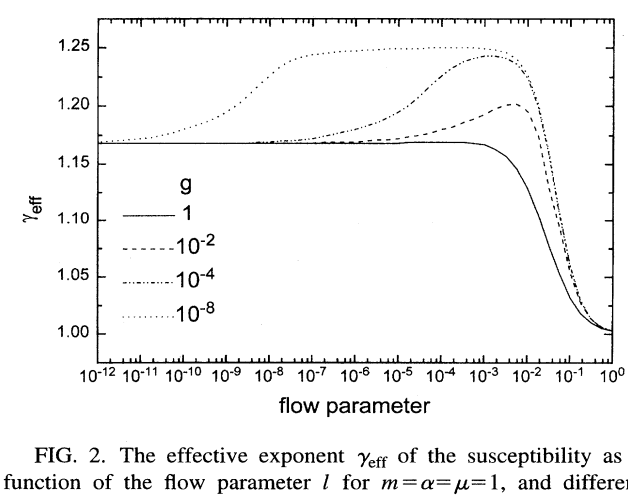

FIG. 2. The effective exponent $\gamma_{\rm eff}$ of the susceptibility as a function of the flow parameter l for $m=\alpha=\mu=1$, and different values of g.

$$I_1\left(\frac{r}{\mu^2},\frac{\tilde g}{\mu},\alpha\mu^2\right)=C\left[\frac{\tilde g}{4\mu}+\frac{\sqrt{r}}{2\mu}\right]^{-2\beta+m/2}\times(\alpha\mu^2)^{-m/4}\,{}_2F_1\left(2\beta-\frac{m}{2},\beta-\frac{m}{2}+\frac{1}{2};\beta+\frac{1}{2};\tilde Q\right),\tag{38}$$

where

$$C=\frac{1}{4}K_{d-m,m}\Gamma(\beta)B\left(\frac{m}{2},2\beta-\frac{m}{2}\right),\quad \tilde Q=\frac{\tilde g-2\sqrt{r}}{\tilde g+2\sqrt{r}},$$

and

$$\beta=1-\left(\frac{d-m}{2}\right)=\frac{\tilde\epsilon}{2}+\frac{m}{4}-1.$$

B and ${}_2F_1$ are the normal beta and hypergeometric functions. Substituting in Eq. (37) for Z_r, and making a suitable $\tilde\epsilon$ expansion, one can find (see Appendix A):

$$\hat\Gamma_R^{(2,0)}\left(\frac{r}{\mu^2},\frac{\tilde g}{\mu},\alpha\mu^2,\tilde u\right)=\frac{r}{\mu^2}\left(1+\frac{\tilde u}{2}\left(\frac{\tilde g}{\mu}+2\sqrt{r/\mu^2}\right)^{-\tilde\epsilon}\left[1+\frac{1}{c_2}\left(\frac{\tilde g}{\mu}\right)^{m/2}\right]\times\left\{\frac{1}{8r}f_2\left(\frac{r}{\mu^2},m,\frac{\tilde g}{\mu}\right)+f_3\left(\frac{r}{\mu^2},m,\frac{\tilde g}{\mu}\right)\right\}\right).\tag{39}$$

The effective exponent $\gamma_{\rm eff}$ is related to the susceptibility $\chi^{-1}(=\Gamma_R^{(2,0)})$ by the relation

$$\gamma_{\rm eff}=\frac{d\ \ln\chi^{-1}(r/\mu^2,\tilde g/\mu,\alpha\mu^2,\tilde u)}{d\ \ln r},\tag{40}$$

with

$$\chi^{-1}\left(\frac{r}{\mu^2},\frac{\tilde g}{\mu},\alpha\mu^2,\tilde u\right)=l^2\chi^{-1}\left[\frac{r(l)}{\mu^2 l^2},\frac{\tilde g}{\mu l},\alpha\mu^2 l^2,\tilde u(l)\right].\tag{41}$$

To first order in $\tilde u$ and $\tilde\epsilon$, the one-loop order $\gamma_{\rm eff}$ is given by

$$\gamma_{\rm eff}=1+\frac{1}{2}\zeta_r[\tilde u(l)]+\frac{d\ \ln\chi^{-1}[1,\tilde g/\mu l,\alpha\mu^2 l^2,\tilde u(l)]}{d\ \ln r},$$
$$=1+\frac{1}{4}\tilde u(l)+\frac{1}{4}\tilde u(l)\left(\frac{\tilde g}{\mu l}+2\right)^{-\tilde\epsilon}l\frac{d}{dl}\left[1+\frac{1}{c_2}\left(\frac{\tilde g}{\mu l}\right)^{m/2}\right]\left\{\frac{1}{8}f_2\left(1,m,\frac{\tilde g}{\mu l}\right)+f_3\left(1,m,\frac{\tilde g}{\mu l}\right)\right\},\tag{42}$$

where the matching condition $r(l)/\mu^2l^2=1$ has been applied.

