

Universal equilibrium currents in the quantum Hall fluid

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The equilibrium current distribution in a quantum Hall fluid that is subjected to a slowly varying confining potential is shown to generally consist of strips or channels of current, which alternate in direction, and which have universal integrated strengths. A measurement of these currents would yield direct independent measurements of the proper quasiparticle and quasihole energies in the fractional quantum Hall states.

I. INTRODUCTION

Since the discovery of the integral and fractional quantum Hall effects, a tremendous experimental and theoretical effort has been made to understand the nonequilibrium transport current in a two-dimensional (2D) interacting electron gas subjected to a strong perpendicular magnetic field. It is now known that the essential feature leading to the quantization of the Hall conductance is the existence of stable incompressible states at certain Landau-level filling factors. In contrast, relatively little attention has been given to the *equilibrium* current distribution in a 2D electron gas in the quantum Hall regime, and we shall show here that the existence of incompressible states at integral or fractional filling factors leads to some remarkable properties of the equilibrium current as well.

Central to the progress in understanding the quantum Hall fluid has been the ability to fabricate semiconductor nanostructures with highly controlled composition and doping, and the ability to subsequently pattern them, or provide them with metal gates, or both. The electron sheet density ρ in a GaAs heterojunction is typically between 10^{11} and 10^{12} cm^{-2} . The quantum Hall regime occurs at low temperatures and at field strengths where the magnetic length $\ell \equiv (\hbar c/eB)^{1/2}$ satisfies $\ell \approx \rho^{-1/2}$. At these high field strengths, the magnetic length is often small compared with the length scale over which the confining potential—produced by the remote donor centers and gates as well as by the electron gas itself—changes by a bulk energy gap. When this condition is satisfied, the confining potential is said to be *slowly varying*.

In this paper, we derive a general expression for the low-temperature equilibrium current distribution in a disorder-free 2D interacting electron gas, subjected to a strong perpendicular magnetic field, and in a slowly varying confining potential. Our expression, which becomes *exact* in the slowly varying limit, has two components: One is an *edge current* or a transverse “diffusion” current, which is proportional to the local density gradient, and the other is a *bulk current* or Hall current, which is proportional to the gradient of the self-consistent con-

fining potential. The edge current is a consequence of an inhomogeneous population of the orbitals in a given Landau level, whereas the bulk current originates from the electric-field-induced mixing of the Landau levels. At low temperatures, the confined quantum Hall fluid separates into compressible edge channels, where the density varies, and incompressible bulk regions, where the density is uniform.¹ As we shall show, the screening in the compressible regions becomes *perfect* at zero temperature, in the sense that the self-consistent confining potential, which includes exchange and correlation, becomes constant there. Hence, the edge current term contributes exclusively to the compressible regions, whereas the bulk component contributes only to the incompressible regions. The resulting current distribution therefore consists of strips or channels of current, and we shall show that the direction of these currents generally displays a striking alternating pattern. Furthermore, we shall show that the integrated current in each channel is *universal*, independent of the width, position, and shape of the channel, as well as the details of the confining potential.

We begin by calculating the current distribution with current-density functional theory, which rigorously accounts for the effects of electron-electron interaction through an exchange-correlation scalar potential V_{xc} and vector potential \mathbf{A}_{xc} . A study of the equilibrium current distribution that neglects exchange and correlation effects has been reported by us elsewhere,² and several other authors³ also have contributed to the understanding of equilibrium and nonequilibrium current distributions.

II. EQUILIBRIUM CURRENT DISTRIBUTION IN THE QUANTUM HALL FLUID

Let H be the effective single-particle Hamiltonian of current-density functional theory,⁴ which contains functionals of the density ρ and current density \mathbf{j} . We shall use the self-consistent solutions of the corresponding Kohn-Sham equations, $H\psi_\alpha = E_\alpha\psi_\alpha$, to define an

effective single-particle Green's function for the confined quantum Hall fluid,

$$G(\mathbf{r}, \mathbf{r}', s) \equiv \sum_{\alpha} \frac{\psi_{\alpha}(\mathbf{r}) \psi_{\alpha}^*(\mathbf{r}')}{s - E_{\alpha}}, \quad (1)$$

where s is a complex energy variable and ψ is a spinor with components ψ_{σ} ($\sigma = \uparrow, \downarrow$). Although this effective Green's function is generally different from the actual single-particle Green's function of the 2D interacting electron gas, it nevertheless yields the *exact* equilibrium number density,

$$\rho(\mathbf{r}) = \text{Tr} \oint \frac{ds}{2\pi i} f(s) G(\mathbf{r}, \mathbf{r}, s), \quad (2)$$

