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Elastic theory of flux lattices in the presence of weak disorder

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The effect of weak impurity disorder on flux lattices at equilibrium is studied quantitatively in the absence of free dislocations using both the Gaussian variational method and the renormalization group. Our results for the mean-square relative displacements $\tilde{B}(x) = \overline{\langle u(x) - u(0)\rangle^2}$ clarify the nature of the crossovers with distance. We find three regimes: (i) a short distance regime ("Larkin regime") where elasticity holds, (ii) an intermediate regime ("random manifold") where vortices are pinned independently, and (iii) a large distance, quasiordered regime where the periodicity of the lattice becomes important. In the last regime we find universal logarithmic growth of displacements for $2 < d < 4$: $\tilde{B}(x) \sim A_d \ln|x|$ and persistence of algebraic quasi-long-range translational order. The functional renormalization group to $O(\epsilon = 4 - d)$ and the variational method, when they can be compared, agree within 10% on the value of A_d. In $d = 3$ we compute the function describing the crossover between the three regimes. We discuss the observable signature of this crossover in decoration experiments and in neutron-diffraction experiments on flux lattices. Qualitative arguments are given suggesting the existence for weak disorder in $d = 3$ of a "Bragg glass" phase without free dislocations and with algebraically divergent Bragg peaks. In $d = 1 + 1$ both the variational method and the Cardy-Ostlund renormalization group predict a glassy state below the same transition temperature $T = T_c$, but with different $\tilde{B}(x)$ behaviors. Applications to $d = 2 + 0$ systems and experiments on magnetic bubbles are discussed.

I. INTRODUCTION

The interest in the pinning of the Abrikosov vortex lattice by impurities was revived recently with the discovery of high-T_c superconductors. Impurity disorder conflicts with the long range translational order of the flux lattice and some glassy state is generally expected to appear. Understanding the precise nature of this new thermodynamic state and how it depends on the type of disorder existing in the system is very important for the determination of the transport properties of these materials, such as critical currents and I-V characteristics.[1–5] This problem, however, is only one aspect of the more fundamental and broader question of the effect of quenched impurities on any translationally ordered structure, such as a crystal. This question arises in a large number of physical systems under current active experimental study. Examples are charge density waves,[6] Wigner crystals,[7–10] magnetic bubbles,[11,12] Josephson junctions,[13,14] the surface of crystals with quenched bulk or substrate disorder,[15] and domain walls in incommensurate solids.[16] All these systems have in common a perfectly ordered underlying structure modified by elastic distortions and possibly by topological defects such as dislocations, due to temperature or disorder. The effect of thermal fluctuations alone on three-dimensional and especially on two-dimensional structures is by now well understood, and it was shown that topological defects are not important in the low-temperature solid phase.[17] Much less is known, however, on the additional effects of quenched disorder. In particular, the important question of precisely how quenched disorder destroys the translational long range order of the lattice is far from being elucidated. If disorder is strong, the underlying order is *a priori* destroyed at every scale and an analytical description of the problem starting from the Abrikosov lattice is difficult. One then has to use a more macroscopic approach based on phenomenological models such as the gauge glass models.[2,18,19] The success of these approaches then crucially depends on whether these effective models are indeed a good representation of the system at large scale, a largely uncontrolled assumption. If disorder is weak enough, however, one expects the perfect lattice to survive at short scales. Thus a natural first step for a theoretical description is to neglect dislocations and to treat the simpler problem of an elastic medium submitted to weak impurities. In that case one can consider a Gaussian random potential created by many weak impurities with short range correlations. In this paper we will focus on pointlike disorder. The problem of correlated disorder, such as columnar or twin boundary pinning in superconductors,[3,4] which is also relevant for any type of *quantum* problems can be treated by similar methods, and is examined in detail in Ref. 20.

This simpler problem of an elastic lattice in the presence of weak disorder is already quite nontrivial. Despite several attempts, its physics has not been completely understood. An important quantity, which measures how fast translational order decays, is the translational correlation function $C_{K_0}(r) = \overline{\langle e^{iK_0\cdot[u(r)-u(0)]}\rangle}$, where $u(r)$ is the displacement from the perfect lattice and K_0 one of the first reciprocal lattice vectors. We denote by $\langle\rangle$ and — the thermodynamic average and the disorder average, respectively. $C_{K_0}(r)$ can be extracted from the Fourier transform of the density-density correlation function at wave vectors near $q = K_0$, or directly measured by imaging the deformed lattice, and is thus a quantity easily accessible in experiments.

A calculation of $C_{K_0}(r)$ was performed by Larkin[21] using a model in which weak *random forces* act independently on each vortex. These forces are correlated over a small length ξ_0, of the order of the superconducting coherence length. This model predicts that weak disorder destroys translational order below four dimensions[21] and $C_{K_0}(r) \sim \exp(-r^{4-d})$. The destruction of the long range order in this simple Gaussian model can be understood easily from the standard Imry-Ma argument. To accommodate the random forces a region of size R will undergo relative deformations of order u. The cost in elastic energy is $R^{d-2}u^2$, while the gain in potential energy is $uR^{d/2}$. The optimal is thus $u = R^{(4-d)/2}$ leading to the above decay of $C_{K_0}(r)$. Using similar arguments in the presence of an external Lorentz force, Larkin and Ovchinikov[22] constructed a theory of collective pinning of the flux lattice. In this theory the critical current is determined from the length scale R_c at which relative displacements are of order $u \approx \xi_0$. This theory was very successful in describing conventional superconductors. However, a need to reconsider this theory was prompted by high-T_c superconductors where the flux lattice is usually probed at larger scales. It turns out that the Larkin model, while it is useful for estimating critical currents, cannot be used to study large scale quantities such as translational order.

In fact the purely Gaussian model with random forces, and the resulting linear elasticity, becomes inadequate beyond the Larkin-Ovchinikov length R_c. It has only one trivial equilibrium state and responds linearly to external force. It is thus too simple to approximate correctly the full nonlinear problem and grossly overestimates the effect of disorder. At larger scales the lattice starts behaving collectively as an elastic manifold in a *random potential* with many metastable states; thus the exponential decay of $C_{K_0}(r)$ in $d = 3$ cannot hold beyond $r > R_c$. Using known results on the so-called "random manifold" problem, Feigelman *et al.*[1] showed that the system presents glassy behavior and computed transport properties. This was also pointed out by Bouchaud, Mézard, and Yedidia (BMY),[23,24] who used the Gaussian variational method (GVM) to study this problem, and found a power-law roughening of the lattice with stretched exponential decay of $C_{K_0}(r)$.

However, the *periodicity* of the lattice was not properly taken into account in all the above works. The periodicity has important consequences for the behavior of correlation functions at large scales. Indeed, it was suggested with the use of qualitative Flory arguments that periodicity leads to logarithmic roughening,[25] rather than a power law.

In this paper we develop a quantitative description of the static properties of a lattice in the presence of disorder. A short account of some of the results of this paper was presented in a recent Letter.[26] We take into account both the existence of many metastable states and the periodicity of the lattice. One of the difficulties in the physics of this problem is that the disorder varies at a much shorter length scale than the lattice spacing. As a consequence the elastic limit has to be taken with some care. Indeed in this limit the displacement varies slowly, but the density still consists in a series of peaks. To couple the density to the random potential it is thus important to distinguish between its slowly varying components $\Delta_{q=0}$ and its Fourier component Δ_{K_0} close to the periodicity of the lattice. This separation of harmonics exposes clearly the physics and allows us to treat all the regimes in length scales in a simple way. To study the resulting model, we mainly use the Gaussian variational method, developed to study manifolds in random media.[27] We also use the renormalization group (RG) close to $d = 4$ dimensions and in $d = 2$ dimensions. Comparison of the two methods provides a confirmation of the accuracy of the GVM.

One of the main results of the present study, which is somewhat surprising in view of conventional wisdom based on Larkin's original calculation, is that quasi-long-range order survives in the system. This means that $C_{K_0}(r)$ decays as a power law at large distance. Such a property for a disordered lattice is similar to the quasiorder found for clean two-dimensional solids. This state, however, has the peculiar property of being a glass with many metastable states, and at the same time shows Bragg peaks as a solid does. For these reasons we would like to call it a "Bragg glass." Note that this is a much stronger property than the so-called "hexatic glass"[28,29] since hexatic order in the elastic limit is a straightforward consequence of the absence of dislocations. In the Bragg glass, two important length scales control the crossover towards the asymptotic decay, a consequence of the fact that the disorder varies at a much smaller scale than the lattice spacing a. When the mean square of the relative displacement $\overline{\langle[u(x) - u(0)]^2\rangle}$ of two vortices as a function of their separation x is shorter than the square of the Lindemann length $l_T^2 = \langle u^2\rangle$, the thermal wandering of the lines averages enough over the random potential and the model becomes equivalent to the random force Larkin model. At low enough temperature, l_T is replaced by the correlation length of the random potential ξ_0, which is of the order of the superconducting coherence length. In that case the crossover length is R_c.[1,24,23] When the relative displacement is larger than the correlation length of the random potential but smaller than the lattice spacing a, this is the random manifold regime where each line experiences effectively an independent random potential. When the relative displacement becomes larger than the lattice spacing, one enters the asymptotic quasiordered regime. This occurs for separations of order ξ. In general the two lengths R_c and ξ are widely different. The theory developed here can be applied to any elastic system in the elastic limit $\xi \gg a$. In relation with experimental systems we focus particularly on the triangular

Abrikosov lattice in $d = 2+1$, point vortices in thin films, and magnetic bubbles in $d = 2+0$, and will also mention lines in a plane $d = 1+1$.

In this paper we will not treat topological defects quantitatively. Although a full description of a lattice in the presence of disorder should also include topological defects, their effect might not be as severe as commonly believed from misleading Imry-Ma type arguments. Indeed the fact that within the elastic theory quasi-long-range order is preserved at large distances makes the system much more stable to dislocations. Since in $d = 3$ the core energy of a dislocation increases as its size L, it is actually very possible that a phase without unbound dislocations exists in $d = 3$ in the presence of weak disorder. Indeed, Bitter decoration experiments[29] at the highest fields available, about 70 G for these low-field experiments, show remarkably large regions free of dislocations. In recent neutron experiments[30] it was shown that the degree of order depends on the way the system is prepared. A more perfect lattice with a smaller number of dislocations was prepared by first driving the system at a velocity high enough for translational order to heal, and then slowing it back down to zero velocity. It is thus conceivable that in $d = 3$ the presence of dislocations is overall a nonequilibrium feature. In two dimensions ($d = 2+0$), dislocations are energetically less costly and will probably appear at large scales, although this has not yet been firmly established. However, as we will discuss here, the length scale between unbound dislocations ξ_D can be much larger than ξ in a low-temperature regime. In that regime the main cause of the decay of translational order is elastic deformations due to disorder.

The paper is organized as follows. For convenience we have separated the mostly technical sections III and V from the ones discussing applications to physical systems IV and VI. In Sec. II, we introduce the model and derive the correct elastic limit. Simple dimensional arguments like those of Fukuyama and Lee are given to identify the relevant length scales. In Sec. III we apply the Gaussian variational method to a simplified isotropic version of the model. Thus the method can be exposed without being obscured by unnecessary technical complications specific to real vortex lattices such as anisotropy and nonlocal elasticity, while the essential physics is retained. This section contains most of the technical details and methods used. We explain why a previous application of the variational method by BMY led to erroneous conclusions about the behavior at large scales. In Sec. IV we apply the theory to $d = 2+1$ solids such as the vortex lattices using a realistic elastic Hamiltonian. We compute the translational order correlation function $C_{K_0}(r)$ with the full crossover between the three regimes in distance. We discuss the experimental signatures for decoration and neutron-diffraction experiments. In particular, the results of a comparison between decoration images and theoretical predictions are mentioned and detailed predictions are made for the neutron-diffraction intensities. We then give a simple physical interpretation of the various regimes in distance and also argue that dislocations are less likely to appear than commonly believed. In Sec. V we apply the functional renormalization group in $d = 4-\epsilon$, and compare its findings with those of the variational method. In Sec. VI we examine two-dimensional systems, for which thermal fluctuations play a more important role. We first apply the variational method which shows that below a critical temperature T_c the system is glassy with logarithmic growth of displacements. This is compared to predictions of the renormalization group in $d = 1+1$. We then give a physical discussion of what should be expected for $d = 2+0$ systems, such as magnetic bubbles, where dislocations have to be considered. Conclusions can be found in Sec. VII. Finally, the reader can look at the Appendixes to which most of the most tedious technicalities are relegated.

II. DERIVATION OF THE MODEL AND PHYSICAL CONTENT

A. A general elastic Hamiltonian

In the absence of disorder the vortices form, at equilibrium, a perfect lattice of spacing a whose sites are labeled by an integer i and position will be denoted by R_i. Since we want to apply this theory also to a lattice of vortex lines, we consider the more general case where the R_i are m-component lattice vectors and there are in addition $d-m$ transverse directions denoted by z so that the total spatial dimension is d. Throughout this paper we will denote the d-dimensional coordinates by $x \equiv (r, z)$ and similarly the Fourier space (momentum) coordinates by $q \equiv (q_\perp, q_z)$. For example, the Abrikosov lattice corresponds to $m = 2$ and $d = 3$ and z is along the direction of the magnetic field. The displacements relative to the equilibrium positions are denoted by the m-component vector $u(R_i, z) \equiv u_i(z)$. For weak disorder ($a/\xi \ll 1$ where ξ is defined below) and in the absence of dislocations, it is legitimate to assume that $u(R_i, z)$ is slowly varying on the scale of the lattice and to use a continuum elastic energy, as a function of the continuous variable $u(x)$. We consider the simple elastic Hamiltonian

$$H_{\rm el} = \frac{1}{2}\sum_{\alpha,\beta}\int_{\rm BZ}\frac{d^dq}{(2\pi)^d}u_\alpha(q)\Phi_{\alpha\beta}(q)u_\beta(-q), \qquad (2.1)$$

where $\alpha, \beta = 1, \ldots, m$ label the coordinates, and BZ denotes the Brillouin zone. The $\Phi_{\alpha\beta}$ are the elastic matrix. Such an elastic description is valid as long as the *relative* displacement of two neighboring points remains small, i.e., $|u(R_i) - u(R_{i+1})| \ll a$, but does *not* suppose the individual displacements themselves to be small. We defer the study of the realistic elastic Hamiltonian (2.1) until Secs. IV and VI, and in order to illustrate the method in Secs. II and III we use the fully isotropic elastic Hamiltonian

$$H_{\rm el} = \frac{c}{2}\int d^dx[\nabla u(x)]^2 \qquad (2.2)$$

corresponding to the case $\Phi_{\alpha\beta}(q) = cq^2\delta_{\alpha\beta}$.

In the limit where many weak impurities act collectively on a vortex the disorder can be modeled by a Gaussian random potential $V(x)$ with correlations $\overline{V(x)V(x')} = \Delta(r - r')\delta(z - z')$, where $\Delta(r)$ is a short range function of range ξ_0 (the superconducting coherence length) and Fourier transform $\Delta_{q_\perp}$. The other limit, corresponding to a few strong pins, can be modeled by a Poissonian distribution and will not be considered here. Since the density of vortices at a given point is given by

$$\rho(x) = \sum_i \delta(r - R_i - u(R_i, z)) \tag{2.3}$$

the total Hamiltonian is therefore

$$H = H_{\rm el} + \int d^d x V(x)\rho(x). \tag{2.4}$$

The simplest way to recover translational invariance is to use the well-known replica trick.[31] This amounts to introducing n identical systems by replicating the original Hamiltonian. It is then possible to average over disorder, the proper quenched average being recovered in the limit $n \to 0$. After replicating, the interaction term becomes

$$H_{\rm pin} = -\frac{1}{2T}\sum_{a,b}\int d^d x d^d x' \Delta(r - r') \times \delta(z - z')\rho^a(x)\rho^b(x'), \tag{2.5}$$

where a, b are the replica indices. Equation (2.5) can be rewritten

$$\begin{aligned} H_{\rm pin} &= -\frac{1}{2T}\sum_{a,b}\int d^d x d^d x' \sum_{i,j}\Delta(x - x')\delta(z - z') \\ &\quad\times \delta(x - R_i - u_i^a(z))\delta(x - R_j - u_j^b(z)) \\ &= -\frac{1}{2T}\sum_{a,b}\sum_{i,j}\int d^{d-m} z \Delta(R_i - R_j \\ &\quad + u_i^a(z) - u_j^b(z)), \end{aligned} \tag{2.6}$$

where $a, b = 1, \ldots, n$ are the replica indices. As we show in the following section it is extremely important to keep the discrete nature of the lattice in (2.6), and the continuum limit of $H_{\rm pin}$ should be done with some care.

B. Decomposition of the density

Using the form (2.6) for the Hamiltonian in terms of the displacement fields u_i is rather cumbersome. Equation (2.6) leads to a nonlocal theory, even in the limit where the disorder is completely uncorrelated $\Delta(r-r') = \delta(r - r')$. Indeed, vortices belonging to two different replica sets can be *a priori* at the same point in space $r = R_i + u_i^a(z) = R_j + u_j^b(z)$ while having very different equilibrium positions $R_i \neq R_j$. This occurs when the displacements of the vortices are large enough. Since R_i, the equilibrium position of the vortices, have clearly no physical significance, except as an internal label, it is much more convenient to use instead a label that is a function of the actual position of the vortices. This can be achieved by introducing the slowly varying field

$$\phi(r, z) = r - u(\phi(r, z), z). \tag{2.7}$$

Such a field will allow the continuum limit of (2.6) to be taken easily. For each configuration of disorder, or alternatively for each replica set, one introduces a different field ϕ. Such a labeling is always exact when the transverse dimension m is $m = 1$.[32] In more than one transverse dimension, this representation assumes the absence of dislocations in the system. In a self-consistent manner, we will justify *a posteriori* both assumptions of elasticity and absence of dislocations using the solution in Secs. III D and IV C.

Using $\phi(x)$ the density can be rewritten as (see Appendix A)

$$\rho(x) = \rho_0 \det[\partial_\alpha \phi_\beta] \sum_K e^{iK\cdot\phi(x)}, \tag{2.8}$$

where the K are the vectors of the reciprocal lattice and ρ_0 is the average density. In the elastic limit one can expand (2.8) to get

$$\rho(x) \simeq \rho_0 \left(1 - \partial_\alpha u_\alpha(\phi(x)) + \sum_{K\neq 0} e^{iKr}\rho_K(x)\right), \tag{2.9}$$

where

$$\rho_K(x) = e^{-iK\cdot u(\phi(x))} \tag{2.10}$$

is the usual translational order parameter defined in terms of the reciprocal lattice vectors K. Expression (2.9) respects the periodicity of the lattice, i.e., is obviously invariant by a global translation $u \to u + a$. Another advantage of the decomposition (2.9) is that the various Fourier components of the density relative to the periodicity of the ordered lattice appear clearly in $H_{\rm pin}$,

$$\begin{aligned} \int dx V(x)\rho(x) &= -\rho_0 \int dx V(x)\partial_\alpha u_\alpha \\ &\quad + \rho_0 \int dx \sum_{K\neq 0} V_{-K}(x)\rho_K(x), \end{aligned} \tag{2.11}$$

where

$$V_K(x) = V(x)e^{-iKr} \tag{2.12}$$

is the part of the random potential with Fourier components close to K. Since the energy is invariant when changing $u \to u + a$, u itself cannot appear in the expression (2.11), but only ∂u, and in principle higher derivatives or a periodic function of u are allowed.

The first term in the right-hand side of (2.11) is the part of the deformation of the lattice that couples to the long wavelength of the disorder potential. It results in an increase or decrease of the average density in regions where the potential is favorable or unfavorable. The second term couples to the higher Fourier components of the disorder. The average density is not affected but the lattice can be shifted so that the lines sit in the minimum of the disorder potential. In the usual elasticity theory, one takes the continuum limit for the displacement field *and* assumes that the density itself is smooth on the scale of the lattice. This allows one to keep only the gradient term in (2.11). Here, although it is possible to take the continuum limit for the displacements u since they vary slowly on the scale of the vortex lattice ($\nabla u \ll 1$), it is imperative to retain the discrete nature of the density. This is because the scale at which the disorder varies (for superconductors it is comparable to the scale of the real atomic crystal) is usually shorter than the lattice spacing of the vortex lattice itself.

