Aharonov-Bohm effect induced by mutual inductance for an array of mesoscopic rings

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The effect of mutual inductance on the persistent currents in N mesoscopic rings placed periodically on a plane is investigated. The persistent currents for the situations with mutual inductance between the nearest-neighbor rings are calculated in detail. We find that while the self-inductance suppresses the persistent current, the effect of the mutual inductance is to enhance it. For example, for a metal ring (Au or Cu) of length $L_0 = 1$ μm and cross section $A = 0.02 \times 0.02$ μm^2, the persistent current due to the self-inductance is $I_L = 0.2 I_0$ where I_0 is the persistent current without the self-inductance. For a pair of rings, the contribution of the persistent current due to the mutual inductance is about 10%. The results can be understood as entirely due to self-consistency of the equilibrium properties in the nanostructure.

There has been a great deal of interest recently in the phenomenon of the persistent equilibrium current occurring in isolated mesoscopic normal rings penetrated by an Aharonov-Bohm (AB) flux. A very interesting effect in the realm of mesoscopic transport is the AB effect.[1] For a one-dimensional ring, the existence of circulating currents was proposed by Büttiker, Imry, and Landauer[2] and a more detailed quantitative analysis was given by Cheung and co-workers.[3] Since then a great deal of theoretical effort has been devoted to the understanding of this phenomena. In recent experiments, the persistent currents have been detected in an ensemble of 10^7 Cu rings,[4] in three single gold rings,[5] and in a single ring of diameter 2.7 μm.[6] In the experiment[4] the magnetic response of a system of 10^7 isolated, identically patterned copper rings has been shown to oscillate as a function of the enclosed magnetic flux on the scale of half a flux quantum. However, the measured result for the persistent currents was two orders of magnitude larger than what had been predicted theoretically. Ambegaoker and Eckern[7] presented a calculation of persistent currents by taking into account the electron coherence along time-reversed paths. Since then a lot of theoretical work has been done to understand this experimental enhancement of the persistent current. For a one-dimensional spinless model the theoretical calculations have indicated that no significant enhancement of the persistent currents is yielded by considering the effect of electron-electron interactions.[8] For a two-dimensional spinless model[9] or one-dimensional models with spins,[10] however, large enhancement of the persistent current is found due to the electron-electron interaction. For the experiment on two-dimensional, few-channel, quasiballistic semiconductor rings[6] the measured persistent current is in good agreement with the predictions of noninteracting electron theory.[3] More recently, Kirczenow[11] concentrated on the enhancement of persistent current by analyzing the electron scattering due to grain boundaries through three-dimensional computer simulations, and the result he obtained is in good agreement with the experimental data.

In this paper we study the persistent currents of N rings placed periodically on the same plane and the AB fluctuations due to the presence of the mutual inductance. Because of the mutual inductance between two rings, the electric current in one ring would produce an induced flux in another ring. When the system is at equilibrium the currents in rings associated with the mutual inductance among the rings may be observable. Qualitatively, if an electric current is made to pass through a mesoscopic normal ring, it is well known that its magnitude will oscillate as a function of the magnetic flux threading the ring. We will analyze the effect of the mutual inductance among the rings on the energies as well as the persistent currents in the rings. In the following discussion, we will assume that the rings are weakly coupled to each other and any coherence among the rings will not be considered. Such a coherence effect will be presented elsewhere.

Let us start generally to consider N ideally mesoscopic rings of radii R_i with $i = 1, \ldots, N$, which are set in the same plane and parallel to each other. Two rings are separated with a distance r between their centers. Since two rings are not contacted and there is no tunneling of the electrons among the rings, the particles moving in these rings are independent. The Hamiltonian of the system can be presented in the form

$$\hat{H} = \sum_i \frac{1}{2mR_i^2} \left(\hat{p}_{\theta_i} + \frac{e\Phi_i}{2\pi c} \right)^2 , \tag{1}$$

