# Frequency-dependent Cooper-pair tunneling in ultrasmall superconductor-insulator-superconductor junctions

A. Hädicke and W. Krech
*Institut für Festkörperphysik, Friedrich-Schiller-Universität Jena, Max-Wien-Platz 1, D-07743 Jena, Germany*
(Received 15 March 1995; revised manuscript received 16 June 1995)

Single-charge tunneling in ultrasmall voltage biased superconductor-insulator-superconductor junctions in a high-impedance electromagnetic environment is considered. The Cooper-pair current is calculated at $T=0$ on the basis of the elementary tunnel Hamiltonian for quasiparticles where the transfer of Cooper pairs emerges automatically as an effect of higher-order perturbation theory. The frequency dispersion of the supercurrent is taken into account because it depends on the frequency-dependent pair current amplitude $I_p(\omega)$.

## I. INTRODUCTION

Effects of single-charge tunneling in ultrasmall capacitance junctions has become of much experimental and theoretical interest. For a review see, for instance, Ref. 1. Modern nanolithography allows the fabrication of ultrasmall junctions with capacitances $C<10^{-16}$ F, where the electrostatic energy differences dominate thermal fluctuations at the 1 K scale. This opens the door to a new kind of electronics.

We consider the dc current through a voltage-biased superconductor-insulator-superconductor (SIS) junction which can be carried by quasiparticles (quasiparticle current $\langle I\rangle_{\rm qp}$) and by Cooper pairs (supercurrent $\langle I\rangle_s$). The current is calculated perturbatively basing on the elementary tunneling Hamiltonian for quasiparticles. Since this Hamiltonian describes only the tunneling of single quasiparticles ($1e$) the Cooper-pair tunneling ($2e$) corresponds to a process of higher order. In leading order the known result of quasiparticle tunneling emerges.[2] The essential feature of this paper is to take into account the frequency dependence of the supercurrent. This is reached due to an expression of the supercurrent in terms of the so-called pair current amplitude $I_p(\omega)$.

The current-voltage characteristic of small tunnel junctions is essentially influenced by the external circuit. This electromagnetic environment is able to absorb energy which is for superconducting electrodes at zero temperature the only possibility to transfer the energy gain of the tunneling process. Because Cooper pairs live in the condensate they cannot absorb this energy. To simplify matters we assume that the external circuit is characterized by an Ohmic resistance $R_E$. Furthermore, we restrict ourselves to the limiting cases of low- and high-resistance environments ($R_E \ll R_Q$ and $R_E \gg R_Q$) at $T=0$ where the quasiparticle currents are suppressed for voltages lower than the thresholds $2\Delta/e$ and $(2\Delta+E_c)/e$, respectively. $R_Q=h/e^2$ is the quantum resistance and $2\Delta$ labels the superconducting energy gap. The additional part $E_c=\hbar\omega_c=e^2/(2C)$ corresponds to the Coulomb energy. For single junctions it is known that the Coulomb blockade can only be observed if the junction is sufficiently decoupled from the voltage bias by a high-resistance environment. Beyond those thresholds Cooper pairs can break up into quasiparticles and the tunnel current is carried mainly by quasiparticles.

Without exception Cooper-pair tunneling is described in literature by using the model of an effective Hamiltonian[3–7] with the perturbation term

$$H_T=E_J\cos\Psi, \tag{1}$$

where the operator $\exp(\pm i\Psi)$ changes the macroscopic charge $Q$ on the junction by the value $\pm 2e$ corresponding to the charge of a Cooper pair. This means that simultaneously tunneling of two electrons (Cooper pair) is introduced by hand. Then the calculated supercurrent reads[3–7]

$$\langle I\rangle_s(V)=\frac{\pi e E_J^2}{\hbar^2}\left\{P'\left(\frac{2eV}{\hbar}\right)-P'\left(-\frac{2eV}{\hbar}\right)\right\}=\frac{\pi I_c^2}{4e}\left\{P'\left(\frac{2eV}{\hbar}\right)-P'\left(-\frac{2eV}{\hbar}\right)\right\}, \tag{2}$$

with

$$P'(\omega)=\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{4J(\tau)+i\omega\tau}\,d\tau. \tag{3}$$