If one uses the representation (2.9) of the density and (2.6) and discards spatial averages of rapidly oscillating terms, the replicated Hamiltonian becomes, in the isotropic case,

$$H_{\text{eff}} = \frac{c}{2}\int d^dx[\nabla u(x)]^2 - \int d^dx \sum_{a,b}\left[\frac{\rho_0^2\Delta_0}{2T}\partial_\alpha u_\alpha^a \partial_\beta u_\beta^b + \sum_{K\neq 0}\frac{\rho_0^2\Delta_K}{2T}\cos\{K\cdot[u^a(x)-u^b(x)]\}\right]. \tag{2.13}$$

To be rigorous the last terms in (2.13) should be written in terms of $u(\phi(x))$ rather than $u(x)$, but this has no effect on our results. It leads only to corrections of higher order in ∇u which we neglect since we work in the elastic limit $a/\xi \ll 1$. The Hamiltonian (2.13) will be our starting model, and from now on we absorb the coefficient ρ_0^2 in Δ_K, $\rho_0^2\Delta_K \to \Delta_K$.

A general property of the Hamiltonian (2.13) is the invariance of the disorder term under the transformation $u_a(x) \to u_a(x) + w(x)$ where $w(x)$ is an arbitrary function of x. This statistical invariance *guarantees* that the elastic term in (2.13) is unrenormalized by disorder. Note that in the *original* nonlocal model (2.6) this symmetry is only approximate, and indeed one would find there a small [of order $(a/\xi)^2$] and unimportant renormalization of the elastic coefficients by disorder.

The principal quantities of interest are the mean squared relative displacement $\tilde{B}(x)$ of two vortices, averaged over disorder, which is determined by the correlation of u diagonal in replicas

$$\tilde{B}(x) = \frac{1}{m}\overline{\langle[u(x)-u(0)]^2\rangle} = 2T\int\frac{d^dq}{(2\pi)^d}[1-\cos(qx)]\tilde{G}(q), \tag{2.14}$$

$$\langle u_\alpha^a(q)u_\beta^a(-q)\rangle = \delta_{\alpha\beta}T\tilde{G}(q),$$

and the translational order correlation function $C_K(x)$

$$C_K(x) = \overline{\langle\rho_K^*(x)\rho_K(0)\rangle}. \tag{2.15}$$

In the Gaussian theory that will be considered below, the two correlation functions are simply related by

$$C_K(x) = e^{-\frac{K^2}{2}\tilde{B}(x)}. \tag{2.16}$$

The Hamiltonian (2.13) can be applied directly to study quantum models with a time dependent disorder. A more physical disorder for quantum systems would be only space dependent. This would correspond to correlated disorder in one (the "time") direction for the classical system and can be studied by the same method as the one used in this paper.[20] For completeness we also give here in Appendix B the connection between the quantum mechanics of interacting bosons and fermions and an elastic system in $d = 1 + 1$ dimensions.

C. Dimensional arguments

Before starting the full calculation, let us estimate the effects of the different terms in (2.11) in a way similar to Ref. 33. In the presence of many weak pins, u cannot distort to take advantage of each of them, due to the cost in elastic energy. One can assume that u varies of $\sim a$ over a length $\xi \gg a$. The density of kinetic energy is $\sim c(a/\xi)^2$, where c is an elastic constant. The various Fourier components of the disorder will give different contributions. The long wavelength part of the disorder gives

$$H^{\text{dis}}_{q\sim 0} \sim \rho_0\int d^dx V(x)\partial_\alpha u_\alpha(x) \sim \Delta_0^{1/2}a/\xi^{1+d/2}. \tag{2.17}$$

For the higher Fourier components the disorder term

$$H^{\text{dis}}_{q\sim K} = \rho_0\int d^dx V(x)e^{iKx}e^{-iKu(x)} \tag{2.18}$$

can be estimated over the volume ξ^d as $e^{-iKu}\int_{\xi^d} d^dx V(x)e^{iKx}$. This sum can be viewed as a random walk in the complex plane[33] and the value of u adjusts itself to match the phase of the random potential. Therefore the gain in energy density due to the disorder term is of order

$$H^{\text{dis}}_{q\sim K} \sim \Delta_K^{1/2}/\xi^{d/2}. \tag{2.19}$$

Optimizing the gain in potential energy versus the cost in elastic energy determines ξ. One can therefore associate to each Fourier component a length scale above which the corresponding disorder will be relevant and destroy the perfect lattice

$$\xi_{q\sim 0} \sim a\left(c^2a^d/\Delta_0\right)^{1/(2-d)} \quad \text{if } d<2 \tag{2.20}$$

$$\xi_{q\sim K} \sim a\left(c^2a^d/\Delta_K\right)^{1/(4-d)}. \tag{2.21}$$

The $q \sim 0$ component of the disorder is relevant only for $d \leq 2$ and the second term in (2.13) can be dropped for $d > 2$ if one is interested in the asymptotic regime. In fact, the $q \sim 0$ part of the disorder can be eliminated exactly from (2.13) and leads only to trivial redefinitions of the correlation functions. One can perform a translation of the longitudinal displacement field u by

$$u_\alpha(x) \to u_\alpha(x) + f_\alpha(x), \tag{2.22}$$

where the Fourier transform of f is

$$f_\alpha(q) = \frac{i\rho_0 q_\alpha V_{q\sim 0}}{cq^2}. \tag{2.23}$$

The translation (2.22) when performed on the replicated form (2.13) eliminates the long wavelength term but does leave the cosine term invariant since it is a local transformation. Note that such a transformation is only possible due to the fact that the various Fourier components of the disorder are uncorrelated. The mean squared relative displacement $\tilde{B}(x)$ becomes

$$\tilde{B}(x) = \tilde{B}(x)\Big|_{\Delta_0=0} + \frac{\Delta_0}{m}\int \frac{d^dq}{(2\pi)^d}\frac{q_\perp^2}{(cq^2)^2}[1-\cos(qx)]. \tag{2.24}$$

As expected, the additional term produces only a subdominant finite correction above two dimensions. In the following we simply set $\Delta_0 = 0$.

As is obvious from (2.20), higher Fourier components $V_{q\sim K}$ disorder the lattice below $d = 4$. We will now examine the effect of these Fourier components more quantitatively.

III. VARIATIONAL METHOD

We now study the Hamiltonian (2.13) using the variational method introduced by Mezard and Parisi.[27] Hamiltonians with more realistic elastic energy terms, directly relevant for experimental systems, will be considered in Secs. IV and VI.

A. Derivation of the saddle point equations

We now look for the best trial Gaussian Hamiltonian H_0 in replica space which approximates (2.13). It has the general form[27]

$$H_0 = \frac{1}{2}\int \frac{d^dq}{(2\pi)^d} G_{ab}^{-1}(q) u_a(q)\cdot u_b(-q), \tag{3.1}$$

where the $[G^{-1}]_{ab}(q)$ is an $n\times n$ matrix of variational parameters. Without loss of generality, the matrix $G_{ab}^{-1}(q)$ can be chosen of the form $G_{ab}^{-1} = cq^2\delta_{ab} - \sigma_{ab}$ where the self-energy σ_{ab} is simply a matrix of constants. The connected part is defined as $G_c^{-1}(q) = \sum_b G_{ab}^{-1}(q)$. We obtain by minimization of the variational free energy $F_{\rm var} = F_0 + \langle H_{\rm eff} - H_0\rangle_{H_0}$ the saddle point equations

$$G_c(q) = \frac{1}{cq^2}, \quad \sigma_{a\neq b} = \sum_K \frac{\Delta_K}{mT} K^2 e^{-\frac{K^2}{2}B_{ab}(x=0)}, \tag{3.2}$$

where m is the number of components of u. One defines

$$\begin{aligned} B_{ab}(x) &= \frac{1}{m}\langle [u_a(x) - u_b(0)]^2\rangle \\ &= T\int \frac{d^dq}{(2\pi)^d}[G_{aa}(q) + G_{bb}(q) \\ &\quad -2\cos(qx)G_{ab}(q)]. \end{aligned} \tag{3.3}$$

Note that the connected part is unchanged by disorder, a direct consequence of the statistical symmetry of (2.13) noted above. Two general classes of solutions can exist for (3.2). One preserves the symmetry of permutations of the replica, and amounts to mimicking the distribution (thermal and over disorder) of each displacement mode $u(q)$ by a simple Gaussian. The other class, which is a better approximation in the glassy phase, breaks replica symmetry and approximates the distribution of displacements by a hierarchical superposition of Gaussians centered at different points in space.[27] Each Gaussian at the lowest level of the hierarchy corresponds to a different metastable "pinned" position of the manifold.

B. Replica symmetric solution

Let us first examine the replica symmetric solution $G_{a\neq b}(q) = G(q)$ and $B_{a\neq b}(x) = B(x)$. Using (3.3) one has

$$B(x=0) = 2T\int \frac{d^dq}{(2\pi)^d} G_c(q). \tag{3.4}$$

For $d \leq 2$, $B(x=0)$ is infinite, and the off-diagonal part of σ_{ab} is zero. The $K \neq 0$ Fourier components of the disorder do not contribute. This solution turns out to be the correct solution for $d < 2$, as shown in Appendix C. This can be explained physically by the fact that for $d < 2$ thermal fluctuations are strong enough to disorder the system.

For $d \geq 2$, $\tilde{G}$ is given by

$$\tilde{G}(q) = \frac{1}{cq^2} + \frac{1}{c^2q^4mT}\sum_K \Delta_K K^2 e^{-K^2 l_T^2/2}. \tag{3.5}$$

l_T is the Lindemann length and measures the strength of thermal fluctuations. It is defined as

$$l_T^2 = 2\langle u^2\rangle_T = 2T\int \frac{d^dq}{(2\pi)^d}\frac{1}{cq^2} \simeq \frac{2TS_d}{c(d-2)}(2\pi/a)^{d-2}, \tag{3.6}$$

where $1/S_d = 2^{d-1}\pi^{d/2}\Gamma[d/2]$. Due to the term in $1/q^4$ in (3.5), the relative displacement correlation function grows as

$$\tilde{B}(x) \sim x^{4-d}. \tag{3.7}$$

The replica symmetric solution is therefore equivalent at large distances to the Larkin result based on a model of independent random forces acting on each vortex. As explained in the Introduction, this solution does not contain the right physics to describe the long distance behavior. In this variational approach this shows by the fact that the replica symmetric solution is unstable towards replica symmetry breaking for $2 < d < 4$. This can be checked from the eigenvalue λ of the replicon mode[27]

$$\lambda = 1 - \sum_K \frac{\Delta_K K^4}{m} e^{-K^2 T\int \frac{d^dp}{(2\pi)^d}G_c(p)} \int \frac{d^dq}{(2\pi)^d} G_c^2(q). \tag{3.8}$$

A negative eigenvalue λ indicates an instability of the replica symmetric solution. We introduce a small regularizing mass in G_c, $G_c(q)^{-1} = cq^2 + \mu^2$, and take the limit $\mu \to 0$. It is easy to see from (3.8) that for $d < 2$ the replica symmetric solution is always stable (see also Appendix C). In that case disorder is in fact irrelevant, due to the strong thermal fluctuations. For $d = 2$ the condition becomes $\mu^{-2(1-\frac{TK_0^2}{4\pi c})} < 1$ for small μ. Thus there is a transition at $T = T_c = 4\pi c/K_0^2$ between a replica symmetric stable high-temperature phase where disorder is irrelevant and a low-temperature (glassy) phase where the symmetric saddle point is unstable. We will examine the physics in $d = 2$ in detail in Sec. VI. For $2 < d < 4$ the replica symmetric solution is *always unstable* and disorder is therefore always relevant.

C. Replica symmetry breaking for $2 < d < 4$

Since for $2 \le d < 4$ the replica symmetric solution is unstable, to obtain the correct physics one has to look for a replica symmetry broken solution. We will focus here on the case $2 < d < 4$, the $d = 2$ case being discussed in Sec. VI. Following Ref. 27 we denote $\tilde{G}(q) = G_{aa}(q)$, similarly $\tilde{B}(x) = B_{aa}(x)$, and parametrize $G_{ab}(q)$ by $G(q, v)$ where $0 < v < 1$ and $B_{ab}(x)$ by $B(x, v)$. Physically, v parametrizes pairs of low-lying states in the hierarchy of states, as described in Ref. 27, $v = 0$ corresponding to states further apart. The saddle point equations become

$$\sigma(v) = \sum_K \frac{\Delta_K}{mT} K^2 e^{-\frac{1}{2}K^2 B(0,v)}, \tag{3.9}$$

where

$$B(0, v) = 2T \int \frac{d^d q}{(2\pi)^d} [\tilde{G}(q) - G(q, v)]. \tag{3.10}$$

$B(0, v)$ corresponds physically to the mean squared relative displacement between the position of the same vortex ($x = 0$) when the manifold is in two different low-lying metastable states. The large distance behavior of disorder-averaged correlators will be determined by the small v behavior of $B(0, v)$.

As we will show in Sec. III C 2, to discuss the large distance behavior $x \gg \xi$ it is enough to keep the smallest reciprocal lattice vectors with $K^2 = K_0^2$ in (3.9) since $B(0, v) \gg a^2$. We will thus first study a single cosine model obtained by keeping only $K = K_0$,

$$H_{\rm eff} = \frac{c}{2} \int d^d x \sum_a [\nabla u^a(x)]^2 - \frac{\Delta}{2T} \cos\{K_0[u^a(x) - u^b(x)]\}. \tag{3.11}$$

Each time we consider this particular model, e.g., in the following subsection (III C 1), we will denote $N_b \Delta_{K_0}$ by Δ, where N_b is the coordination number, i.e., the number of vectors K_0 with minimal norm.

1. Asymptotic behavior (single cosine model)

We look for a solution such that $\sigma(v)$ is constant for $v > v_c$, v_c itself being a variational parameter, and that has an arbitrary functional form below v_c. The algebraic rules for inversion of hierarchical matrices[27] give

$$B(0, v) = B(0, v_c) + \int_v^{v_c} dw \int \frac{d^d q}{(2\pi)^d} \frac{2T\sigma'(w)}{\{G_c(q)^{-1} + [\sigma](w)\}^2}, \tag{3.12}$$

where $[\sigma](v) = u\sigma(v) - \int_0^v dw \sigma(w)$ and

$$B(0, v_c) = \int \frac{d^d q}{(2\pi)^d} \frac{2T}{G_c(q)^{-1} + [\sigma](v_c)}. \tag{3.13}$$

In that case, taking the derivative of (3.9) (keeping only $K{=}K_0$) with respect to v, using $[\sigma]'(v) = v\sigma'(v)$, (3.12), and (3.9) again one finds

$$1 = \sigma(v) \int \frac{d^d q}{(2\pi)^d} \frac{K_0^2 T}{\{cq^2 + [\sigma](v)\}^2} \simeq \sigma(v) \left(\frac{TK_0^2 c_d}{c^{d/2}}\right) [\sigma(v)]^{(d-4)/2}. \tag{3.14}$$

Since the integral is ultraviolet convergent, we have taken the short distance momentum cutoff $\Lambda = 2\pi/a$ to infinity, a limit discussed below. Taking the derivative one more time one gets for the effective self-energy

$$[\sigma](v) = (v/v_0)^{2/\theta}, \tag{3.15}$$

where $v_0 = 2K_0^2 T c_d c^{-d/2}/(4 - d)$ and

$$c_d = \int \frac{d^d q}{(2\pi)^d} \left(\frac{1}{q^2 + 1}\right)^2 = \frac{(2 - d)\pi^{1-d/2}}{2^{d+1}\sin(d\pi/2)\Gamma(d/2)} \tag{3.16}$$

with $c_{d=3} = 1/(8\pi)$ and $c_{d=2} = 1/(4\pi)$.

The behavior of $[\sigma](v)$ controls the scaling of the energy fluctuations[27] $\Delta F \propto L^\theta \propto T/v$, with the scale L, and the large scale behavior is controlled by small v. Equation (3.15) thus gives an energy fluctuation exponent $\theta = d - 2$.

Using (3.15) in (2.14) one can now compute the correlation functions. Larger distances will correspond to less massive modes, and one obtains

$$\overline{\langle [u(x) - u(0)]^2 \rangle} = 2mT \int \frac{d^d q}{(2\pi)^d} [1 - \cos(qx)] \tilde{G}(q), \tag{3.17}$$

$$\tilde{G}(q) = \frac{1}{cq^2}\left(1 + \int_0^1 \frac{dv}{v^2} \frac{[\sigma](v)}{cq^2 + [\sigma](v)}\right) \sim \frac{Z_d}{q^d}, \tag{3.18}$$

with $Z_d = (4 - d)/(TK_0^2 S_d)$ and $1/S_d = 2^{d-1}\pi^{d/2}\Gamma[d/2]$. Thus for $2 < d < 4$ we find *logarithmic* growth:[26,34]

$$\overline{\langle[u(x)-u(0)]^2\rangle} = \frac{2m}{K_0^2} A_d \ln|x| \tag{3.19}$$

with $A_d = 4-d$. Note that the amplitude is independent of temperature and disorder.

The solution (3.15) is *a priori* valid up to a breakpoint v_c, above which $[\sigma]$ is constant, since $\sigma'(v) = 0$ is also a solution of the variational equations. To obtain the behavior at shorter scales for the single cosine model, we need to determine the breakpoint v_c. For $v > v_c$ using $[\sigma](v) = \Sigma$ one can rewrite (3.15) as

$$[\sigma](v) = \Sigma \left(\frac{v}{v_c}\right)^{\frac{2}{\theta}} \tag{3.20}$$

with $v_c = v_0 \Sigma^{\frac{d-2}{2}}$. Using (3.9) and (3.14) the equation determining Σ is

$$\Sigma^{\frac{(4-d)}{2}} = \frac{\Delta K_0^4 c_d}{mc^{d/2}} e^{-\frac{1}{2}K_0^2 B(0,v_c)}; \tag{3.21}$$

in terms of the nonuniversal quantity $B(0, v_c)$

$$B(0,v_c) = 2T \int \frac{d^d q}{(2\pi)^d} \frac{1}{cq^2+\Sigma}. \tag{3.22}$$

One can define a length l such that $cl^{-2} = \Sigma$. Since $x \ll l$ corresponds to $v \gg v_c$ where $[\sigma](v)$ is a constant, the solution is similar to a replica symmetric one. l is therefore the length below which the Larkin regime will be valid. When $l \gg a$, one finds $B(0, v_c) \approx l_T^2$. For instance, in $d = 3$, $B(0, v_c) = l_T^2\{1 - [a/(2\pi l)] \arctan(2\pi l/a)\}$. Equation (3.21) can be rewritten

$$1 - \frac{\Delta K_0^4}{m} e^{-TK_0^2 \int \frac{d^dq}{(2\pi)^d} \frac{1}{cq^2+\Sigma}} \int \frac{d^d q}{(2\pi)^d} \frac{1}{(cq^2+\Sigma)^2} = 0, \tag{3.23}$$

which is equivalent to $\lambda_{\text{replicon}}(\sigma) = 0$. Assuming $l \gg a$ one finds

$$l \simeq \xi e^{-K_0^2 l_T{}^2/(4-d)} \simeq \xi \tag{3.24}$$

with

$$\xi = (mc^2/\Delta K_0^4 c_d)^{1/(4-d)} \tag{3.25}$$

and

$$v_c = \frac{2K_0^2 T c_d}{(4-d)cl^{d-2}}. \tag{3.26}$$

Note that, although the breakpoint $v_c \to 0$ when $T \to 0$, the length l which is associated with the transition between the two regimes remains finite.

For the single cosine model, the characteristic length l below which the replica symmetric part of the solution $\{[\sigma](u) = \Sigma\}$ determines the physics, is equal to ξ, the length for which the relative displacements are of order a. For this model one has a direct crossover between the Larkin regime and the logarithmic growth of the displacements. This is to be expected since the disorder is here characterized by a single harmonic. It has therefore no fine structure for distances smaller than a, the lattice spacing. This will not be the case any more if higher harmonics are included. The disorder will be able to vary strongly for distances smaller than a, and one expects l and ξ to be different, and a third regime to appear in between: the so-called random manifold regime.

2. Study of the crossover

We study now the full model (2.13). Since this model contains all the harmonics of the disorder, it can describe correctly the short distance regimes. In particular, we will examine here the crossover from the random manifold regime to the logarithmic one.

