

Anderson-Mott transition as a quantum-glass problem

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We combine a recent mapping of the Anderson-Mott metal-insulator transition on a random-field problem with scaling concepts for random-field magnets to argue that disordered electrons near an Anderson-Mott transition show glasslike behavior. We first discuss attempts to interpret experimental results in terms of a conventional scaling picture, and argue that some of the difficulties encountered point towards a glassy nature of the electrons. We then develop a general scaling theory for a quantum glass, and discuss critical properties of both thermodynamic and transport variables in terms of it. Our most important conclusions are that for a correct interpretation of experiments one must distinguish between self-averaging and non-self-averaging observables, and that dynamical or temperature scaling is not of power-law type but rather activated, i.e., given by a generalized Vogel-Fulcher law. Recent mutually contradicting experimental results on Si:P are discussed in light of this, and new experiments are proposed to test the predictions of our quantum-glass scaling theory.

I. INTRODUCTION

The metal-insulator transition that is observed in doped semiconductors and other disordered solids is not fully understood, despite almost 20 years of intense experimental and theoretical efforts.¹ There is strong evidence for both disorder and electron-electron interactions to play an important role at this transition, which is called an Anderson-Mott transition (AMT) to distinguish it from a disorder dominated pure localization or Anderson transition on one hand, and from a correlation dominated pure Mott transition on the other hand.² This interplay between disorder and interactions makes the AMT a very hard problem in statistical mechanics.

On the theoretical side, until very recently virtually all approaches studied the problem in the vicinity of two dimensions ($d = 2$) by generalizing Wegner's theory for the Anderson transition.³ These theories¹ have led to a classification of the AMT into various universality classes that depend, *inter alia*, on the presence or absence of spin-orbit scattering, magnetic impurities, magnetic fields, etc. For most of these universality classes, perturbative renormalization-group methods lead to a critical fixed point in $d = 2 + \epsilon$ dimensions, and standard critical behavior with power-law scaling is found. However, the framework of these theories does not allow for an order parameter (OP) description of the AMT, and does not lead to a simple Landau or mean-field theory. As a result, the physics driving the AMT remains relatively obscure in this approach, compared to standard theories for other phase transitions. An alternative line of attack has recently been explored by the present authors.⁴⁻⁶ In these papers we showed that the same model⁷ that was

used for the $2 + \epsilon$ expansion allows for an OP description of the AMT with the tunneling density of states (DOS) as the OP, and for a simple Landau theory that yields the critical behavior exactly above the upper critical dimension $d_c^+ = 6$. Furthermore, it was shown that the problem has random-field aspects and is closely related to a random-field Ising model. The structure of that theory also suggests that for certain parameter values, in particular for weak effective electron-electron interactions, the OP driven AMT can be pre-empted by a different kind of metal-insulator transition. The most obvious candidate is the Anderson transition, where the DOS or OP is un-critical. These results lead to the suspicion that in large parts of parameter space important physical features of the AMT have been missed both in the low-dimensional theories, and in the interpretations of experiments that were based on these theories.

On the experimental side, the best studied systems are doped semiconductors, and some of the most detailed and careful experiments have been done on Si:P. In what became a benchmark experiment in the field, Paalanen, Rosenbaum, Thomas and collaborators⁸⁻¹¹ (to be referred to as the Bell experiment) performed transport measurements at temperatures, T , down to 2.7 mK, and used a stress tuning technique to drive a barely insulating sample through the metal-insulator transition. These experiments concluded that the critical P concentration, n_c , in this system is close to $3.7 \times 10^{18} \text{ cm}^{-3}$, and that the conductivity, extrapolated to $T = 0$, vanishes with a critical exponent $s \approx 0.5$, with error bars of about 3% for n_c and about 14% for s . The measured value of s proved hard to understand theoretically, and the critical behavior of Si:P is still considered enigmatic.¹ More

disturbingly, however, a similar experiment on the same system¹² (to be referred to as the Karlsruhe experiment) has recently produced results that are inconsistent with those of Refs. 8–10, viz., n_c close to $3.5 \times 10^{18} \text{ cm}^{-3}$, and $s \approx 1.3$. These disagreements are far greater than the error bars quoted, and have not been settled between the respective experimental groups.¹³ Measurements of the Hall coefficient have also led to mutually contradicting results. Dai, Zhang, and Sarachik¹⁴ have reported evidence for a divergent Hall coefficient in Si:P, in contrast to previous results¹⁵ that had found the Hall coefficient to remain finite. Finally, different experimental results have been obtained for the crossover exponent of the (longitudinal) conductivity in an external magnetic field,¹⁶ but in this case no direct comparisons on the same system have been done.

The fact that the most careful experiments on the best studied systems lead to contradictory results is, taken at face value, extremely discouraging. An important question is what the source of these discrepancies is. One relevant consideration is the temperature range covered by the respective experiments, and the lowest temperature reached. This is important since any measurement of static critical exponents at the AMT involves an extrapolation to $T = 0$. For instance, the discrepancy concerning the Hall effect results has been blamed in Ref. 14 on the earlier experiments not having reached sufficiently low temperatures. In the case of the conductivity exponent in Si:P, however, extrapolation problems by themselves are not sufficient to explain the disagreement. References 8, 9, and 12 agree that the extrapolated conductivity shows a “tail” at phosphorus concentrations $n < 3.7 \times 10^{18} \text{ cm}^{-3}$, but disagree about whether or not this tail contains the salient physics. The Bell group observed strong sample-to-sample variations of the conductivity in the tail region at the lowest temperatures, and concluded that the tail was due to sample inhomogeneities and should be discarded. The Karlsruhe group, on the other hand, claims that the tail represents the asymptotic critical region of the AMT. The fact that no strong sample-to-sample fluctuations were observed in this case should, however, not be overweighted: The lowest temperature reached in the Karlsruhe experiment was $T = 62.5 \text{ mK}$, while the strong fluctuations in the Bell data set in only at lower temperatures. Also, sample-to-sample variations in the T dependence of the conductivity does seem to set in just at the lowest T reached in the Karlsruhe experiment, a phenomenon attributed in Ref. 12 to “thermal decoupling,” i.e., problems in reaching and maintaining equilibrium. To summarize the experimental situation regarding the conductivity in Si:P, one might say that unusual features, variably described as “rounding,” “smearing,” “thermal decoupling,” etc. were observed at low temperatures close to the critical point. These anomalies became stronger upon approaching the critical point and lowering the temperature, and the main discrepancy between the Bell and Karlsruhe experiments can be traced to different assumptions concerning their significance, i.e., whether or not the “tail” should be taken seriously.

Apart from the conductivity, unusual behavior has also

been observed in thermodynamic properties of doped semiconductors. Both the magnetic susceptibility, χ_m , and the specific heat, c_V , show a pronounced non-Fermi-liquid behavior.^{17–19} This behavior is observed near the transition as well as far away from it, is not obviously related to any critical phenomena near the AMT, and is usually explained in terms of local moments.¹⁸ However, Bhatt and Fisher²⁰ have argued that once local-moment/local-moment interactions are taken into account, one obtains singularities in χ_m and c_V that are significantly weaker than those observed experimentally. This suggests that phenomena other than local moments might contribute to the observed anomalies in the thermodynamic susceptibilities.

If one insists on a conventional theoretical interpretation of the conductivity data in terms of power-law scaling,¹ then the inescapable conclusion is that either the Bell experiment erroneously discarded the data in the true critical region, or the Karlsruhe experiment mistook spurious effects for the critical behavior. If this was the case, then a careful scaling analysis of both data sets and, if necessary, new and more accurate experiments, should be able to show that the conductivity, and possibly other quantities, show scaling behavior in one region but not in the other, thus settling the issue. There is, however, another possibility. If the theoretical suggestion^{5,6} that the AMT has random-field aspects is correct, then one would expect glasslike features and unconventional scaling similar to what has been predicted^{21,22} and observed²³ in classical random-field magnets.

In this paper we explore these possibilities. In Sec. II we assume conventional scaling, and check whether a scaling analysis of the existing data can settle the disagreement between the experimentalists. We find that it cannot, mostly due to an insufficient temperature range and the lack of precision experiments that probe the critical behavior of more than one quantity in a given material. Our analysis suggests, however, various experiments that might be able to tell which, if any, of the two doping regions that have been suggested to be the critical one, displays conventional scaling behavior. In Sec. III we assume instead that the AMT features activated scaling of the type found in random-field magnets, appropriately modified for a quantum problem. Accordingly, we first develop a general scaling description of a quantum glass, and then work out predictions for the critical behavior of various observables. We check to what extent the existing data are consistent with these predictions and propose new experiments to further investigate this issue.