The function $J(\tau)$ contains the information about the environment $[\beta=1/(k_BT)]$[8,9]

$$J(\tau)=\frac{1}{R_Q}\int_{-\infty}^{\infty}\frac{\mathrm{Re}Z_t(\omega)}{\omega}\left\{\coth\frac{\beta\hbar\omega}{2}[\cos\omega\tau-1]-i\sin\omega\tau\right\}d\omega, \tag{4}$$

where $Z_t(\omega)=1/(i\omega C+1/R_E)$. Note the following relations at $T=0$:

$$\frac{R_E}{R_Q}\to 0 \;\Rightarrow\; J(\tau)=0,\; P'\left(\frac{2eV}{\hbar}\right)=\delta(2eV/\hbar), \tag{5}$$

$$\frac{R_E}{R_Q}\to\infty \;\Rightarrow\; J(\tau)=-i\omega_c\tau,\; P'\left(\frac{2eV}{\hbar}\right)=\delta(2eV/\hbar-4\omega_c). \tag{6}$$

This peak structure actually has been seen in experiment.[10] The result is said to be correct if the Josephson coupling energy $E_J=\hbar/(2e)\cdot I_c$ is much smaller than $E_c$. By use of the known formula for the critical current $I_c$ one gets the inequality

$$E_c \gg \frac{R_Q}{8R_T}\Delta. \tag{7}$$

$R_T$ is the normal tunnel resistance which obeys the relation $R_T \gg R_Q$. This model corresponds to a rough approximation because it does not contain any frequency dispersion of Cooper-pair tunneling which should be important for voltages of the order of the gap voltage. Nevertheless, this approximation seems to be reasonable for lower voltages. Instead of the dependence on the constant critical Josephson current $I_c$ an improved model should show a connection with the frequency-dependent pair current amplitude $I_p(\omega)$ of the Werthamer theory. For instance, the quasiparticle current ($T>0$)

$$\langle I\rangle_{\rm qp}(V)=\int_{-\infty}^{\infty} {\rm Im}I_q(\omega)P\left(\frac{eV}{\hbar}-\omega\right)\frac{1-e^{-\beta eV}}{1-e^{-\beta\hbar\omega}}\,d\omega \tag{8}$$

is expressed in terms of the quasiparticle current amplitude ${\rm Im}I_q(\omega)$. The definition of $P(\omega)$ differs from that of $P'(\omega)$ [Eq. (3)] by the lack of the factor 4 in front of the function $J(\tau)$. For ${\rm Im}I_q(\omega)$ according to standard BCS theory at $T=0$ in the case of a symmetric junction see, e.g., Ref. 11.

The dependence of the supercurrent on the factor $I_c^2$ in Eq. (2) indicates that the supercurrent has something to do with the squared pair current amplitude $I_p(\omega)$. To show this connection we calculate the supercurrent by means of a perturbation theory of higher order in the elementary tunneling Hamiltonian. In this way the special features of Cooper-pair tunneling (frequency dependence, transfer of charges $2e$, energy transfer only to the environment) arise automatically. In other words, we do not consider tunneling particles with charge $2e$ from the beginning. Rather, we start with elementary particles (electrons) with charge $1e$ and the supercurrent arises as an effect of higher order.