In Fig. 2 the effective exponent $\gamma_{\rm eff}$ is displayed for $m=\mu=\alpha=1$ as a function of the flow parameter l, and for different values of $\tilde g(l=1)$ as indicated in the graph. One can see from Fig. 2 the pronounced effect of small values of $\tilde g$ on $\gamma_{\rm eff}$. At small values of $\tilde g$ and in the interval $(10^{-1}\geq l\geq 10^{-3})$, $\gamma_{\rm eff}$ starts to increase to the Lifshitz point value of one loop ($\simeq$1.25). At small l value ($l<10^{-4}$), $\gamma_{\rm eff}$ starts to go back to the usual fixed-point value [$\simeq$1(1/6)]. The correction due to the shape function, i.e., the third term in Eq. (42), turns out to be small in our case. The solution of the matching condition leading to the relation between l and r, itself shows a crossover between $l\approx r^2$ in the Gaussian region and $l\approx r^{1.16}$ in the asymptotic region as shown in Fig. 3.

VI. TEMPERATURE DEPENDENCE OF THE SPECIFIC HEAT

The specific heat C_B could also be calculated by using the cumulant (C) definition:

$$C_B=\left\langle\frac{1}{2}\varphi_0^2(p=0)\,\frac{1}{2}\varphi_0^2(p=0)\right\rangle_C.\tag{43}$$

Above T_c, this quantity is related directly to the vertex function $\Gamma_B^{(0,2)}(p=0)$ with two φ insertions:

$$C_B=\Gamma_B^{(0,2)}(p=0).$$

This vertex function has to be normalized additively by means of the relation

$$\Gamma_R^{(0,2)}=Z_{\varphi 2}^2\Gamma_B^{(0,2)}-[Z_{\varphi 2}^2\Gamma_B^{(0,2)}]_{\rm sing},\tag{44}$$

where the renormalization constant $Z_{\varphi 2}^2$ is identical to the mass renormalization, i.e., $Z_{\varphi 2}=Z_r$, and $\Gamma_R^{(0,2)}$ is given by Eq. (14).

Since there is no zero-loop contributions to the specific heat, the vertex function $\Gamma_B^{(0,2)}$ is given to one-loop order by

$$\Gamma_B^{(0,2)}(p=0)=-\frac{1}{2}I_2(r_0,g_0,\alpha_0)=-\frac{1}{2}\left[-\frac{\partial}{\partial r_0}I_1(r_0,g_0,\alpha_0)\right].\tag{45}$$

See Appendix B for the calculation of the I_2 integral. Using Eq. (29), then the additive renormalization is found to be

$$\hat B_{\varphi 2}(g,\alpha,\mu)=\frac{1}{2}K_{d-m,m}c_1(\alpha\mu^2)^{-m/4}\left[1+\frac{1}{c_2}\left(\frac{\tilde g}{\mu}\right)^{m/2}\right]^{-1},\tag{46}$$

For the renormalized scaling function of the specific heat, one finds (see Appendix B)

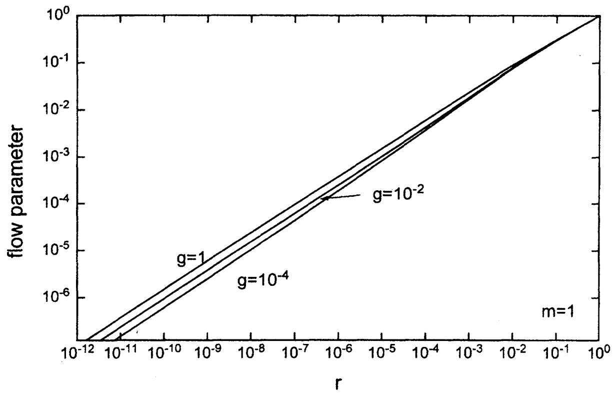

FIG. 3. Flow parameter l as a function of r according to the matching condition for $m=\alpha=\mu=1$, and different values of g.

