# Laughlin wave function and one-dimensional free fermions

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Making use of the well-known phase-space reduction in the lowest Landau level, we show that the Laughlin wave function for the $\nu=1/m$ case can be obtained exactly as a coherent-state representation of a one-dimensional (1D) wave function. The 1D system consists of $m$ copies of free fermions associated with each of the $N$ electrons, confined in a common harmonic well potential. Interestingly, the condition for this exact correspondence is found to incorporate Jain's parton picture. We argue that this correspondence between the free fermions and quantum Hall effect is due to the mapping of the 1D system under consideration to the Gaussian unitary ensemble in the random matrix theory.

The Laughlin wave function, which describes the incompressible quantum fluid phase of a two-dimensional (2D) interacting electron gas in a strong magnetic field, has enjoyed tremendous success in explaining the observed features of the fractional quantum Hall effect (FQHE).[1,2] Jain has given an interpretation of FQHE in terms of the integral quantum Hall effect (IQHE) of fractionally charged "partons"[3] and also as the IQHE of composite fermions—fermions with an even number of flux quanta attached.[4] In recent times there have been attempts[5] to relate the quantum Hall effect with a certain 1D model—the Calegero-Sutherland model.[6] This is due to the similarities between the structures of the ground state and the excited states of the two systems. Azuma and Iso, in particular, have shown that a close relationship exists between the two wave functions for a quantum Hall system under certain restrictions[7] and Iso also has argued for a universality between the two in the long-wavelength limit.[8] It is interesting to inquire if such a construction is possible under more general conditions and what is the role of Jain's picture in this context. In this paper, we obtain an exact mapping between the Laughlin wave function for the $\nu=1/m$ case and the wave function for 1D free fermions, making use of an approach analogous to Jain's parton picture.

In arriving at the Laughlin wave function, it is assumed that the electrons are restricted to the lowest Landau level (LLL) due to the effect of the strong magnetic field and low temperatures. It is well known,[9] as we shall also show below, that this restriction converts the configuration space of the electrons to a phase space of a 1D system. Viewed in this way, it is pertinent to ask if the Laughlin wave function itself can be considered as a coherent-state representation of a 1D system. We show that the ground-state wave function for $m$ noninteracting partons associated with each of the $N$ electrons confined in a common harmonic well has as its coherent-state representation the Laughlin wave function when certain restrictions are imposed. Interestingly, these conditions implement Jain's parton picture quite elegantly. In contrast to Ref. 7, we do not have to take the limit of magnetic field going to infinity.

Consider a particle of change $e$ and mass $m_0$ in two dimensions (2D) in a transverse magnetic field $B$ and a potential $V(x,y)$. In the gauge $\mathbf{A}=(B/2)(-x,y)$,

$$L=\tfrac{1}{2}m_0(\dot{x}^2+\dot{y}^2)+\frac{eB}{2c}(-y\dot{x}+x\dot{y})-V(x,y)\ . \tag{1}$$

Since the spectrum is equally spaced with spacings $\hbar\omega_c=\hbar eB/m_0c$, when the potential is weak the restriction to LLL takes place when zero mass limit is taken. In this limit, it follows from the Lagrangian[9] that

$$[x,y]=-il_B^2\ , \tag{2}$$

where $l_B^2=\hbar c/eB$. Thus the phase space is reduced from the four variables $p_x$, $p_y$, $x$, $y$ to two variables $X_1$, $X_2$ defined below. These can also be seen as the guiding center coordinates of the cyclotron orbit given by $X_i=x_i+(l_B^2/\hbar)\epsilon_{ij}\pi_j$, where $\pi_j=p_j-(e/c)A_j$, satisfying the same relation as in (2). Here, $x_1$ and $x_2$ are the coordinates $(x,y)$ of the electrons before the LLL restriction is made. When restricted to the LLL, the coordinates of the electron in 2D are identified with that of the guiding center coordinates. The two coordinates thus behave like canonically conjugate variables of a 1D system. The combination $X_1\mp iX_2=\sqrt{2}b(b^\dagger)$, obeying the oscillator algebra, connects the degenerate angular momentum states. Thus the wave function of the LLL state is given by $\Psi(z)=\langle\Psi|z\rangle$, where $z=x+iy$ is the eigenvalue of $X_1+iX_2$ in the state $|z\rangle$. $|z\rangle$ is the coherent state associated with the angular momentum algebra. The coordinate space representation of the coherent state, $\langle q|z\rangle$, where $|q\rangle$ is the 1D coordinate basis, follows easily. Taking the inner product of $\langle q|$, with the defining coherent-state relation and using a suitable representation for $X_1,X_2$ in terms of 1D variables, the following result follows:

$$\langle q|z\rangle=\frac{1}{[l_B\sqrt{\pi}]^{1/2}}\exp-\frac{1}{2l_B^2}(\bar{z}z+z^2+q^2-2\sqrt{2}qz)\ . \tag{3}$$

The Laughlin wave function for $\nu=1/m$ is given by

$$\langle\psi|z_i\rangle=\prod_{i\geq j}(z_i-z_j)^m\exp-\sum_i\bar{z}_iz_i\ , \tag{4}$$

 

where we have expanded the state $\langle\psi|$ in the coherent-state basis. We wish to identify the 1D system whose coherent-state representation gives exactly (4).