II. CONVENTIONAL SCALING

A. Homogeneity laws

Let us recall the homogeneity laws for the tunneling or single-particle DOS, N , and the conductivity, σ , that express conventional power-law scaling,¹

$$N(t, T) = b^{-\beta/\nu} N(tb^{1/\nu}, Tb^z) \quad , \quad (2.1)$$

$$\sigma(t, T) = b^{-s/\nu} \sigma(tb^{1/\nu}, Tb^z) \quad . \quad (2.2)$$

Here $t = n/n_c - 1$ denotes the dimensionless distance from the critical point, and T is the temperature. Since in a quantum problem temperature and frequency scale the same way,²⁴ one obtains the same homogeneity laws at $T = 0$ with T replaced by ω , where ω is the external frequency in the case of the conductivity, and the distance in energy space from the Fermi level, i.e., the bias voltage in a tunneling experiment, in the case of the DOS. β and s are the critical exponents for the DOS and the conductivity, respectively. ν is the correlation length exponent, z is the dynamical critical exponent, and b is an arbitrary length scale factor. Analogous homogeneity laws can be written down for all other quantities of interest.¹ Here we focus on the DOS, since it is the order parameter for the AMT according to our recent theory⁴⁻⁶ and since it is easily measured, and on the conductivity since it is the most obviously interesting observable in the context of a metal-insulator transition. Note that throughout this paper we ignore the possibility of significant corrections to scaling.^{1,25}

Putting b equal to the correlation length, $b = \xi \sim t^{-\nu}$, we obtain

$$N(t, T) = t^\beta F_N(T/t^{\nu z}) \quad , \quad (2.3)$$

and

$$\sigma(t, T) = t^s F_\sigma(T/t^{\nu z}) \quad , \quad (2.4)$$

where F_N and F_σ are scaling functions.

These homogeneity laws predict that N and σ are functions of a particular combination of their two arguments t and T , and they have been derived from the traditional description of the AMT.¹ They also follow from a perturbative treatment of the order parameter theory put forward recently.^{5,6} As has been mentioned in these references, it is likely that in the second case the perturbative results are misleading, and that the dynamical scaling in the physical dimension $d = 3$ is of activated rather than of power-law type. We will explore the consequences of activated scaling in Sec. III below. For now we assume that the power-law scaling expressed by Eqs. (2.3) and (2.4) is correct asymptotically close to the transition observed in Si:P. This could be due to either the OP theory of the AMT not being applicable to Si:P, or to the perturbative analogy between the AMT and classical random-field magnets being misleading. In this section we investigate the experimental consequences of this assumption. In doing so it is important to realize that the theoretical values of the critical exponents are *not* known.²⁶

Another quantity of interest is the correlation function of the local, unaveraged DOS,

$$C(t, T; \mathbf{x} - \mathbf{y}) = \langle N(\mathbf{x})N(\mathbf{y}) \rangle \quad , \quad (2.5)$$

which according to Refs. 4-6 is the order parameter susceptibility for the AMT. In Eq. (2.5), $N(\mathbf{x})$ is the local DOS at the Fermi energy, and $\langle \dots \rangle$ denotes the impurity average. The Fourier transform of C obeys the homogeneity law

$$C(t, T; q) = b^{2+\theta-\eta} C(tb^{1/\nu}, Tb^z; qb) \quad . \quad (2.6)$$

Here η is the usual critical exponent that governs the spatial dependence of the OP susceptibility, and $\theta \geq 0$ is minus the scale dimension of a dangerous irrelevant variable in the random-field problem.^{5,6} In the presence of random-field effects, the perturbative value of θ is 2, while in the absence of random-field effects one has $\theta = 0$. Even though the perturbative result cannot be correct, at least not in $d = 3$, it is very likely that the presence of θ will overcompensate η , and make the scale dimension of C larger than 2. This leads to a strong divergence of $C(t = 0, T = 0; q \rightarrow 0)$, the analogue of which has been observed in classical random-field magnets.²⁸ Alternatively, one can consider the homogeneous correlation to find,

$$C(t, T; q = 0) = T^{-(2+\theta-\eta)/z} F_C(T/t^{\nu z}) \quad , \quad (2.7)$$

with F_C a scaling function. Equation (2.7) predicts a strong divergence as $T \rightarrow 0$, the presence of which can be checked experimentally. We will come back to this point.

B. Scaling analysis of the Bell experiment

Let us now check whether the data of Ref. 10 are consistent with the above homogeneity law, Eq. (2.4). From a double logarithmic plot of the extrapolated $T = 0$ conductivity the experimentalists determined the critical value of the stress, S_c , in their stress-tuning experiment to be $S_c = 6.5 \pm 0.2$ kbar, and the conductivity exponent $s \approx 0.5$. From the T dependence of σ at a stress value estimated to be close to the critical one, they also inferred the value of $\nu z \approx 2.7$. In Ref. 1 it was shown that with these exponent values the data do not obey scaling. It must be emphasized, however, that the error in the value of νz is at least 30%.¹⁰ Accordingly, let us keep $S_c = 6.5$ kbar and $s = 0.5$ fixed, but let νz float freely to produce the best scaling plot. The result is shown in Fig. 1, which replots data from Fig. 1 of Ref. 10 in a way suggested by Eq. (2.4). With $\nu z = 2.13$ one obtains a scaling plot that is better than the one in Ref. 1, although its absolute quality is not very good.

We next check whether the quality of the scaling plot can be improved by changing S_c . The largest value of S_c that is consistent with the error bars given in Ref. 10 is $S_c = 6.7$ kbar. Indeed, inspection of Fig. 1 in Ref. 10 shows that at $S = 6.59$ kbar the data still show the curvature at the lowest temperatures that the authors considered characteristic of the insulating regime. Let us therefore assume that the next higher stress value, $S = 6.71$ kbar, was the critical one. With this value for S_c , we found that the best scaling plot is achieved with $s = 0.29$ and $\nu z = 1.82$, which is shown in Fig. 2. The quality of the scaling plot is now much better.

As Figs. 1 and 2 show, the dynamical scaling plot favors a large value of the critical stress, S_c , and a correspondingly small value of the conductivity exponent s . This requires some comments in the light of the determi-

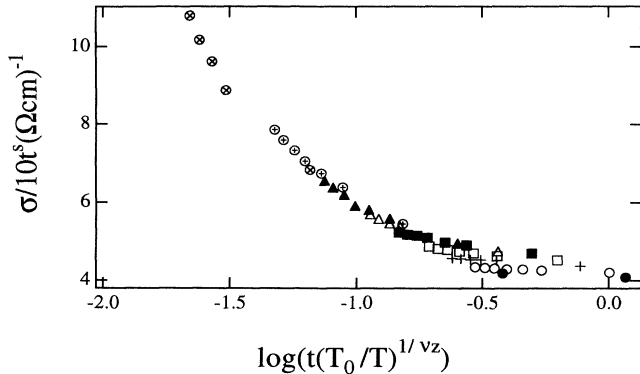


FIG. 1. Dynamical scaling plot of the conductivity data from Fig. 1 of Ref. 10. The plot assumes a critical stress $S_c = 6.5$ kbar, and exponent values $s = 0.5$, $\nu z = 2.13$. Only data in the temperature range $T < 60$ mK have been included in the plot, and different symbols denote different stress values, from $S = 6.59$ kbar to $S = 8.03$ kbar. We have chosen $T_0 = 100$ K, and the relation between t and $S - S_c$ was taken from Ref. 8, viz., $t = (S - S_c) 5.4 \times 10^{-3} (\text{kbar})^{-1}$.

nation $s = 0.51 \pm 0.05$ in Ref. 10. The determination of S_c and s in Refs. 8–10 was guided by the desire to achieve a good static scaling plot, i.e., a straight line of $\ln \sigma$ vs $\ln t$, over as large a t interval as possible. While this is a legitimate and often used criterion for determining the critical point, in the case of Si:P it leads to a rather peculiar result: It is found that an exponent of $s = 0.5$ fits the behavior of $\sigma(T = 0)$ very well out to $t \approx 1.0$. While this is remarkable and may well have interesting (and presently unknown) reasons, it is very unlikely that the critical region in Si:P is that large. Indeed our dynamical scaling plot, which is a more sophisticated test of scaling than the static one, shows that it is not. On the

other hand, a static scaling plot over a more restricted t range, viz., the data from Fig. 1 of Ref. 10 (which is the same data set that was used to produce the dynamical scaling plots in Figs. 1 and 2), shows that $S_c = 6.71$ kbar and $s = 0.29$ fits the data close to the transition very well, as shown in Fig. 3. This interpretation suggests a size of the critical region of about 1%, which is comparable with the corresponding value for most thermal phase transitions.²⁹

We conclude that the Bell experiment allows for a reasonably good dynamical scaling plot, consistent with a conventional power-law scaling interpretation, provided that the location of the critical point is adjusted upward, and the value of the conductivity exponent s downward, compared to the values given in Ref. 10. The resulting small value of s aggravates the problem that results from the inequality $\nu \geq 2/3$ in conjunction with the exponent relation $s = \nu$. This interpretation is therefore only feasible within theories that allow for $s \neq \nu$, as, e.g., the order parameter description of Ref. 5 and 6. Any theory that yields power-law scaling with $s = \nu$, like those reviewed in Ref. 1, is inconsistent with the Bell experiment unless there are large corrections to scaling.²⁵ For later reference we also note that the dynamical scaling observed in this experiment is restricted to a small dynamical range of just a bit over one decade: For the plot in Fig. 2 only data at temperatures $T \leq 60$ mK were included. Data at higher temperature do *not* scale, as can be seen from the inset in Fig. 2.