In order to rewrite the equations in term of dimensionless quantities, we introduce the rescaling

$$\sigma(v) = \frac{c\xi^{-2}}{v_\xi} s(v/v_\xi), \tag{3.27}$$
$$B(v) = \frac{a^2}{2\pi^2} b(v/v_\xi).$$

As will be obvious later ξ is the crossover length between the random manifold regime and the logarithmic one, and v_ξ corresponds to the value of v for which the crossover occurs. One chooses ξ and v_ξ such that (3.9) and (3.12) become, in terms of the dimensionless quantities (3.27),

$$s(y) = \sum_p p^2 e^{-p^2 b(y)}, \tag{3.28}$$

$$b(y) = b(y_c) + \int_y^{y_c} dy \frac{s'(y)}{[s](y)^{(4-d)/2}}, \tag{3.29}$$

where $y_c = v_c/v_\xi$ and the integration over momentum in (3.12) has been performed. We have introduced the dimensionless variable p such that $K = 2\pi p/a$. When using the definition (3.28) one gets

$$\xi = \left(\frac{ma^4c^2}{16\pi^4 \Delta c_d}\right)^{\frac{1}{4-d}}, \tag{3.30}$$
$$v_\xi = \frac{2\pi^2}{a^2}\left(\frac{2Tc_d a^{2-d}}{c}\right)\left(\frac{a}{\xi}\right)^{d-2} \sim \left(\frac{l_T}{a}\right)^2 \left(\frac{a}{\xi}\right)^{d-2}.$$

Thus v_ξ is always very small compared to 1. In (3.30), for simplicity, we have assumed that all Δ_K have the same value $\Delta = \Delta_{K_0}$.

The equations (3.28) and (3.29) can be solved in a parametric form. We introduce the variable $z = b(y)$ and define

$$h(z) = \sum_p (p^2)^2 e^{-zp^2}, \qquad H(z) = \sum_p p^2 e^{-zp^2}. \tag{3.31}$$

It is possible to keep different Δ_K, for instance, $\Delta_K \sim \exp(-K^2\xi_0^2)$, to describe the effect of the finite correlation length of the random potential, by just modifying the functions H and $h = -H'$ to

$$h(z) = \sum_p \frac{\Delta_K}{\Delta_{K_0}}(p^2)^2 e^{-zp^2}. \tag{3.32}$$

Using the variable z and taking the derivative of (3.29), Eqs. (3.28) and (3.29) become

$$s(y) = H(z), \tag{3.33}$$
$$b'(y) = -\frac{s'(y)}{[s](y)^{(4-d)/2}}. \tag{3.34}$$

Taking the derivative of (3.33) one gets

$$s(y) = H(z), \tag{3.35}$$
$$[s](y)^{\frac{4-d}{2}} = h(z). \tag{3.36}$$

Finally, using $y = \frac{[s]'(y)}{s'(y)} = \frac{d[s]}{dz}/\frac{ds}{dz}$ we obtain the solution in a parametric form:

$$[s](y)^{\frac{4-d}{2}} = h(z), \tag{3.37}$$
$$y = -\frac{2}{4-d}h'(z)h(z)^{\frac{2d-6}{4-d}}, \tag{3.38}$$

with $z_c = b(y_c) < z < \infty$.

Let us examine first the various asymptotic behaviors of the solution (3.37). As will be obvious later, large z correspond to large scales and small z to small scales. At large z only the smallest p contributes in the sum (3.31) for $h(z)$, giving

$$h(z) = 2me^{-z} \quad \text{for a square lattice} \tag{3.39}$$
$$h(z) = \frac{32}{3}e^{-4z/3} \quad \text{for a triangular lattice.}$$

In that case the high harmonics are irrelevant and (3.37) and (3.38) give back formula (3.15), for the single cosine model. One recovers the quasiordered large distance logarithmic regime.

We now study the behavior at small z. In that case all harmonics must be kept and it is convenient to use the following duality transformation of formula (3.31):

$$I(z) = \sum_p e^{-p^2 z} = \frac{1}{\Omega}\sum_R \left(\frac{\pi}{z}\right)^{m/2} e^{-\frac{\pi^2}{z}R^2}, \tag{3.40}$$

where the vectors p have been defined above. Ω is the volume of the unit cell in the space of the vector p ($\Omega = 2/\sqrt{3}$ for a triangular lattice and 1 for a square lattice). The vectors R are the reciprocal vectors of the p which themselves are normalized in units of $2\pi/a$. The R thus correspond to the original lattice with a spacing unity. For small z, only $R = 0$ contributes and

$$I(z) \sim \frac{1}{\Omega}\left(\frac{\pi}{z}\right)^{m/2}. \tag{3.41}$$

Therefore

$$h(z) \simeq \frac{\pi^{m/2}}{\Omega}\frac{m(m+2)}{4}\frac{1}{z^{(m+4)/2}}, \tag{3.42}$$

which gives

$$[s](y) = \Xi y^{2/\theta_{\rm RM}}, \tag{3.43}$$

where the fluctuation energy exponent of the random manifold regime is $\theta = (2d - 2m + dm)/(4 + m)$ and the amplitude

$$\Xi = \left(\frac{\pi^{m/2}}{\Omega}\frac{m(m+2)}{4}\right)^{\frac{4}{2d-2m+dm}}\left(\frac{4-d}{4+m}\right)^{2/\theta_{\rm RM}}. \tag{3.44}$$

Equations (2.14) and (3.18) once rescaled using (3.30) give

$$\tilde{B}(x) = \frac{a^2}{2\pi^2}\tilde{b}\left(\frac{x}{\xi}\right) \tag{3.45}$$

and

$$\tilde{b}(x) = \frac{\pi\theta_{\rm RM}}{2\sin(\pi\theta_{\rm RM}/2)c_d}\Xi^{\theta_{\rm RM}/2} \times \int\frac{d^dq}{(2\pi)^d}\frac{1}{q^{2+\theta_{\rm RM}}}[1-\cos(qx)] \tag{3.46}$$

with the useful intermediate formula

$$\int_0^\infty \frac{dy}{y^2}\frac{y^a}{1+y^a} = \frac{\pi/a}{\sin\pi/a}. \tag{3.47}$$

Using the asymptotic expression

$$\int\frac{d^dq}{(2\pi)^d}\frac{1}{q^{d+2\nu}}[1-\cos(qx)] = I_{d,\nu}x^{2\nu} \tag{3.48}$$

with

$$I_{d,\nu} = \frac{\pi^{1-d/2}}{2^{d+2\nu}\Gamma(d/2+\nu)\Gamma(1+\nu)\sin(\nu\pi)}, \tag{3.49}$$

one gets

$$\tilde{b}(x) \sim \left(\frac{1}{c_d}\frac{\pi\theta_{\rm RM}}{2\sin(\pi\theta_{\rm RM}/2)}\Xi^{\theta_{\rm RM}/2}I_{d,\nu}\right)x^{2\nu} \tag{3.50}$$

with $2\nu = 2+\theta_{\rm RM}-d$. Thus the exponent entering in the relative displacement growths is $\nu = \frac{4-d}{4+m}$ with $\nu = 1/6$ for $m = 2$. This corresponds to the random manifold regime.[24,23] In this regime each vortex is held by the elastic forces of the other vortices and experiences an independent random potential. The mean squared displacement grows more slowly than in the Larkin regime (the exponent is 1/3, compared to 1 for the Larkin regime in $d = 3$). In $d = 3$ and $m = 2$ one gets $\theta_{\rm RM} = 4/3$ and for a triangular lattice in $d = 3$,

$$\Xi = \frac{1}{6}\sqrt{\frac{\pi}{3\Omega}}, \tag{3.51}$$

the amplitude in (3.50) is $\simeq 2.3817$.

As for the single cosine model of Sec III C 1, the solution (3.43) is valid up to a breakpoint v_c above which the self-energy $[\sigma](v)$ is constant. This corresponds to scales such that $\tilde{B}(x)$ is smaller than l_T^2 and ξ_0^2. One then recovers the replica symmetric propagator $\tilde{G}(q) \sim 1/q^4$ for $q^2 \gg [\sigma](v_c)$, and Larkin's model behavior. To compute the crossover and to determine the breakpoint $v_c = y_c v_\xi$, we proceed similarly to Sec. III C 1. The equation determining Σ is now

$$\Sigma^{\frac{(4-d)}{2}} = \sum_K \frac{\Delta_K K^4 c_d}{m c^{d/2}} e^{-\frac{1}{2}K^2 B(0,v_c)}, \tag{3.52}$$

where $B(0, v_c)$ is given in (3.22). Note that keeping the correlation length of the disorder using $\Delta_K = \Delta \exp(-1/2K^2{\xi_0}^2)$ amounts to changing $B(0, v_c)$ into $B(0, v_c) + {\xi_0}^2$ and thus l_T^2 into $l_T^2 + {\xi_0}^2$. We will thus take $\Delta_K = \Delta$, keeping in mind this change.

Solving (3.52) and using the small z expansion of $h(z)$ one gets for the length scale l such that $\Sigma = cl^{-2}$ (assuming $l \gg a$ and $l_T \ll a$)

$$l = C_s \xi \left(\frac{l_T}{a}\right)^{\frac{1}{\nu}} \tag{3.53}$$

where

$$C_s = (2\pi^2)^{1/(2\nu)} \left(\frac{4\Omega}{\pi^{m/2} m(m+2)}\right)^{1/(4-d)}. \tag{3.54}$$

The breakpoint v_c can obtained using (3.37) with the value $z = z_c = 2\pi^2 l_T^2/a^2$. This gives using (3.30)

$$y_c \sim \left(\frac{l_T}{a}\right)^{-\frac{\theta_{\rm RM}}{\nu}}, \tag{3.55}$$

which leads to

$$v_c \sim \left(\frac{a/\xi}{(l_T/a)^{1/\nu}}\right)^{d-2} \sim \left(\frac{a}{l}\right)^{d-2}. \tag{3.56}$$

The characteristic length l separates the Larkin regime from the random manifold regime, and is in that case much smaller than ξ. Lowering the temperature reduces the range over which the Larkin regime occurs. This is because the thermal wandering responsible for smoothing the random potential on a scale l_T decreases. The relative displacements of two vortices separated by l is of order $\max(l_T, \xi_0)$, giving $\tilde{B}(l) \sim \max(l_T, \xi_0)^2$. As $T \to 0$, l becomes identical to the Larkin-Ovchinikov length R_c. Using the expression of $\tilde{B}(x) \sim Cx^{2\nu}$ in the random manifold regime and the additional relation $\tilde{B}(\xi) \sim a^2$ one recovers the expression (3.53) for l. When $v_c = 1$ the Larkin regime disappears. This occurs when $l \sim a$. The criterion $l \gg a$ for which the Larkin regime exists is equivalent to

$$\frac{a}{\xi} \ll \left(\frac{l_T}{a}\right)^{\frac{1}{\nu}} \tag{3.57}$$

and corresponds therefore to extremely weak disorder and intermediate temperatures. The absence of a Larkin regime means that the disorder-induced relative displacement of two neighbors in the lattice is already larger than $\max(l_T, \xi_0)$.

3. Crossover in $d = 3$

In $d = 3$, it is possible to solve the equations describing the crossover analytically, and thus to obtain the full crossover function between the random manifold and the quasiordered regime. We will examine $d = 3$ and $m = 2$ for the model (2.13). Such a case is physically relevant for the case of vortex lattices. The crossover lengths are given by (3.30):

$$\xi = \frac{a^4 c^2}{\pi^3 \Delta}, \qquad v_\xi = \frac{\pi^4}{2} T\Delta/(a^6 c^3) \sim l_T^2/(a\xi). \tag{3.58}$$

Using (3.45) one has

$$\tilde{b}(x) = \frac{1}{c_3} \int \frac{d^3k}{(2\pi)^3} [1 - \cos(kx)] \times \int_0^{y_c} \frac{dy}{y^2} \frac{[s](y)}{k^2\{k^2 + [s](y)\}}. \tag{3.59}$$

Performing the angular integration over momentum in (3.59) we find

$$\tilde{b}(x) = \frac{1}{2\pi^2 c_3} \int_0^\infty dk \left(1 - \frac{1}{k|x|}\sin(k|x|)\right) \times \int_0^\infty \frac{dy}{y^2} \frac{[s](y)}{k^2 + [s](y)}, \tag{3.60}$$

where we have extended the integral over y to infinity, assuming $v_\xi \ll v_c$ or equivalently $l_T \ll a$, in which case there is a wide random manifold regime. Performing the remaining integration over k one gets

$$\tilde{b}(x) = \frac{1}{4\pi c_3} \int_0^\infty \frac{dy}{y^2} \left([s]^{1/2}(y) - \frac{1}{|x|}(1 - e^{-|x|[s]^{1/2}(y)})\right). \tag{3.61}$$

Using the parametric solution (3.37) and (3.38) for $[s](y)$ we obtain the final expression

$$\tilde{b}(x) = \int_0^\infty dt \frac{h''(t)h(t)}{h'(t)^2} f(xh(t)), \tag{3.62}$$

$$f(x) = 1 - \frac{1}{x}(1 - e^{-x}). \tag{3.63}$$

Expression (3.62) gives the full relative displacement correlation function as a function of distance. To recover the asymptotic expression of Sec. III C 1, for large distance x, one notices that in (3.62) $h(t) \sim Ae^{-\alpha t}$ as shown in (3.39) for large t. Thus the large x behavior will be controlled by small t. One obtains the asymptotic expression

$$\tilde{b}(x) = \frac{1}{\alpha} \left[f(\infty)[\ln(Ax) + 1] + \int_0^\infty dz \ln(1/z) f'(z)\right], \tag{3.64}$$

where we have used $[h/h']_0^\infty = -1/\alpha$. Using (3.62), one finds

$$\tilde{b}(x) = \frac{1}{\alpha}\left[\ln(Ax) + \gamma\right], \quad (3.65)$$

where $\alpha = [K_0 a/(2\pi)]^2$ and $\gamma \simeq 0.57721$ is the Euler constant. This implies that the translational order correlation function

$$C_{K_0}(x) = \langle e^{iK_0\cdot u(x)} e^{-iK_0\cdot u(0)}\rangle \quad (3.66)$$

behaves for large x as

$$C_{K_0}(x) = \frac{e^{-\gamma\xi}}{A|x|}. \quad (3.67)$$

One recovers the power-law behavior of Sec. III C 1 as well as the amplitude. The intermediate distance behavior will be examined in more detail for more realistic elastic Hamiltonians in connection with vortex lattices in Sec. IV.

D. Self-consistence of the physical assumptions

Finally, for our solution to be valid, one has to check self-consistently that even in the presence of disorder the basic assumption that elastic theory was applicable remains valid. One has therefore to check that $\frac{1}{m}\overline{\langle(\nabla u)^2\rangle} \ll 1$.

For simplicity we will make the analysis for the single cosine model (for which $l \sim \xi$) but similar results can be derived for the full Hamiltonian. Using the variational solution of Sec. III C 1, one obtains

$$\frac{1}{m}\overline{\langle(\nabla u)^2\rangle} = \frac{2T}{c}\int\frac{d^dq}{(2\pi)^d}\left[\int_0^{v_c}\frac{dv}{v^2}\frac{\Sigma(v/v_c)^{2/\theta}}{cq^2+\Sigma(v/v_c)^{2/\theta}} + \left(\frac{1}{v_c}-1\right)\frac{\Sigma}{cq^2+\Sigma}+1\right]. \quad (3.68)$$

Using (3.6) one gets, for the case $l \gg a$,

$$\frac{1}{m}\overline{\langle(\nabla u)^2\rangle} \simeq \Sigma\frac{l_T^2}{cv_c}\frac{2}{4-d} + \left(\frac{2\pi l_T}{a}\right)^2\frac{d-2}{d}, \quad (3.69)$$

where the last contribution is due to thermal fluctuations only. Replacing v_c by (3.26) one finds

$$\overline{\langle(\nabla u)^2\rangle} = \frac{8\sin[\pi(d-2)/2]}{\pi(d-2)^2}\left(\frac{a}{2\pi l}\right)^{4-d} + \left(\frac{2\pi l_T}{a}\right)^2\frac{d-2}{d}.$$

When $d \to 2$ one cannot neglect Σ in the denominator of (3.68), and the expression (3.70) becomes

$$\overline{\langle(\nabla u)^2\rangle} \sim \left(\frac{a}{l}\right)^2\ln(l/a). \quad (3.70)$$

As is obvious from (3.70), one has always $\overline{\langle(\nabla u)^2\rangle} \ll 1$, provided that $l \gg a$ or, equivalently, using (3.24) for the single cosine model, provided that one is far from the melting temperature $l_T \ll a$ and that the disorder is weak $\xi \gg a$. In that case one can indeed use an elastic theory in the absence of dislocation, even in the presence of disorder, and our solution is valid in such a regime.

E. Comparison with BMY

The previous application of the variational method by BMY (Refs. 24 and 23) led to the erroneous conclusion that the fluctuations are enhanced at large scale. They find for $d = 3$ $B(x) \sim x^{1/2}$ instead of the logarithmic behavior found here. Although they want to describe the same physical situation as the one studied here, they in fact consider a model which turns out to be fundamentally different, in which each vortex experiences a different disorder. In their model the random potential is also dependent on the line index i such that

$$\overline{V(R_i,r,z)V(R_j,r',z')} = \Delta\delta(r-r')\delta(z-z')|R_i-R_j|^{-\lambda}. \quad (3.71)$$

This amounts to introducing an extra disorder in the original model (2.4) with correlations decaying as $1/|R_i - R_j|^\lambda$. Then BMY retain only the long wavelength part of this disorder, which indeed for a fixed $\lambda > 0$ dominates the contribution of higher harmonics. They then look at the limit of the exponents when $\lambda \to 0$. The result they obtain with this procedure is incorrect (although their derivation is technically sound) and comes from the following artifact: by assuming that different lines experience different random potentials, they make it possible to optimize the pinning energy by a global translation of the whole lattice. In that case the pinning energy will obviously be dependent on u, even for a uniform u. On the other hand, for genuine disorder, which is only dependent on the space position of the lines, it is obvious that a translation by one of the vectors of the lattice cannot change the energy, and therefore the $q \sim 0$ part of $H_{\rm pin}$ cannot depend on u but only on ∂u. By regularizing the integrals with a disorder dependent on the line index they introduce an extra and nonphysical disorder which is relevant and changes the long range behavior of the correlation function compared to the physical case. Indeed, there is a crossover length ξ_λ associated with this disorder above which the long distance behavior is the one given by BMY. Below this distance the vortices all experience the same random potential. To recover the physical model one has to take $\lambda \to 0$ and in that case $\xi_\lambda \to \infty$.

In more mathematical terms, the variational method gives three types of contributions for the self-energy as shown in Appendix D:

$$\sigma(q,v) \sim c_1 q^2 + c_2 e^{-\frac{K_0^2}{2}B(0,v)} + \lambda B(0,v)^{-1-\lambda/2}. \quad (3.72)$$

The first term is the long wavelength contribution of the genuine disorder which is irrelevant. The second one is the higher harmonic contribution which is responsible for the logarithmic growth at large distances, and the third term is the long wavelength contribution of this extra disorder giving a $B(x) \sim x^{1/2}$ for $x > \xi_\lambda$. Only the third term was kept by BMY, artificially taking the limit $\lambda \to 0$ in the exponent only but *not* in the amplitude of such a term. Note that if one takes the limit $\lambda \to 0$ (which corresponds to the physical situation) before taking the limit $x \to \infty$ one recovers that the $q \simeq 0$ part of the disorder does not play any role, and the amplitude they obtain vanishes.

A simple Flory argument can be made to estimate the effect of the long wavelength part of the disorder on the displacements. This confirms that it is irrelevant above $d = 2$ (see also Sec. II C). Let u be the typical relative displacement over a length scale L. The elastic energy cost is u^2L^{d-2} while the typical energy gain due to the disorder is

$$\sqrt{L^d + uL^{d-1}} - \sqrt{L^d} \propto uL^{d/2-1}, \tag{3.73}$$

which comes from the change of density of the vortices. Since the vortices in the center are unaffected the gain of energy can come only from *boundary* terms. Balancing the two terms one finds $u \sim L^{(d-2)/2}$ which is obviously irrelevant above two dimensions.