and orbital current density,

$$\begin{aligned} \mathbf{j}(\mathbf{r}) = & -\frac{e}{m^*} \text{Tr} \oint \frac{ds}{2\pi i} f(s) \\ & \times \lim_{\mathbf{r}' \rightarrow \mathbf{r}} \text{Re} \left(-i\hbar \nabla + \frac{e}{c} \mathbf{A} \right) G(\mathbf{r}, \mathbf{r}', s), \end{aligned} \quad (3)$$

at fixed chemical potential μ . Here e is the magnitude of the electron charge, m^* is the effective mass, and \mathbf{A} is a vector potential associated with the uniform external magnetic field $\mathbf{B} = B\mathbf{e}_z$. The contour in the complex energy plane is to be taken in the positive sense around the poles of G , avoiding the poles of the Fermi distribution function $f(\epsilon) \equiv [e^{(\epsilon - \mu)/k_B T} + 1]^{-1}$.

Next, we let V be a slowly varying potential from the remote donor centers and gates, and we write the Hamiltonian as $H = H^0 + H^1$, where

$$H^0 \equiv \frac{\Pi^2}{2m^*} + \frac{1}{2} g\mu_B \sigma_z B, \quad (4)$$

$\Pi \equiv \mathbf{p} + \frac{e}{c} \mathbf{A}$, and where

$$H^1 \equiv \frac{e}{2m^*c} (\mathbf{A}_{\text{xc}} \cdot \Pi + \Pi \cdot \mathbf{A}_{\text{xc}}) + V + V_H + V_{\text{xc}}. \quad (5)$$

Here $\mu_B \equiv e\hbar/2mc$ is the Bohr magneton, V_H is the Hartree potential, V_{xc} is a 2×2 diagonal matrix with elements $V_{\text{xc}}^{\sigma} \equiv (\delta E_{\text{xc}}/\delta \rho_{\sigma})_{\mathbf{v}}$, and

$$\mathbf{A}_{\text{xc}} \equiv -\frac{c}{\rho} \nabla \times \left(\frac{\delta E_{\text{xc}}}{\delta \mathbf{v}} \right)_{\rho_{\sigma}}. \quad (6)$$

We have omitted a term in H proportional to $\mathbf{j} \cdot \mathbf{A}_{\text{xc}}$ which is irrelevant in the slowly varying limit. The exchange-correlation energy $E_{\text{xc}}[\rho_{\sigma}, \mathbf{v}]$ is a functional of the ρ_{σ} and of the gauge-invariant vorticity $\mathbf{v} \equiv \nabla \times (\mathbf{j}_p/\rho)$, where \mathbf{j}_p is the paramagnetic part of the current density.

In the low-temperature ($k_B T \ll e^2/\kappa\ell$) phase of the uniform quantum Hall fluid, $\delta E_{\text{xc}}/\delta \rho$ and $\delta E_{\text{xc}}/\delta \mathbf{v}$ have discontinuities at certain field-dependent densities ρ_i . Analysis of the low-temperature density and current distributions is most conveniently carried out by separating the exchange-correlation potentials into regular and singular parts,

$$V_{\text{xc}}(\rho) = \bar{V}_{\text{xc}}(\rho) + \Delta V_{\text{xc}}(\rho), \quad (7)$$

$$\mathbf{A}_{\text{xc}}(\rho) = \bar{\mathbf{A}}_{\text{xc}}(\rho) + \Delta \mathbf{A}_{\text{xc}}(\rho). \quad (8)$$

Here $\bar{V}_{\text{xc}}(\rho)$ is simply $V_{\text{xc}}(\rho)$ with the discontinuities removed, and

$$\Delta V_{\text{xc}}(\rho) \equiv \sum_i \Delta V_{\text{xc},i} \theta(\rho - \rho_i) \quad (9)$$

is the remainder, where $\Delta V_{\text{xc},i}$ is the discontinuity in $V_{\text{xc}}(\rho)$ at ρ_i , and where $\theta(x)$ is the unit step function. Similarly,

$$\bar{\mathbf{A}}_{\text{xc}} \equiv -\frac{c}{\rho} \nabla \times \left(\frac{\delta E_{\text{xc}}}{\delta \mathbf{v}} \right)_{\rho_{\sigma}} \quad (10)$$

is the regular part of \mathbf{A}_{xc} , where $(\delta E_{\text{xc}}/\delta \mathbf{v})$ is equal to $(\delta E_{\text{xc}}/\delta \mathbf{v})$ with the discontinuities removed, and $\Delta \mathbf{A}_{\text{xc}}(\rho)$, a sum of delta functions, is the remainder.