C. Scaling analysis of the Karlsruhe experiment

For the Karlsruhe data a dynamical scaling plot has been given by Stupp *et al.* in Ref. 13. For a direct comparison with the Bell data, we have digitized the data

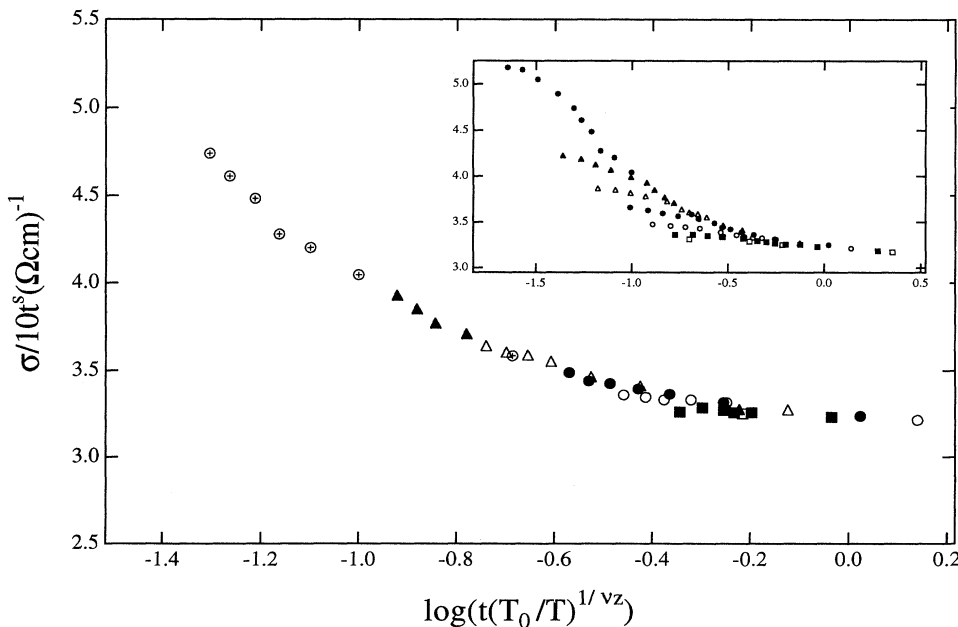


FIG. 2. Same as Fig. 1, but with $S_c = 6.71$ kbar, $s = 0.29$, and $\nu z = 1.82$. The inset shows that the data cease to scale once the temperature region $60 \text{ mK} < T < 225 \text{ mK}$ is taken into account.

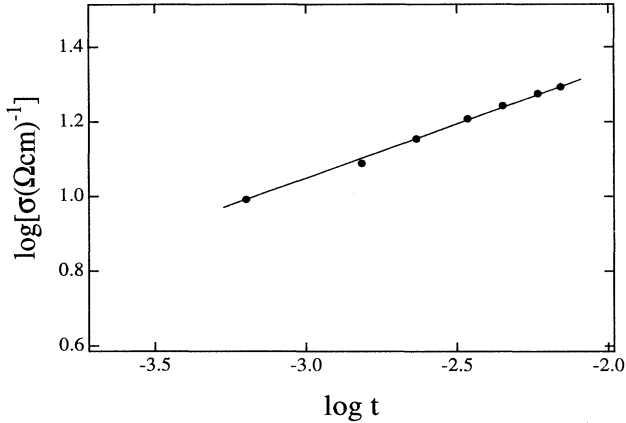


FIG. 3. Static scaling plot of the conductivity at $T = 0$ with the same parameter values as for the dynamical scaling plot in Fig. 2. The line corresponds to an exponent $s = 0.29$.

from Fig. 1 of Ref. 12, and plot them in Fig. 4 in the same way as the Bell data in Figs. 1 and 2. We have included data for $T < 160$ mK, but have left out the lowest temperature points on some samples that showed obvious rounding effects. We will discuss this rounding in the next subsection. We have assumed a critical phosphorus concentration of $3.52 \times 10^{18} \text{ cm}^{-3}$, and exponent values $s = 1.3$, and $\nu z = 2.7$. This yields the plot shown in Fig. 4. The slight differences between our dynamical scaling plot and that of Stupp *et al.* (who found the optimal value of νz to be 3.5) are due to Stupp *et al.* optimizing over a larger range of t values, and possibly due to some errors introduced by redigitizing the data. Overall, however, the two plots are of comparable quality. Figure 5 shows a static scaling plot analogous to the one shown in Fig. 3. Again, the assumed value of s fits the $T = 0$ conductivity well over one decade of t .

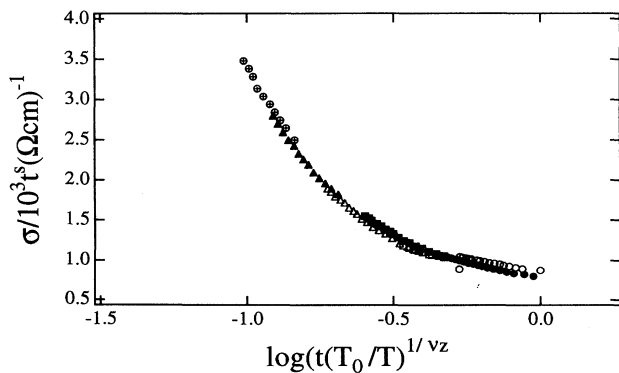


FIG. 4. Dynamical scaling plot of the conductivity data from Fig. 1 of Ref. 12. The plot assumes a critical P density $n_c = 3.52 \times 10^{18} \text{ kbar}$, and exponent values $s = 1.3$, $\nu z = 2.7$. Only data in the temperature range $T < 160$ mK have been included in the plot, and different symbols denote different P densities, from $n = 3.55 \times 10^{18} \text{ cm}^{-3}$ to $n = 3.69 \times 10^{18} \text{ cm}^{-3}$. We have chosen $T_0 = 100$ K.

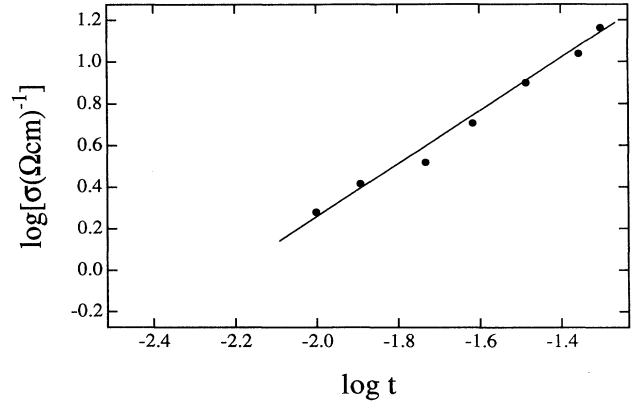


FIG. 5. Static scaling plot of the conductivity at $T = 0$ with the same parameter values as for the dynamical scaling plot in Fig. 4. The line corresponds to an exponent $s = 1.3$.

D. Discussion of the conventional scaling interpretation

Sections IIB and IIC and the accompanying figures show that the data from both the Bell and the Karlsruhe experiment allow for dynamical scaling plots of equal and satisfactory quality, even though their results are mutually inconsistent. A check for dynamical scaling is therefore not sufficient to distinguish between them, and it is necessary to consider additional experimental information to settle the issue.

At the heart of the discrepancy lie the different values of n_c , which differ by 6% between the two groups. An independent measurement of n_c with an error of about 1% or less would therefore suffice to rule out at least one of the two interpretations. Unfortunately, both groups claim to have done just that, but again do not agree on the results. Paalanen *et al.*¹¹ have measured the dielectric polarizability and in at least one sample have found insulating behavior for values of n as little as 1% below their value of n_c . Lakner and von Löhneysen³⁰ have found the thermopower to exhibit a metallic characteristic for values of n above their n_c , but below that of the Bell group.