$$\hat{\Gamma}_R^{(0,2)}\left(\frac{r}{\mu^2},g,\alpha\mu^2,u\right) = \frac{1}{2}K_{d-m,m}c_1(\alpha\mu^2)^{-m/4}\left(\frac{\tilde{g}}{\mu}+2\sqrt{r/\mu^2}\right)^{-\bar{\epsilon}} \times\left\{\frac{1}{8}\left[\frac{\tilde{g}}{\sqrt{r}}(f_{14}-2f_{12})+2f_{14}\right]+f_3\right\}. \quad (47)$$

The renormalized dimensionless specific heat (C) is defined by $C=\mu^\epsilon\Gamma_R^{(0,2)}$. In Fig. 4, the $\log_{10}[C]$ has been plotted versus $\log_{10}[l]$ for $m=\alpha=\mu=1$, and for different values of $\tilde{g}(l=1)$ as indicated in the graph. One can see from Fig. 4 the asymptotic behavior $C\approx l^{-2/7}$ at small l value ($l<10^{-6}$). On approaching the Lifshitz point, experimental measurements[13] show an increase in the effective exponent of the specific heat. These results are in fair agreement, qualitatively, with the changing in the slope of the specific heat (see Fig. 4) at a fixed temperature region with decreasing g. Theoretically, at $d=3$ and one-loop order, the specific heat exhibits the usual crossover from Gaussian behavior (l^{-1}), to the asymptotic behavior ($l^{-2/7}$), and, in between the Lifshitz behavior ($l^{-2/5}$), extends over a small l region. Therefore, it is quite hard (experimentally) to distinguish the Lifshitz behavior from the usual crossover even by looking at the effective exponent. Consequently, the decision about the type of crossover should not rely on the specific heat alone.

VII. SUMMARY

The present work is an attempt to study crossover behavior in the neighborhood of a Lifshitz point. Using the field theory, incorporated with the minimal subtraction scheme, we were able to calculate the crossover functions for the susceptibility and the specific heat. Mainly, we analyzed the uniaxial Lifshitz point (where the wave vector instability occurs in one dimension only, i.e., $m=1$), since it is the most relevant case for the experimentalists. The crossover behavior in the cases of $m=2$ and 3 turns out to be more pronounced. This is related to the higher critical dimension d_c for larger m, and therefore stronger fluctuation effect at $d=3$.

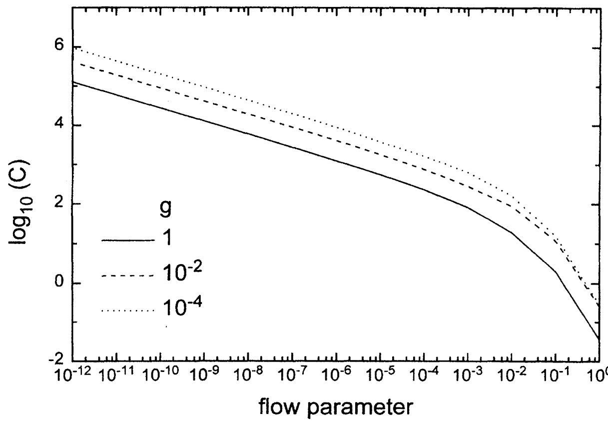

FIG. 4. The logarithmic of the dimensionless specific heat C as a function of the flow parameter ln[l] with $m=\alpha=\mu=1$, and for different values of g.

In a comparison of the results with experiment one has to bear in mind that our results are calculated in one-loop order, usually not verified by experiments. E.g., the asymptotic value of the $n=1$ susceptibility exponent γ is to the best knowledge $\gamma_{\text{Borel}}=1.240$,[19] whereas the one-loop value is $\gamma_{\text{one loop}}=1.167$. At the Lifshitz point such an accurate value and even the two-loop value is not available (for $m=1$). However we expect a shift of $\gamma_{\text{Lifshitz}}=1.25$ of similar size as at the usual critical point (but see, on the other hand, the experimental values of the crossover exponent ϕ which is compatible with the one-loop value.[20]). Nevertheless one could treat our results by a phenomenological procedure as suggested in Ref. 21. Similar considerations could be made for the specific-heat exponent α (not to be confused with the parameter α from above), which at the usual critical point takes the respective values $\alpha_{\text{Borel}}=0.107$, $\alpha_{\text{one loop}}=0.167$ and $\alpha_{\text{Lifshitz}}=0.25$. In the magnetic system MnP a large value of the specific-heat exponent about 0.45 has been found experimentally,[9] which would not be in accordance with a "shift" procedure as just mentioned. So the situation remains unclear and not suitable for phenomenological procedures and further experimental work addressed to the crossover behavior seems to be necessary.