In fact one can simplify the the saddle point equations of Refs. 24 and 23 by noting that the x dependence of $B(x,u)$ in these equations is unimportant, up to higher-order terms in ∇u. Such a calculation is performed in Appendix D. One then recovers the local model (2.13) which is simple and physically transparent enough to allow for the exact solution of Sec. III C.

IV. FLUX LATTICES

A. Model

The theory developed in Sec. III, when specialized to $m = 2$ and $d = 3$, can be applied to describe the effects of weak disorder on the Abrikosov phase of type II superconductors. High-T_c superconductors can be modeled by stacks of coupled planes. The system is therefore described by layers of two-dimensional triangular lattices of vortices. The displacements u are two-dimensional vectors, hence $m = 2$ (the vortex can only move within the plane). We denote by R_i the equilibrium position of the vortex labeled by an integer i in the x-y plane, and by $u(R_i, z)$ their in-plane displacements. z denotes the coordinate perpendicular to the planes and along the magnetic field. The total energy is

$$H = \frac{1}{2}\int d^2r dz[(c_{11} - c_{66})(\partial_\alpha u_\alpha)^2 + c_{66}(\partial_\alpha u_\beta)^2 + c_{44}(\partial_z u_\alpha)^2] + \int d^2r dz V(r,z)\rho(r,z), \tag{4.1}$$

where α, β denote in-plane coordinates. The Hamiltonian (4.1) is identical to (2.1) with $\Phi_{\alpha\beta}(q) = G_c^{-1,T}(q)P^T_{\alpha\beta} + G_c^{-1,L}(q)P^L_{\alpha\beta}$ where

$$G_c^{-1,T}(q) = c_{44}q_z^2 + c_{66}q_\perp^2, \quad G_c^{-1,L}(q) = c_{44}q_z^2 + c_{11}q_\perp^2 \tag{4.2}$$

and

$$P^T_{\alpha\beta}(q) = \delta_{\alpha\beta} - q_\alpha q_\beta/q_\perp^2, \quad P^L_{\alpha\beta}(q) = q_\alpha q_\beta/q_\perp^2 \tag{4.3}$$

are the transverse and longitudinal propagators. $q_\perp$ denotes the in-plane vector, whereas q_z is the out-of-plane component. Equation (4.1) corresponds to a local elastic theory, but nonlocal elasticity can also be considered at the expense of introducing q-dependent coefficients c. This point will be considered in greater detail below. For the moment we restrict ourselves to dispersionless elastic constants.

Weak pointlike disorder such as oxygen vacancies, or defects introduced artificially in a controlled way, e.g., by electron irradiation,[35] can be modeled by a Gaussian random potential of correlation length of order ξ_0. Here the disorder will be taken as completely uncorrelated from plane to plane, $\Delta(x - x') = \Delta(r - r')\delta(z - z')$. Such a description will be valid as long as each pinning center is weak enough so that the pinning length l [also called R_c in formula (51) of the Larkin-Ovchinikov paper[22]] is much larger than the average distance between impurities.

The disorder term in (4.1) is transformed into a form similar to (2.13). We can now use the methods of Sec. III with the realistic elastic Hamiltonian (4.1) to get the physical properties of a vortex lattice. Most of the theoretical calculations are confined to Sec. IV B, whereas a simple physical interpretation of the results is given in Sec. IV C. The experimental consequences are discussed in detail in Sec. IV D.

B. Theoretical predictions

One can then perform a variational ansatz identical to (3.1) but for the introduction of the longitudinal $G^L_{\alpha\beta}$ and transverse part $G^T_{\alpha\beta}$,

$$H_0 = \frac{1}{2}\int \frac{d^dq}{(2\pi)^d} G^{-1}_{\alpha\beta,ab}(q)u^a_\alpha(q)u^b_\beta(-q) \tag{4.4}$$

and

$$G_{\alpha\beta,ab}(q) = G^T_{ab}(q)P^T_{\alpha\beta} + G^L_{ab}(q)P^L_{\alpha\beta}. \tag{4.5}$$

The saddle point equation (3.9) for the self-energy now becomes

$$\sigma_{\alpha\beta}(v) = \frac{\Delta}{mT}\sum_K K_\alpha K_\beta e^{-\frac{1}{2}K_\gamma K_\delta B_{\gamma\delta}(x=0,v)}. \tag{4.6}$$

The correlation function $B_{\alpha\beta}$ is defined by (3.3) with the replacement of G by $G_{\alpha\beta}$.

Since $B_{\gamma\delta}(x = 0, v)$ is a purely local quantity it is isotropic and $B_{\gamma\delta}(x = 0, v) = \delta_{\gamma\delta}B(x = 0, v)$. This implies that $\sigma_{\alpha\beta} = \sigma(v)\delta_{\alpha\beta}$, i.e., an isotropic self-energy. Thus $B(v) = \frac{1}{2}[B^L(v) + B^T(v)]$, where by definition $B^{L,T}(v)$ satisfy Eq. (3.12) with respect to $G_c^{L,T}$. Integration over q leads to

$$B(v) = B(v_c) + Tc_d\frac{1}{c_{44}^{1/2}}\left(\frac{1}{c_{66}} + \frac{1}{c_{11}}\right)\int_u^{u_c}\frac{\sigma'(v)dv}{[\sigma]^{1/2}(v)}. \tag{4.7}$$

The solution of (4.6) can trivially be obtained from the isotropic solution with the replacements

$$c \to c_{44}, \tag{4.8}$$
$$c_d \to c_d' = c_d\frac{1}{2}\left(\frac{c_{44}}{c_{66}} + \frac{c_{44}}{c_{11}}\right).$$

One can now compute the correlation functions

$$\begin{aligned}\tilde{B}_{\alpha\beta}(x) &= \overline{\langle[u_\alpha(x) - u_\alpha(0)]\,[u_\beta(x) - u_\beta(0)]\rangle}\\ &= \tilde{B}^T P^T_{\alpha\beta}(r) + \tilde{B}^L P^L_{\alpha\beta}(r),\end{aligned} \tag{4.9}$$

where the longitudinal and transverse propagators have been defined in (4.3), and similarly to (3.17)

$$\begin{aligned}\tilde{B}^L(r,z) = 2T\int\frac{d^3q}{(2\pi)^3}&(\hat{q_\perp}\hat{r})^2[1 - \cos(q_\perp r + q_z z)]\tilde{G}^L(q)\\ &+[1 - (\hat{q_\perp}\hat{r})^2][1 - \cos(q_\perp r + q_z z)]\tilde{G}^T(q),\end{aligned} \tag{4.10}$$

and a similar equation for $\tilde{B}^T$ obtained from (4.10) by permuting L, T. $\tilde{G}^{L,T}$ are defined similarly to Eq. (3.18) with the replacement of cq^2 by $G^{-1,LT}$ defined in (4.2). One then rescales q and r to obtain isotropic integrals over momenta. Equation (4.10) then takes the form

$$\begin{aligned}\tilde{B}^L(r,z) = \frac{a^2}{2\pi^2}\Bigg[&\frac{c_{44}}{c_{11}}\tilde{F}^L\left(r\sqrt{\frac{c_{44}}{c_{11}}}, z\right)\\ &+\frac{c_{44}}{c_{66}}\tilde{F}^T\left(r\sqrt{\frac{c_{44}}{c_{66}}}, z\right)\Bigg].\end{aligned} \tag{4.11}$$

Those integrals contain the self-energy $[\sigma](v)$ which is determined itself from Eqs. (4.6) and (4.7). These equations are rescaled similarly to (3.27) with the replacement (4.8). This defines two crossover lengths ξ and ξ_z for in-plane and z directions, given by

$$\xi_z = \frac{2a^4c_{44}}{\pi^3\Delta(\frac{1}{c_{66}} + \frac{1}{c_{11}})}, \quad \xi = \sqrt{\frac{c_{66}}{c_{44}}}\xi_z = \frac{2a^4c_{44}^{1/2}c_{66}^{3/2}}{\pi^3\Delta(1 + \frac{c_{66}}{c_{11}})}, \tag{4.12}$$

where, as we recall, Δ is the disorder strength $\rho_0\overline{V(q)V(-q)} = \Delta$.

Rescaling by the lengths ξ and ξ_z one gets

$$\begin{aligned}\tilde{B}^L(r,z) = \frac{a^2}{2\pi^2}\Bigg[&\frac{c_{44}}{c_{11}}\tilde{F}^L\left(\frac{r}{\xi}\sqrt{\frac{c_{66}}{c_{11}}}, \frac{z}{\xi_z}\right)\\ &+\frac{c_{44}}{c_{66}}\tilde{F}^T\left(\frac{r}{\xi}, \frac{z}{\xi_z}\right)\Bigg],\end{aligned} \tag{4.13}$$

where the functions F are given for $z = 0$ by

$$\begin{aligned}\tilde{F}^L(r) = \frac{1}{(2\pi)^3c_3'}\int_0^1\frac{dy}{y^2}[s](y)\int &d^2q_\perp dq_z\cos^2(\theta)\\ &\times\{1 - \cos[q_\perp r\cos(\theta)]\}\\ &\times\frac{1}{q_\perp^2 + q_z^2}\frac{1}{q_\perp^2 + q_z^2 + [s](y)},\end{aligned} \tag{4.14}$$

$$\begin{aligned}\tilde{F}^T(r) = \frac{1}{(2\pi)^3c_3'}\int_0^1\frac{dy}{y^2}[s](y)\int &d^2q_\perp dq_z[1 - \cos^2(\theta)]\\ &\times\{1 - \cos[q_\perp r\cos(\theta)]\}\\ &\times\frac{1}{q_\perp^2 + q_z^2}\frac{1}{q_\perp^2 + q_z^2 + [s](y)}.\end{aligned} \tag{4.15}$$

Performing the $q_\perp, q_z$ integrations, one gets

$$\tilde{F}^{T,L} = \frac{1}{4\pi c_3'}\int_0^\infty\frac{dy}{y^2}f^{T,L}(y), \tag{4.16}$$

where

$$\begin{aligned}f^L(x) &= \Bigg[\frac{[s]^{1/2}}{2} - \frac{1}{[s]^{1/2}|x|^2}\\ &\qquad+\left(\frac{1}{|x|} + \frac{1}{[s]^{1/2}|x|^2}\right)e^{-[s]^{1/2}|x|}\Bigg],\\ f_T(x) &= f_I(x) - f_L(x),\\ f_I(x) &= \left[[s]^{1/2} - \frac{1}{|x|} + \frac{1}{|x|}e^{-[s]^{1/2}|x|}\right].\end{aligned} \tag{4.17}$$

Expressing again the $[s]$ in terms of the functions h, one gets

$$\begin{aligned}\tilde{B}^{L,T}(r) = \frac{a^2}{2\pi^2}\Bigg[&\frac{2c_{11}}{c_{11} + c_{66}}\tilde{b}^{T,L}\left(\frac{r}{\xi}\right)\\ &+\frac{2c_{66}}{c_{11} + c_{66}}\tilde{b}^{L,T}\left(\sqrt{\frac{c_{66}}{c_{11}}}\frac{r}{\xi}\right)\Bigg],\end{aligned} \tag{4.18}$$

where $\tilde{b}^{L,T}$ have an expression similar to (3.62):

$$\tilde{b}^{L,T}(x) = \int_0^\infty dt\frac{h''(t)h(t)}{h'(t)^2}f^{L,T}[xh(t)] \tag{4.19}$$

with $f^{L,T}$ given by

$$\begin{aligned}f^L(x) &= \frac{1}{2} - \frac{1}{x^2} + \left(\frac{1}{x} + \frac{1}{x^2}\right)e^{-x},\\ f^T(x) &= f(x) - f^L(x).\end{aligned} \tag{4.20}$$

Equations (4.18)–(4.20) give the complete expression of the displacement correlation function for equal z as a function of the distance in the transverse plane.

In the large distance regime one obtains an expression similar to (3.64), with the f replaced by $f^{T,L}$. This gives

$$\begin{aligned}\tilde{b}^L(x) &= \frac{1}{2\alpha}\left[\ln(Ax) + \gamma + \frac{1}{2}\right],\\ \tilde{b}^T(x) &= \frac{1}{2\alpha}\left[\ln(Ax) + \gamma - \frac{1}{2}\right],\end{aligned} \tag{4.21}$$

where $\alpha = [K_0 a/(2\pi)]^2$, $\gamma \simeq 0.57721$ is the Euler constant, and $A = 32/3$ for the triangular lattice. This leads for the $\tilde{B}^{L,T}$ functions at large distance to

$$\tilde{B}^{L,T}(r) = \frac{2}{K_0^2}\left[\ln(Ar/\xi) + \gamma + \frac{\epsilon^{L,T}}{2}\frac{c_{11}-c_{66}}{c_{11}c_{66}} + \frac{c_{66}}{2(c_{11}+c_{66})}\ln\left(\frac{c_{66}}{c_{11}}\right)\right], \tag{4.22}$$

where $\epsilon^L = -1$ and $\epsilon^T = +1$. Note that $\tilde{B} = (\tilde{B}^L + \tilde{B}^T)/2$.

It is interesting to note that complete isotropy in the displacement correlation functions is recovered at large scales. The translational correlation function is

$$C_{K_0}(r) = \frac{e^{-\gamma}\xi}{Ar} e^{-\frac{c_{66}\ln(c_{66}/c_{11})}{2(c_{11}+c_{66})}} \times e^{[(\hat{K}_0\cdot\hat{r})^2 - \frac{1}{2}]\left(\frac{c_{11}-c_{66}}{c_{11}+c_{66}}\right)}. \tag{4.23}$$

For the vortex lattice (2.4), shear deformations dominate ($c_{66} \ll c_{11}$) in most of the phase diagram. The expression for the function $\tilde{B}^{L,T}(r)$ which describes the crossover between the random manifold (intermediate distance) regime and the large distance regime then simplifies:

$$\tilde{B}^{L,T}(r) \simeq \frac{a^2}{\pi^2}\tilde{b}^{T,L}\left(\frac{r}{\xi}\right). \tag{4.24}$$

In the limit of weak disorder $\xi \gg a$ we find that there should be a well-defined crossover function, i.e., all curves should scale when plotted in units of x/ξ. The relative displacement correlation functions $\tilde{B}^{T,L}$, as predicted by the variational method, are plotted in Fig. 1 and Fig. 2 for the triangular lattice, by numerically integrating (4.19). The crossover between the random manifold regime and the asymptotic quasiordered asymptotic regime is apparent, and occurs at a scale of order ξ. At the length scale $r = \xi$ where the random manifold regime ceases to be valid, the translational order correlation function $C_K(r) = e^{-\frac{1}{2}K_\alpha K_\beta \tilde{B}_{\alpha,\beta}(r)}$ is of order $C_{K_0} \sim 0.1$. Therefore the crossover should be experimentally observable. In Fig. 3, we have shown the ratio R of the transverse to longitudinal displacements. As was shown by BMY,[24,23] its value is $2\nu + 1$ in the random manifold regime (the variational method gives $R = 4/3$). At large scale, we find that this ratio decreases to $R = 1$, and in that sense isotropy is restored. However, if one looks at the correla-

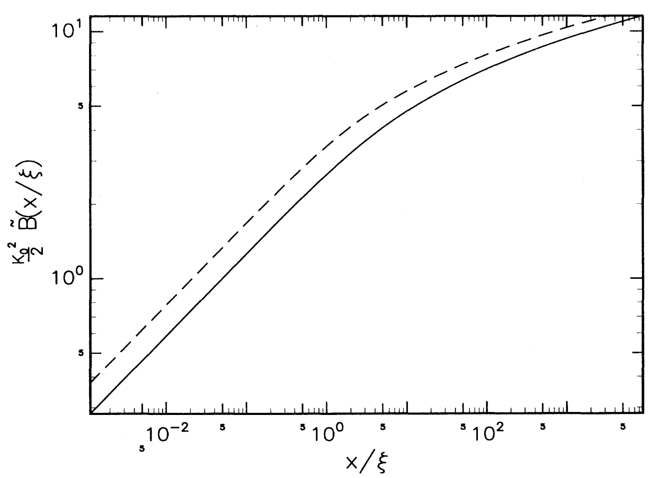

FIG. 1. Plot of $\frac{K_0^2}{2}\tilde{B}_{L,T}$ for the triangular Abrikosov lattice, in the limit $c_{66} \ll c_{11}$. The longitudinal (solid line) and transverse (dashed line) relative displacement correlation functions $\tilde{B}_{L,T}$ are defined in (4.9). K_0 is one of the first reciprocal lattice vectors, and ξ is the crossover length defined in (4.12). When $x < \xi$ one sees a power law with $2\nu = 1/3$ characteristic of the random manifold regime.

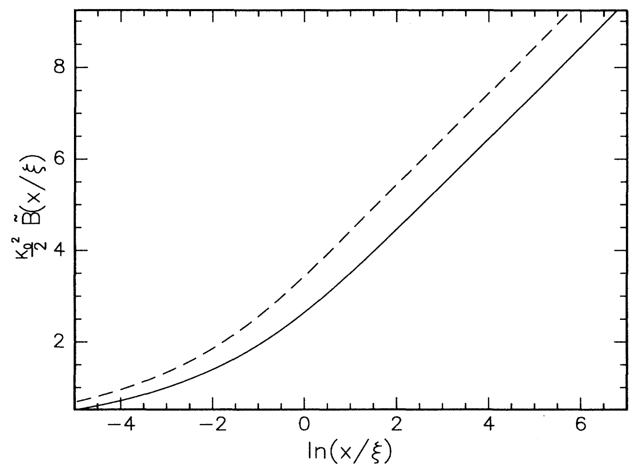

FIG. 2. Plot of $\frac{K_0^2}{2}\tilde{B}_{L,T}$ for the triangular Abrikosov lattice, in the limit $c_{66} \ll c_{11}$. The longitudinal (solid line) and transverse (dashed line) relative displacement correlation functions $\tilde{B}_{L,T}$ are defined in (4.9). K_0 is one of the first reciprocal lattice vectors, and ξ is the crossover length defined in (4.12). When $x > \xi$ one sees the logarithmic regime.

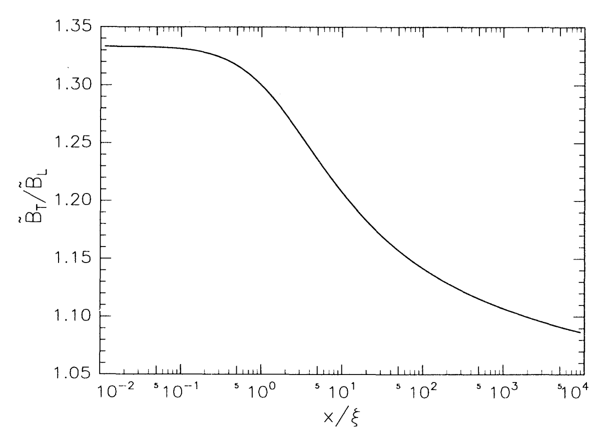

FIG. 3. Plot of the ratio $\tilde{B}_T/\tilde{B}_L$ for the triangular Abrikosov lattice, in the limit $c_{66} \ll c_{11}$, where $\tilde{B}_{L,T}$ are defined in (4.9). ξ is the crossover length defined in (4.12). When $x < \xi$ the ratio takes the random manifold value $2\nu+1$. It decrease slowly to 1 in the asymptotic regime $x \gg \xi$.