Clearly, what is needed in this situation is an independent determination of n_c . The existing measurements of the spin susceptibility¹⁷ and the specific heat¹⁸ are not suitable for this purpose since they showed singular behavior in the metallic phase far from the critical point. One therefore expects these thermodynamic susceptibilities to show a superposition of critical behavior and some noncritical, but nevertheless singular, background, which makes them unsuitable for the present purpose. The most obvious observable that should be free from such complications, and that is easy to measure, is the DOS. If the conventional scaling scenario is correct, then the DOS as a function of t and T should show dynamical scaling, as expressed by Eq. (2.3), and the quality of the dynamical scaling plot should be equal to that of the conductivity with the same parameter values.

Even such an additional measurement, however, may turn out to be inconclusive unless it is carried out at sufficiently low temperatures, and unless the nature of the sample-to-sample fluctuations that were observed in the tail region in Refs. 8 and 9 is clarified. The Karlsruhe experiment did not observe such fluctuations, and seemed to yield a smooth conductivity as a function of doping. However, this may be misleading since the lowest T reached in this experiment was 62.5 mK, while the sample dependence in the Bell experiment became obvious only at lower temperature. In this context it is interesting to note that in the latter there is a clear break in the temperature dependence of the conductivity around $T = 60$ mK; see Fig. 1 of Ref. 10 and the inset in our Fig. 2. Furthermore, in the same temperature range sample-dependent problems did start to arise in the Karlsruhe experiment, which were attributed to thermal decoupling; see Fig. 1 of Ref. 12. Finally, we note that all of the work done by the group at CUNY, both on Si:P (Ref. 31) and on Si:B,³² which also yielded a smooth behavior of the conductivity, was at temperatures higher than 60 mK.

We conclude that the existing experimental data on Si:P provide evidence that close to the critical point (whose location is only imprecisely known), and at temperatures $T \leq 60$ mK there are large sample-to-sample fluctuations, and possibly equilibration problems. While the data at higher temperatures, and the low-temperature data with the tail region discarded, are not inconsistent with conventional scaling behavior, this poses the question whether the critical behavior at the AMT might be more exotic than is suggested by Eqs. (2.1)–(2.7). Indeed, our recent work on an order parameter description of the AMT shows that the AMT has random-field aspects.^{5,6} Classical random-field magnets are well known to display glassy behavior with exponentially long equilibration times, unconventional scaling, etc.²⁸ It is natural to assume that similar phenomena can characterize the AMT. While any conventional scaling interpretation of the AMT in Si:P will necessarily imply that one of the disagreeing experimental groups made a gross error in the determination of the critical concentration, an interpretation in terms of random-field physics has the potential for explaining the unusual features observed, and the disagreements between the experimentalists, in terms of real physical effects that are germane to the AMT. We consider this a very appealing possibility. In the remainder of this paper we therefore take this suggestion seriously, develop it, and then come back to a discussion of the experimental situation.

III. QUANTUM GLASSY BEHAVIOR AND ACTIVATED SCALING

A. Scaling theory of a quantum glass transition

An important characteristic of a glass transition, as opposed to an ordinary phase transition, is the occurrence of extremely long time scales. While critical slowing down at an ordinary transition means that the crit-

ical time scale τ grows like a power of the correlation length, $\tau \sim \xi^z$, with z the dynamical critical exponent,³³ at a glass transition the critical time scale grows exponentially with ξ ,³⁴

$$\ln(\tau/\tau_0) \sim \xi^\psi, \quad (3.1)$$

with τ_0 a microscopic time scale, and ψ a generalized dynamical exponent. As a result of such extreme slowing down, the system's equilibrium behavior near the transition becomes inaccessible for all practical purposes: Even at the smallest feasible frequencies (i.e., the inverse of the longest feasible waiting times, say, days), finite frequency effects become noticeable (“the system falls out of equilibrium”) at modest values of ξ or t . It has been proposed that the phase transition in classical random-field magnets is of this type,^{21,22} and experiments have confirmed this conjecture.²³ The physical picture behind this model of random-field magnets is as follows. The frustration induced by the competition between the exchange interaction and the random field leads to large clusters of misaligned spins, that is, locally ordered spins that “point the wrong way,” within the ordered phase. Even though aligning these clusters leads overall to a lower free energy, it requires a large free energy barrier to be overcome. These free-energy barriers grow like L^ψ as a function of some length scale L , and near the critical point they diverge like the correlation length ξ to the power ψ . The exponent ψ is therefore often referred to as the “barrier exponent.” Via the Arrhenius law, this leads to Eq. (3.1).

In a quantum system one expects time and inverse temperature to show the same scaling behavior, irrespective of whether the critical slowing down follows an ordinary power law, or Eq. (3.1).²⁴ Quantum mechanics thus makes it even harder to observe the static scaling behavior, since in addition to exponentially long times or small frequencies it requires exponentially small temperatures as well. Under realistic experimental conditions the system will either fall out of equilibrium, or pick up finite temperature effects. This is a crucial point, which we will have to keep in mind for any discussion of experimental consequences of our ideas.

The role played by temperature in a glassy quantum system can be seen explicitly in Fisher's recent study of a quantum-mechanical Ising spin chain in a transverse random magnetic field.³⁵ This system is closely related to the classical McCoy-Wu model, for which a number of exact results have been obtained.^{36,37} Since the AMT has been shown to be a quantum phase transition with random-field aspects⁶ we believe that these results are qualitatively relevant for our purposes, and will often use them for comparisons. The physical idea analogous to the one explained above for spin systems is that while a repulsive electron-electron interaction always leads to a decrease in the local DOS, the random potential can in general lead to an increase in the local DOS as well. The competition between these two effects leads to frustration and to, for example, large insulating clusters within the metallic phase. Delocalizing these large clusters requires energy barriers to be overcome, which again grow like ξ^ψ

as the transition is approached.

Another question that arises is the scaling behavior of an external magnetic field. In a spin system the magnetic field, just like the temperature, obviously sets an energy scale and thus should depend exponentially on the length scale. In an itinerant electron system, on the other hand, the magnetic field plays a dual role: It influences the orbital motion of the electrons, and in that capacity it acts like a length. However, it also couples to the electron spin via the Zeeman term, and there it acts like an energy. We therefore expect the slowest dependence of a given observable on the magnetic field to be a logarithmic one, with power-law corrections. A further consequence of the barrier model is that the frequency or temperature argument of scaling functions is expected to be $\ln(\tau/\tau_0)/\ln(T_0/T)$, rather than τT as in Eqs. (2.1) and (2.2). The reason is that one expects a very broad distribution of energy barriers and hence of relaxation times τ . The natural variable is therefore $\ln \tau$ rather than τ .³⁸ This is often referred to as activated scaling.

We next face the question of which observables can be expected to obey homogeneity laws analogous to the ones given by Eqs. (2.1) and (2.2). First of all, we have to remember that the homogeneity laws hold for averaged quantities, with the average including both a quantum-mechanical and an impurity or ensemble average, and that we have to distinguish between self-averaging and non-self-averaging quantities.³⁹ For the former, their probability distribution for the ensemble average becomes normal with a vanishing variance in the thermodynamic limit, so their average value coincides with the most probable or typical one. For the latter, either the probability distribution remains broad (e.g., log-normal), or the most probable value is different from the average one, or both. If the observable in question is non-self-averaging due to a log-normal distribution, then one expects its logarithm to be a self-averaging quantity.

It is well known that in a system with quenched disorder the free energy is self-averaging, while the partition function is not, and correlation functions in general are not, either.^{39,40} Therefore, all thermodynamic quantities, which can be obtained as partial derivatives of the free energy, are self-averaging. For a general thermodynamic quantity, Q , one might therefore expect a homogeneity law

$$Q(t, T) = b^{-x_Q} F_Q \left(tb^{1/\nu}, \frac{b^\psi}{\ln(T_0/T)} \right), \quad (3.2)$$

where x_Q is the scale dimension of Q , F_Q is a scaling function, and T_0 is a microscopic temperature scale, e.g., the Fermi temperature. A finite frequency will have the same effect as a finite temperature; as mentioned above, frequency and temperature are interchangeable in a scaling sense. A magnetic field dependence will be added later, when we discuss the magnetization.