where $\hat{p}_{\theta_i} = -i\hbar\partial/\partial\theta_i$ and $i = 1,\ldots,N$ denotes the rings. The exact solution of the stationary-state Schrödinger equation $\hat{H}\Psi = \mathcal{E}\Psi$ can then be written as $\Psi = (2\pi)^{-N/2}\Pi_{i=1}^{N}\exp\,(i\ n_i\theta_i)$ with the energy eigenvalue being

$$\mathcal{E} = \sum_{i=1}^{N} \frac{\hbar^2}{2mR_i^2}\left(n_i + \frac{\Phi_i}{\Phi_0}\right)^2, \tag{2}$$

where $n_i = 0, \pm1, \pm2, \ldots$, which determines the energy levels and $\Phi_0 = hc/e$. The characteristic feature of this spectrum is that the levels are equidistant. The energy levels $\mathcal{E}$ and other related physical quantities are therefore periodic in Φ_0. For a time-independent flux Φ, the equilibrium currents (at $T = 0$) associated with the state n_i in ith ring is $I_{n_i}^{(i)} = -(e/2\pi R_i)v_{n_i}^{(i)} = -c(\partial\mathcal{E}_{n_i}/\partial\Phi_i)$, where $v_{n_i}^{(i)} = \partial\mathcal{E}/\hbar\partial k_{n_i} = (2\pi c/e)\partial\mathcal{E}/\partial\Phi_i$ is the velocity of the state n_i. The total current for each ring is the sum over the contributions $I_{n_i}^{(i)}$ of all states weighted with the appropriate occupation probability $I^{(i)} = \sum_{n_i} f_{n_i} I_{n_i}^{(i)}$ and can be written as

$$I^{(i)} = -\frac{e\hbar}{2\pi mR_i^2}\sum_{n_i}\left(n_i + \frac{\Phi_i}{\Phi_0}\right), \tag{3}$$

where we have set the temperature $T = 0$. An important condition for $I_{n_i}^{(i)}$ to be nonzero is that the wave functions of the charged and spin carriers should stay coherent along the circumference $2\pi R_i$ of the rings $i = 1,\ldots,N$. Note that the flux Φ_i in Eq. (3) includes the external flux and the contributions from the self-inductance and the mutual inductance among rings.

In Fig. 1, we plot three rings a, b, and c, where the first ring a carries a current I and the magnetic-field lines are also plotted. The second ring b is arranged so that it is concentric to the first ring. The third ring c is placed on the same plane of the first ring. Note that the magnetic-field lines penetrating ring b and ring c are opposite in direction. As a result, the induced currents are also opposite in sign. This means that the coefficients of the mutual inductance of rings a and b and rings a and c are opposite in sign. To calculate the mutual inductance of two rings on a plane, we first consider a more general situation where two current rings whose orientation in space is fixed, but whose relative separation r can be changed. It has been shown[12] that the mutual inductance $W_{12}(\mathbf{r}) = \oint_{l_1}\oint_{l_2}(1/r)d\boldsymbol{l}_1 d\boldsymbol{l}_2$ is the solution of the Laplace equation,

$$\nabla^2 W_{12}(\mathbf{r}) = 0, \tag{4}$$

or

$$W_{12}(r,\theta) = \sum_l (A_l r^l + B_l r^{-l-1})P_l(\cos\theta), \tag{5}$$

where $P_l(\cos\theta)$ is the Legendre polynomials of order l. The coefficients A_l and B_l can be determined once a solution of Eq. (4) is known. To get a special solution of Eq. (4), we arrange two rings concentric and parallel to each other and assume that the centers of two rings are separated by a distance r (which corresponds to rings a and b in Fig. 1). The mutual inductance of this configuration is easy to calculate and has an analytic form,

$$W_{12} = \frac{8\pi}{k}\sqrt{R_1R_2}\left\{\left(1 - \frac{k^2}{2}\right)\mathcal{K} - E\right\}, \tag{6}$$

where $k^2 = 4R_1R_2/[(R_1+R_2)^2 + r^2]$, and $\mathcal{K}$ and E are the first class and second class complete elliptic integrals, respectively, given by

$$\mathcal{K} = \int_0^{\pi/2}\frac{d\alpha}{\sqrt{1 - k^2\sin^2\alpha}},$$
$$E = \int_0^{\pi/2}\sqrt{1 - k^2\sin^2\alpha}\,d\alpha. \tag{7}$$