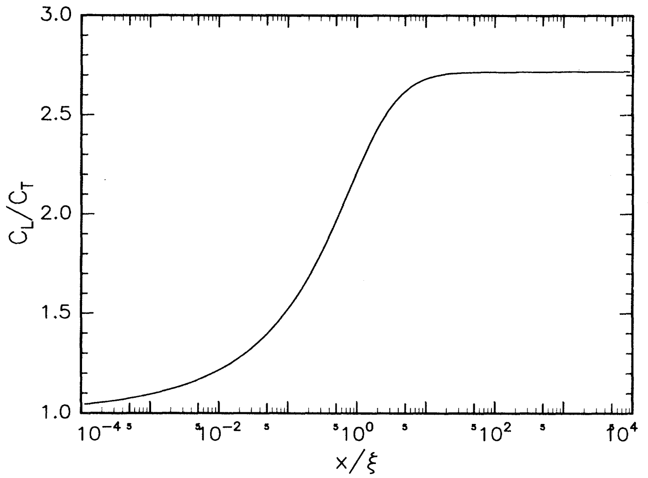

FIG. 4. Plot of the ratio of the translational order correlation functions $C_L(r,z=0)/C_T(r,z=0)$ for the triangular Abrikosov lattice, in the limit $c_{66} \ll c_{11}$. $\tilde{C}_{L,T}$ are defined in (4.25). ξ is the crossover length defined in (4.12). The ratio increases from 1 at short distance $x < \xi$ and saturates rapidly to a universal number in the asymptotic regime $x \gg \xi$. The variational method predicts this number to be e.

tion functions for translational order, one finds that the difference between the longitudinal and transverse parts persists at large scales. Defining the longitudinal and transverse translational correlation functions by

$$C^{L,T}(r) = e^{-\frac{K_0^2}{2}\tilde{B}^{L,T}(r)}, \tag{4.25}$$

$C^{L,T}(r)$ correspond to correlation functions with a separation r, parallel and perpendicular, respectively, to the vector K_0. As is seen in Fig. 4, the ratio $R_C = C^L/C^T$ increases from 1 at short distances and saturates at a finite value at large distances. This value depends on the elastic constants, as seen from (4.23). In the limit $c_{66} \ll c_{11}$, this number takes a value which the variational method gives as the universal constant $e = 2.7182\ldots$. The fact that R_C saturates at large distance is a consequence of the existence of the quasiorder. Had a random manifold or a Larkin regime been valid up to large distances, this ratio would increase indefinitely. On the other hand, if the system was genuinely ordered R_C would saturate at a much smaller value than e, a value which would go to 1 when $T \to 0$. As is discussed in more detail in Sec. IV D, this should have observable experimental consequences.

C. Physical discussion: Crossover lengths, dislocations

One can give a simple physical explanation for the three regimes found here. Consider two flux lines separated in the ideal lattice by x. In the presence of disorder the mean squared relative displacement is $\tilde{B}(x)$. There is a length at which the potential experienced by a line is smooth. This length is the greatest of the correlation lengths of the random potential ξ_0, or the Lindemann length $l_T = \sqrt{\langle u^2\rangle_T}$. This defines a separation between vortices which we have denoted by l in this paper such that $\tilde{B}(l) \simeq \max(\xi_0^2, l_T^2)$. At zero temperature it equals the length defined by R_c, the Larkin-Ovchinikov length (and L_c in the z direction), and in general l can be thought of as the Larkin-Ovchinikov length renormalized by temperature. Below this length the elastic manifold experiences a smooth potential with well-defined derivatives, thus a local random force can be defined. Indeed, expanding in u the disorder potential energy in (2.18) gives a random force term $f\cdot u$ with $f(x) = \sum_K V(x)K\exp(-iKx) = \nabla V(R_i)$. In the sum over harmonics the maximum K is $K_{\max} = 2\pi/\xi_0$. Thus this expansion is valid only as long as $u \ll \xi_0$. This defines the range of validity of the Larkin regime, i.e., at $T=0$ $x < R_c$ and more generally $x < l$ (we assume $R_c > a$).

For separations larger than R_c but such that $\tilde{B}(x) \ll a^2$ each flux line explores only its immediate vicinity and experiences different disorder. This is the regime explored by BMY which is identical to the random manifold. This can be seen, on a more mathematical level, from our model by summing over all the harmonics, for instance on the replicated Hamiltonian (2.13). One gets $V(u) \sim \sum_{R_i}\delta(u_a - u_b - R_i)$. For $u \ll a$ only the $R=0$ term contributes and each line experiences an independent random potential. This intermediate random manifold regime holds up to the length $x=\xi$ such that $\tilde{B}(\xi) \approx a^2$ at which periodicity becomes important. There is no gain in energy to shift the lattice by a. In this regime displacements grow much more slowly and only the lowest harmonics contribute. This is the quasiordered regime.

In order to apply this theory to experimental systems one has in principle to worry about topological defects, such as dislocations. Although the influence of dislocations is still a controversial question, their influence has been clearly overestimated in the past. Let us mention some arguments, which we believe are incorrect, put forward to argue that unbound dislocations will proliferate even at weak disorder. An Imry-Ma type argument is the following. The core energy cost of a dislocation cannot be avoided and scales as L^{d-2}. A dislocation loop of size L creates extra displacements of order $O(1)$ up to logarithms, in a region of size L^d. By adjusting the position of the loop one can hope to gain an energy from disorder $L^{d/2}$. Thus below $d=4$ dislocation will be favorable. Such an argument is incorrect because it is again based on the Larkin random force model for which the disorder energy is linear in the displacement. For the real model the energy varies as $\cos(2\pi u/a)$ and adding a dislocation displacement will not necessarily gain enough disorder energy. In fact if the Larkin or the random manifold regime were true up to infinite scales it would indeed be favorable to create dislocations. The energy fluctuation due to disorder is $L^\theta \sim L^{d-2+2\nu} \gg L^{d-2}$. If $\nu > 0$ dislocations will occur because it will always be energetically favorable to replace an elastic distortion by a dislocation.[36] However, in the case of a lattice or if quasi-long-range order is preserved in the system as is the case here, both energies scale the same way, since $\nu = 0$, maybe up to logarithms. The prefactor of the disorder term can then be made arbitrarily small at weak disorder while the core energy of the dislocation is a given finite number. Thus if disorder is weak enough it is likely that dislocations will not appear. Even if they do the scale will be huge, and the effects associated with disorder that we discuss in this paper should be observable over a wide range of distances. If the disorder is Gaussian one could also argue that rare fluctuations will eventually lead to dislocations at exponentially large scale. Realistic disorder, however, is bounded and thus such an effect should be absent.

D. Experimental consequences

Let us now discuss in more detail the experimental consequences of our findings. Two main types of experiments exist at the moment to probe the translational order of the vortex lattice: magnetic decoration experiments and neutron-diffraction experiments. One would expect for these two experiments that the results of Secs. IV B and IV C would apply. However, direct comparison with experiments could be complicated due to the effects of the nonlocal elasticity and three-dimensional (3D) anisotropy. These effects can be included, in principle, in the variational calculations by simply changing the elastic Hamiltonian, at the price of extremely tedious calculations. Even if a detailed treatment of such effects is beyond the scope of this paper, their importance can be estimated by the following simple arguments.

In the high-T_c Abrikosov lattice, the elastic constants vary by orders of magnitude when the wave vector goes from $1/\xi_0$ to $1/\lambda$. A good approximation of the elastic moduli for $H_{c1} \ll B \ll H_{c2}$ is[37,38]

$$\begin{aligned} c_{44}(q) &= \frac{B^2}{4\pi}\frac{1}{1+\lambda^2 q_z^2+\lambda_c^2 q_\perp^2} + c_{44}', \\ c_{11}(q) &= \frac{1+\lambda_c^2 q^2}{1+\lambda^2 q^2}, \\ c_{66}(q) &= \frac{\Phi_0 B}{(8\pi\lambda)^2}, \end{aligned} \tag{4.26}$$

where λ is the London penetration depth in the ab plane, $\lambda_c = \Gamma\lambda$ along the c axis with $\Gamma = \sqrt{M_z/M}$, and $c_{44}'(q_z)$ is the single flux line contribution to the tilt modulus. One must have also $B < H_{\rm dec} = \Gamma^2\Phi_0/d^2$ to avoid further effects of decoupling between planes, where d is the distance between CuO planes. Since in this regime c_{66} is dispersionless and much smaller than c_{11}, most of the effects of nonlocal elasticity come from the $q_\perp$ momentum dependence of c_{44}. In the region of Fourier space where c_{44} varies strongly, i.e., for $1 \ll \lambda_c q_\perp \ll \lambda_c a^2$, a good approximation to c_{44} is

$$c_{44} \approx \frac{B^2}{4\pi\lambda_c^2 {q_\perp}^2}. \tag{4.27}$$

This new momentum dependence of c_{44} will lead to very different lattice displacements. Using (3.14) with the proper $c_{44}(q_\perp)$ (4.27), one finds now $[\sigma](v) \sim v/\ln^2(v)$. This leads, using (3.17), to a very slow growth of the relative displacements, $\overline{\{\langle[u(x)-u(0)]\rangle\}^2} \propto \ln(x)$ or constant, for $\lambda < x < \lambda_c$. If the translational correlation length $\xi > \lambda_c$, the random manifold regime will survive, whereas if $\xi < \lambda_c$ one would expect the nonlocal elasticity effects to dominate the random manifold regime. Note, however, that the asymptotic large distance regime for $\xi > \lambda_c$ will be completely unchanged.

In decoration experiments, however, one is usually in a regime of very small fields. In particular $B < H_{c1}$ (bulk), giving $a > \lambda$. For example, in the experiments of Grier *et al.*[29] performed on 10 μm thick samples of Bi-Sr-Ca-Cu-O, one has $\lambda = 0.3$ μm and $\lambda_c \approx 60\lambda$. The highest field pictures (69 G) have very large regions free of dislocations, for which one can hope to apply the theory of the present paper. For such fields $a \sim 2\lambda$, and thus $\lambda < a < \lambda_c$. It is likely, since the interactions between vortices are less important than in the regime $B > H_{c1}$, that single vortex contributions will dominate. Thus the $q_\perp$ dependence of c_{44} will be weaker, and one can hope nonlocal elasticity to be unimportant.

Recently, we have carefully reanalyzed[39] the data of Ref. 29, performing a Delaunay triangulation of the larger field images which do not contain dislocations. This allows one to compute $\tilde{B}(r)$ directly. Preliminary results indicate a very good fit to a power-law behavior $\tilde{B}(r) \sim r^{2\nu}$ from $r = a$ up to $r = 30a$ with an exponent $2\nu = 0.4 \pm 0.05$. Thus, assuming dispersionless elastic constants, this is a strong indication that one is seeing the random manifold regime. The exact exponent 2ν for the random manifold regime is unknown, but the (Flory) value $2\nu = 0.33$ predicted by the variational method is expected[40] to be a (relatively good) *lower bound.* Another prediction[1,41] for the exponent using refined scaling arguments, which might turn out to be more accurate, is $2\nu = 4(4-d)/(8+m)$, i.e., $2\nu = 0.40$ for $d = 3$ and $m = 2$. The data exclude a Larkin-type behavior $\tilde{B}(x) \sim x$, and in fact there seems to be no measurable Larkin regime for small r, indicating that $l \sim a$. One does clearly observe a saturation in $\tilde{B}(r)$ around $r = 30a$–$40a$ at a value of $\tilde{B}(r)$ consistent with the predicted saturation to the slower logarithmic growth. However, larger pictures would be necessary to make conclusions unambiguously on the crossover itself, as well as the large distance regime. The main obstacle is of a statistical nature, i.e., there are not enough pairs of points uncorrelated statistically to perform the necessary translational average. Larger pictures would allow such an average to be taken.

Clearly, both the understanding of the short distance regime, of the importance or not of nonlocal elasticity, and of the existence of the quasiordered regime deserve further studies. Other difficulties in interpreting data from decoration experiments can come from the fact that surface interactions may be different from the bulk ones.[36] It has been argued recently, however, that the effects of the surface interactions may be visible only at scales much larger than the size of the decoration pictures.[42]

Another good probe of the correlations in the vortex lattice, which is free of potential surface problems, is neutron-scattering experiments. Detailed neutron-diffraction studies are now available for different type

II superconductors such as $NbSe_2$ (see Ref. 30) as well as Bi-Sr-Ca-Cu-O. Neutron experiments measure (up to a form factor taking into account the field distribution created by a single vortex line) the Fourier transform at $k = K_0 + q$ of the density correlation around a reciprocal lattice vector K_0. The structure factor which is measured is given by

$$S(q) = \int d^3x e^{iqx} e^{-\frac{1}{2}K_\alpha K_\beta \tilde{B}_{\alpha\beta}(x)}, \tag{4.28}$$

where $\tilde{B}$ is given in (4.9) and (4.18)–(4.20). The full calculation of $S(q)$ requires a numerical integration of (4.28), but the main features can be given analytically. Let us recall that ξ is the translational correlation length due to disorder defined in (4.12) and that trivial anisotropy has been taken into account by proper rescalings of z versus r directions.

At small q, $q < 1/\xi$, the integral (4.28) is dominated by the large distance regime where $\tilde{B}(x) = A_3 \ln(x)$, where $A_3 = 1$ according to the variational method. The structure factor is therefore

$$S(q) \sim (1/q)^{3-A_3} \tag{4.29}$$

and thus *diverges* at small q, a consequence of the persistence of quasi-long-range order in the system. True Bragg peaks therefore exist. This is in sharp contrast with previous predictions assuming simple or stretched-exponential decay of the translational correlation function up to large distance. At a wave vector of order $q \sim 1/\xi$, the behavior of $S(q)$ will cross over to a slower decay, controlled by the random manifold regime. In this regime

$$S(q) \sim (1/q)^{3+2\nu}. \tag{4.30}$$

A clear signature that one is indeed in the regime (4.29) described here should show in the neutron experiments by the fact that $S(q)$ has no true half-width. The maximum value of $S(q)$ will be limited either by the experimental resolution or by the distance between unpaired dislocations ξ_D. Varying these parameters should leave the rest of the curve nearly unchanged (this is valid as long as $\xi_D \gg \xi$). The distance between dislocations could be controlled by annealing the lattice, using either a driving force or a field-cooling procedure similarly to what is done in Ref. 30.

Another interesting prediction can be made for the in-plane, $q_\perp$-dependent ratio $S(q_\perp \parallel K_0)/S(q_\perp \perp K_0)$. After integration over q_z the structure factor becomes, at small $q_\perp$,

$$S(q_\perp, z=0) = \int dq_z S(q_\perp, q_z) \sim \frac{2\pi}{q_\perp} e^{\frac{1}{2}-(\hat{K_0}\cdot\hat{r})^2}. \tag{4.31}$$

Thus the ratio goes for small $q_\perp$ to the value

$$\frac{S(q_\perp \perp K_0, z=0)}{S(q_\perp \parallel K_0, z=0)} = e. \tag{4.32}$$

V. FUNCTIONAL RENORMALIZATION GROUP

Another method widely used to study disordered problems is the functional renormalization group (FRG). It turns out that its application to the present problem is simple due to the periodicity in (2.13). It provided a good complement to the variational method, none of the methods being rigorous. The functional RG can only give results in an $\epsilon = 4 - d$ expansion, which does not have presently the status of rigor of the standard ϵ expansions of the usual critical phenomena for pure systems. In particular, the effects of multiple minima will affect higher orders in perturbation theory and could very well result in replica symmetry breaking instability in the FRG flow, as found recently in Ref. 43 (see also Sec. VI). On the other hand, the functional RG should include fluctuations more accurately than the variational method, provided it does not miss another part of the physics. Comparison of the two methods near four dimensions should allow one to test their accuracy.

A. One-component model

For simplicity we confine our study to a model with isotropic elasticity as in (2.13). Let us first consider u to be a scalar field ($m = 1$) and set $c = 1$. The full replicated Hamiltonian is

$$H/T = \frac{1}{2T}\int d^dx [\nabla u(x)]^2 - \frac{1}{2T^2}\sum_{ab}\int d^dx \Delta(u_a(x) - u_b(x)). \tag{5.1}$$

For simplicity we take $K_0 = 2\pi$, so that the function $\Delta(z)$ is periodic of period 1. In the original Hamiltonian $\Delta(z)$ is a sum of cosines given in (2.13). One then performs the standard rescaling $x \to e^l x$ and $u \to e^{\zeta l} u$. The idea of this renormalization is to perform an expansion around a classical solution at zero temperature. One should keep the whole function Δ in the renormalization procedure. The RG equations to order $\epsilon = 4 - d$ have been derived by Fisher[44] for the random manifold problem[46] (see also Refs. 41 and 45

$$\frac{d\Delta}{dl} = (\epsilon - 4\zeta)\Delta + \zeta z \Delta' + \frac{1}{2}(\Delta'')^2 - \Delta''\Delta''(0), \tag{5.2}$$

$$\frac{dT}{dl} = (2-d)T.$$

A factor $1/S_d = 2^{d-1}\pi^{d/2}\Gamma[d/2]$ has been absorbed into Δ in (5.2). The temperature is an irrelevant variable and flows to zero. The correlation function

$$\tilde{\Gamma}(q) = T\tilde{G}(q) = \langle u^a(q) u^a(-q)\rangle \tag{5.3}$$

satisfies the RG flow equation

$$\tilde{\Gamma}(q, T, \Delta) = e^{(d+2\zeta)l}\tilde{\Gamma}(qe^l, Te^{(2-d)l}, \Delta(l)). \tag{5.4}$$

The periodicity of the function Δ implies that the roughening exponent is $\zeta = 0$ for the large distance behavior. This allows us to obtain the only periodic fixed point function $\Delta^*(z)$ in the interval $[0,1]$:

$$\Delta^*(z) = \frac{\epsilon}{72}\left(\frac{1}{36} - z^2(1-z)^2\right). \quad (5.5)$$

Values for other z are obtained by periodicity. The fixed point is stable except for a constant shift, which corresponds to a change in the free energy. The linearized spectrum is discrete and the eigenvectors can be obtained using hypergeometric functions. The fixed point function is nonanalytic at the origin. It has a singular part which behaves as $z^2|z|$ for small z and leads to $\Delta^{*(4)}(0) = \infty$. As discussed below this is a general feature of fixed points for this type of disordered systems.[44] For a periodic fixed point, i.e., $\zeta = 0$, one can set $e^{l^*}q = 1/a$ in (5.4). This allows one to obtain perturbatively, provided $l^* \gg 1$,

$$\tilde{\Gamma}(q,T,\Delta) = \left(\frac{1}{qa}\right)^d \tilde{\Gamma}\left(\frac{1}{a}, T \simeq 0, \Delta^*\right). \quad (5.6)$$

One can evaluate the correlation in (5.6), at a scale of the order of the cutoff, perturbatively in Δ. One then gets

$$\tilde{\Gamma}(q,T,\Delta) = \left(\frac{1}{qa}\right)^d \left.\frac{-\Delta''^*(0)}{k^4}\right|_{k=1/a}. \quad (5.7)$$

Using (5.5), and remembering the factor $1/S_d$ in Δ, one obtains

$$\tilde{\Gamma}(q) = \frac{a^{4-d}\epsilon}{36 S_d q^d}. \quad (5.8)$$

This gives

$$\overline{\langle [u(x)-u(0)]^2\rangle} = \frac{\epsilon}{18}\ln|x|, \quad (5.9)$$

whereas the variational method gives

$$\overline{\langle [u(x)-u(0)]^2\rangle} = \frac{\epsilon}{2\pi^2}\ln|x|. \quad (5.10)$$

Equation (5.9) gives a power-law decay for the translational correlation functions with an exponent $A_{d,\mathrm{RG}} = \epsilon(2\pi)^2/36 = 1.10\epsilon$ against $A_{d,\mathrm{RG}} = \epsilon$ for the variational method. The agreement of the two methods on the exponent A_d is within 10%, which is very satisfactory. The fact that $A_{d,\mathrm{var}} \le A_{d,\mathrm{RG}}$ is not surprising since the variational method underestimates *a priori* the effect of fluctuations. One can remark that omitting the term $(\Delta'')^2/2$ in (5.2) leads to a fixed point $\Delta^*(z) = \cos(2\pi z)/(2\pi)^2$, which gives exactly the same exponent as the variational method.

At intermediate distance it is enough to focus on the small u behavior of the function Δ, and thus to forget in effect the periodicity. At short distances the function Δ is analytic. In that case[44]

$$\frac{d\Delta''(0)}{dl} = (\epsilon - 2\zeta)\Delta''(0). \quad (5.11)$$

Setting $\zeta = \epsilon/2$ allows one to get a fixed point $\Delta(z) = Az^2 - 2A^2$. Using (5.4) one obtains

$$\tilde{\Gamma}(q) \sim \frac{\Delta''(0)}{q^4}, \quad (5.12)$$

which corresponds to the Larkin random force regime. This, however, holds only at short scales. This fixed point is unstable, and a nonanalyticity at $z = 0$ develops, corresponding to an algebraic decay of the Δ_K in (2.13). Δ eventually flows towards the long distance regime described by the fixed point (5.5). There might be an intermediate random manifold regime.