It is obvious, however, that Eq. (3.2) cannot hold for all self-averaging quantities, or even for all thermodynamic ones. Suppose Q is some self-averaging quantity, and suppose Eq. (3.2) holds for Q . Then $P \equiv Q/T$ is also self-averaging, but since F_Q is not a homogeneous function of T , Eq. (3.2) will *not* hold for P . While this is

a rather trivial “breakdown of scaling,” it has observable consequences as we will see below. Here we assume that whether or not a given thermodynamic quantity obeys Eq. (3.2) can be decided by dimensional analysis: If the quantity contains scale dimensions of time or energy, then it does not obey Eq. (3.2), otherwise it does. For instance, the free-energy density cannot be written in this form, while the entropy density can, etc. However, the question whether or not a particular energy acts like an inverse time in a scaling sense can be a nontrivial one. We will further consider this point when we explicitly discuss various observables in Sec. III B below.

A rather different class of observables is formed by transport coefficients, like, e.g., the charge or heat diffusivity. Since they are directly related to a relaxation time, we expect them to be non-self-averaging, while their logarithms should be self-averaging. Let $\tilde{\Xi}$ be an *unaveraged* transport coefficient, i.e., its value for a particular sample or impurity arrangement. Then we expect its logarithm to obey

$$\langle \ln \tilde{\Xi} \rangle (t, T) = b^{\pm\psi} F_{\tilde{\Xi}} \left(tb^{1/\nu}, \frac{b^\psi}{\ln(T_0/T)} \right), \quad (3.3)$$

with $F_{\tilde{\Xi}}$ a scaling function. Notice that Eq. (3.3) describes only the leading, i.e., logarithmic, scaling behavior, and neglects power-law corrections to scaling. The “scale dimension” of $\langle \ln \tilde{\Xi} \rangle$ is necessarily plus or minus ψ , with the sign depending on whether the quantity vanishes or diverges at the transition.

As with conventional scaling at ordinary phase transitions, Eqs. (3.2) and (3.3) hold only for the singular parts of the respective quantities, and in general there will be nonvanishing, analytic background contributions. In the case of Eq. (3.3) another complication is to be expected. A general transport coefficient $\tilde{\Xi}$ is related, by means of an Einstein relation, to the corresponding diffusivity $\tilde{\Delta}$ via $\tilde{\Xi} = \tilde{\chi} \tilde{\Delta}$, with $\tilde{\chi}$ an unaveraged susceptibility. Since $\tilde{\chi}$ is a thermodynamic quantity, $\ln \tilde{\chi}$ is not expected to show scaling behavior. If χ is critical, one therefore expects a critical, nonscaling background contribution to $\langle \ln \tilde{\Xi} \rangle$, in addition to the scaling part given by Eq. (3.3). We will come back to this.

In the above paragraphs we have stated all of the assumptions that enter our scaling theory of a quantum glass. In the remainder of this paper we explicitly discuss the behavior of a number of specific observables that are of interest in the context of the AMT. We work out the consequences of our assumptions, and compare the results with those obtained from conventional scaling, Sec. II, and with the experiments on Si:P discussed above.

B. Discussion of observables

In this section we discuss explicitly the consequences of our scaling assumptions for various observables. We start with thermodynamic quantities, for which Eq. (3.2) and the related discussion above are relevant. Then we turn to the electrical and the thermal conductivity as examples of transport coefficients which realize the scaling behavior shown in Eq. (3.3).

1. Tunneling density of states

Let us first discuss the tunneling density of states, N , as a function of t and T . We restrict ourselves to the density of states at the Fermi level, since according to the discussion in connection with Eq. (3.2) a finite frequency or energy, which measures the bias voltage or the distance from the chemical potential, will have the same effect as T . In the theory put forward in Refs. 4–6, N is the order parameter for the AMT, and its scale dimension follows directly from the order parameter field theory to be $x_N = d - \theta - 2 + \eta$. Comparison with Eq. (2.1) gives the exponent relation^{5,6}

$$\beta = \frac{\nu}{2}(d - \theta - 2 + \eta) \quad (3.4a)$$

From Eq. (3.2) we then obtain the generalization of Eq. (2.1) to the case of activated scaling as

$$N(t, T) = b^{-\beta/\nu} F_N \left(tb^{1/\nu}, \frac{b^\psi}{\ln(T_0/T)} \right) \quad (3.4b)$$

Here F_N is a scaling function, and θ is an exponent related to a dangerous irrelevant variable that is characteristic of the random-field problem.⁴¹ In a classical random-field problem thermal fluctuations are dangerously irrelevant, and θ is minus the scale dimension of temperature. In the present context θ expresses the fact that quantum fluctuations are dangerously irrelevant at the AMT.⁵

A fundamental question that arises in this context is how many independent exponents are needed to describe the AMT. In a classical random-field problem there are three independent static exponents, e.g., ν , η , and θ .⁴² In the AMT theory developed in Refs. 5 and 6 it turned out that the dynamical exponent z was not independent, reflecting the irrelevancy of quantum fluctuations, so that there were still three independent exponents. Here ψ has taken over the role of z , and the question is whether or not it is independent. In order to decide this, let us recall the physical meaning of the two exponents θ and ψ . As noted above, θ is related to a dangerous irrelevant variable, u , which vanishes as a function of length scale L like $u \sim L^{-\theta}$. The random-field fixed point is characterized by $u\Delta$ scaling to a constant, with Δ the random potential energy scale.⁴¹ Hence Δ must diverge as L^θ . The free-energy landscape of the random-field problem is a complicated one, with many near-degenerate valleys that are separated by energy barriers with saddle points. As a function of length scale L , one expects a typical valley elevation to be related to the random potential and therefore grow like L^θ . The typical saddle-point elevation grows like L^ψ , which defines the exponent ψ . In order for this picture to be consistent, one must have $\psi \geq \theta$, as pointed out by Fisher in Ref. 38. Classically, only the valleys contribute to the free energy, so one expects the exponent characterizing the dangerously irrelevant thermal fluctuations to be θ , and the barrier exponent to be ψ , and in general the two will be independent. Quantum mechanically, however, the saddle points also contribute to the free energy, and one cannot distinguish between the barrier exponent and the exponent that expresses

the fact that quantum fluctuations are dangerously irrelevant. We therefore expect $\theta = \psi$ in the quantum case, although in our notation we will continue to distinguish between the two exponents. This leaves us with three independent exponents, e.g., ν , η , and $\psi = \theta$. A fourth one will be necessary when we discuss systems in external magnetic fields in Secs. III B 4 and III B 5 below.

As in the case of ordinary power-law scaling, we can eliminate the arbitrary parameter b from Eq. (3.4b) and write $N(t, T)$ in two different scaling forms to emphasize either the static or the dynamic aspects of the scaling law. The crossover between the two types of scaling occurs at a temperature T_\times for which the two arguments of F_N are equal. Ordinarily, this criterion would lead to a power-law dependence of T_\times on t , but activated scaling implies $T_\times \sim \exp(-1/t^{\nu\psi})$. As a result, the static scaling region will be very small unless $\nu\psi$ is very small. Since ν is bounded from below, $\nu \geq 2/d$,²⁷ this would require ψ to be very small.⁴³ Whether or not the static scaling behavior, $N(t, T) \approx N(t, T=0) \sim t^\beta$, is observable will then strongly depend on the precise value of $\nu\psi$, on the size of the critical region, and, to a lesser extent, on the value of the microscopic temperature scale T_0 . We thus put $b^\psi = \ln(T_0/T)$, and write Eq. (3.4b) as

$$N(t, T) = \frac{1}{[\ln(T_0/T)]^{\beta/\nu\psi}} G_N [t^{\nu\psi} \ln(T_0/T)] \quad (3.5)$$

The scaling function G_N is related to the function F_N in Eq. (3.4b) by $G_N(x) = F_N(x^{1/\nu\psi}, 1)$, and has the properties $G_N(x \rightarrow \infty) \sim x^{\beta/\nu\psi}$, and $G_N(x \rightarrow 0) \rightarrow \text{const.}$

Equation (3.5) makes a qualitative prediction that can be used to check experimentally for glassy aspects of the AMT: Measurements of the tunneling density of states very close to the transition should show an anomalously slow temperature dependence, i.e., N should vanish as some power of $\ln T$ rather than as a power of T . While this is a straightforward check in principle, in practice it may require a very large T range to distinguish between the two possibilities. For instance, in classical magnets the frequency had to be varied over seven decades in order to convincingly demonstrate the presence of activated scaling.²³ However, measurements over a smaller dynamic range would also be of interest, since they would put experimental bounds on possible values of ψ . This is particularly important since in the absence of any information about the value of ψ it is impossible to tell whether at a given temperature one is in the static or the dynamic scaling regime. Unfortunately, to our knowledge all measurements of N close to metal-insulator transitions have been performed at fixed (and rather high) temperatures, so that no T -dependent data are available for analysis.

2. Order parameter susceptibility

We now turn to fluctuations of the order parameter. We first give a statistical argument that N is indeed, as assumed above, a self-averaging quantity. This also sheds some light on the crucial role played by the electron-

electron interaction in our theory for the AMT.