In ferroelectrics, the situation is complicated by the presence of uniaxial dipolar forces.[22] Then a competition between several possible crossovers may lead to a much more structural of γ_{eff}. Similar complicated situations have been studied for the case of anisotropic uniaxial dipolar ferromagnets.[23] Another complication in the application to $Sn_2P_2(Se_xS_{1-x})_6$ ferroelectrics might be the influence of the virtual tricritical point near the Lifshitz point,[13] leading to a possible tricritical uniaxial dipolar Lifshitz behavior.[24]

ACKNOWLEDGMENTS

I.N. is grateful to KFUPM for the financial support during the sabbatical year. The authors acknowledge helpful discussions with Dr. E. Frey.

APPENDIX A

In the following, we calculate the parameter integral $J(r,\tilde{g},\alpha)=I_1(r,\tilde{g},\alpha)-I_1(0,\tilde{g},\alpha)$, suppressing μ for simplicity, which we used in calculating $\Gamma_R^{(2,0)}$. Starting our procedure by Eq. (38), using the duplication formula (AS p. 256, 6.1.18) (Ref. 25) $[\Gamma(\beta)/\Gamma(2\beta)]\Gamma(m/2)=2^{2-\tilde{\epsilon}}(\beta+1/2)\Gamma(m/4)$, and $\Gamma(2\beta-m/2)=\Gamma(\tilde{\epsilon}-2)=1/2\tilde{\epsilon}$, one can find

$$I_1(r,\tilde{g},\alpha)=\frac{c_1}{8\tilde{\epsilon}}\,\alpha^{-m/4}t_1t_2, \tag{A1}$$

where $t_1=(\tilde{g}+2\sqrt{r})^{2-\tilde{\epsilon}}$, $t_2=\{2^{\tilde{\epsilon}}\xi_2F_1(\tau,\kappa;\xi;\tilde{Q})\}$, $\tau=\tilde{\epsilon}-2$, $\kappa=(\tilde{\epsilon}-m/2-1)/2$, and $\xi=(\tilde{\epsilon}+m/2-1)/2$. Note that, the factor of ξ in t_2, used to cancel the accidental singularity at $m=2$ in the expansion of ${}_2F_1$.

Using the abbreviation

$$\xi_2F_1(\tau,\kappa;\xi;\tilde{Q})=a(r)+\tilde{\epsilon}b(r), \tag{A2}$$

then t_2 could be expanded to first order in $\tilde{\epsilon}$, to give

$$t_2=a(r)+\tilde{\epsilon}c(r), \tag{A3}$$

where $c(r)=b(r)+a(r)\ln(2)$. In (A2), we defined $a(r)=\Sigma_{i=0}^{2}(a_i/4)\tilde{Q}^i$, where $a_0=-2+m$, $a_1=4+2m$, and $a_2=-2+m$, and $b(r)=\Sigma_{i=0}^{n}b_i\tilde{Q}^i$, where the first few terms could be read as

$$b_0=\frac{1}{2},$$

$$b_1=-\frac{(m+6)}{4},$$

$$b_2=\frac{(20-12m-3m^2)}{8(2+m)},$$

$$b_3=\frac{(6-m)(m-2)}{12(6+m)},$$

$$b_4=\frac{(m-10)(m-6)(m-2)}{48(60+16m+m^2)},$$

$$b_5=\frac{(14-m)(m-10)(m-6)(m-2)}{120(840+284m+30m^2+m^3)},$$

$$b_6=\frac{(m-18)(14-m)(m-10)(m-6)(m-2)}{240(15210+5952m+824m^2+48m^3+m^4)},$$

and

$$b_7=\frac{(22-m)(m-18)(14-m)(m-10)(m-6)(m-2)}{420(6+m)(10+m)(14+m)[(18+m)(22+m)]}.$$

In the above definitions of $a(r)$ and $b(r)$, the other arguments such as m and $\tilde{g}$, have been suppressed for simplicity.