## II. THE MODEL

We follow the usual approach (see, e.g., Ref. 9) and start with the total Hamiltonian

$$H=H_0+H_T=QV+H_{\rm res}+H_T, \tag{9}$$

where the interaction part corresponds to the tunnel Hamiltonian

$$H_T=H_++H_-, \quad H_-=H_+^\dagger,$$

$$H_+=\sum_{l,r,\sigma} T_{lr}c_{r,\sigma}^\dagger c_{l,\sigma}e^{i\Phi}. \tag{10}$$

$c_{l,\sigma}$ and $c_{r,\sigma}$ stand for quasiparticle annihilation operators of the left and right electrode satisfying anticommutation relations. $T_{lr}$ are the tunneling matrix elements and the spin is labeled by the subscript $\sigma$. In the case of superconducting electrodes one can assume that the macroscopic phase is already contained in the phase operator $\Phi$.[12] The phase operator changes the charge on the junction by one elementary charge $e$ according to the relation[13]

$$H_\pm\cdot F(Q)=F(Q\pm e)\cdot H_\pm, \tag{11}$$

where $F$ is an arbitrary function of the junction charge $Q$. This algebra corresponds to the elementary commutation relation

$$[Q,\Phi]=ie. \tag{12}$$

The convention is chosen in such a way that a positive voltage favors tunneling from left to right which reduces the junction charge $Q$ by $e$.

The reservoir Hamiltonian $H_{\rm res}$ consists of terms corresponding to the left and right electrodes and the environment which can be described in standard way.[8,9] Due to the phase operators, tunneling is connected with elementary excitations of the electromagnetic environment.

Now the stationary mean current can be calculated as usual (see, e.g., Ref. 14). In first-order perturbation theory one gets the quasiparticle current

$$\langle I\rangle_{\rm qp}=-\frac{2e}{\hbar^2}{\rm Re}\int_{-\infty}^{t} dt'\langle[H_+^{(I)}(t),H_-^{(I)}(t')]\rangle_0, \tag{13}$$

where the superscript $(I)$ means the interaction representation. The dc current in the next nonvanishing (third) order of perturbation theory reads

$$\langle I\rangle^{(3)}=\frac{2e}{\hbar^4}{\rm Re}\int_{-\infty}^{t}dt'\int_{-\infty}^{t'}dt''\int_{-\infty}^{t''}dt'''\{\langle[[[H_+^{(I)}(t),H_+^{(I)}(t')],H_-^{(I)}(t'')],H_-^{(I)}(t''')]\rangle_0$$
$$+[[[H_+^{(I)}(t),H_-^{(I)}(t')],H_+^{(I)}(t'')],H_-^{(I)}(t''')]\rangle_0+[[[H_+^{(I)}(t),H_-^{(I)}(t')],H_-^{(I)}(t'')],H_+^{(I)}(t''')]\rangle_0\}. \tag{14}$$

It turns out that the supercurrent is a part of this expression. By splitting off the voltage dependence by means of Eq. (11) and using new time variables

$$\tau\equiv t-t', \quad \tau'\equiv t'-t'', \quad \tau''\equiv t''-t''',$$

one gets

$$\langle I\rangle^{(3)}=\frac{2e}{\hbar^4}{\rm Re}\int_0^\infty d\tau\int_0^\infty d\tau'\int_0^\infty d\tau''\{e^{-\frac{i}{\hbar}eV(\tau+2\tau'+\tau'')}\kappa_1(\tau,\tau',\tau'')+e^{-\frac{i}{\hbar}eV(\tau+\tau'')}\kappa_2(\tau,\tau',\tau'')+e^{-\frac{i}{\hbar}eV(\tau-\tau'')}\kappa_3(\tau,\tau',\tau'')\}. \tag{15}$$

## III. CALCULATION OF CORRELATION FUNCTIONS

Now one has to deal with the three correlation functions $\kappa_1$, $\kappa_2$, and $\kappa_3$. The operators $H_\pm^{(I)}(t)$ can be written as $\tilde{H}_\pm(t)\exp(\pm i\Phi(t))$ where the operators $\tilde{H}_\pm(t)$ carry only the time dependence with respect to the electrodes and the phase-dependent operators $\exp(\pm i\Phi(t))$ carry those with respect to the environment. For instance, the function $\kappa_1$ reads, therefore,

$$\kappa_1=\langle\tilde{H}_+(t)\tilde{H}_+(t-\tau)\tilde{H}_-(t-\tau-\tau')\tilde{H}_-(t-\tau-\tau'-\tau'')\rangle_0$$
$$\times\langle e^{i\Phi(t)}e^{i\Phi(t-\tau)}e^{-i\Phi(t-\tau-\tau')}e^{-i\Phi(t-\tau-\tau'-\tau'')}\rangle_0+7 \text{ further terms.} \tag{16}$$