Let us write the unaveraged order parameter, \tilde{N} , as its average plus fluctuations, $\tilde{N} = N + \delta N$, and consider the mean-square fluctuation $\langle (\delta N)^2 \rangle$. At $T = 0$ in a system of size L one has

$$\langle (\delta N)^2 \rangle = N \Phi(t, L) \quad , \quad (3.6)$$

with Φ some function of t and L . Far away from the critical point we expect $\Phi = \text{const}$, which leads to $[\langle (\delta N)^2 \rangle / N^2]^{1/2} \sim 1/\sqrt{N}$ as usual. In the critical region, on the other hand, we expect Φ to scale,

$$\Phi(t, L) = b^{\gamma/\nu} \Phi(t b^{1/\nu}, L b^{-1}) \quad , \quad (3.7)$$

with γ the critical exponent for the order parameter susceptibility. At criticality, Eq. (3.7) implies $\langle (\delta N)^2 \rangle \sim N L^{2-\eta} \sim L^{d+2-\eta}$, where we have used the exponent relation $\gamma = \nu(2 - \eta)$, and $N \sim L^d$. For the root-mean-square order parameter fluctuations this means

$$\left[\langle (\delta N)^2 \rangle / N^2 \right]^{1/2} \sim L^{(2-d-\eta)/2} \sim L^{-2\beta/\nu} \quad . \quad (3.8)$$

For the last relation in Eq. (3.8) we have assumed hyperscaling to be valid. In its absence the argument needs a trivial modification, but the end result is still given by the far right-hand side of Eq. (3.8).

Equation (3.8) implies that order parameter fluctuations at the critical point become small in large systems provided $\eta > 2 - d$, or $\beta > 0$ (the first condition depends on hyperscaling, while the second one does not). This has some interesting consequences. In the noninteracting localization problem, one has $\eta = 2 - d$, and $\beta = 0$.^{1,44} As can be seen from the above discussion, this means more than simply that the order parameter is uncritical in the localization problem: It indicates that the fluctuations of the density of states are independent of the system size, and as large as the average. Consequently, one expects that the density of states in a system of noninteracting disordered electrons has a very broad distribution, and that the Anderson transition is pathological from a statistical mechanics point of view. All of this is consistent with explicit studies of the Anderson transition.⁴⁵ Our order parameter description of the AMT, on the other hand, leads to $\beta > 0$. Therefore the density of states will be self-averaging in accord with our assumptions in Sec. III A above, and there are no obvious obstacles for the description of the AMT in terms of the standard concepts for continuous phase transitions.

We now discuss the order parameter correlation function, which we define as

$$C(t, T; \mathbf{x} - \mathbf{y}, \omega) = \left\langle \tilde{N}(\mathbf{x}, \epsilon_F + \omega/2) \tilde{N}(\mathbf{y}, \epsilon_F - \omega/2) \right\rangle \quad , \quad (3.9)$$

or its spatial Fourier transform, $C(t, T; q, \omega)$. Here $\tilde{N}(\mathbf{x}, \epsilon)$ denotes the unaveraged, local density of states at energy ϵ . From an analogy with Eq. (2.6) one expects C to scale, and to show an anomalously strong divergence as $q \rightarrow 0$. However, in random-field systems the

correlation function at nonzero wave number is likely to not be normally distributed,³⁹ while at $q = 0$ no such complications are expected to occur. Moreover, in our quantum system a small- q divergence would be cut off by a finite temperature. We therefore restrict ourselves to a discussion of the homogeneous correlation function, i.e., the order parameter susceptibility, which obeys

$$C(t, T; q = 0, \omega) = b^{2+\theta-\eta} F_C \left(t b^{1/\nu}, \frac{b^\psi}{\ln(T_0/T)}; \frac{b^\psi}{\ln(\omega_0/\omega)} \right) \quad . \quad (3.10)$$

At criticality, the order parameter susceptibility diverges as the temperature goes to zero, but only as a power of $\ln T$,

$$C(t = 0, T; q = 0, \omega = 0) \sim [\ln(T_0/T)]^{(2+\theta-\eta)/\psi} \quad . \quad (3.11)$$

The local density of states is measurable (on a perfect surface) with a scanning tunneling microscope, or STM. By measuring the density of states both by means of a tunnel junction and by means of an STM one can therefore check whether it is indeed a self-averaging quantity: If it is, then in a large system fluctuations of the local density of states should be small, and both the local and the junction measurements should give the same result. Furthermore, by measuring $\tilde{N}(\mathbf{x})$ across a sample at pairs of points with a fixed separation it should be possible to measure the correlation function C , and to check the prediction of Eq. (3.11).

We conclude this subsection by noting that C , at least away from the AMT, contains an uncritical, ‘‘mesoscopic’’ power-law singularity due to hydrodynamic diffusion modes.⁴⁶ In order to distinguish this singularity from the one given by Eq. (3.11) one can use arguments like those used for the classical random-field problem with a conserved order parameter.⁴⁷ The key idea is to consider a small but finite wave number q , subject to a number of constraints. First we require a self-averaging quantity, which according to the discussion above Eq. (3.10) requires $q\xi \ll 1$. Second, we require the critical part of C to effectively be at $q = 0$, which leads to the restriction $\ell q \ll 1/(\ln T_0/T)^{1/\psi}$, with ℓ a microscopic length on the order of the inverse Fermi wave number. Third, the noncritical mesoscopic contribution should be T independent. This will be the case if $q\ell \gg (T/T_0)^{1/2}$, since this singularity is due to diffusion. It can be readily verified that these three conditions can be simultaneously satisfied.

3. Specific heat

Let us now consider the entropy density, $s(t, T) = \partial f / \partial T$, where f is the free-energy density. As a thermodynamic quantity, s is expected to be self-averaging, and since its dimension is that of an inverse volume (in units chosen such that $k_B = 1$) we can write for the singular part of s ,

$$s(t, T) = b^{-d+\theta} F_s \left(tb^{1/\nu}, \frac{b^\psi}{\ln(T_0/T)} \right). \quad (3.12)$$

The specific heat is obtained from s by means of a logarithmic derivative with respect to T , $c_V = \partial s / \partial \ln T$, which yields for the singular part of c_V ,

$$\begin{aligned} c_V(t, T) &= b^{-(d-\theta+\psi)} F_{c_V} \left(tb^{1/\nu}, \frac{b^\psi}{\ln(T_0/T)} \right) \\ &= \frac{1}{[\ln(T_0/T)]^{1+(d-\theta)/\psi}} G_{c_V} \left[(T/T_0)^{t^{\nu\psi}} \right]. \end{aligned} \quad (3.13)$$

Here the scaling functions G_{c_V} and F_{c_V} are related by $G_{c_V}(x) = F_{c_V} [(-\ln x)^{1/\nu\psi}, 1]$.

The fact that the specific heat must vanish at $T = 0$ puts constraints on the possible behaviors of the function G_{c_V} at small values of its argument. One possibility is $G_{c_V}(x \rightarrow 0) \rightarrow \text{const}$. However, a more natural assumption is that $G_{c_V}(x \rightarrow 0)$ vanishes like a power of its argument. This possibility is realized, e.g., in the model studied recently by Fisher.³⁵ In that case, the specific heat in the critical region goes like

$$c_V(t, T) = \frac{T^{\text{const} \times t^{\nu\psi}}}{[\ln(T_0/T)]^{1+(d-\theta)/\psi}}. \quad (3.14)$$

If we substitute the exponent values that are appropriate for the model of Ref. 35, viz., $\nu = 2$, $\psi = 1/2$, $\theta = 0$, $d = 1$ and $\text{const} = 2$, then we recover from Eq. (3.14) Fisher's result for the transverse random-field Ising chain. The most interesting aspect of this result is the continuously varying exponent in the numerator, which leads to successive derivatives of c_V becoming divergent as the critical point is approached. This behavior means that there is a Griffiths phase, or rather Griffiths region, away from the critical point within which certain observables become divergent at various values of t .⁴⁸ Regardless of the behavior of $G_{c_V}(x \rightarrow 0)$, we have a non-Fermi-liquid behavior of the system in a finite region around the critical point: The specific-heat coefficient $\gamma(t, T) \equiv c_V(t, T)/T$ diverges as $T \rightarrow 0$, even away

from criticality.