The effective Green's function (1) may be written (again suppressing spin indices) as

$$\begin{aligned} G(\mathbf{r}, \mathbf{r}', s) = & G^0(\mathbf{r}, \mathbf{r}', s) \\ & + \int d^2 r'' G^0(\mathbf{r}, \mathbf{r}'', s) H^1(\mathbf{r}'') G(\mathbf{r}'', \mathbf{r}', s), \end{aligned} \quad (11)$$

where G^0 is the Green's function for the unconfined non-interacting electron gas. For large $|\mathbf{r} - \mathbf{r}'|$, the magnitude of G^0 falls off as $e^{-|\mathbf{r} - \mathbf{r}'|^2/4\ell^2}$, except at its poles. The Dyson equation (11) and the short-ranged nature of G^0 may then be used to evaluate the effective Green's function by a *gradient expansion* in the self-consistent confining potential. At each point in the fluid, the confining potential is approximated by a local potential plus a gradient. We sum the local confining potential terms to all orders and the gradients to first order. The orbital current density is found to be $\mathbf{j} = \mathbf{j}_{\text{edge}} + \mathbf{j}_{\text{bulk}}$, where

$$\begin{aligned} \mathbf{j}_{\text{edge}}(\mathbf{r}) = & -e\omega_c \ell^2 \sum_n \left(n + \frac{1}{2} \right) \nabla \rho_n(\mathbf{r}) \times \mathbf{e}_z \\ & + e\omega_c \rho(\mathbf{r}) \frac{\bar{\mathbf{A}}_{\text{xc}}(\mathbf{r})}{B}, \end{aligned} \quad (12)$$

$$\mathbf{j}_{\text{bulk}}(\mathbf{r}) = -\frac{e}{m^* \omega_c} \sum_{\sigma} \rho_{\sigma}(\mathbf{r}) \nabla V_{\text{eff}}^{\sigma}(\mathbf{r}) \times \mathbf{e}_z, \quad (13)$$

and $\omega_c \equiv eB/m^*c$ is the cyclotron frequency. The electron density is given by

$$\begin{aligned} \rho(\mathbf{r}) = & \frac{1}{2\pi\ell^2} \text{Tr} \sum_n f \left[\left(n + \frac{1}{2} + \frac{1}{2} \gamma \sigma_z \right) \hbar \omega_c \right. \\ & \left. + \Delta V_{\text{xc}}(\rho(\mathbf{r})) + V_{\text{eff}}(\mathbf{r}) \right], \end{aligned} \quad (14)$$

where $\gamma \equiv g\mu_B B/\hbar\omega_c$ is the dimensionless spin splitting, and $\rho_n(\mathbf{r})$ is simply the n th term in (14). These expressions differ from those obtained in Ref. 2 by the new edge current term proportional to $\bar{\mathbf{A}}_{\text{xc}}$, and by a self-consistent confining potential,

$$V_{\text{eff}}(\mathbf{r}) \equiv V(\mathbf{r}) + V_H(\mathbf{r}) + \bar{V}_{\text{xc}}(\mathbf{r}), \quad (15)$$

which is modified by exchange and correlation. The ΔV_{xc}

term in (14) leads to strips of density with fractional filling factor $\nu \equiv 2\pi\ell^2\rho$, in the same way that the discontinuities by $\hbar\omega_c$ in the chemical potential of the noninteracting system lead to strips at integral ν .

Because ρ and \mathbf{v} are slowly varying, (10) may be written as

$$\bar{\mathbf{A}}_{\text{xc}} = \left(\frac{m^*c^2}{e^2} \right) \frac{1}{\rho} \nabla \times \bar{\mathbf{M}}_{\text{xc}}, \quad (16)$$

where $\bar{\mathbf{M}}_{\text{xc}}$ is the exchange-correlation contribution to the orbital magnetization of a uniform 2D electron gas, with the discontinuities removed. The second edge current term in (12) may therefore be rewritten as $c\nabla \times \bar{\mathbf{M}}_{\text{xc}}$.

In a uniform Hall fluid with no confining potential, the orbitals in each Landau level are equally populated, so their associated current densities cancel each other and the equilibrium current density vanishes everywhere. The edge contribution (12) occurs because the Kohn-Sham orbitals ψ_α for a given Landau level are not equally populated in a nonuniform system. The bulk term (13) is the well-known transverse Hall current responding to a local electric field $\nabla V_{\text{eff}}/e$. This contribution occurs because the electric field changes the form of the orbitals and hence their associated current densities, which prevents the aforementioned cancellation, even if the density is uniform.