These other terms arise due to the resolution of the interlaced commutators. The decisive step of the identification of the contributions which describe Cooper-pair tunneling is to reduce the 4-correlators with respect to $\tilde{H}$ into 2-correlators containing only operators with the same signature, namely,

$$\langle\tilde{H}_+(t_1)\tilde{H}_+(t_2)\tilde{H}_-(t_3)\tilde{H}_-(t_4)\rangle_0=\langle\tilde{H}_+(t_1)\tilde{H}_+(t_2)\rangle_0\langle\tilde{H}_-(t_3)\tilde{H}_-(t_4)\rangle_0.$$

This decomposition guarantees that only condensate states corresponding to Cooper pairs are taken into account. One can prove this from the point of view of the elementary operators $c_{l,r}^\dagger$ and $c_{l,r}^\dagger$. The decomposition is equivalent to

$$\langle c_r^\dagger(t_1)c_r^\dagger(t_3)\rangle_0\langle c_r(t_2)c_r(t_4)\rangle_0\langle c_l(t_1)c_l(t_3)\rangle_0\langle c_l^\dagger(t_2)c_l^\dagger(t_4)\rangle_0$$

and one can see that on both banks of the junction only the condensate properties contribute. The terms which have been neglected in this decomposition belong to quasiparticle tunneling of higher order and processes including both quasiparticles and Cooper pairs. Now the correlators $\langle\tilde{H}_+(t_1)\tilde{H}_+(t_2)\rangle_0$ can be expressed in terms of the pair current amplitude $I_p(\omega)$ as follows:[13]

$$\kappa_\pm(\tau)\equiv\langle\tilde{H}_\pm(t)\tilde{H}_\pm(t-\tau)\rangle_0=\langle\tilde{H}_\pm(\tau)\tilde{H}_\pm(0)\rangle_0=-\frac{\hbar^2}{2\pi e}\int_{-\infty}^{\infty}d\omega\ \mathrm{Im}I_p(\omega)e^{-i\omega\tau}\frac{1}{1-e^{-\hbar\omega/(k_BT)}}\ . \tag{17}$$

In this way the supercurrent becomes frequency dependent. Note the symmetry $\kappa_+(\tau)=\kappa_-(\tau)$. The phase correlations in Gaussian approximation can be calculated, for instance, by generalizing the method presented in Ref. 5.

Then the correlation function $\kappa_1(\tau,\tau',\tau'')$ reads

$$\kappa_1(\tau,\tau',\tau'')=\kappa_+(\tau)\kappa_-(\tau'')e^{J(\tau+4\tau'+\tau'')}-\kappa_+(-\tau)\kappa_-(-\tau'')e^{J(-\tau-4\tau'-\tau'')}-\kappa_+(-\tau)\kappa_-(\tau'')e^{J(3\tau+4\tau'+\tau'')}$$
$$+\kappa_+(\tau)\kappa_-(-\tau'')e^{J(-3\tau-4\tau'-\tau'')}-\kappa_+(\tau)\kappa_-(\tau'')e^{J(-\tau+\tau'')}+\kappa_+(-\tau)\kappa_-(-\tau'')e^{J(\tau-\tau'')}$$
$$+\kappa_+(-\tau)\kappa_-(\tau'')e^{J(\tau+\tau'')}-\kappa_+(\tau)\kappa_-(-\tau'')e^{J(-\tau-\tau'')}. \tag{18}$$

Further analytical results are only possible for the low- or high-impedance environments [Eq. (5) or (6)]. We consider in the following the high-impedance case ($\omega_c\neq 0$). The other case of a low-resistance environment corresponds to the limit $\omega_c\to 0$.