For a discussion of the experimental consequences of Eqs. (3.13) and (3.14) one must keep in mind that the singular contribution to c_V is additive to the noncritical Fermi-liquid background that is linear in T , and that the singular part will dominate only at sufficiently low temperatures. Measurements of the specific heat in Si:P (Refs. 18 and 19) have indeed observed non-Fermi-liquid behavior both near the critical point, and rather far away from it in either phase. This has been interpreted in terms of local magnetic moments.¹⁸ The relation between the local moment and quantum glass pictures is currently unclear. It is interesting to note that both predict singular behavior of thermodynamic quantities away from the critical point, and it is conceivable that the glass picture is related to an interacting local moment description. Also, from the discussion in Sec. II we suspect that the lowest temperatures reached in the Si:P experiments (≈ 30 mK) were not low enough to be in the critical region. Since going to substantially lower temperatures is not realistic, it would be desirable to have similar measurements performed on a system with a higher Fermi temperature than Si:P, which has $T_F \approx 100$ K near the critical P concentration.

4. Magnetization

In order to discuss the magnetization, and the magnetic susceptibility in the next subsection, we need to add an external magnetic field H to our discussion. As discussed in Sec. III A, the leading effect of a magnetic field will come from its coupling to the electron spin, and will scale the same way as the temperature does, viz. $T \sim \exp(b^\psi)$, $H \sim \exp(b^\psi)$, both up to multiplicative power-law corrections. In general one therefore expects $T/H \sim b^{\phi\psi}$, with ϕ an exponent that characterizes differences in the corrections to scaling of T and H .⁴⁹ Dimensionally, the magnetization $m = \partial f / \partial H$ is an inverse volume times a temperature divided by a magnetic field, and therefore the scale dimension of m is $d - \theta - \phi\psi$. Therefore we have

$$m(t, T, H) = b^{-d+\theta+\phi\psi} F_m \left(tb^{1/\nu}, \frac{b^\psi}{\ln(T_0/T)}, (H/T)b^{\phi\psi}, \frac{b^\psi}{\ln(T_0/(T + Hb^{\phi\psi}))} \right). \quad (3.15)$$

Here the last argument of the scaling function F_m expresses the fact that the Zeeman energy provided by the magnetic field helps the system to overcome free energy barriers, and its functional form is motivated by the fact that the effect of a nonzero H will always be cut off by a nonzero T . However, the reverse is not true, which is why one still needs the second argument containing only T . The third argument contains the physics due to fluctuations within a given free-energy valley, with no attempts to climb over barriers.

The most interesting consequence of Eq. (3.15) is the leading H dependence of m at $T = 0$ at criticality, which is

$$m(t = 0, T = 0, H) \sim \frac{1}{[\ln(T_0/H)]^{-\phi+(d-\theta)/\psi}}, \quad (3.16)$$

where we have assumed that $F_m(0, 0, \infty, 1)$ is a finite number. If we substitute the exponent values that are appropriate for the model of Ref. 35, namely, $\phi = (1 + \sqrt{5})/2$ and d, θ , and ψ as given after Eq. (3.14), then we recover Fisher's result for the transverse Ising chain.

The physical interpretation of Eq. (3.16) is as follows. Equation (3.15) says that the magnetic degrees of freedom are glassy, and relax slowly just as the singlet or DOS degrees of freedom. If the system is cooled in a magnetic field then as the magnetic field is turned off,

the magnetization vanishes slowly as a function of both field and time. This behavior is reminiscent of that found in random-field magnets. The difference is that here no long-range magnetic order develops across the transition. However, we do expect that experiments that examine the difference between field cooling and zero-field cooling will be very interesting, as they are in random-field magnetic systems.²⁸

5. Magnetic susceptibility

So far all quantities we have discussed have been both self-averaging (except for the order parameter correlation function at nonzero wave number) and scaling; i.e., they obey homogeneity laws of the type given in Eq. (3.2). The magnetic susceptibility, $\chi_m = \partial m / \partial H$, is clearly self-averaging, but does not obey a homogeneity law for the reasons explained after Eq. (3.2). Nevertheless, we can obtain the functional form of χ_m by differentiating Eq. (3.15) with respect to H . The leading behavior of the zero-field susceptibility, which is produced by the third argument of the scaling function F_m , is

$$\chi_m(t, T) = [\ln(T_0/T)]^{-(d-\theta)/\psi+2\phi} \times \frac{1}{T} G_{\chi_m} \left[(T/T_0)^{t^\nu \psi} \right]. \quad (3.17)$$

In an ordinary, power-law scaling scenario one would expect the scaling function G_{χ_m} to behave such that at $T = 0$ the susceptibility is finite for $t \neq 0$. Here this is not possible since χ_m is not a homogeneous function of T . Rather, we conclude that the magnetic susceptibility will diverge as $T \rightarrow 0$ in a region of finite size around the critical point. This divergence is power law with logarithmic corrections, and the exponent of the power law is a continuous function of t ,

$$\chi_m(t, T) = \frac{T^{-1 + \text{const} \times t^\nu \psi}}{[\ln(T_0/T)]^{(d-\theta)/\psi-2\phi}}. \quad (3.18)$$

Clearly, this is another manifestation of the Griffiths phenomenon discussed above in Secs. III B 3 and III B 4, and again our result is consistent with that of Ref. 35 for the transverse random-field Ising chain.

As mentioned below Eq. (3.14), there is some similarity between our results and those obtained from the local moment picture.¹⁸ Both lead to a divergent magnetic susceptibility and specific-heat coefficient in the metallic phase, in qualitative agreement with experiments on Si:P and Si:P,B.¹⁷⁻¹⁹ The main difference is that we predict a critical singularity for either quantity, while the local moment picture yields thermodynamic anomalies that are decoupled from the AMT. Several points should be kept in mind, however. First, the coupling, or absence of it, of local moments to the AMT is an unsolved problem.¹ Second, Bhatt and Fisher²⁰ have pointed out that interactions between local moments may considerably weaken the effects found in Ref. 18.

6. Density susceptibility

The thermodynamic density susceptibility, $\partial n / \partial \mu$, is not directly measurable in a three-dimensional system. However, it is of interest since it enters the Einstein relation between the electrical conductivity and the mass diffusion coefficient and can therefore influence the critical behavior of the conductivity.

As a thermodynamic quantity, $\partial n / \partial \mu = \partial^2 f / \partial \mu^2$ is self-averaging, but it is not a scaling quantity in the sense of Eq. (3.2) since f is not. $\partial n / \partial \mu$ thus belongs in the same category as the magnetic susceptibility, namely, that of self-averaging, nonscaling observables. In order to determine the critical behavior of $\partial n / \partial \mu$, we first note that the chemical potential μ in this derivative does not scale like an energy, but rather like the correlation length to some power. This can be seen from the explicit formulation of the order parameter field theory for the AMT,⁶ whose only dependence on the chemical potential is in the μ dependence of t . We thus write

$$\partial n / \partial \mu = \partial^2 f / \partial \mu^2 = (\partial^2 f / \partial t^2) (\partial t / \partial \mu)^2 + (\partial f / \partial t) (\partial^2 t / \partial \mu^2).$$

At an ordinary quantum phase transition one would expect $f \sim b^{-(d+z)}$. At glassy quantum phase transitions, z effectively diverges so that the free energy does not satisfy a simple homogeneity law. Instead, one expects, schematically, $b^{-z} \sim T \sim \exp(-b^\psi) \sim \exp(-1/t^\nu \psi)$, i.e., the singular part of the free energy has an essential singularity in t at zero temperature. This argument is consistent with the results in Secs. III B 3–III B 5 where we showed that the thermodynamics of our model is qualitatively the same as that of the transverse random-field Ising chain.³⁵ At $T = 0$, the latter in turn is equivalent to the model considered by Shankar and Murthy,³⁷ who found the singular part of the free energy to behave like $f(t, T = 0) \sim t^{1/t}$. We therefore expect this behavior to qualitatively hold in our case as well, which means that $\partial f / \partial t$ and $\partial^2 f / \partial t^2$ vanish exponentially as $t \rightarrow 0$. $\partial t / \partial \mu$ and $\partial^2 t / \partial \mu^2$, on the other hand, can diverge at most like a power of $1/t$. This can be seen as follows. Let μ_c be the critical value of μ at a given value of the disorder. t must vanish as $\mu \rightarrow \mu_c$, and it can do so either as a power or as an exponential function of $\mu - \mu_c$. In the latter case all of the derivatives of t with respect to μ also vanish exponentially, while in the former case they may at most diverge like a power of $1/(\mu - \mu_c)$. Consequently, the singular part of $\partial n / \partial \mu$ at zero temperature must vanish exponentially as $t \rightarrow 0$. This is in contrast to the conventional scaling scenario, which yields a power-law dependence of $\partial n / \partial \mu$ on t in the framework of the order parameter field theory,⁶ and an uncritical $\partial n / \partial \mu$ in the $2 + \epsilon$ expansion.^{1,7}

In addition to this singular part, one expects in general a nonvanishing analytic background contribution to $\partial n / \partial \mu$.⁶ The electrical conductivity will then have the same critical behavior as the charge or mass diffusivity. However, if in a particular system, or for particular parameter values, that background contribution should vanish, then the resulting exponential vanishing of $\partial n / \partial \mu$

will lead to complications in the scaling description of the conductivity, as was mentioned in the discussion of Eq. (3.3) above.