Another renormalization method that has been used was a real space RG by Villain and Fernandez.[47] For $2 < d < 4$ this method, which is approximate, also predicts a logarithmic growth of the correlations. It does not allow one, however, to compute the universal prefactor A_d or the crossover function. The agreement between these methods, none being rigorous, lends credibility to the additional results in $d = 3$ obtained using the variational method.

B. General case

Let us consider now the more general case of an m-component vector u, and isotropic elasticity (2.2). The equation giving the fixed point function becomes instead of (5.2)

$$\Delta[u] + \frac{1}{2}(\partial_\alpha\partial_\beta\Delta[u])^2 - \partial_\alpha\partial_\beta\Delta[u=0]\partial_\alpha\partial_\beta\Delta[u] = 0, \quad (5.13)$$

while the displacement correlation function becomes, similarly to (5.7),

$$\overline{\langle u_\alpha(q)u_\beta(-q)\rangle} = (\partial_\alpha\partial_\beta\Delta^*[u=0])\frac{1}{q^d}, \quad (5.14)$$

where ϵ has been included in Δ. For the case $m = 1$, (5.13) reduces to (5.2). For the case $m = 2$, the analysis depends on the symmetry of the lattice. For a square lattice a separable function

$$\Delta[u_x, u_y] = \Delta^*[u_x] + \Delta^*[u_y], \quad (5.15)$$

where Δ^* is a solution of (5.2), satisfies (5.13) and gives (3.19) with the same exponent $A_{d,\mathrm{RG}}$ as for the case $m = 1$. A rectangular lattice would give the same exponent. The triangular lattice is more difficult to treat and no simple solution of (5.13) can be found. We have performed a numerical solution of (5.13). There is a nontrivial solution which has the full symmetry of the triangular lattice. In that case one would expect in general a different exponent A_d than for the square lattice, unless there is a hidden symmetry reason for which the exponent does not depend on the lattice symmetry. It is difficult to get a high precision for the exponent because of the nonanalytic nature of the solution. The numerical value found for A_d was within 5% of the one for the square lattice, but we were unable to decide within our accuracy whether the two exponents were equal or different. The exponent is again very close to the one predicted by the variational method, which is independent of the lattice symmetry. Once again, neglecting the term $\frac{1}{2}(\partial_\alpha\partial_\beta\Delta[u])^2$ in (5.13), one recovers exactly the result of the variational method.

VI. $d = 2$

In $d = 2$, thermal fluctuations are expected to play a more important role. Already in the case of the pure system, they change the true long range order of the lattice into a power-law decay of the correlation functions, with an exponent controlled by the temperature. One can therefore expect a stronger competition between disorder and temperature than in higher dimensions. In addition, standard renormalization group techniques are available in $d = 2$ and can be compared with the variational method. In Sec. VI A we examine the $d = 2$ problem using both the variational method and the renormalization group. We will focus mainly on $d = 1 + 1$ (flux lines in a plane). The results are mostly relevant there since the starting model (2.13) becomes exact, due to the fact that dislocations cannot exist in $d = 1 + 1$. The physical consequences for various experimental systems in both $d = 1 + 1$ and $d = 2 + 0$ will be discussed in Sec. VI B, together with the effects of dislocations.

A. Theoretical results

In $d = 2$ the variational method applied to the starting model (2.13) leads to a solution which belongs to the class of "one-step" replica symmetry breaking[26,34] in some extended sense, i.e., such that $[\sigma](v) = 0$ vanishes for $v < v_c$ and $[\sigma](v) > 0$ for $v_c < v < 1$. This can be seen readily by taking the limit $d \to 2^+$ in (3.15), a limit which vanishes for $v < v_0 = TK_0^2/(4\pi c)$. This solution represents a glass phase. Since v_0 cannot be larger than 1, the glass phase exists in $d = 2$ for $T < T_c = 4\pi c/K_0^2$. For $T > T_c$ the disorder is irrelevant, and the replica symmetric solution is stable, as already discussed in Sec. III B. The detailed behavior below T_c will again depend on whether one considers simply a single cosine model, or takes into account all the harmonics present in (2.13). For simplicity we will focus here on the single cosine model, which has been simulated numerically and is interesting in itself. The main effect of the higher harmonics is again to allow for a random manifold crossover regime at low enough temperature in the glass phase. It is examined in detail in Appendix E.

In the case of the single cosine model (3.11) one can look simply for a constant solution $[\sigma](v) = \Sigma_1$ for $v > v_c$. The details of the calculations can be found in Appendix C, where the saddle point equations (C9) for Σ_1 and v_c are derived. These equations are solved in (C12) to give

$$v_c = \frac{TK_0^2}{4\pi c}\frac{\Lambda}{\Sigma_1}\ln\left[1 + \frac{\Sigma_1}{\Lambda}\right], \tag{6.1}$$
$$\Sigma_1 = K_0^2\frac{\Delta v_c}{mT}\left(\frac{\Lambda}{\Sigma_1} + 1\right)^{-K_0^2 T/(4\pi c)},$$

where the cutoff $\Lambda \simeq c(2\pi/a)^2$. Assuming the cutoff to be very large (i.e., $\xi \ll a$) in (6.1) one gets

$$v_c = \frac{TK_0^2}{4\pi c}, \tag{6.2}$$
$$\Sigma_1 = \Lambda\left(\frac{\Delta K_0^4}{4\pi m\Lambda}\right)^{1/(1-T/T_c)}.$$

As we will show below Σ_1 defines a characteristic length

$$\xi(T) = \sqrt{c/\Sigma_1} = a(c^2a^2/\Delta)^{1/(2-TK_0^2/2\pi c)}$$
$$= a(c^2a^2/\Delta)^{1/(2-2T/T_c)} \tag{6.3}$$

above which there is a crossover to the asymptotic regime dominated by the disorder. The above expression for $\xi(T)$ in $d = 2$ is valid for $\xi \gg a$. At zero temperature it coincides with the length found using simple Fukuyama-Lee arguments (see Sec. II C) and it is renormalized downwards by thermal fluctuations at finite temperature, an effect specific to two dimensions.

Using (2.14) and the form (C2) for $\tilde{G}(q)$ one obtains for the relative displacement

$$\tilde{B}(x) = \frac{1}{m}\overline{\langle[u(x) - u(0)]^2\rangle}$$
$$= \tilde{B}_0(x) + 2T\left(\frac{1}{v_c} - 1\right)\frac{1}{(2\pi)}\int_0^\infty kdk$$
$$\times\left(\frac{1}{ck^2} - \frac{1}{ck^2 + \Sigma_1}\right)[1 - J_0(kx)], \tag{6.4}$$

where $\tilde{B}_0(x)$ is the value of $\tilde{B}$ in the absence of disorder,

$$\tilde{B}_0(x) = \frac{T}{c\pi}\ln(\Lambda x). \tag{6.5}$$

Equation (6.4) is convergent although each term is individually divergent, but is easily regularized by multiplying by $J_0(\epsilon k)$ with $\epsilon \to 0$. This leads to

$$\tilde{B}(x) = \frac{T}{c\pi}\ln(\Lambda x) + \frac{T_c - T}{c\pi}\left[\ln(x/\xi) + K_0(x/\xi)\right.$$
$$\left.+\gamma + \ln(1/2)\right], \tag{6.6}$$

where γ is the Euler constant. The expression (6.6) gives the following crossover for $\tilde{B}(x)$:

$$\tilde{B}(x) \simeq \frac{T}{c\pi}\ln(x), \qquad x \ll \xi \tag{6.7}$$
$$\tilde{B}(x) \simeq \frac{T_c}{c\pi}\ln(x) \qquad x \gg \xi,$$

where $T_c = 4\pi c/K_0^2$. Note that for $a \ll x \ll \xi$ there is in principle a Larkin regime where one has algebraic growth of the disorder part of the correlation function with $2\nu_L = (4 - d) = 2$ plus logarithmic corrections:

$$\tilde{B}(x) = \tilde{B}_0(x) + \frac{T_c - T}{4c\pi}\{\ln[\xi/(2x)] + 1 - \gamma\}\left(\frac{x}{\xi}\right)^2. \tag{6.8}$$

However, except at very low temperatures, the thermal part always exceeds the disorder part and thus disorder effects are masked by thermal effects at short distances.

The variational method predicts therefore a simple logarithmic growth of the displacements at large distances, both above T_c and below. The effects of disorder are limited to the freezing of the prefactor to the value T_c for temperatures below T_c. Note that T_c is a universal quantity, independent of the strength of the disorder. Such a result is valid in the limit where the ultraviolet cutoff Λ is very large. The disorder strength enters the crossover length ξ above which the asymptotic behavior for $T < T_c$ can be observed. Of course $\xi \to \infty$ when $T \to T_c$ as can be seen from (6.3).

Note that the effect of the cutoff, which could be important for a numerical simulation not at small disorder, led to some temperature dependence of the amplitude of the logarithm. The amplitude in (6.7) becomes

$$A = T/(v_c c\pi) = T_c/(c\pi)\frac{\Sigma_1/\Lambda}{\ln(1 + \Sigma_1/\Lambda)}, \tag{6.9}$$

where one can use Σ_1 from (6.2) or better from Eq. (C13) for y in Appendix C. One finds an increase of the amplitude A when the temperature decreases.

In $d = 1+1$, it is also possible to write renormalization group equations for the disorder.[47,48,15] To lowest order, and on a square lattice for simplicity, such RG equations were derived by Cardy and Ostlund[48] and read

$$\frac{d\Delta}{dl} = \left(2 - \frac{K_0^2 T}{2\pi c}\right)\Delta - C\Delta^2, \tag{6.10}$$

where we put $\Delta = \Delta_{K_0}$ and

$$\frac{d\Delta_0}{dl} = \frac{{K_0}^2 a^4}{T^2}\Delta^2. \tag{6.11}$$

Both Δ and Δ_0 are defined in (2.13). C is a constant[43] which is unimportant for our purposes. For $T > T_c$ the disorder is irrelevant, in agreement with the variational method. For $T < T_c$ the disorder term becomes relevant and there is a new nontrivial fixed point, at $\Delta = (T_c - T)$. This fixed point, however, has the unusual feature that the variable Δ_0 flows to infinity $\Delta_0(l) \propto l$. However, since this variable does not feed back at any order in perturbation theory [only averages of the type $\overline{(\sum_a C_a\phi_a)^2}$ with $\sum_a C_a = 0$ appear] it has been assumed that this fixed point was correct.

Using the RG one can again define a short and large distance regime. At short distances the RG is certainly correct, and is more accurate than the variational method since it treats the fluctuations correctly. As was noted in Sec. III C 1, at short scales x such that $|u(x) - u(0)| \ll \xi$ (for the single cosine model), it is possible to expand the cosine to equivalently recover the Larkin random force model. The correlation function for that model reads simply

$$\tilde{G}(q) \propto \frac{1}{q^2} + \frac{\Delta}{q^4}. \tag{6.12}$$

This usually leads to $\delta\tilde{B}(x) \sim x^2$, but here one must take into account renormalization by thermal fluctuations. This is done by integrating the RG equation in the small distance regime where we note that (6.10) is correct as long as $\Delta a^2/T^2 \ll 1$, which is equivalent to $(a/\xi)^2/(l_T/a)^2 \ll 1$. One can integrate (6.10) to obtain

$$\Delta(l) = \Delta(0)e^{l(2-2\frac{T}{T_c})}. \tag{6.13}$$

Applying the RG flow equation

$$\tilde{G}(q, T, \Delta_0, \Delta) = e^{2l}\tilde{G}(qe^l, T, \Delta_0(l), \Delta(l)), \tag{6.14}$$

where $\tilde{G}$ has been defined in (2.14), immediately leads to

$$\delta\tilde{G}(q) \sim \frac{\Delta(l^*)}{q^2} \sim \frac{1}{q^{(4-2T/T_c)}}. \tag{6.15}$$

The RG therefore predicts that the Larkin regime is in fact *anomalous* with an exponent continuously varying as a function of the temperature,

$$\delta\tilde{B}(x) = \tilde{B}(x) - \tilde{B}_0(x) \sim a^2(x/\xi)^{2-\frac{TK_0^2}{2\pi c}}, \tag{6.16}$$

instead of (6.8), for $x < \xi$ (in the low-T regime ξ is replaced by another length). There are also corrections coming from the renormalization of Δ_0. Integrating (6.11) one gets

$$\Delta_0(l) = \Delta_0(0) + \frac{K_0^2 a^4 \Delta(0)^2}{2T^2(2 - \frac{T}{T_c})}(e^{2l(2-\frac{T}{T_c})} - 1) \tag{6.17}$$

but such corrections are obviously smaller at short distances.

At large distance $\Delta_0(l) \gg 1$, in order to obtain the correlation functions, one has to assume that the unusual Cardy-Ostlund (CO) fixed point is indeed correct. If one does so, correlation functions can be computed[15] using RG flow equation (6.14). Iterating until l^* such that $e^{l^*}q = 1/a$ allows one to obtain for large l

$$\tilde{G}(q, T, \Delta_0, \Delta) = \frac{1}{q^2}\tilde{G}(a, T, \Delta_0(l), \Delta^*) \sim \ln(1/q)/q^2 \tag{6.18}$$

which leads to $\tilde{B}(x) \sim \ln^2(x)$. In (6.18) it has been assumed that simple perturbation theory could be done for the correlation functions at scale $x = a$. The RG approach would therefore predict a $\ln^2(x)$ growth of the displacements, at variance with the predictions of the variational method, which gives a simple logarithmic growth. In fact the RG result is based on the assumption of replica symmetry. As we have shown recently,[43] a careful analysis of the Cardy-Ostlund fixed point and of the RG flow shows that it is unstable to replica symmetry breaking (RSB). When RSB is allowed one obtains a runaway flow of the RG which is consistent with the findings of the variational method. Two recent numerical calculations on this model[49,50] seem to confirm that the GVM does describe the correct physics at large distance. None of them is compatible with a $\ln^2(x)$ growth of the displacements. In Ref. 49, no change in the static correlation functions was observed in the presence of small disorder, whereas a transition occurring in the dynamic correlation functions was observed at T_c. A careful comparison was performed with the predictions of the RG calculations, and the results were found incompatible. These numerical results are, however, consistent with the prediction of the variational method. Indeed, for such weak disorder the length $\xi(T)$ is very large, especially near T_c, and simulations performed on a too small system will show no deviations as is obvious from (6.7). However, in Ref. 50 the disorder is much larger, and $\xi(T=0)$ is of the order of the lattice spacing a. This simulation indeed shows quite clearly a freeze of the amplitude of the logarithm below T_c at the value $A = T_c/(c\pi)$.

B. Physical realization in 2 + 0 dimensions: Magnetic bubbles

Elastic models in two dimensions have been studied for some time. The Hamiltonian (2.13) describes several physical disordered systems, such as randomly pinned flux arrays in a plane,[2,51,26] the surface of crystals with quenched bulk or substrate disorder,[15] planar Josephson junctions,[13] and domain walls in incommensurate solids. In addition, a very nice realization, investigated in detail recently, is provided by magnetic bubbles.[12] In such systems one should be able to test the predictions of the previous section. However, if the elastic objects are not lines ($d = 1+1$) but points ($d = 2+0$) it is now important *a priori* to take into account the effects of free dislocations. In two dimensions, dislocations are expected to be much more important than in three-dimensional systems, and to have observable consequences on the destruction of translational and/or orientational order. A very naive argument in $d = 2$ is that while the energy of a dislocation pair of separation r increases as $\sim T_0 \ln(r/a)$ in the absence of disorder (a slowly growing function but still allowing for a quasisolid phase for $T < T_0$), in the presence of disorder, the energy should saturate to a finite value $E_{\max} \simeq T_0 \ln(\xi/a)$ when $r > \xi$. This is because long range order is supposed to be destroyed beyond this length, effectively screening the elastic pair interaction. If it is the case, unpaired dislocations will be thermally excited at any finite temperature with weight $e^{-E_{\max}/T}$. Note that the same type of argument in $d = 3$ excludes dislocation loops of size r whose energy scales as $r\ln(\xi/a)$ and thus cannot proliferate so easily. This was discussed in Sec. IV C.

As was discussed in Sec. VI A, in the absence of dislocations quasi-long-range order persists in the system even in the presence of disorder, and the above naive arguments need to be reexamined. A quantitative theory including both disorder and dislocations is difficult. One can, however, get an idea of the effects of dislocations by using the renormalization equations of Cardy and Ostlund in the presence of topological defects.[48] These equations were derived by Cardy and Ostlund for a one-component field (XY model) with both disorder (random field) and vortices. They correspond to the modification of (6.10) and (6.11) to include the dislocation fugacity, and an additional equation for the renormalization of the dislocation fugacity itself.[48] Generalizing them to a triangular lattice beyond linear order goes beyond the scope of this paper, but we do not expect radically new conclusions, rather a change by a small factor of the parameters (temperature, etc.) (note that extension to $n = 2$ components on a square lattice is obvious). There are several reasons why we believe that the CO equations might *overestimate* the effect of dislocations. One is that, as we saw before (Sec. VI A), these RG equations implicitly assume replica symmetry. They lead to a faster decay of the translational correlation function in the absence of dislocations, of the type $\exp[-\ln^2(r)]$ and thus overestimate the effect of disorder compared to the GVM, which includes replica symmetry breaking. The GVM predicts that quasi-long-range order exists in the system. Thus the CO equations should also overestimate the effects of dislocations.

A stability diagram can be constructed by examining the renormalization equations to linear order, for the fugacity of topological defects and for the disorder. In the system considered by Cardy and Ostlund[48] the fugacity of vortices y satisfies to lowest order in y

$$\frac{dy}{dl} = 2y\left(1 - \frac{\pi c}{2K_0^2 T} + \frac{\pi\rho_0^2\Delta_0}{2K_0^2T^2}\right), \tag{6.19}$$

while the $q = K_0$ component of the disorder Δ_{K_0} renormalizes to linear order as

$$\frac{d\Delta_{K_0}}{dl} = 2\Delta_{K0}\left(1 - \frac{K_0^2 T}{4\pi c}\right). \tag{6.20}$$

The influence of *internal* disorder on a two-dimensional crystal was studied in Ref. 52. In that work the disorder was taken as a quenched random stress coupling to the strain, i.e., a term $-\sigma(x)\nabla\cdot u$ was added to the Hamiltonian with $\overline{\sigma(x)\sigma(x')} = \rho_0^2\Delta_0\delta(x-x')$. As discussed in Sec. II this corresponds to including only the *long wavelength* part of the disorder, and to neglecting the $q = K_0$ component of the disorder, which is very important for the present problem of *substrate* disorder. For the 2D elastic triangular lattice the fugacity of dislocations y renormalizes, to linear order, in a way very similar to (6.19):

$$\frac{dy}{dl} = 2y\left[1 - \frac{a^2}{4\pi T}\frac{c_{66}(c_{11}-c_{66})}{c_{11}} + \frac{\rho_0^2\Delta_0 a^2}{\pi T^2}\left(\frac{c_{66}}{c_{11}}\right)^2\right]. \tag{6.21}$$

One easily sees that, to linear order, the $q = K_0$ disorder renormalizes also very similarly to (6.20):

$$\frac{d\Delta_{K_0}}{dl} = 2\Delta_{K_0}\left(1 - \frac{K_0^2(c_{11}+c_{66})T}{8\pi c_{11}c_{66}}\right). \tag{6.22}$$

The stability diagram is shown in Fig. 5. In the presence of dislocations there are thus various possible regimes. There are two critical temperatures. One is the melting temperature $T_m = a^2 c_{66}(c_{11} - c_{66})/4\pi c_{11}$, above which dislocations unbind for the pure system which melts into a hexatic. The other one is the glass transition temperature $T_G = 8\pi c_{11}c_{66}/K_0^2(c_{11}+c_{66})$, below which disorder becomes relevant in the absence of dislocations. The ratio between these temperatures is equal to half of the exponent η_{K_0} of the pure system at T_m. For the CO model it is thus universal and equal to $T_m/T_G = 1/8$. For the triangular lattice it depends on the (renormalized) elastic constants[17] but cannot exceed 1/6. In Fig. 5 we have shown schematically how y and $\overline{y} \sim \Delta_{K_0}$ are renormalized at linear order as a function of temperature and long wavelength disorder Δ_0 from (6.21) and (6.22). At high temperature dislocations are relevant and disorder is washed away by thermal fluctuations. Below T_G disorder becomes relevant leading, in the absence of dislocations, to the Cardy-Ostlund line of fixed points. This line however is *unstable* to dislocations. In the shaded region in Fig. 5, the dislocations would be perturbatively irrelevant for the system with $\Delta_{K_0} = 0$ and the solid would survive (in fact in that case if one includes nonlinearities in y the solid is stable in an even greater region[52]). However, in this region the disorder $\overline{y} \sim \Delta_{K_0}$ is relevant. It will eventually increase Δ_0 as can be seen from (6.11), and drive the flow towards the region where y itself increases and unpaired dislocations will eventually appear at large scale.