It is tempting to regard the edge current (12) as being diamagnetic and the bulk current (13) as paramagnetic, but this is not generally correct. However, in a simple quantum dot with a confining potential that is a non-decreasing function of radius, this identification is correct. In more complicated geometries, the edge or bulk current may be diamagnetic in one region and paramagnetic in another. We also note that the total conserved current in a magnetic field also includes a spin contribution, $\mathbf{j}_{\text{spin}} = -\frac{1}{2}cg\mu_B\nabla(\rho_\uparrow - \rho_\downarrow) \times \mathbf{e}_z$, which will not be discussed here further.

The precise nature of the compressible regions of the slowly confined quantum Hall fluid as $T \rightarrow 0$ is made evident by a remarkably simple *perfect screening theorem*. In a slowly varying system, (12) and (13) may be used to obtain a total energy functional $E[\rho_\sigma]$ of the density only. Minimizing this functional with respect to the ρ_σ ($\sigma=\uparrow, \downarrow$) under the constraint of fixed total particle number leads to the conditions

$$\mu_0^\sigma(\rho(\mathbf{r})) + \Delta V_{\text{xc}}(\rho(\mathbf{r})) + V_{\text{eff}}^\sigma(\mathbf{r}) = \text{const}, \quad (17)$$

where $\mu_0^\sigma(\rho)$ is the spin- σ chemical potential of a noninteracting Hall fluid of uniform density ρ in the same field \mathbf{B} . Of course, (17) does not apply to incompressible regions because μ_0 or ΔV_{xc} is discontinuous there. In fact, (17) shows that the discontinuity in $\mu_0 + V_{\text{eff}}$ at an incompressible strip is precisely equal to the electron chemical potential gap $\Delta\mu$ there. Because $\mu_0^\sigma(\rho)$ becomes piecewise constant as $T \rightarrow 0$, and $\Delta V_{\text{xc}}(\rho)$ is constant in each compressible region, *the self-consistent potentials $V_{\text{eff}}^\sigma(\mathbf{r})$ in each compressible region must also become uniform in this limit*. The screening is perfect in the compressible regions in the sense that the self-consistent potential be-

comes constant there. Therefore, we see that there exists a *complementarity* in the low-temperature phase of the slowly confined quantum Hall fluid: In the incompressible regions, ρ is uniform and V_{eff} varies, whereas in the compressible regions, ρ varies and V_{eff} is uniform.

The largest contributions to the current density come from the first term in (12) and from (13), and these have opposite signs because $\nabla\rho$ and ∇V_{eff} are antiparallel. Furthermore, the perfect screening in the edge regions makes the bulk current vanish there, and the incompressibility of the bulk regions causes the density gradient and also $\bar{\mathbf{A}}_{\text{xc}}$ to vanish in those regions. *Therefore, the current distribution generally consists of a series of strips or channels of distributed current, which follow the equipotentials of the self-consistent confining potential, and which alternate in direction*. The origin of this striking alternating pattern is the oscillations in the low-temperature orbital magnetization of a 2D electron gas, in a fixed magnetic field, as a function of density. These oscillations are of course caused by the same competition between the energy of a Landau level and its degeneracy that leads to the de Haas-van Alphen effect.

III. UNIVERSAL EQUILIBRIUM CURRENTS

The integrated currents follow straightforwardly from (12) and (13). For example, the magnitude of the integrated current at the edge of the filled Landau level n , assuming there is no bulk current present from incompressible strips at fractional filling factors, is

$$I_{\text{edge}} = (2n+1) \frac{e\omega_c}{4\pi} + c\Delta\bar{\mathbf{M}}_{\text{xc}}, \quad (18)$$

where $\Delta\bar{\mathbf{M}}_{\text{xc}}$ is the change in the z component of $\bar{\mathbf{M}}_{\text{xc}}$ across the edge channel. Similarly, the integrated equilibrium current in a bulk region of integral or fractional filling factor $\nu \equiv 2\pi\ell^2\rho$ has magnitude

$$I_{\text{bulk}} = \nu \frac{e}{h} \Delta\mu = c\Delta M, \quad (19)$$

where $\Delta\mu$ is the energy gap and ΔM the discontinuity in the magnetization there. In (19), we have used the fact that, according to the Maxwell relation $(\partial M/\partial\rho)_B = -(\partial\mu/\partial B)_\rho$, the discontinuities in μ and M are generally related by

$$\left(\frac{\Delta M}{\mu_B^*/2\pi\ell^2} \right) = 2\nu \left(\frac{\Delta\mu}{\hbar\omega_c} \right), \quad (20)$$

where $\mu_B^* \equiv (m/m^*)\mu_B$. The integrated currents (18) and (19) in the confined Hall fluid are clearly *universal*, depending only on properties of the uniform Hall fluid, and independent of the details of the confining potential.