With the integrals

$$I_1(r,\tilde{g},\alpha)=\frac{c_1}{8\tilde{\epsilon}}\,\alpha^{-m/4}(\tilde{g}+2\sqrt{r})^{2-\tilde{\epsilon}}\{a(r)+\tilde{\epsilon}c(r)\}, \tag{A4}$$

and

$$I_1(0,\tilde{g},\alpha)=\frac{c_1}{8\tilde{\epsilon}}\,\alpha^{-m/4}\tilde{g}^{2-\tilde{\epsilon}}\{a(0)+\tilde{\epsilon}c(0)\}, \tag{A5}$$

we can get

$$J(r,\tilde{g},\alpha)=\frac{c_1}{8\tilde{\epsilon}}\,\alpha^{-m/4}(\tilde{g}+2\sqrt{r})^{-\tilde{\epsilon}}\times\{f_1(r,m,\tilde{g})+\tilde{\epsilon}f_2(r,m,\tilde{g})\}, \tag{A6}$$

where

$$f_1(r,m,\tilde{g})=(\tilde{g}+2\sqrt{r})^2a(r)-\tilde{g}^2a(0)=-8r,$$

and

$$f_2(r,m,\tilde{g})=\left\{(\tilde{g}+2\sqrt{r})^2c(r)-\tilde{g}^2\times\left[c(0)-a(0)\ln\left(\frac{\tilde{g}}{\tilde{g}+2\sqrt{r}}\right)\right]\right\}.$$

Also, to the first order in $\tilde{\epsilon}$, Z_r could have the following form:

$$Z_r=1+\frac{1}{2}\,uc_1\,\frac{\alpha^{-m/4}}{\tilde{\epsilon}}\,[c_2+\tilde{g}^{m/2}]^{-2\tilde{\epsilon}/m}$$
$$=1+\frac{1}{2}\,\tilde{u}(\tilde{g}+2\sqrt{r})^{-\tilde{\epsilon}}\left(1+\frac{\tilde{g}^{m/2}}{c_2}\right)\frac{1}{\tilde{\epsilon}}\times\{1+\tilde{\epsilon}f_3(r,m,\tilde{g})\},$$

where

$$f_3(r,m,\tilde{g})=\ln[(\tilde{g}+2\sqrt{r})(c_2+\tilde{g}^{m/2})^{-2/m}]. \tag{A7}$$

APPENDIX B

In this appendix, we evaluate the integral $I_2(r,\tilde{g},\alpha)$, which is related to $\Gamma_R^{(0,2)}$ by Eq. (43). By means of Eq. (A1), then

$$I_2(r,\tilde{g},\alpha)=-\frac{\partial}{\partial r}\,I_1(r,\tilde{g},\alpha)=-\frac{c_1}{8\tilde{\epsilon}}\,\alpha^{-m/4}\,\frac{\partial}{\partial r}\,(t_1t_2). \tag{B1}$$

With a simple differentiation, one can find to the first order of $\tilde{\epsilon}$

$$I_2(r,\tilde{g},\alpha)=-\frac{c_1}{8\tilde{\epsilon}}\,\alpha^{-m/4}(\tilde{g}+2\sqrt{r})^{-\tilde{\epsilon}}\times\left\{f_{11}+\tilde{\epsilon}\left[\frac{\tilde{g}}{\sqrt{r}}\,(f_{14}-2\,f_{12})+2f_{14}\right]\right\}, \tag{B2}$$

where

$$f_{12}=\left\{\ln(2)\,\frac{d}{d\tilde{Q}}\,a(r)+\frac{d}{d\tilde{Q}}\,b(r)\right\},$$

$$f_{14}=2b(r)+a(r)[2\,\ln(2)-1],$$

and

$$f_{11}=2\left[a(r)-\frac{\tilde{g}}{\sqrt{r}}\,\frac{d}{d\tilde{Q}}\,a(r)\right]+4a(r)=-8.$$

Also, to the first order of $\tilde{\epsilon}$, we have

$$[Z_{\varphi 2}^{2}\Gamma_{B}^{(0,2)}]_{\rm sing}=\frac{c_1}{8\tilde{\epsilon}}\,\alpha^{-m/4}(\tilde{g}+2\sqrt{r})^{-\tilde{\epsilon}}[1+\tilde{\epsilon}f_3(r,m,\tilde{g})]. \tag{B3}$$

The functions $a(r)$, $b(r)$, and $f_3(r,m,\tilde{g})$ are defined in Appendix A.

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*On sabbatical leave at: Institute of theoretical Physics, Johannes Kepler University at Linz, A-4040 Linz-Auhof, Austria.

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