Since Eq. (6) is a solution (with $\theta = 0°$) of Eq. (4), the mutual inductance of two rings on the same plane ($\theta = 90°$), which corresponds to rings a and c in Fig. 1, can be calculated in terms of a power series of R/r. For example, when $R_1 = R_2 = R$ and $r > 2R$, one obtains[10]

$$W_{12}(r) = -\pi^2 R\left[\left(\frac{R}{r}\right)^3 + \frac{9}{4}\left(\frac{R}{r}\right)^5 + \frac{375}{64}\left(\frac{R}{r}\right)^7 + \cdots\right], \tag{8}$$

where r is the distance between the centers of two rings. As expected we obtained a negative coefficient of the mutual inductance. The coefficient of mutual inductance $|W_{12}|/R$ as a function of r/R is depicted in Fig. 2, from which we see that $|W_{12}|/R$ drops sharply as r becomes larger. The coefficient of self-inductance L cannot be obtained from W_{12} by taking the limit $r \to 0$. It has to be calculated separately by assuming that the ring has finite width so that L will not diverge. For a thin ring of wire having a radius R and radius R_0 for the wire, the coefficient of self-inductance L can be written as[13]

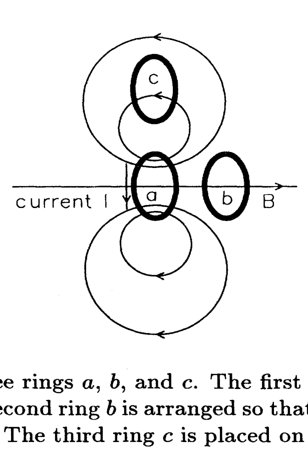

FIG. 1. Three rings a, b, and c. The first ring a carries a current I. The second ring b is arranged so that it is concentric to the first ring. The third ring c is placed on the same plane of the first ring. The magnetic-field lines generated by the first ring are also shown.

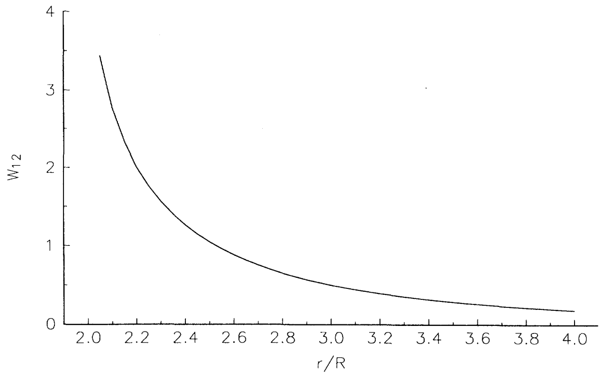

FIG. 2. The magnitude of the coefficient of the mutual inductance of two rings $|W_{12}|$ as a function of their separation r/R, where R is the radius of the ring.

$$L = 4\pi R\left(\ln\frac{8R}{R_0} - \frac{7}{4}\right), \tag{9}$$

where the magnetic susceptibility has been set to 1 and the condition $R_0 \ll R$ has been used in deriving the above equation.