Consider the ground-state wave function of $m$ noninteracting fermions associated with each of the $N$ particles, confined in a harmonic well of frequency $\omega$ given by the cyclotron frequency. These $m$ particles are referred to as "partons" following Jain and the reason for that will be clear later.

$$\langle\psi|q\rangle \equiv \psi(q_1^{(1)}\cdots q_1^{(m)}\cdots q_N^{(1)}\cdots q_N^{(m)}) = \prod_{i\geq j,a}(q_i^{(a)}-q_j^{(a)})\exp-\left[\frac{m_0\omega^2}{2\hbar}\sum_{i,a}q_i^{2(a)}\right] . \quad (5)$$

This wave function, which is the ground-state wave function of free fermions in a common harmonic well, is antisymmetric in particle index $i,j$ and symmetric in the "parton" index $a$. The number of partons $m$ has to be odd for it to describe electrons in the LLL in 2D. We show that the coherent-state representation of (5) is (4) when the partons are constrained to be the same in the phase space, for each of the $N$ particles and their charges chosen to be $1/m$ of electrons.

To find the coherent-space representation of (5), we need

$$\langle\psi|z\rangle = \int\langle\psi|q\rangle\langle q|z\rangle dq . \quad (6)$$

The overlap between the coherent state and the coordinate space is given by

$$\langle q|z\rangle \equiv \prod_{i,a}\langle q_i^{(a)}|z_i^a\rangle . \quad (7)$$

This is a generalization of (3) from the one-particle case to the many-particle case. This expression in conjunction with (5) gives the coherent-space representation of a given wave function. $z_i^a$ now has the meaning of coordinates of partons in the LLL in 2D. The chirality associated with the Laughlin wave function, due to the presence of the magnetic field, enters in (7): if the direction of the magnetic field is reversed, then $z\to\bar{z}$, as $|z\rangle$ is the coherent state associated with the angular momentum lowering operator. Now $z_i^a$ are chosen to be the same for all $(a)$ for a given $i$:

$$z_i^{(a)} = z_i . \quad (8)$$

Only with this restriction does the Jastrow factor of the coherent-state representation match that of the Laughlin wave function. This choice has the physical meaning of constraining the coordinates of the $m$ partons to be the same in the 2D coordinate space of the LLL system. This restriction is analogous to Jain's picture of treating electrons as composed of $m$ partons. This is the reason for the term parton used in this paper.

Also as in Jain's picture, each parton having coordinate $z_i$ in the LLL is taken to have charge $1/m$ of electrons. This is needed, as we shall see, to obtain the Laughlin wave function with the correct Gaussian width.

Evaluating (6), using (8) in (7), requires the following result:

$$\int\prod_i dq_i\prod_{i<j}(q_i-q_j)\exp-\sum_i(q_i^2-\sqrt{2}q_iz_i) = \text{const}\prod_{i>j}\left[\frac{d}{dz_i}-\frac{d}{dz_j}\right]\exp\sum_i(z_i^2) = \text{const}\prod_{i>j}(z_i-z_j)\exp\sum_i(z_i^2) . \quad (9)$$

Using (9) to evaluate (7) by integrating over each 1D coordinate of the partons for all of the $N$ particles, we get

$$\langle\psi|z\rangle = \text{const}\prod_{i>j}(z_i-z_j)^m\exp-\frac{m\sum_i|z_i|^2}{2l_B^2} . \quad (10)$$

By choosing the charge of each parton to be $1/m$ of that of the electron's charge, the Laughlin wave function results.

We have thus shown, using the LLL restriction, that the Laughlin wave function is the holomorphic representation of a 1D system of free fermions in a harmonic well. The fact that it is the noninteracting fermions which are related to FQHE wave functions can possibly be understood as follows. It is well known that the probability distribution of free fermions in harmonic confinement in 1D is isomorphic to the probability distribution of the eigenvalues in a Gaussian unitary ensemble in the random matrix theory.[10] The latter corresponds to time-reversal-noninvariant Hamiltonian systems. In recent times,[11] an intriguing connection has been established between static and dynamic correlations of eigenvalues in random matrix theory and particle coordinates in the Calegero-Sutherland model for certain values of the coupling constants, which includes free fermions. Viewed in this light, therefore, it is not surprising that the eigenvalue distribution in *time-reversal-noninvariant* ensemble corresponds to the probability distribution of coordinates of electrons in a *magnetic field*.

Our construction of the Laughlin wave function from 1D fermionic theory is different from that of the construction in Ref. 7: in our construction 1D fermions are free fermions, but with $m$ of them associated with each of the 2D electrons; it is an exact correspondence, valid without taking the $B\to\infty$ limit. The restrictions needed to obtain it and the value of the parton's charge correspond to Jain's picture. This relation of the Laughlin wave function and 1D system is possibly related, in general, to the edge states of the quantum Hall effect and specifically to Ref. 12, where the FQHE is related to 1D free fermions.

Extension to other filling factors involves, in Jain's picture, the filling up of higher Landau levels. Hence, their identification to 1D systems through the phase-space picture is not obvious. It should be interesting to find such an extension for other filling factors. This correspondence may also be useful to study the symmetry aspects of the FQHE like $W_\infty$ symmetry,[13] since the 1D system we have is known to have such a symmetry.[14] This can also possibly offer a better calculational procedure to compute expectation values in quantum Hall states by converting them to 1D problems.

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