The origin of this universality lies in the relation between the equilibrium orbital current and orbital magnetization. The equilibrium density is stationary, so $\nabla \cdot \mathbf{j} = 0$, and we may write the orbital current as

$$\mathbf{j} = c\nabla \times M\mathbf{e}_z, \quad (21)$$

where M is the z component of the local thermodynamic magnetization, and \mathbf{e}_z is a unit vector in the z direction. Therefore, between any two regions in the 2D electron gas having slow density variation, the magnitude of the integrated equilibrium current is simply

$$I = c \Delta M. \quad (22)$$

This result is valid regardless of whether there are additional incompressible channels present, and regardless of whether the system is slowly varying in the intermediate region.

Let $M = M_0 + M_{xc}$, where M_0 is the kinetic contribution. The ground-state kinetic energy per unit area, ignoring spin, is

$$\mathcal{E}_0 = f_0(\nu) \frac{\hbar\omega_c}{2\pi\ell^2}, \quad (23)$$

where

$$f_0(\nu) \equiv \frac{1}{2}[\nu]^2 + ([\nu] + \frac{1}{2})(\nu - [\nu]), \quad (24)$$

and where $[x]$ denotes the integer part of x . The kinetic chemical potential $\mu_0 \equiv (\partial\mathcal{E}_0/\partial\rho)_B$ is discontinuous at all integral fillings factors by an amount $\hbar\omega_c$. At $T = 0$, the orbital magnetization of noninteracting spinless electrons is

$$M_0 = \{[\nu] - (2[\nu] + 1)(\nu - [\nu])\} \frac{\mu_B^*}{2\pi\ell^2}, \quad (25)$$

as plotted in the dashed curve of Fig. 1. Note the discontinuity in $M_0/\mu_B^*\rho$ at integral filling factors, which is equal to twice the discontinuity in $\mu_0/\hbar\omega_c$ there. The

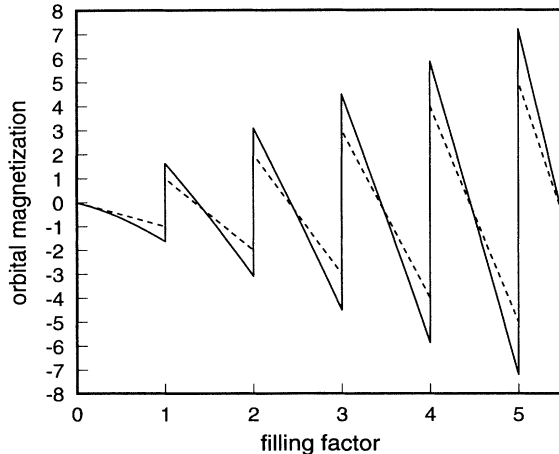


FIG. 1. Orbital magnetization of the zero-temperature Hall fluid as a function of filling factor $\nu \equiv 2\pi\ell^2\rho$, in units of $\mu_B^*/2\pi\ell^2$. Here $\mu_B^* \equiv e\hbar/2m^*c$ is the effective Bohr magneton, and ℓ is the magnetic length. The dashed curve is the magnetization of noninteracting spinless electrons, M_0 . The solid curve includes the effects of exchange at a field strength corresponding to $\alpha = 1$, where $\alpha \equiv e^2/\hbar\omega_c\kappa\ell$ is the dimensionless Coulomb interaction strength.

change in M_0 across the edge of a filled Landau level leads to the first term in (18), and the change in the regular part of M_{xc} leads to the second term.

The interaction energy per area of a quantum Hall fluid generally depends on ρ and B separately, but with the assumption of negligible Landau-level mixing by the interactions, we may write

$$\mathcal{E}_{xc} = f_{xc}(\nu) \frac{e^2/\kappa\ell}{2\pi\ell^2}, \quad (26)$$

where κ is the bulk dielectric constant. In terms of f_{xc} ,

$$M_{xc} = \alpha(-3f_{xc} + 2\nu f'_{xc}) \frac{\mu_B^*}{2\pi\ell^2}, \quad (27)$$

where

$$\alpha \equiv \frac{e^2/\kappa\ell}{\hbar\omega_c} \quad (28)$$

is the dimensionless Coulomb interaction strength. The zero-temperature orbital magnetization of spinless electrons, including the effects of exchange, is plotted in the solid curve of Fig. 1. The exchange contribution to the magnetization is calculated in the Appendix.

Note that the interaction strength varies with magnetic field as $\alpha \sim B^{-\frac{1}{2}}$. The interactions will cause negligible Landau-level mixing if $\alpha \ll 1$; that is, in the strong-magnetic-field limit.