We now arrange N rings on the same plane. Because of the self-inductance and the mutual inductance among the rings, the total flux threading the ith ring is given by

$$\Phi_i = \Phi_i^{\rm ext} + \frac{1}{2c}L_i I^{(i)} + \frac{1}{c}{\sum_j}' W_{ij} I^{(j)}, \tag{10}$$

where L_i is the coefficient of the self-inductance for the ith ring, W_{ij} is the coefficient of the mutual inductance between rings i and j, and $\Phi_i^{\rm ext}$ is the external flux for the ith ring. By substituting Φ_i into Eq. (3), the current can be divided into two parts $I^{(i)} = I_0^{(i)} + I_M^{(i)}$, where

$$I_0^{(i)} = -\frac{4\pi e c^2 \hbar}{e^2 L_i N_e^{(i)} + 2(2\pi)^2 c^2 m R_i^2} X_i \tag{11}$$

and

$$I_M^{(i)} = -\frac{2e^2 N_e^{(i)}}{e^2 L_i N_e^{(i)} + 2(2\pi)^2 c^2 m R_i^2} {\sum_j}' W_{ij} I^{(j)}, \tag{12}$$

where $N_e^{(i)}$ is the number of electrons in ith ring, $I_M^{(i)}$ is the correction contributed from the mutual inductance, and X_i is given by

$$X_i = \sum_{n_i}\left(n_i + \frac{\Phi_i^{\rm ext}}{\Phi_0}\right) = \begin{cases} N_e^{(i)}\frac{\Phi_i^{\rm ext}}{\Phi_0} & \text{for } N_e^{(i)} \text{ odd}, \quad -0.5 \le \frac{\Phi_i^{\rm ext}}{\Phi_0} < 0.5 \\ N_e^{(i)}\left(\frac{\Phi_i^{\rm ext}}{\Phi_0} - \frac{1}{2}\right) & \text{for } N_e^{(i)} \text{ even}, \quad 0.0 \le \frac{\Phi_i^{\rm ext}}{\Phi_0} < 1.0. \end{cases}$$

The ratio $I_M^{(i)}/I_0^{(i)}$ is given by

$$\frac{I_M^{(i)}}{I_0^{(i)}} = \frac{N_e^{(i)}\sum_j W_{ij} I^{(j)}}{\Phi_0 c X_i} \sim \frac{e\hbar N_e^{(i)}}{2\pi m R^2 c \Phi_0}\sum_j W_{ij}, \tag{13}$$

and the persistent current $I_L^{(i)}$ due to the self-inductance can be calculated from Eq. (11) and is equal to

$$I_L^{(i)} \sim \frac{e\hbar N_e^{(i)} L}{4\pi m R^2 c \Phi_0} I_0^{(i)}. \tag{14}$$

Now we estimate the current contribution of the self-inductance and the mutual inductance for a metal ring. In most conceivable experiments the wires making the ring have a finite cross section A. Thus, the number of transverse states (across the wire) below the Fermi energy E_F is on the order of

$$N_\perp \sim k_F^2 A, \tag{15}$$

and the total number of electrons is[14]

$$N_e \sim k_F L_0 N_\perp, \tag{16}$$

or more accurately

$$N_e = \frac{L_0 A}{3\pi^2} k_F^3, \tag{17}$$

where k_F is the Fermi wave vector and $L_0 = 2\pi R$ is the circumference of the ring. Equation (14) can be written as

$$I_L^{(i)} \sim \frac{e\hbar N_e^{(i)} L}{6\pi^2 m c \Phi_0} A \bar{L} k_F^3 I_0^{(i)}, \tag{18}$$

where $\bar{L} = L/R$. Hence the persistent current due to the self-inductance is proportional to the cross-section area of the ring, the Fermi vector, and the coefficient of the self-inductance. Similar expressions can be obtained for the mutual inductance. For a metal ring ($k_F \approx 1.2 \times 10^{10}$ m^{-1}, e.g., Au or Cu) of $L_0 = 1$ μm and $A = 0.02 \times 0.02$ μm^2,[3] one finds $N_e \approx 2.3 \times 10^7$. The coefficient of the self-inductance can be estimated from Eq. (9), which gives $L \approx 39R$. Finally, we obtain the persistent current due to the self-inductance $I_L = 0.2 I_0$ where I_0 is the persistent current without the self-inductance and the mutual inductance. For the contribution of the mutual inductance, we have $I_M \approx 0.01\sum_j (W_{ij}/R) I_0$ from Eq. (13). In Fig. 3, we plot $|I_M/I_0|$ as a function of the separation for a square array of rings. We see that when two rings are close together, i.e., $r < 2.2R$, the contribution from the mutual inductance is over 10%. However, when they are far apart, say, $r > 4R$, the effect is small, i.e., about 1%. For the gold ring studied in Ref. 5 the ring dimensions are the average diameter $2R \approx 3$ μm, the ring line width 0.09 μm, and