## IV. SUPERCURRENT

Now the first contribution to the supercurrent in Eq. (15) ($\langle I\rangle_s^{(\kappa_1)}$) can be calculated. Using the definition

$$f(\omega)\equiv\frac{\mathrm{Im}I_p(\omega)}{1-e^{-\beta\hbar\omega}}\overset{T\to 0}{\to}\mathrm{Im}I_p(\omega)\Theta(\omega), \tag{19}$$

where $\Theta$ is the unit step function as well as the definition of the function $\delta_+$,

$$\delta_+(x)\equiv\frac{1}{2\pi}\int_0^\infty e^{ikx}dk,$$

the result can be written as

$$\langle I\rangle_s^{(\kappa_1)}=\frac{2\pi}{\hbar}\mathrm{Re}\int_{-\infty}^{\infty}d\omega\int_{-\infty}^{\infty}d\omega'\ f(\omega)f(\omega')\{\delta_+(-v-\omega-\omega_c)\delta_+(-v-\omega'-\omega_c)\delta_+(-v-2\omega_c)$$
$$-\delta_+(-v+\omega+\omega_c)\delta_+(-v+\omega'+\omega_c)\delta_+(-v+2\omega_c)-\delta_+(-v+\omega-3\omega_c)\delta_+(-v-\omega'-\omega_c)\delta_+(-v-2\omega_c)$$
$$+\delta_+(-v-\omega+3\omega_c)\delta_+(-v+\omega'+\omega_c)\delta_+(-v+2\omega_c)-\delta_+(-v-\omega+\omega_c)\delta_+(-v-\omega'-\omega_c)\delta_+(-v)$$
$$+\delta_+(-v+\omega-\omega_c)\delta_+(-v+\omega'+\omega_c)\delta_+(-v)+\delta_+(-v+\omega-\omega_c)\delta_+(-v-\omega'-\omega_c)\delta_+(-v)$$
$$-\delta_+(-v-\omega+\omega_c)\delta_+(-v+\omega'+\omega_c)\delta_+(-v)\}. \tag{20}$$

Here, the variable $v=eV/\hbar$ is used. The same procedure has to be employed with respect to the terms including the other correlation functions $\kappa_2(\tau,\tau',\tau'')$ and $\kappa_3(\tau,\tau',\tau'')$. The real part of the sum of these terms can be calculated by means of the Dirac formula

$$\delta_+(x)=\frac{1}{2}\left(\delta(x)+\mathscr{P}\frac{1}{x}\right). \tag{21}$$

Finally, at least one integration [e.g., with respect to $\omega'$, see Eq. (20)] can be carried out and one gets after a lengthy calculation

$$\langle I\rangle_s(v)=\frac{\pi}{2e}\left\{\left[\frac{1}{\pi}\int_{-\infty}^{\infty}d\omega\,\frac{f(\omega)}{\omega-\omega_c}\right]^2+f(\omega_c)\right\}\delta(v-2\omega_c)$$
$$-\frac{f(v-\omega_c)}{2\pi e}\,\frac{2\omega_c}{v(v-2\omega_c)}\int_{-\infty}^{\infty}d\omega\,\frac{f(\omega)(\omega+\omega_c)}{(\omega+v-\omega_c)(\omega-v+\omega_c)}-[v\rightarrow -v]. \tag{22}$$

The dash in the integral sign means that one has to take the principal value of the integral. Equation (22) is our main result.

## V. DISCUSSION

One can make the following remarks:

(i) The supercurrent is an antisymmetric function of the applied voltage which reflects the expectation that a reversed voltage leads to a reversed current.

(ii) The current shows a $\delta$-like singularity at $2eV=4E_c$ corresponding to the fact that the energy $2eV$ connected with the tunneling of a Cooper pair has to be transferred to the environment. Because Cooper pairs live in the condensate they cannot absorb this energy. Of course, this singular expression will be smoothed due to both finite temperatures and finite environment resistances.