Let us now estimate at which scale dislocations will appear, assuming weak disorder. The disorder $\Delta_{K_0} = 0$ becomes of order 1 at length scale of order ξ. The key point is that up to this length scale the fugacity of dislocation has been renormalized *downwards* and is now much smaller with $y(\xi)/y(a) \sim (a/\xi)^{2(1-T_m/T)}$. At this length scale, one ends up with a system for which the disorder is of order 1 and the fugacity of dislocations, is extremely small. One can therefore predict that the typical distance between unpaired dislocations ξ_D in the shaded region is much larger than ξ, the distance above which the effects of disorder become manifest and (6.7) can be observed. It is impossible to compute rigorously the ratio ξ_D/ξ using perturbative RG since beyond ξ one of the coupling constants (Δ_{K_0}) is large, but one can still estimate this ratio by the following heuristic argument. If one assumes that, when Δ_{K_0} has become of order 1, (6.11) is still valid, then above lengths of order ξ

$$\Delta_0(l) = \Delta_0 + \alpha l, \tag{6.23}$$

where α is a coefficient of order 1. Then estimating the distance between dislocations as the scale at which y becomes of order 1, one gets using (6.19)

$$\xi_D \sim \xi \exp\left(C \ln^{1/2}[(\xi/a)^{2(T_m/T-1)}/y(a)]\right), \tag{6.24}$$

where C is a constant of order unity and $y(a)$ the bare value of the dislocation fugacity.

The resulting prediction is that in the shaded region of Fig. 5 dislocations appear only at scales large compared to ξ, and the main reason for decay of translational correlations is the pinning of the elastic manifold. Thus the elastic theory developed in this paper should apply up to distances up to order ξ_D. In that low-temperature region perturbative RG does not allow one to obtain the correlation functions precisely even at not too large distance since one is far from the perturbative region $T \sim T_G$. The variational method, on the other hand, predicts the following results: as shown in Appendix E, there is a crossover temperature $T^* = T_G/\ln(\xi/a)$. For temperatures $T > T^*$ there is no intermediate random manifold regime and the displacement correlation function should show the logarithmic growth of (6.7). However, since the disordered solid regime corresponds to $T < T_m \simeq T_G/8$, one is likely to be in the regime $T < T^*$. In that case there is a random manifold regime at short distances where the displacement correlation function grows as $\tilde{B}(x) \sim (x/\xi)^{2/3}$. At large distance one recovers the logarithmic asymptotic regime. The anisotropy ratio between the longitudinal and transverse displacements $R = \tilde{B}_T/\tilde{B}_L$ should cross over from $2\nu + 1 \simeq 1.7$ in the random manifold regime to 1 in the asymptotic regime.

Experimentally, in the system of magnetic bubbles[53,12] some regimes are found indeed where the distance between dislocations ξ_D is of the order of up to five times the translational correlation length. The authors of Ref. 12 observe a crossover between a regime where $\xi_D \sim \xi$ and a regime where $\xi_D \gg \xi$, which probably corresponds to the transition to the shaded region discussed above. Indeed, the measured ratio ξ_D/ξ keeps increasing with density and is limited there only by the experimental setting. It would be very interesting to increase the range experimentally accessible. In Ref. 12 some comparison with the random manifold result was also performed. However, the definition of the anisotropy ratio used in Ref. 12 (ξ_T/ξ_L) does not seem appropriate, since the data were first fitted to a simple Lorentzian shape. New insight in to the physics of such disordered 2D sys-

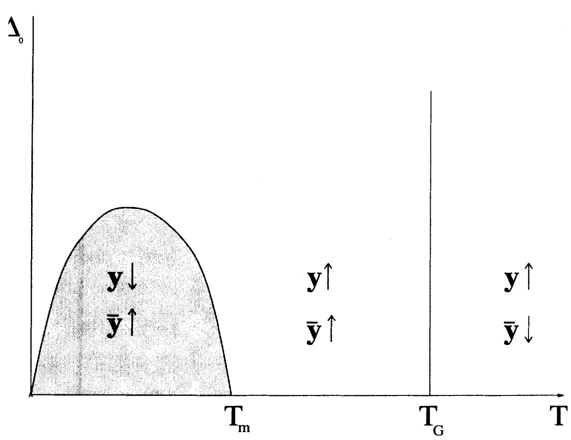

FIG. 5. Stability diagram of a two-dimensional solid with weak quenched substrate disorder, as a function of the temperature T and the long wavelength (i.e., harmonic) part of the disorder Δ_0. The diagram indicates schematically the stability of the harmonic part of the free energy to dislocations (fugacity y) and short wavelength disorder ($\overline{y} \sim \Delta_{K_0}$) by showing the relevance of these variables to linear order.

tems could be gained by a reanalysis of these data. More theoretical work is also needed since the above analysis of the dislocations is very crude. It has been generally assumed that the nonperturbative RG flow in the presence of dislocations was towards a kind of hexatic, but nothing is really known on this phase. In particular, in the Cardy-Ostlund approach, the vortices are supposed to be completely thermalized and to experience only a uniform potential, and feel no effect of the disorder. This is physically incorrect, since in a disordered system the fugacity will also depend on the position. One should therefore take also into account terms which will pin the vortices. This is another reason why we believe that the CO analysis overestimates the effect of topological defects.

To conclude, $d = 2$ elastic disordered systems have therefore a very rich behavior. Many other experimental systems could be investigated, among them colloids[54,55] and thin superconductors. A realization which would exclude dislocations could be to adsorb a polymerized membrane, with very high dislocation core energy, on a disordered substrate.

VII. CONCLUSION

In this paper we have developed a quantitative description of the static properties of a lattice in the presence of weak disorder. We have derived a model (2.13) valid in the elastic limit $\nabla u \sim a/\xi \ll 1$ which contains the proper physics at all length scales, and allows us to describe the various regimes as functions of distance. We have applied the Gaussian variational method (GVM) to this model and computed the correlation functions of the relative displacements. This method has the advantage of being applicable in any dimension. The comparison with the renormalization group study that we have also performed, which is possible in $d = 4 - \epsilon$ and in $d = 2$, indicates that the GVM is an accurate variational ansatz for this problem. It also shows that the GVM should be a good tool to explore other disordered problems where different length scales are present. Contrary to previous studies the present analysis includes the effect both of metastable states and of the intrinsic periodicity of the lattice, both effects being found to be very important.

We found that the effect of impurities on the translational order of the lattice is weaker than was previously thought and that quasi-long-range order persists at large scales. The resulting phase, which we call a "Bragg glass" has the properties both of being a glass and of being quasiordered. The analogy with the quasiorder which subsists at low temperature in pure two-dimensional solids is puzzling. The Bragg glass possesses several intrinsic length scales. At very short scales it behaves as predicted by Larkin's random force model. This regime should be quite limited in $d = 3$ at low temperature and for a potential rough at scales smaller than the lattice. In $d = 2$, however, thermal effects being stronger, this regime is wider. At intermediate scales, and low temperature in $d = 2$, the system behaves as a random manifold of flux lines independently pinned by impurities with stretched exponential decay of translational order. At distances larger than ξ, we have shown that due to the periodicity of the lattice the pinning by impurities becomes less effective and quasiorder survives.

We have not attempted to include explicitly topological defects in the present quantitative study. However, we have given evidence in both $d = 3$ and $d = 2$ that for weak impurity disorder the effect of dislocations is less important than is usually believed. This is due to the fact that quasi-long-range order subsists even in the presence of disorder for the elastic system. In $d = 2$ it is possible, but by no means established, that unpaired dislocations will appear at large scale. We have estimated conservatively the length scale at which this happens and found that it can be much larger than ξ. Thus in $d = 2$ there is a regime where the decay of translational order is due primarily to the elastic displacements induced by disorder. In $d = 3$, we have argued that there is probably a phase *without unpaired dislocations* for weak disorder and at low temperature.

Thus in the three-dimensional high-temperature superconductors there could generically be (at least) *two* types of glass phase caused by point impurities, i.e., two types of vortex glasses. The first one is the strong disorder vortex glass phase which is described by the gauge glass type of model[2,18] and presumably contains a lot of dislocations. The second one is the quasiordered Bragg glass described in the present paper. A natural speculation then is that the tricritical point recently observed in experiments,[56] whose position in the H-T plane can be raised by adding more point impurities in a controlled fashion,[35] has something to do with the relevance of dislocations. It could mark the separation between these two phases since it is natural that the quasiordered glass melts through a sharper transition,[57] much like a pure solid, while the vortex glass transition of Refs. 2 and 18 is believed to be continuous. In fact, it is not even clear whether, strictly speaking, the strongly disordered vortex glass phase of Refs. 2 and 18 is a true thermodynamic phase rather than a very long crossover from the flux liquid. Indeed, recent simulations indicate that $d = 3$ could be slightly below its lower critical dimension.[19] The Bragg glass, on the other hand, should not suffer from such existential problems.

The results of the present study suggest some comments concerning experiments. First, decoration experiments should be reanalyzed. Clearly a fit to a simple exponential as inspired by Refs. 28 and 58 is certainly inadequate for pictures where no dislocation is present. Since we predict that the crossover to the asymptotic quasiordered regime occurs when the correlation function $C_{K_0}(r)$ is of order 0.1 this crossover should be observable, in principle. One must keep in mind that Abrikosov lattices might not be the ideal system to compare with the theory, since they are very complicated objects with additional intrinsic scales. In particular the effect of nonlocal elasticity, 2D to 3D crossover effects, and surface interactions, although they can in principle be incorporated into the method, complicate the analysis by introducing new crossover lengths such as the London length. It would be therefore highly desirable, in order to test the above predictions, to investigate simpler elastic systems.

One example are colloids which have been very successful to investigate translational order in pure systems.[54,55] In addition, these experiments could help decide whether dislocations are a thermodynamic property of the state or simply a nonequilibrium feature which can be eliminated. Neutron-diffraction experiments could also help decide on these issues. We obtained several characteristic predictions for neutron diffraction, such as the existence of algebraically diverging Bragg peaks and predictions for the ratio of transverse to longitudinal scattering intensities.

We have addressed here only the statics of a lattice in the presence of disorder, the quantities of interest in that case being mainly related to positional order. It would be clearly very interesting to obtain information on the *dynamics* and especially the driven dynamics of such systems. This is indeed particularly important for superconductors where most experiments measure only dynamical quantities. Several remarks are in order.

There exists presently a qualitative approach to the dynamics of the vortex glass based mostly on scaling arguments for energy barriers.[1] On such a qualitative level one usually assumes a single scale for the energy landscape, i.e., that barriers $E_b(L) \sim L^\theta$ scale with the same exponent θ which describes energy fluctuations in the statics. Assuming that there is a regime of transport where plastic deformations can be neglected, i.e., that energy barriers are controlled only by elastic motion, the present study of the statics can be used to describe the creep regime at low temperature. The argument goes that in the presence of an applied external force f, such as the Lorentz force created by an external current $|f| \sim |j|$, the barrier for an optimal deformation $u \sim L^\nu$ away from a low-energy configuration on a scale L is lowered and becomes $E_b(L) \sim L^\theta - fuL^d$. To unpin locally the manifold thus needs to go over an Arrhenius barrier $E_b \sim 1/f^\mu$ with $\mu = \theta/(d+\nu-\theta)$. This and the exponents found here lead to a nonlinear voltage-current relation $V \sim \exp[-1/(Tj^\mu)]$ in the flux creep regime, where μ crosses over from $\mu \approx 0.7$–0.8 to $\mu = 1/2$ as j decreases.

On a more fundamental level one anticipates that the methods used here for the statics and the results they yielded will set the ground for the study of the dynamics. As was discovered recently[59] the hierarchical structure of states which is encoded in the replica symmetry breaking solution of the statics finds an exact translation in the dynamics. There it corresponds to the existence of a long time dynamics for which the fluctuation dissipation theorem breaks down and nonequal time correlation and response functions become highly nontrivial. It would be interesting to generalize these studies to the flux lattice. Let us finally note that the quantities computed here using the statics are equal time disorder averages and they will coincide with the observed translational averages in a given experiment provided equilibrium has been reached, which will always hold below a certain length scale.

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APPENDIX A: RELABELING OF THE LINES

In this Appendix we detail the derivation of expression (2.8) for the decomposition of the density in terms of the relabeling field (2.7). We denote d-dimensional positions $x = (r,z)$ where r belongs to the m-dimensional transverse space. The density is given by (2.3)

$$\rho(x) = \sum_i \delta(r - R_i - u_i(z)). \tag{A1}$$

In order to take the continuum limit, one can introduce a smooth displacement field $u(r,z)$ by

$$u(r,z) = \int_{\mathrm{BZ}} \frac{d^d m}{(2\pi)^m} e^{iq_\perp r} \sum_j e^{-iq_\perp R_j} u_j(z) \tag{A2}$$

such that $u(R_i,z) = u_i(z)$ which has no Fourier components outside of the Brillouin zone (BZ). In terms of the smooth field (A2) one can introduce the relabeling field

$$\phi(r,z) = r - u(\phi(r,z),z). \tag{A3}$$

In the absence of dislocations there is a unique solution of (A3) giving $u(r,z)$ as a function of $\phi(r,z)$. ϕ is an m-component smooth vector field labeling the lines, and which takes integerlike values at each location of a line

$$\phi(R_i + u(R_i,z),z) = R_i. \tag{A4}$$

Substituting (A3) in (A1) one gets

$$\rho(x) = \sum_i \delta(R_i - \phi(r,z)) \det[\partial_\alpha \phi_\beta(r,z)]. \tag{A5}$$

Using the integral representation of the δ function, (A5) becomes

$$\rho(x) = \det[\partial_\alpha \phi_\beta] \int \frac{d^d q}{(2\pi)^d} \rho_0(q) e^{iq\phi(x)}, \tag{A6}$$

where

$$\rho_0(q) = \sum_i e^{iqR_i}. \tag{A7}$$

For the case of a perfect lattice $\rho_0(q)$ is

$$\rho_0(q) = \rho_0 (2\pi)^d \sum_K \delta(q - K). \tag{A8}$$

Using (A8) in (A6) one gets formula (2.8),

$$\rho(x) = \rho_0 \det[\partial_\alpha \phi_\beta] \sum_K e^{iK\cdot\phi(x)}. \tag{A9}$$

Assuming that we are in the elastic limit $\partial_\alpha u_\beta \ll 1$ one can expand (A9) to get

$$\rho(x) \simeq \rho_0 \left[1 - \partial_\alpha u_\alpha[\phi(x)] + \sum_{K\neq 0} e^{iK[r-u(\phi(x),z)]} \right] . \tag{A10}$$

In (A10) one can replace $u(\phi(r,z),z)$ by $u(r,z)$ up to terms of order $\partial_\alpha u_\beta \ll 1$. Note that in doing so u has negligible (suppressed by powers of a/ξ) Fourier components outside the Brillouin zone, and thus there is a complete decoupling between the gradient term and higher K terms.

The same procedure can be carried on in the case where the equilibrium lattice of the R_i contains topological defects such as dislocations, vacancies, etc. Suppose for instance that one wants to study a lattice with a fixed number of dislocations at prescribed positions in the internal coordinate of the lattice, i.e., a network with a fixed topology (connectivity), but which is now allowed to fluctuate in the embedding space due to coupling to disorder and thermal noise. This is relevant for the physical situation of a flux lattice with *quenched-in* dislocations whenever one can neglect dislocation motion, i.e., glide and climb. The equilibrium positions R_i now correspond to a minimum of the elastic energy with the constraint of prescribed connectivity. The problem at hand is to analyze the extra small elastic displacements around the equilibrium position due to disorder. The density is still given by (A6) where $\rho_0(q)$ is now the Fourier transform of the lattice of the R_i (A7).

Upon coupling to disorder, the equivalent of the last term in (2.13) will be generated. After averaging over the random potential it reads

$$\int \frac{d^dq}{(2\pi)^d} \frac{\Delta_q}{2T} \cos\{q[u^a(x) - u^b(x)]\} S(q), \tag{A11}$$

where $S(q) = \rho_0(q)\rho_0(-q)$ (with no averages) is the structure factor of the lattice without disorder, with fixed connectivity, and in its equilibrium position. If in addition one wants to allow for topological defects to equilibrate one needs to perform some further average over connectivities, which is a difficult task.

APPENDIX B: LINK WITH QUANTUM MODELS IN 1 + 1 DIMENSIONS

The variational method can also be applied to study disordered one-dimensional (space) interacting bosons or fermions. To see that fact we use a representation of operators in terms of phase fields introduced by Haldane.[32] This representation maps the system into an elastic Hamiltonian similar to the one used to describe flux lattices in Sec. II.

The single particle creation operator is written

$$\Psi^\dagger(x) = [\rho(x)]^{\frac{1}{2}} e^{i\theta(x)}, \tag{B1}$$

where $\rho(x)$ is the particle density operator and $\theta(x)$ the phase of the Ψ field. To take into account the discrete nature of the particle density, one introduces an operator Φ which increases by π at each particle's location. The density operator then is

$$\rho(x) = \frac{1}{\pi}\frac{\partial\Phi(x)}{\partial x} \sum_{m=-\infty}^{+\infty} \exp[2im\Phi(x)]. \tag{B2}$$

Φ can be expressed in terms of another operator ϕ by $\Phi = \pi\rho_0 x + \phi$, and ρ_0 is the average density. The ϕ and θ fields obey the canonical commutation relations

$$\left[\phi(x), \frac{1}{\pi}\theta(x')\right] = i\delta(x - x'). \tag{B3}$$

Using (B1), the single particle operator becomes[32]

$$\Psi^\dagger(x) = \left[\rho_0 + \frac{1}{\pi}\nabla\phi(x)\right]^{1/2} \sum_m e^{im[\pi\rho_0 x+\phi(x)]} e^{i\theta(x)}. \tag{B4}$$

For fermions the sum over m in (B4) is only over odd m, whereas for bosons the sum is only over even values of m. For fermions $\pi\rho_0$ can be replaced in (B4) by k_F, where k_F is the Fermi momentum. The long-wavelength low-energy properties of the interacting boson or fermion gas are described by the Hamiltonian[32]

$$H = \frac{1}{2\pi} \int dx \left\{ \left(\frac{v}{K}\right) (\partial_x\phi)^2 + (vK)(\partial_x\theta)^2 \right\}. \tag{B5}$$

When going to the Lagrangian, one gets

$$\mathcal{L} = \frac{1}{2\pi K} \int dx d\tau \left[\frac{1}{u}(\partial_\tau\phi)^2 + u(\partial_x\phi)^2 \right], \tag{B6}$$

which is obviously the same as the classical Hamiltonian (2.2).

From Galilean invariance one has $(vK)/\pi = \rho_0/m$ and $v^2 = 1/(\kappa\rho_0 m)$, where κ is the compressibility. v and K are therefore functions of the interactions and incorporate all the interaction effects. The excited states of H are sound waves with phase velocity v. For the fermion problem the noninteracting case corresponds to $v = v_F$ and $K = 1$. For repulsive interactions $K < 1$, whereas $K > 1$ for attractive ones. For the boson problem $K \to \infty$ when the repulsion between bosons goes to zero, and K decreases for increasing repulsive interactions. For the case of a δ function repulsion, K varies from ∞ to 1. $K = 1$ would correspond to an infinite onsite repulsion. $K < 1$ can be obtained only if longer range interactions are considered. The coefficient K determines the asymptotic behavior of the correlation functions.[32] For the bosons one gets

$$\langle\Psi_B^\dagger(r)\Psi_B(0)\rangle = B\rho_0(\rho_0 r)^{-1/(2K_b)},$$

$$\langle\rho(r)\rho(0)\rangle = 2K_b(2\pi\rho_0 r)^{-2} + A\rho_0^2(\rho_0 r)^{-2K_b}\cos(2\pi\rho_0 r), \tag{B7}$$

with some numerical constants A and B, whereas the fermion correlation functions are

$$\langle \Psi_F^\dagger(r)\Psi_F(0)\rangle = B\rho_0(\rho_0 r)^{-(K_f+K_f^{-1})/2}e^{i\pi\rho_0 r}$$

$$\langle \rho(r)\rho(0)\rangle$$
$$= 2K_f(2\pi\rho_0 r)^{-2} + A\rho_0^2(\rho_0 r)^{-2K_f}\cos(2k_F r), \quad \text{(B8)}$$

up to an angular part. The coupling to a random potential uncorrelated in space and time will give again a Lagrangian identical to (2.13). Terms of the form

$$\mathcal{L}_{\rm dis} = \sum_p \sum_{a\neq b}\int dx d\tau D_p \cos\{2p[\phi^a(x,\tau) - \phi^b(x,\tau)]\}, \quad \text{(B9)}$$

where p is an integer and D_p some constants proportional to the disorder, will appear. They correspond to Fourier components of the random potential close to $2\pi p\rho_0$. The Fourier components close to $q = 0$ correspond to forward scattering and give terms similar to the gradient terms in (2.13).