IV. INTEGRATED CURRENTS IN THE LOWEST LANDAU LEVEL

We now calculate the integrated current at the edge of the lowest spin-polarized Landau level ($n\sigma = 0\downarrow$), between $\nu = 1^-$ and $\nu = 0$, to first order in α . We shall consider for convenience a long Hall bar oriented in the y direction, with a confining potential $V(x)$ that varies in the x direction only (except near the two ends of the Hall bar, which we avoid). The current is then directed along the Hall bar in the y direction. Near $\nu = 0$, the ground-state energy is expected to be close to that of a Wigner crystal, so $f_{xc} \propto -\nu^{\frac{3}{2}}$ there. Particle-hole symmetry in the lowest Landau level implies

$$f_{xc}(1-\nu) = f_{xc}(\nu) + (1-2\nu)f_{xc}(1), \quad (29)$$

where $f_{xc}(1) = -(\pi/8)^{\frac{1}{2}}$. Hence, we find

$$M_{xc}(1^-) = -\alpha \left(\frac{\pi}{8}\right)^{\frac{1}{2}} \frac{\mu_B^*}{2\pi\ell^2}, \quad (30)$$

and therefore

$$I_{0\downarrow} = -\left(1 + \alpha\sqrt{\frac{\pi}{8}}\right) \frac{e\omega_c}{4\pi}. \quad (31)$$

Expressions for the integrated currents at the edge of higher filled Landau levels, obtained by a similar analysis, are presented in Table I.

TABLE I. Magnitude of the integrated orbital current at the edge of the filled Landau level n , apart from a spin-degeneracy factor of 2, for the case of negligible spin splitting ($\gamma \ll 1$). Currents are in units of $e\omega_c/4\pi$.

n			
0	1	2	3
$1 + \alpha \frac{1}{2} \left(\frac{\pi}{2}\right)^{\frac{1}{2}}$	$3 + \alpha \frac{1}{8} \left(\frac{\pi}{2}\right)^{\frac{1}{2}}$	$5 + \alpha \frac{265}{128} \left(\frac{\pi}{2}\right)^{\frac{1}{2}}$	$7 + \alpha \frac{1371}{512} \left(\frac{\pi}{2}\right)^{\frac{1}{2}}$

Next, we shall assume that there is an incompressible strip at an odd-denominator filling factor $\nu_0 = \frac{1}{q}$ present in this edge channel. At filling factors very close to ν_0 , the ground state is expected to be a Laughlin state plus a Wigner crystal of fractionally charged quasiparticles or quasiholes. Let ϵ_L be the interaction energy per electron in the Laughlin state at ν_0 , in units of $e^2/\kappa\ell$, and let ϵ_{qp} and ϵ_{qh} be the associated gross quasiparticle and quasihole energies in the same units. Then close to ν_0 ,

$$f_{xc}(\nu) = \begin{cases} \epsilon_L/q + q\epsilon_{qh}(\nu_0 - \nu) + \dots & \text{for } \nu \leq \nu_0 \\ \epsilon_L/q + q\epsilon_{qp}(\nu - \nu_0) + \dots & \text{for } \nu \geq \nu_0. \end{cases} \quad (32)$$

For example, Morf and Halperin⁵ have evaluated ϵ_L , ϵ_{qp} , and ϵ_{qh} using trial wave functions at $\nu_0 = \frac{1}{3}$; they find $\epsilon_L = -0.410$, $\epsilon_{qp} = -0.132$, and $\epsilon_{qh} = 0.231$. According to (32), $\mu_{xc} \equiv (\partial\mathcal{E}_{xc}/\partial\rho)_B$ is discontinuous by an amount $\Delta\mu = qE_{\text{gap}}$, where $E_{\text{gap}} \equiv \epsilon_{qp} + \epsilon_{qh}$ is the energy required to create a single well-separated quasiparticle-quasihole pair. The interaction contribution to the orbital magnetization near ν_0 is therefore

$$M_{xc} = \begin{cases} -2\alpha\tilde{\epsilon}_{qh}(\mu_B^*/2\pi\ell^2) & \text{for } \nu = \nu_0^- \\ 2\alpha\tilde{\epsilon}_{qp}(\mu_B^*/2\pi\ell^2) & \text{for } \nu = \nu_0^+, \end{cases} \quad (33)$$

where $\tilde{\epsilon}_{qh} = \epsilon_{qh} + 3\epsilon_L/2q$ and $\tilde{\epsilon}_{qp} = \epsilon_{qp} - 3\epsilon_L/2q$ are the *proper* quasihole and quasiparticle energies defined in Ref. 5. For example, $\tilde{\epsilon}_{qh} = 0.026$, $\tilde{\epsilon}_{qp} = 0.073$, and $E_{\text{gap}} = 0.099$ are the calculated values for $\nu_0 = \frac{1}{3}$. The discontinuity in $M_{xc}/\mu_B^*\rho$ at ν_0 is equal to twice the discontinuity in $\mu_{xc}/\hbar\omega_c$, as expected.