(iii) There is an additional current contribution in Eq. (22) which is proportional to $f(v-\omega_c)$. Because of Eq. (19) and of the known structure of $\mathrm{Im}I_p(\omega)$ in standard BCS theory $[\mathrm{Im}I_p(\omega)=-\mathrm{Im}I_p(-\omega)]$[11]

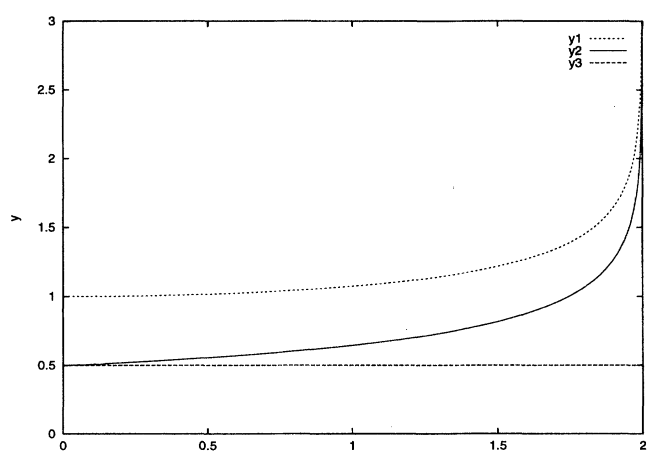


FIG. 1. Plot of $y_1$, $y_2$, and $y_3$ in units of $I_c$ versus $x=\hbar\omega_c/\Delta$ in the subgap region $(0<\omega_c<2\Delta/\hbar)$. The strength of the $\delta$-like singularity in our approach and in the model (1) is proportional to $(y_2)^2$ and $(y_3)^2$, respectively.

$$\mathrm{Im}I_p(\omega)=\frac{2}{\pi}I_c\begin{cases}0 & \text{for}\quad 0<\dfrac{\hbar\omega}{\Delta}<2\\ \dfrac{2\Delta}{\hbar\omega}K\left(\sqrt{1-\left(\dfrac{2\Delta}{\hbar\omega}\right)^2}\right) & \text{for}\quad \dfrac{\hbar\omega}{\Delta}>2\end{cases} \tag{23}$$

this contribution only exists if $v-\omega_c\geqslant 2\Delta/\hbar$ or $eV\geqslant 2\Delta+E_c$ which is just the condition for the onset of the quasiparticle current. The symbol $K$ stands for the complete elliptic integral of the first kind.[15] Since our approach is based on higher-order perturbation theory we are only interested in effects which occur in the gap region of quasiparticle tunneling. Therefore, only the first term of Eq. (22) has to be considered.

(iv) It is reasonable to discuss the first term of Eq. (22) in the case $\omega_c<2\Delta/\hbar$ where the position of the supercurrent peak is in the region between zero and the threshold of quasiparticle tunneling. This means $f(\omega_c)\equiv 0$ and the supercurrent reads $(0<v<2\Delta/\hbar+\omega_c)$

$$\langle I\rangle_s(v)=\frac{\pi}{2e}\left[\frac{1}{\pi}\int_{-\infty}^{\infty}d\omega\,\frac{f(\omega)}{\omega-\omega_c}\right]^2\delta(v-2\omega_c)\quad . \tag{24}$$

The comparison between Eqs. (2) and (24) shows that our approach corresponds to the substitution

$$\frac{I_c}{2}\rightarrow\left|\frac{1}{\pi}\int_{-\infty}^{\infty}d\omega\,\frac{f(\omega)}{\omega-\omega_c}\right|. \tag{25}$$

The integral in Eq. (24) reminds one of the definition of $\mathrm{Re}I_p(\omega_c)$ according to the Kramers-Kronig relation. The only difference is the $\Theta$ function in the integrand.