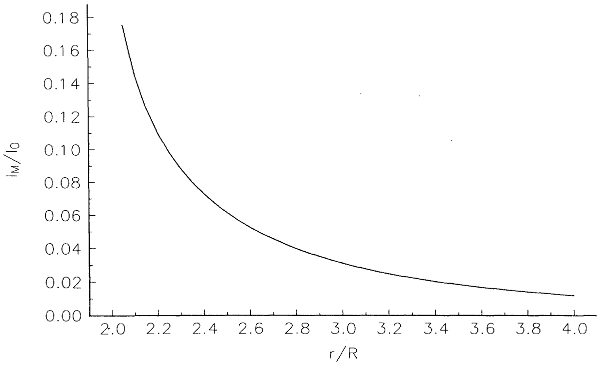

FIG. 3. The ratio of the persistent current due to the mutual inductance to the original persistent current $|I_M/I_0|$ as a function of their separation r/R for a square array of identical rings.

the thickness of the ring, 0.06 μm. In this case, the coefficient of self-inductance $L = 50R$ and the contribution due to the self-inductance is $I_L = 3.5 I_0$, which is much larger because of the larger cross-section area. Similarly, we obtain the persistent current due to the mutual inductance $I_M = 0.14 \sum_j (W_{ij}/R) I_0$. For an array of rings with separation $r \approx 3R$ (Ref. 4) we have $I_M \sim 0.4 I_0$.

To illustrate the self-consistent persistent currents in the rings, we first examine the contribution of the mutual inductance of two identical rings where Eqs. (11) and (12) can be written as

$$\left(\frac{LN_e^{(1)}}{2c\Phi_0} + \frac{2\pi mR^2}{e\hbar}\right) I_1 + \frac{WN_e^{(1)}}{c\Phi_0} I_2 = -X_1, \tag{19}$$

$$\left(\frac{LN_e^{(2)}}{2c\Phi_0} + \frac{2\pi mR^2}{e\hbar}\right) I_2 + \frac{WN_e^{(2)}}{c\Phi_0} I_1 = -X_2, \tag{20}$$

where R is the radius of the ring, L is the coefficient of self-inductance, and W is that of the mutual inductance. The solution of the above two equations is given by

$$I_1 = -\frac{1}{\Delta}\left\{\left(\frac{LN_e^{(2)}}{2c\Phi_0} + \frac{2\pi mR^2}{e\hbar}\right) X_1 - \frac{WN_e^{(1)}}{c\Phi_0} X_2\right\} \tag{21}$$

and

$$I_2 = -\frac{1}{\Delta}\left\{\left(\frac{LN_e^{(1)}}{2c\Phi_0} + \frac{2\pi mR^2}{e\hbar}\right) X_2 - \frac{WN_e^{(2)}}{c\Phi_0} X_1\right\}, \tag{22}$$

where Δ is given by

$$\begin{aligned}\Delta &= \frac{N_e^{(1)} N_e^{(2)}}{c^2\Phi_0^2}\left(\frac{L^2}{4} - W^2\right) + \frac{2\pi mR^2 L}{ec\hbar\Phi_0}\left(N_e^{(1)} + N_e^{(2)}\right) + \frac{4\pi^2 m^2 R^4}{e^2\hbar^2} \\ &= \left[\frac{2\pi mR^2}{e\hbar} + \frac{N_e^{(1)}}{c\Phi_0}\left(\frac{L}{2} \mp W\right)\right]\left[\frac{2\pi mR^2}{e\hbar} + \frac{N_e^{(2)}}{c\Phi_0}\left(\frac{L}{2} \pm W\right)\right] \mp \frac{2\pi mR^2 W}{ec\hbar\Phi_0}\Delta N_e,\end{aligned} \tag{23}$$

where $\Delta N_e = N_e^{(1)} - N_e^{(2)}$.