Let I_1 be the integrated current in the edge channel between $\nu = 1^-$ and $\nu = \nu_0^+$, I_2 be the current in the incompressible strip at $\nu = \nu_0$, and I_3 be the current in the edge channel between $\nu = \nu_0^-$ and $\nu = 0$. According to (12) or (22),

$$I_1 = - \left[1 - \frac{1}{q} + \alpha \left(\sqrt{\frac{\pi}{8}} + 2\tilde{\epsilon}_{qp} \right) \right] \frac{e\omega_c}{4\pi}. \quad (34)$$

Similarly, I_2 is given by $2\alpha E_{\text{gap}}(e\omega_c/4\pi)$. Restoring units to E_{gap} leads to

$$I_2 = \frac{e}{h} E_{\text{gap}} = \nu \frac{e}{h} \Delta\mu, \quad (35)$$

as in (19). *The magnitude of the integrated equilibrium current in any incompressible strip is $\nu(e/h)\Delta\mu$, where $\Delta\mu$ is the electron chemical potential gap in the uniform quantum Hall fluid at filling factor ν .* Finally,

$$I_3 = - \left[\frac{1}{q} + 2\alpha\tilde{\epsilon}_{qh} \right] \frac{e\omega_c}{4\pi}. \quad (36)$$

Note the alternating signs of the integrated currents, and that their sum agrees with (31), even though there is now an incompressible strip at ν_0 . Furthermore, a measurement of the currents (34–36) would provide direct *independent* measurements of the fundamental quantities $\tilde{\epsilon}_{qp}$, $\tilde{\epsilon}_{qh}$, and, of course, E_{gap} .

V. DISCUSSION

We would like to correct here a common misconception in the literature regarding the equilibrium currents in the two-dimensional electron gas. For definiteness, consider a noninteracting electron gas in a channel oriented along the y direction, with a confining potential $V(x)$ that varies in the x direction only. It is commonly believed that the sign of the current density is determined by the gradient of the local confining potential or the gradient of the local Landau bands, but this is incomplete, as there is also an edge current contribution (12) proportional to the density gradient, which has the opposite sign. The source of the confusion is that although the net integrated current

$$I_\alpha \equiv \int_{-\infty}^{\infty} dx j_\alpha^y(x) \quad (37)$$

associated with a *single* given Landau-gauge orbital ψ_α , which is centered about x , is in fact proportional to $V'(x)$, the current density $j_\alpha^y(x)$ is distributed over a region of the order of the magnetic length and has spatial oscillations. Because the equilibrium current density $j(x)$ at a point x includes contributions from *different parts* of many nearby orbitals, $j(x)$ may have either sign.

We now discuss the strong-magnetic-field limit of the equilibrium current distribution. Because the interaction energy of the Hall fluid scales with the magnetic field as $e^2/\kappa\ell$, whereas the kinetic energy scales as $\hbar\omega_c$, the exchange-correlation contribution to the current is smaller than the kinetic contribution by a factor of α [see Eq. (28)]. Therefore, in the strong-magnetic-field limit, where $\alpha \rightarrow 0$, the current distribution approaches that of a noninteracting Hall fluid. If, in addition, the density remains fixed as $B \rightarrow \infty$, and no confining potential is present, the many-body wave function will lie entirely within the lowest Landau level, and the current density, given by (12), becomes

$$\mathbf{j} = -\frac{e\hbar}{2m} \nabla\rho \times \mathbf{e}_z. \quad (38)$$

This formula, first derived by Girvin and MacDonald,⁶ is *exact* for lowest Landau-level many-body states.

Finally, we would like to make a comment regarding our use of current-density functional theory,⁴ which employs functionals of the density and current density, instead of conventional density-functional theory, which uses the density only. Both formulations may be used to calculate the density and total energy in a strong

magnetic field, but only current-density functional theory rigorously determines the current. In fact, the current has an interaction contribution directly proportional to \mathbf{A}_{xc} , the exchange-correlation vector potential unique to current-density functional theory. In general, the density calculated with current-density functional theory will also depend on \mathbf{A}_{xc} , but in the slowly varying limit considered here, this contribution is of second order in the gradient of the confining potential.