In Fig. 1 we have plotted the expressions

$$y_1\equiv\mathrm{Re}I_p(\omega_c)=-\frac{1}{\pi}\int_{-\infty}^{\infty}d\omega'\,\frac{\mathrm{Im}I_p(\omega')}{\omega'-\omega_c}$$
$$=\frac{2}{\pi}I_c\begin{cases}K\left(\dfrac{\hbar\omega_c}{2\Delta}\right) & \text{for}\quad 0<\dfrac{\hbar\omega_c}{\Delta}<2\\ \dfrac{2\Delta}{\hbar\omega_c}K\left(\dfrac{2\Delta}{\hbar\omega_c}\right) & \text{for}\quad \dfrac{\hbar\omega_c}{\Delta}>2\end{cases}$$

(see Ref. 11),

$$y_2 \equiv -\frac{1}{\pi}\int_{-\infty}^{\infty} d\omega' \, \frac{\mathrm{Im} I_p(\omega')\Theta(\omega')}{\omega' - \omega_c},$$

and

$$y_3 \equiv \frac{\mathrm{Re} I_p(0)}{2} = \frac{I_c}{2}$$

for $0 \leqslant \omega_c < 2\Delta/\hbar$. The plot shows that sufficiently far from the position of the Riedel peak ($\omega_c < 2\Delta/\hbar$) the approximation

$$-\frac{1}{\pi}\int_{-\infty}^{\infty} d\omega \, \frac{f(\omega)}{\omega - \omega_c} \approx \frac{1}{2} \mathrm{Re} I_p(\omega_c) \qquad (26)$$

holds, which becomes exact for $\omega_c \to 0$. Hence, for very small $\omega_c$ ($\omega_c \ll 2\Delta/\hbar$) one can write

$$-\frac{1}{\pi}\int_{-\infty}^{\infty} d\omega \, \frac{f(\omega)}{\omega - \omega_c} \approx \frac{1}{2} I_c. \qquad (27)$$

Using this approximation, the supercurrent reads

$$\langle I \rangle_s(v) \approx \frac{\pi}{8e} I_c^2 \{\delta(v - 2\omega_c) - \delta(-v - 2\omega_c)\}, \qquad (28)$$

which corresponds for $T \to 0$ and $R_E/R_Q \to \infty$ exactly to the result of Eq. (2). It has been shown that this formula is valid for $0 < eV < 2\Delta + E_c$ and $E_c \ll 2\Delta$. There is no contradiction to the inequality (7) because $2\Delta \gg E_c \gg R_Q/(8R_T)\Delta$ is satisfied provided that the relation $R_Q \ll R_T$ holds. However, this condition is just necessary for single-charge tunneling because it guarantees that quantum fluctuations can be neglected.

Equations (24) and (26) show that for small $\omega_c$ the strength of the $\delta$-like current peak at the $eV = 2E_c$ is determined by the Josephson current amplitude $\mathrm{Re} I_p(\omega_c)$. This strength increases for growing $\omega_c$ and becomes singular for $\omega_c \to 2\Delta/\hbar$ where the position of the $\delta$ peak tends to the onset position of quasiparticle tunneling. This is the main difference from the effective model of Eq. (1) where this strength is given by the constant $I_c$, which does not feel the vicinity of the Riedel peak and the threshold of quasiparticle current. Equation (22) shows that for $T > 0$ there are also current contributions depending on the dissipative part of the pair current $\mathrm{Im} I_p$ which describes pair transfer processes via thermally excited quasiparticles.

In the case of a finite environment resistance the substitution $J(t) = -i\omega_c t$ in Eq. (18) is not possible. The investigation has shown that the integration over $\tau'$ in Eq. (15) is the origin of the resulting $\delta$ function in Eq. (28). Therefore, this integration would indeed generate the function $P'(2eV)$ known from Eq. (3) provided that the relation $J(\tau' + \tau) = J(\tau') + J(\tau)$ would hold at least approximately. But then there are also functions $J(\tau)$ and $J(\tau'')$ which are modifying the other integrations over $\tau$ and $\tau''$. To sum up it can be said that in this stricter approach the dependence on the environment is much more complicated than in the model (1) which leads to Eq. (2).