These expressions can be further simplified if, for example, we consider when both $N_e^{(1)}$ and $N_e^{(2)}$ are odd. We then have

$$\begin{aligned}I_1 &= -\frac{1}{\Delta}\left[\frac{2\pi mR^2}{e\hbar} + \frac{N_e^{(2)}}{c\Phi_0}\left(\frac{L}{2} - W\right)\right]\frac{N_e^{(1)}}{\Phi_0}\Phi^{\rm ext} \\ &\approx -\frac{4\pi ec^2\hbar}{e^2(L+2W)N_e^{(1)} + 2(2\pi)^2 c^2 mR^2}\frac{N_e^{(1)}}{\Phi_0}\Phi^{\rm ext},\end{aligned} \tag{24}$$

$$\begin{aligned}I_2 &= -\frac{1}{\Delta}\left[\frac{2\pi mR^2}{e\hbar} + \frac{N_e^{(1)}}{c\Phi_0}\left(\frac{L}{2} - W\right)\right]\frac{N_e^{(2)}}{\Phi_0}\Phi^{\rm ext} \\ &\approx -\frac{4\pi ec^2\hbar}{e^2(L+2W)N_e^{(2)} + 2(2\pi)^2 c^2 mR^2}\frac{N_e^{(2)}}{\Phi_0}\Phi^{\rm ext},\end{aligned} \tag{25}$$

where we have assumed $\Delta N_e \ll N_e$ so that the term involving ΔN_e in Eq. (22) can be neglected. Comparing Eq. (23) with Eq. (11), we see that the presence of the self-inductance suppresses the persistent current, which is expected from Lenz's law. However, the mutual inductance tends to enhance the persistent current (since W is negative). Because W is much smaller than L the net effect of the self-inductance and the mutual inductance is to reduce the persistent current. When both $N_e^{(1)}$ and $N_e^{(2)}$ are even, we get the same expression for persistent current except that $\Phi^{\rm ext}/\Phi_0$ is replaced by $\Phi^{\rm ext}/\Phi_0 - 1/2$.

For an array of N metal rings, since X_i in Eq. (11) is a periodic function of $\Phi^{\rm ext}$, the persistent current also has period Φ_0. If the number of electrons N_e is kept fixed but varies randomly from one ring to the other, the periodicity is halved for average magnetization. Finally the energies for the particle in each ring can be expressed as

$$\mathcal{E}^{(i)} = \sum_{n_i} \frac{\hbar^2}{2mR_i^2}\left(n_i + \frac{\Phi_i^{\rm ext}}{\Phi_0} + \frac{1}{2c}\frac{L_i}{\Phi_0} I^{(i)} + \frac{1}{c\Phi_0}\sum_{j\neq i} W_{ij} I^{(j)}\right)^2. \tag{26}$$

In addition, the energy of the system for the magnetic interaction is given by

$$\mathcal{E}^{\text{mag}} = \frac{1}{2c}\int \mathbf{j}\cdot\mathbf{A}dV = \frac{1}{2}\sum_{i,j}^{N} W_{ij}\frac{I^{(i)}I^{(j)}}{c^2}, \qquad (27)$$

where $W_{ii} \equiv L_i$. The total energy of the system is expressed as $\mathcal{E}^T = \sum_i \mathcal{E}^{(i)} + \mathcal{E}^{\text{mag}}$.

In summary, we have proposed a model for induced AB effect by mutual inductance of an array of mesoscopic rings. We evaluate the system in such a way that the motion of the particles is left in each of the rings and the cycle passage is coherent through the mutual inductance. The equilibrium of system persists due to the total flux, partially via mutual inductance, threading the rings. Our detailed calculations of the persistent currents indicate that the self-inductance suppresses the persistent current but the effect of mutual inductance among rings is to enhance it.

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