7. Electrical conductivity

We now turn to the behavior of the electrical conductivity. As explained in Sec. III A, this is not a self-averaging quantity. We consider instead the unaveraged conductivity, $\bar{\sigma}$, and define $l_\sigma \equiv \langle \ln(\sigma_0/\bar{\sigma}) \rangle$, with σ_0 a suitable conductivity scale, e.g., the solution of the Boltzmann equation. According to Eq. (3.3) and Sec. III B 6 we expect l_σ to be self-averaging, and to obey

$$l_\sigma(t, T) = b^\psi F_\sigma \left(tb^{1/\nu}, \frac{b^\psi}{\ln(T_0/T)} \right) \\ = \ln(T_0/T) G_\sigma [t^{\nu\psi} \ln(T_0/T)] \quad . \quad (3.19)$$

This holds if the density susceptibility has a nonvanishing uncritical background contribution, as one usually expects to be the case. If $\partial n/\partial \mu$ vanishes at criticality, then there will be a critical, nonscaling background contribution to l_σ , as explained in connection with Eq. (3.3).

As in the case of Eq. (3.13) or (3.17), the behavior of the scaling function G_σ for large values of its argument is *a priori* unclear. However, we can use physical arguments to determine it. Let us define $\Sigma \equiv \sigma_0 \exp(-l_\sigma)$ as a measure of the conductivity. If $G_\sigma(x \rightarrow \infty)$ vanished faster than $1/x$, then Σ would approach σ_0 even for arbitrarily small $t \neq 0$ as $T \rightarrow 0$. Since σ_0 is a noncritical quantity, this is unphysical. On the other hand, if $G_\sigma(x \rightarrow \infty)$ vanished more slowly than $1/x$, then $\Sigma(T = 0)$ would vanish even for $t \neq 0$. However, for $t \neq 0$ there are no infinite free-energy barriers, and hence density fluctuations are able to relax and Σ must be nonzero. We therefore conclude that $G_\sigma(x \rightarrow \infty) \sim 1/x$. This yields

$$\Sigma(t, T = 0) \sim \exp(-1/t^{\nu\psi}) \quad , \quad (3.20a)$$

and

$$\Sigma(t = 0, T) \sim T^{G_\sigma(0)} \quad . \quad (3.20b)$$

Note that at zero temperature Σ vanishes exponentially with t , and that at the critical point Σ vanishes like a *nonuniversal* power of T .

The conclusion that is most important with respect to the interpretation of experimental results is that the conductivity, $\bar{\sigma}$, is not self-averaging, while $\ln \bar{\sigma}$ is self-averaging and $\langle \ln \bar{\sigma} \rangle$ scales. While in Sec. II we have shown that the existing data for the conductivity do allow for scaling plots, we have also seen that many “strange” features in the experimental results, especially in the ultralow-temperature results of the Bell experiment, must be ignored in order to reach that conclusion. Also, it was necessary to let all exponents and the position of the critical point float. Within our current activated scaling scenario there is no reason to believe that $\bar{\sigma}$ would scale if one obtained bounds on the exponents and on n_c by measuring thermodynamic quantities (which do

scale even under the current scenario) at the same low temperatures as the conductivity. $\langle \ln \bar{\sigma} \rangle$ does scale, but would be hard to measure. In other words, if activated scaling is present but conventional scaling is used for analyzing experiments, then better experiments will make things worse rather than better. Furthermore, $\bar{\sigma}$ is predicted to not be a self-averaging quantity, but to have a broad probability distribution. Measurements of $\bar{\sigma}$ at sufficiently low temperatures should therefore show large sample-to-sample fluctuations.

We propose that the unusual features that were observed in the experiments on Si:P, particularly in the ultralow-temperature Bell experiment, and which we have reviewed in Secs. I and II, are manifestations of the “glassy” behavior that we have derived above. The fact that the observed anomalies became stronger at lower temperatures is certainly consistent with this. The fact that other experiments¹ did not provide any indications for $\bar{\sigma}$ not being a well-behaved quantity is not a valid counterargument, since they all stayed above, or barely got below, the 60 mK where there is a clear break in the T dependence of the conductivity; see Ref. 10 and Fig. 2. In order to further check this proposal, one should measure thermodynamic quantities, preferably the tunneling density of states, together with the conductivity at as low temperatures, and over as wide a temperature range, as possible. A system with a higher Fermi temperature than doped Si would be advantageous, since it would alleviate the need for ultralow temperatures. Ni(S,Se)₂ may be promising in this respect. In a very recent interesting experiment, Jin *et al.* have found that, although conductivity data down to $T = 30$ mK do allow for a conventional dynamical scaling plot, there are hysteresis effects that may be indicative of a glasslike behavior of the electrons.⁵⁰

8. Thermal conductivity

We finally consider the electronic contribution to the thermal conductivity, which is the product of the specific heat and the heat diffusivity. For the same reasons as in the case of the electrical conductivity we expect the thermal conductivity, $\tilde{\kappa}$, not to be a self-averaging quantity. We define $l_\kappa \equiv \langle \ln(\kappa_0/\tilde{\kappa}) \rangle$, with κ_0 the Boltzmann value, and expect⁵¹

$$l_\kappa(t, T) = b^\psi F_\kappa \left(tb^{1/\nu}, \frac{b^\psi}{\ln(T_0/T)} \right) \quad . \quad (3.21)$$

This equation can be discussed analogously to Eq. (3.19) for the conductivity. If we define $K \equiv \exp(-l_\kappa)$ as a measure of the thermal conductivity, then we obtain an interesting prediction for the generalized Wiedemann-Franz ratio K/Σ at criticality:

$$K(t = 0, T)/\Sigma(t = 0, T) \sim T^{G_\kappa - G_\sigma} \quad , \quad (3.22)$$

with G_κ and G_σ *nonuniversal* numbers [see Eq. (3.20b)]. This is in sharp contrast to the conventional scaling description of the AMT, which predicts that the

Wiedemann-Franz law $\kappa/\sigma = \langle \tilde{\kappa} \rangle / \langle \tilde{\sigma} \rangle \sim T$ holds even at the transition.⁵²

IV. SUMMARY

We conclude by briefly summarizing the results of this paper. We have employed both conventional and activated scaling scenarios to analyze experiments on the metal-insulator transition in doped semiconductors, most notably Si:P. Our main goals were to understand the discrepancies between different experimental findings, and to work out and analyze the glassy dynamical features of the transition that are suggested by recent theoretical advances.

In Sec. II conventional scaling ideas were used to interpret existing experimental data. The most important conclusions were that existing experiments for the conductivity are inconsistent with each other and that, at least in Si:P at very low temperatures, there are large sample-to-sample fluctuations, and possibly equilibration problems, sufficiently close to the critical point.

In Sec. III we assumed that the AMT is a quantum glass transition, and we developed a general description of such a transition. Our chief results are as follows: (1) The specific heat and spin susceptibility are singular as

$T \rightarrow 0$ even in the metallic phase; see Eqs. (3.14) and (3.18). These results are consistent with existing experiments, and the theory given here provides an alternative to the previous explanation in terms of noninteracting local moments. (2) The DOS is the order parameter for the quantum glass transition and it is both self-averaging and a scaling quantity; see Sec. III B 1. At criticality, it is predicted to vanish logarithmically with temperature; see Eq. (3.5). The critical behavior of the OP susceptibility has also been discussed. (3) The electrical conductivity $\tilde{\sigma}$ is so broadly distributed that it is not a self-averaging quantity, but $\ln \tilde{\sigma}$ is both self-averaging and a scaling quantity, see Sec. III B 7. This result was used to explain the sample-to-sample fluctuations in $\tilde{\sigma}$ that were observed in Si:P. In Sec. III we also suggested a number of additional experiments to test the hypothesis that the AMT is a quantum glass transition.

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- or $s = \nu = 1 + O(1)$ in $d = 3$. This leaves the exponent values in $d = 3$ completely undetermined, especially since it is known that the $2 + \epsilon$ expansion is very badly behaved. Virtually the only constraint on the exponents in $d = 3$ is the rigorous inequality $\nu \geq 2/3$; see Ref. 27. The $2 + \epsilon$ expansion further yields $s = \nu$ in $d = 3$ (Ref. 1), while in the OP description (Ref. 6) random-field effects relax this condition, and s and ν are independent.
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