In conclusion, we have derived an expression for the low-temperature equilibrium current distribution in a confined quantum Hall fluid. The current distribution has two components, (12) and (13), which contribute exclusively to the compressible and incompressible regions, respectively, and have opposite signs. The current distribution therefore consists of strips or channels of current, which alternate in direction. The integrated current in each channel is also shown to be universal, reflecting the change in local orbital magnetization across its boundaries, and it is noted that measurement of the integrated currents would yield direct independent measurements of the proper quasiparticle and quasihole energies in the fractional quantum Hall states.

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APPENDIX: EXCHANGE CONTRIBUTION TO THE ORBITAL MAGNETIZATION OF THE HALL FLUID

In this appendix, we calculate the exchange contribution to the zero-temperature orbital magnetization of a two-dimensional system of spinless electrons. The exchange energy per area is given by

$$\mathcal{E}_x(\nu) = -\frac{1}{2} \int_0^\infty dx e^{-\frac{1}{2}x^2} \times \left[\sum_{j=0}^{n-1} L_j^0\left(\frac{1}{2}x^2\right) + (\nu - n)L_n^0\left(\frac{1}{2}x^2\right) \right]^2 \frac{e^2/\kappa\ell}{2\pi\ell^2}, \quad (\text{A1})$$

where $L_j^a(x)$ is a Laguerre polynomial and $n \equiv [\nu]$ is the integer part of the filling factor ν . It is convenient for our purposes to write (A1) as

$$\mathcal{E}_x(\nu) = [A_n + (\nu - n)B_n + (\nu - n)^2C_n] \frac{e^2/\kappa\ell}{2\pi\ell^2}, \quad (\text{A2})$$

where

$$\begin{aligned} A_n &= -\frac{1}{2} I_{n-1, n-1}^{11}, \\ B_n &= -I_{n, n-1}^{01}, \\ C_n &= -\frac{1}{2} I_{n, n}^{00}, \end{aligned} \quad (\text{A3})$$

TABLE II. Coefficients of the exchange energy per unit area and the exchange contribution to the orbital magnetization, as defined in the Appendix.

n	A_n	B_n	C_n	D_n	E_n	F_n
0	0.0000	0.0000	-0.6267	0.0000	0.0000	-0.6267
1	-0.6267	-0.6267	-0.4700	0.6267	-1.2533	-0.4700
2	-1.7233	-1.0183	-0.4015	1.0967	-2.1933	-0.4015
3	-3.1431	-1.3219	-0.3598	1.4981	-2.9962	-0.3598
4	-4.8248	-1.5770	-0.3308	1.8579	-3.7159	-0.3308
5	-6.7326	-1.8009	-0.3089	2.1887	-4.3775	-0.3089

and

$$I_{mn}^{ab} \equiv \int_0^\infty dx e^{-\frac{1}{2}x^2} L_m^a\left(\frac{1}{2}x^2\right) L_n^b\left(\frac{1}{2}x^2\right). \quad (\text{A4})$$

The coefficients (A3) may be evaluated analytically by a simple generating function method. The identity

$$\sum_{n=0}^\infty L_n^a(x) y^n = \frac{e^{\frac{xy}{1-y}}}{(1-y)^{a+1}} \quad (\text{A5})$$

may be used to obtain the relation

$$\sum_{m=0}^\infty \sum_{n=0}^\infty I_{mn}^{ab} y^m y'^n = \sqrt{\frac{\pi}{2}} (1-y)^{-a-\frac{1}{2}} (1-y')^{-b-\frac{1}{2}} \times (1-yy')^{-\frac{1}{2}}. \quad (\text{A6})$$

Expanding the right-hand side of (A6) in a double power series leads to

$$I_{mn}^{ab} = \sqrt{\frac{\pi}{2}} \sum_{j=0}^{\min(m,n)} (-1)^{m+n+j} \binom{-a-\frac{1}{2}}{m-j} \times \binom{-b-\frac{1}{2}}{n-j} \binom{-\frac{1}{2}}{j}, \quad (\text{A7})$$

where

$$\binom{a}{j} \equiv \frac{a(a-1)\cdots(a-j+1)}{j!}. \quad (\text{A8})$$

Here $\min(m, n)$ is equal to m if $m \leq n$ or to n otherwise. The exchange energy coefficients for the lowest few Landau levels are presented in Table II.

The exchange contribution to the orbital magnetization, obtained by using (27) and (A1), may be written as

$$M_x(\nu) = \alpha [D_n + (\nu - n)E_n + (\nu - n)^2F_n] \frac{\mu_B^*}{2\pi\ell^2}, \quad (\text{A9})$$

where $D_n \equiv -3A_n + 2nB_n$, $E_n \equiv -B_n + 4nC_n$, and $F_n \equiv C_n$. These exchange magnetization coefficients are also given in Table II.

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