Using some simple assumptions one can reconstruct an effective Hamiltonian which leads in first-order perturbation theory to the same result (28). It turns out that this effective Hamiltonian corresponds just to the Hamiltonian (1). Our starting point is the Hamiltonian (10) and the assumption that the perturbation term $H_T$ can be written as

$$H_T = H_+ + H_- = He^{i\Psi} + He^{-i\Psi}.$$

This ansatz with a real constant $H$ seems to be very likely because the prefactor of the $\delta$ function in the approximated supercurrent (28) is a constant, too. $\Psi$ is a phase operator which is assumed to obey the commutation relation

$$[Q, \Psi] = ike, \qquad (29)$$

where the constant $k$ is for the time being arbitrary. The mean current of this theory reads in first order

$$\langle I \rangle_s = \frac{2keH^2}{\hbar^2} \mathrm{Re} \int_0^{\infty} d\tau \, e^{\frac{i}{\hbar} keV\tau} [e^{k^2 J(\tau)} - e^{k^2 J(-\tau)}], \qquad (30)$$

where we already know the function $J$ from Eq. (4). By comparing Eq. (30) with Eq. (28) in the limit $T = 0$ and for $R_E/R_Q \to \infty$ the unknown constants $H$ and $k$ can be determined

$$k = 2, \quad H = \frac{\hbar}{4e} I_c = \frac{E_J}{2}, \qquad (31)$$

which reproduce expression (1) with $\Psi = 2\Phi$. The value $k = 2$ shows that the effective Hamiltonian describes tunneling of electron pairs (Cooper pairs). In this way the transition to the effective model corresponds to the transition from $[Q, \Phi] = ie$ to $[Q, \Psi] = i2e$. However, due to the chosen assumptions this effective model cannot contain dissipation.

## ACKNOWLEDGMENTS

We would like to thank H.-O. Müller for useful discussions. This work was supported by the Deutsche Forschungsgemeinschaft.


[1] *Single Charge Tunneling: Coulomb Blockade Phenomena in Nanostructures*, Vol. 294 of *NATO ASI Series B: Physics*, edited by H. Grabert and M. H. Devoret (Plenum, New York, 1992).

[2] M. Tinkham, *Introduction to Superconductivity* (McGraw-Hill, New York, 1976).

[3] D. V. Averin, Yu. V. Nazarov, and A. A. Odintsov, Physica B **165&166**, 945 (1990).

[4] G. Falci, V. Bubanja, and G. Schön, Z. Phys. B **85**, 451 (1991); Europhys. Lett. **16**, 109 (1991).

[5] G.-L. Ingold and Yu. V. Nazarov, in *Single Charge Tunneling: Coulomb Blockade Phenomena in Nanostructures* (Ref. 1), pp. 21–107.

[6] G.-L. Ingold, H. Grabert, and U. Eberhardt, Phys. Rev. B **50**, 395 (1994).

[7] W. Krech and A. Hädicke, Phys. Status Solidi B **180**, 175 (1993).
[8] M. H. Devoret, D. Esteve, H. Grabert, G.-L. Ingold, H. Pothier, and C. Urbina, Phys. Rev. Lett. **64**, 1824 (1990).
[9] H. Grabert, G.-L. Ingold, M. H. Devoret, D. Esteve, H. Pothier, and C. Urbina, Z. Phys. B **84**, 143 (1991).
[10] L. S. Kuzmin, Yu. V. Nazarov, D. B. Haviland, P. Delsing, and T. Claeson, Phys. Rev. Lett. **67**, 1161 (1991).
[11] K. K. Likharev, *Dynamics of Josephson Junctions and Circuits* (Gordon and Breach, New York, 1986).
[12] D. Rogovin and D. J. Scalapino, Ann. Phys. (N.Y.) **86**, 1 (1974).
[13] D. V. Averin and K. K. Likharev, J. Low Temp. Phys. **62**, 345 (1986).
[14] G. D. Mahan, *Many Particle Physics* (Plenum, New York, 1993), Chap. 9.
[15] I. S. Gradstein and I. M. Ryshik, *Tables of Series, Products, and Integrals* (Harri Deutsch, Thun, 1981).