As discussed in Sec. VI space and time uncorrelated disorder is relevant when $K \leq 1$. The transition point $K = 1$ corresponds to noninteracting fermions or equivalently to bosons interacting with infinite δ repulsive potential. Disorder is thus always relevant for fermions with repulsive interactions, and always irrelevant for fermions with attractive interactions. Disorder is irrelevant for bosons with only finite δ-function repulsion, while it becomes relevant for bosons with sufficiently strong longer range interactions. One experimental realization corresponds to flux lines in superconductors, confined to a plane, which can be realized by proper alignment of the magnetic field. It was argued in Ref. 51 that this always leads to a glass phase, i.e., $K < 1$. This does not seem correct to us. If the vortex line interaction was the sum of an infinite repulsive δ (i.e., forbidden crossings) and an extra repulsive interaction of finite range λ, then disorder would indeed always be relevant. But this is not the case, and the on-site interaction is finite, which presumably leaves room for both a glass phase ($K < 1$) and a high-temperature phase ($K > 1$) in this experimental system.

APPENDIX C: ONE-STEP SOLUTION FOR $d \leq 2$

In this section we examine a one-step replica symmetry broken solution. For simplicity we look at the model with a single harmonic (3.11). We now search a one-step symmetry breaking solution of the form

$$\begin{aligned} &\sigma(v) = \sigma_0, \quad [\sigma](v) = 0, \quad v < v_c \\ &\sigma(v) = \sigma_1, \quad [\sigma](v) = v_c(\sigma_1 - \sigma_0) = \Sigma_1, \quad v > v_c. \end{aligned} \quad \text{(C1)}$$

With the form (C1) one gets

$$\tilde{G}(q) = \frac{G_c(q)}{v_c} + \left(1 - \frac{1}{v_c}\right)\frac{1}{G_c^{-1}(q) + \Sigma_1} + \sigma_0 G_c^2(q), \quad \text{(C2)}$$

$$G(q,v) = \begin{cases} \frac{\Sigma_1 G_c(q)}{v_c[G_c^{-1}(q)+\Sigma_1]} + \sigma_0 G_c^2(q), & v > v_c \quad \text{(C3)} \\ \sigma_0 G_c^2(q), & v < v_c, \quad \text{(C4)} \end{cases}$$

$$[G] = \begin{cases} 0, & v < v_c \quad \text{(C5)} \\ \frac{\Sigma_1 G_c(q)}{[G_c^{-1}(q)+\Sigma_1]}, & v > v_c. \quad \text{(C6)} \end{cases}$$

To determine v_c one has to minimize the free energy

$$\begin{aligned} F/(nm\Omega) = \frac{T}{2}\int \frac{d^dq}{(2\pi)^d}\Bigg[& cq^2\tilde{G}(q) - \ln[G_c(q)] \\ &- \frac{G(q,0)}{G_c(q)} + \int_0^1 \frac{dv}{v^2}\ln\left(\frac{G_c(q) - [G]}{G_c(q)}\right)\Bigg] \\ &+ \frac{\Delta}{2mT}\int_0^1 dv e^{-\frac{K_0^2}{2}B(0,v)}, \end{aligned} \quad \text{(C7)}$$

where

$$B(0,v) = \begin{cases} 2T\int \frac{d^dq}{(2\pi)^d}\frac{1}{G_c^{-1}(q)+\Sigma_1}, & v \geq v_c \\ B(0, v\geq v_c) + 2T\int \frac{d^dq}{(2\pi)^d}\frac{\Sigma_1}{G_c^{-1}(q)[G_c^{-1}(q)+\Sigma_1]v_c}, & v < v_c. \end{cases} \quad \text{(C8)}$$

To get the saddle point equations one has to differentiate (C7) with respect to $\sigma_0(q)$, G_c, Σ_1, and v_c. Differentiation with respect to $\sigma_0(q)$ gives $G_c^{-1}(q) = cq^2$. Differentiating with respect to $1/v_c$ and Σ_1 and then replacing $G_c^{-1}(q)$ by its value one gets

$$\Sigma_1 = \frac{K_0^2\Delta v_c}{mT}e^{-K_0^2 T\int \frac{d^dq}{(2\pi)^d}\frac{1}{cq^2+\Sigma_1}}, \quad \text{(C9)}$$

$$\Sigma_1 v_c = TK_0^2\int \frac{d^dq}{(2\pi)^d}\left[\ln\left(1 + \frac{\Sigma_1}{cq^2}\right) - \frac{\Sigma_1}{cq^2+\Sigma_1}\right].$$

Note that σ_0 does not appear in these equations. One could in principle determine the value of σ_0 by differentiating (C7) with respect to $G_c^{-1}(q)$. However, it is much easier to use the saddle point equations (3.9). For $d \leq 2$ one has

$$\sigma_1 = K_0^2\frac{\Delta}{mT}e^{-K_0^2 T\int \frac{d^dq}{(2\pi)^d}\frac{1}{cq^2+\Sigma_1}}, \quad \text{(C10)}$$

$$\sigma_0 = \sigma_1 e^{-K_0^2 T\int \frac{d^dq}{(2\pi)^d}\frac{\Sigma_1}{cq^2(cq^2+\Sigma_1)v_c}}. \quad \text{(C11)}$$

Thus $\sigma_0 = 0$ for $d \leq 2$. Note that in order to determine v_c one should not substitute in (C7) the expressions (C10) *before* differentiating with respect to Σ_1 and $1/v_c$. Such expressions are only valid at the saddle point.

In $d = 2$ the above equations read

$$\Sigma_1 v_c = T K_0^2 \frac{\Lambda}{4\pi c} \ln\left[1 + \frac{\Sigma_1}{\Lambda}\right], \tag{C12}$$

$$\Sigma_1 = K_0^2 \frac{\Delta v_c}{mT}\left(\frac{\Lambda}{\Sigma_1} + 1\right)^{-K_0^2 T/(4\pi c)},$$

where we used a circular cutoff $\Lambda \simeq c(2\pi/a)^2$. Let us set $T_c = 4\pi c/K_0^2$ and $y = \Sigma_1/\Lambda$. We then have

$$v_c = \frac{T}{T_c}\frac{\ln(1+y)}{y}, \tag{C13}$$

$$y^{(1-T/T_c)} = \frac{K_0^2 \Delta}{m T_c \Lambda}\frac{\ln[1+y]}{y(1+y)^{T/T_c}}.$$

We are interested in the case $\frac{K_0^2\Delta}{mT_c\Lambda} < 1$ since $\xi \gg a$ and thus there is clearly a one-step solution for $T < T_c$. Note that for arbitrary Λ Σ_1 vanishes when $T \to T_c^-$ and simultaneously v_c goes to 1.

For $d < 2$ the above equations lead to

$$v_c = T K_0^2 \left(\frac{2-d}{d}\right) j_d \Sigma_1^{\frac{d-2}{2}} c^{-d/2}, \tag{C14}$$

$$\Sigma_1 = \frac{K_0^2 \Delta v_c}{mT} e^{-K_0^2 T j_d \Sigma_1^{(d-2)/2} c^{-d/2}}, \tag{C15}$$

where

$$j_d = \int \frac{d^d q}{(2\pi)^d}\frac{1}{q^2+1} = \frac{\pi^{1-d/2}}{2^d \sin(\pi d/2)\Gamma[d/2]}. \tag{C16}$$

We can compare the free energy (C7) of the one-step solution with the free energy of the symmetric one that one obtains by taking $v_c = 1$ and $\Sigma_1 = 0$. The free energy difference $\Delta F = F_{\text{one step}} - F_{\text{RS}}$ is

$$\frac{\Delta F}{mn\Omega} = \frac{T}{2}\left(\frac{1}{v_c} - 1\right) \times \int \frac{d^d q}{(2\pi)^d}\left[\frac{\Sigma_1}{cq^2+\Sigma_1} - \ln\left(\frac{cq^2+\Sigma_1}{cq^2}\right)\right] + (1-v_c)\frac{\Delta}{2mT} e^{-K_0^2 B(0,v_c)/2}, \tag{C17}$$

where $B(0, v_c)$ is obtained from (C8). Using the saddle point equations (C9), (C17) simplifies into

$$\frac{\Delta F}{mn\Omega} = \frac{\Sigma_1}{2K_0^2}\frac{(1-v_c)^2}{v_c}. \tag{C18}$$

Thus whenever there exists a one-step solution it has higher energy than the replica symmetric solution.

Examining Eqs. (C14) one sees that indeed a one-step solution survives in $d = 1$ even while the replica symmetric solution is stable there. Note, however, that the one-step solution changes nature and, as an approximation to the original problem, acquires some unphysical features. In $d = 1$ it does not exist for small disorder at fixed T. When it exists the transition is discontinuous and Σ_1 jumps from zero to some nonzero value. The equations are in fact very similar to a variational approach to roughening in $d = 1$ where it is obvious that the cosine term is irrelevant and the variational approach cannot be trusted beyond weak disorder, where it gives sensible results. Here it is also clear that $q = K_0$ disorder (i.e., the cosine term) is irrelevant at large scale for $d < 2$, in the sense that it will not pin the $d = 1$ manifold. It will, however, renormalize the long wavelength disorder which simply adds to the thermal roughening $u \sim L^{1/2}$. The physical meaning of the one-step solution for $d = 1$ being unclear, although it could be related to transitions in the dynamics, we will not consider it further. Note finally that in $d = 0$ the one-step solution does not exist at all (since the mass is zero).

APPENDIX D: VARIATIONAL CALCULATION ON THE NONLOCAL MODEL

If one does not use the decomposition (2.9) of the density, the variational equation for the off-diagonal self-energy is[24,23] instead of (3.2) [in the presence of the extra disorder (3.71)]

$$\sigma_{a\neq b}(q_\perp) = \frac{\Delta}{T}\sum_i \frac{1}{R_i^\lambda}\frac{1}{B_{ab}^{m/2+1}(R_i)}\left(\frac{m}{2} - \frac{R_i^2}{2B_{ab}(R_i)}\right) \times e^{-\frac{R_i^2}{2B_{ab}(R_i)}}\cos(q_\perp R_i). \tag{D1}$$

As for the local model (2.13) one has to look for a non-replica symmetric solution. Let us use the Poisson formula valid for an arbitrary lattice of R_i:

$$\mathcal{V}(R_i)\sum_{R_i}\delta(x - R_i) = \sum_K e^{iKx}, \tag{D2}$$

where $\mathcal{V}(R_i)$ is the volume of the unit cell and the K are the reciprocal lattice vectors. One can then replace the discrete sum in (D1) by an integral and insert the integration over x in formula (D1). One gets

$$\sigma(q_\perp, v) = \frac{\rho_0\Delta}{T}\sum_K \int d^m r e^{iKr}\frac{1}{r^\lambda}\frac{1}{B^{m/2+1}(r,v)} \times \left(\frac{m}{2} - \frac{r^2}{2B(r,v)}\right)e^{-\frac{r^2}{2B(r,v)}}\cos(q_\perp r). \tag{D3}$$

From this exact expression, BMY kept only the components with $K = 0$. As pointed out by BMY,[24,23] if one takes v small enough, then $B(r = 1, v)$ is large enough so that, for all the r contributing to (D3), one can replace $B(r, v)$ by $B(0, v)$. The idea is that $B(r, v)$ varies much more slowly than r^2, which is true in the elastic limit. Indeed one can easily see that for small v and large r one has $B(r, v) \propto \ln(v_\xi/v) + C\ln(r)$. Thus $B(r, v) \simeq r^2$ implies that $r \sim \ln(v_\xi/v)^{1/2}$, and therefore $B(r, v) \simeq B(0, v)$. If one does so and since $B(0, v) \gg 1$, one can restrict oneself to $K = 0$ or $K = \pm K_0$ in (D3). The off-diagonal part of the self-energy becomes

$$\sigma(q,v) \simeq \lambda B(0,v)^{-1-\lambda/2} + c_1 q^2 e^{-\frac{q^2}{2}B(0,v)} + c_2 e^{-\frac{K_0^2}{2}B(0,v)}. \quad \text{(D4)}$$

The two first terms in (D4) come from the $K=0$ part in (D3). The term in λ is due to the extra unphysical disorder and as long as λ is finite dominates the long range behavior [$q \to 0$ and large $B(0,v)$]. The other terms correspond to the same separation of the Fourier components of the random potential as in the discussion leading to our model (2.13). The $q \simeq 0$ components give the same contribution as the terms in (2.13) in $\nabla u_a \nabla u_b$. Indeed this $K=0$ term comes from

$$\left\langle \int dr dr' \delta(r - r' + u^a(r) - u^b(r')) \right\rangle. \quad \text{(D5)}$$

This term measures the smooth change in local density due to the slowly varying displacement field, and keeping only this term amounts to neglecting the discreteness of the vortices in the original Hamiltonian. The third term in (D4) comes from the $q \simeq K_0$ component of the disorder, i.e., the part which has the periodicity of the equilibrium lattice. For the physical disorder $\lambda = 0$ (D4) is identical to the one obtained from the local model (2.13).

In fact neglecting the x dependence in $B(x,v)$ is equivalent to the replacement $\phi(x) = x - u(x)$, i.e., identifying $u(\phi(x)) - u(\phi(y))$ with $u(x) - u(y)$, which led to the model (2.13). Such a replacement becomes exact in the elastic limit $\overline{\langle(\nabla u)^2\rangle} \ll 1$ which has been checked self-consistently on our solution.

APPENDIX E: RANDOM MANIFOLD IN $d = 2$

Let us consider the original model (2.13) in $d=2$ keeping all the harmonics. We will treat only the simpler case of isotropic elastic matrix, but for arbitrary m and lattice symmetry. Similarly to the study in $d > 2$ we use the rescaled quantities (3.27) which satisfy the rescaled equations (3.37). In $d=2$ one has $\xi = a^2 c\sqrt{m/4\pi^3 \Delta_{K_0}}$ and $v_\xi = \pi T/a^2 c$. The equations read

$$[s](y) = h(z), \qquad y = -h'(z)/h(z), \quad \text{(E1)}$$

where $h(z)$ is defined in (3.31). It appears clearly in (E1) that y has a minimum value which is

$$y_0 = \left(\frac{K_0 a}{2\pi}\right)^2. \quad \text{(E2)}$$

Thus the function $[\sigma](v)$ must vanish for $v < v_0 = v_\xi y_0 = T/T_c$ which is identical to the value of the breakpoint v_c in (6.2) of the one-step solution for the single cosine model. The asymptotic behavior at large distance will thus be identical to the one of the single cosine model studied in Sec. III C 1. The transition temperature also turns out to be exactly the same $T_c = 4\pi c/K_0^2$. However, for $y > y_0$, $[s](y)$ will increase continuously until the breakpoint $y = y_c$ above which it becomes constant again. Using Eq. (3.22) the breakpoint $z_c = b(y_c)$ satisfies

$$z_c = \frac{\pi T}{ca^2}\ln\left[1 + \frac{\Lambda^2\xi^2}{[s](y_c)}\right], \quad \text{(E3)}$$

where we used the circular cutoff $\Lambda = 2\pi/a$. Using Eq. (E1) one sees that z_c is determined as the solution of the equation $z_c = f(z_c)$ where we have defined the function $f(z)$ as

$$f(z) = \frac{\pi T}{ca^2}\ln\left[1 + \frac{\Lambda^2\xi^2}{h(z)}\right]. \quad \text{(E4)}$$

Let us study the function $f(z)$. Using the two limiting forms for $h(z)$, i.e.,

$$h(z) = Ae^{-\alpha z}, \quad z \gg 1 \quad \text{(E5)}$$

$$h(z) = B/z^{(m+4)/2}, \quad z \ll 1 \quad \text{(E6)}$$

from (3.39) and (3.42), one finds that the function $f(z)$ has three different behaviors depending on the values of z:

$$f(z) \simeq \begin{cases} \frac{\pi T}{ca^2}\{\alpha z + \ln[(\xi/a)^2/A]\}, & z \gg 1 \\ \frac{\pi T}{ca^2}\left(\frac{m+4}{2}\ln(z) + \ln[(\xi/a)^2/B]\right), & (a/\xi)^{4/(m+4)} \ll z \ll 1 \\ \frac{\pi T}{Bca^2}(\xi/a)^2, z^{(m+4)/2}, & z \ll (a/\xi)^{4/(m+4)} \end{cases} \quad \text{(E7)}$$

taking into account that $\xi/a \gg 1$.

The first regime corresponds to high temperatures. One finds no solution when $T > T_c$ and when $T < T_c$ one finds $z_c = \pi T \ln[(\xi/a)^2/A]/ca^2(1 - T/T_c)$. In that regime y_c and y_0 are very close to each other and the solution is very similar to the one-step solution of the single cosine model. There is no true random manifold regime.

Solving the equation in the second regime one finds approximately

$$z_c \simeq \frac{\pi T}{ca^2}\ln[(\xi/a)^2/B], \qquad v_c \simeq \frac{m+4}{2\ln[(\xi/a)^2/B]}, \quad \text{(E8)}$$

where we have used that in that regime $y \simeq (m+4)/(2z)$. The condition $z_c \ll 1$ shows that this regime exists only at low temperature:

$$T < T^* = \frac{ca^2}{\pi \ln[(\xi/a)^2/B]}. \quad \text{(E9)}$$

There all the harmonics contribute, y_c and y_0 are very different from each other, and there is a large random manifold regime between the Larkin regime and the asymptotic regime. In this regime one has $\tilde{B}(x) \sim A(x/\xi)^{2\nu}$ with $\nu = \nu_{\rm RM} = 2/(4+m)$.

It is easy to see that the last regime of behavior of $f(z)$ never contributes. This is because one is restricted to values of z such that $v_c = y_c v_\xi < 1$ which corresponds to $z_c > ca^2/(T\pi)$. This regime, where there is no breakpoint ($l \sim a$), and therefore no Larkin regime, as already discussed after formula (3.56), arises at very low temperatures $T < T'^* \sim (ca^2/\pi)(a/\xi)^{4/(4+m)}/\ln[(\xi/a)^2/B]$.

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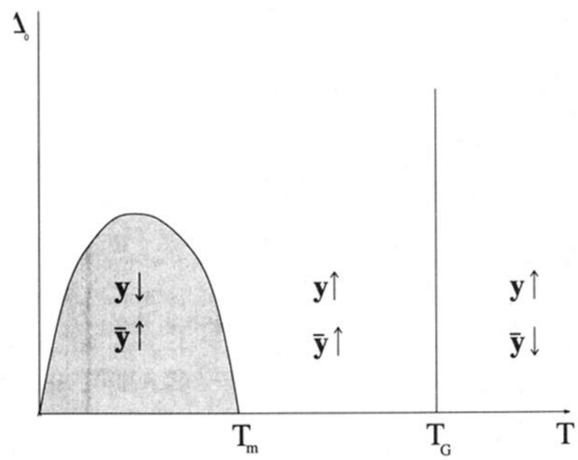

FIG. 5. Stability diagram of a two-dimensional solid with weak quenched substrate disorder, as a function of the temperature T and the long wavelength (i.e., harmonic) part of the disorder Δ_0. The diagram indicates schematically the stability of the harmonic part of the free energy to dislocations (fugacity y) and short wavelength disorder ($\overline{y} \sim \Delta_{K_0}$) by showing the relevance of these variables to linear order.