Line of continuously varying criticality in the Ashkin-Teller quantum chain

Masanori Yamanaka and Mahito Kohmoto
Institute for Solid State Physics, University of Tokyo, 7-22-1, Roppongi, Minato-ku, Tokyo 106 Japan
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We study the line of continuously varying critical exponents found recently in the one-dimensional quantum Ashkin-Teller model using the conformal field theory. On this line, the exponents of the energy density, the polarization, and the magnetization operators are numerically determined by the finite-size scaling method in the conformal field theory. We find that the magnetic exponent is associated with the primary field $(\frac{1}{16}, \frac{1}{16})$, which exists in the coupling constant independent sector of the partition function. Within the existing theory, the numerical results are consistent with the identification of this line with the orbifold line in the Gaussian universality class with the thermal exponent $1 \leq x_T \leq 2$.

I. INTRODUCTION

In the study of two-dimensional critical phenomena, one important goal is to describe all universality classes of critical models.[1,2] Among them, the Gaussian universality class plays a fundamental role in our understandings of these critical phenomena.[3–10] Since the pioneering work of Belavin, Polyakov, and Zamolodchikov,[11,12] there has been considerable progress in this field by the description of conformally invariant field theory. The conformal field theories are classified by the central charge c of the Virasoro algebra. The Gaussian universality class is described by the $c = 1$ theory in which we are interested in this paper.

The continuously varying criticality in the eight-vertex model[13] and the Ashkin-Teller model[14,15] has attracted much interest. The two-dimensional Ashkin-Teller model consists of two Ising models coupled by a four-spin interaction.[14] The classical Hamiltonian is given by

$$H = -\sum_{\langle i,j\rangle} \left[K_2(s_i s_j + t_i t_j) + K_4 s_i s_j t_i t_j\right], \qquad (1.1)$$

where $s_i = \pm 1$ and $t_i = \pm 1$ are the two kinds of Ising spins at site i on a square lattice and the sum is taken over the nearest-neighbor pairs. The parameters K_2 and K_4 denote the two-spin coupling constant and the four-spin coupling constant, respectively. When $K_4 = 0$, the model (1.1) reduces to the two decoupled Ising models with the nearest-neighbor coupling K_2. When $K_2 = K_4$, the model (1.1) is equivalent to the four-state Potts model. On the line $e^{-2K_4} = \sinh 2K_2$ ($K_4 < \frac{1}{4}\ln 3$), the system exhibits the continuously varying criticality.

The Ashkin-Teller quantum chain is obtained by a highly anisotropic limit of the classical Hamiltonian (1.1).[16] The two-dimensional classical system is reduced to a one-dimensional quantum system by taking an extreme lattice anisotropic limit.[17] The quantum Hamiltonian reads (see the Appendix)

$$\mathcal{H} = -\sum_j [(\sigma_j + \sigma_j^\dagger + \lambda\sigma_j^2) + \beta(\Gamma_j\Gamma_{j+1}^\dagger + \Gamma_j^\dagger\Gamma_{j+1} + \lambda\Gamma_j^2\Gamma_{j+1}^2)], \qquad (1.2)$$

where σ_j and Γ_j are matrices,

$$\sigma_j = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -i \end{pmatrix}, \quad \Gamma_j = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}. \qquad (1.3)$$

In the region $\beta \geq 0$, the phase diagram of the Hamiltonian (1.2) was first obtained by Kohmoto, den Nijs, and Kadanoff[16] and it was confirmed by several methods.[18–20] It has a rich structure including the line of the continuously varying criticality [the critical line ($\beta = 1, -1 \leq \lambda \leq 1$)].[21] The critical exponents vary continuously as λ changes. The thermal exponent, the scaling dimension of the energy density operator, takes the form $x_T(\lambda) = 1/2(1 - \frac{1}{\pi}\arccos\lambda)$. The line of the continuously varying critical exponents in the two-dimensional classical Ashkin-Teller model is mapped to this line. Kadanoff and Brown[5] studied the continuously varying criticality in the eight-vertex and Ashkin-Teller models by the description of the Gaussian solution. They identified the operators, except for a magnetization operator, with those of the Gaussian model. The Ashkin-Teller model has a magnetization operator which has a constant scaling dimension $x_H = \frac{1}{8}$ all along the critical line. There is, however, no corresponding operator in the Gaussian model. Von Gehlen and Rittenberg[22] numerically studied this line and found special points where symmetry enhancement occurs. These points are not described by the theory of Kadanoff and Brown. In this sense, the mapping to the Gaussian model was not complete. Yang[23] clarified these points with the description of the S^1/Z_2 orbifold model.[24–26]

Recently, the phase diagram in the negative β region was studied by series expansions.[27] The new line of the continuously varying criticality was found numerically.

It connects the two end points [whose coordinates are $(\lambda, \beta) = (-1, 0)$ and $(1, -\infty)$] through the point $(\lambda, \beta) = (0, -1)$. (It is line $DA'D'$ in Fig. 1 in Ref. 27.) In the present paper, we investigate this line using conformal field theory. We do not have exact solutions except at $\lambda = -1$, 0, and 1. (At $\lambda = \pm 1$ the Hamiltonian is trivially solvable, because it reduces to that of decoupled spins. At $\lambda = 0$ it reduces to the two decoupled Ising models.) Therefore, we numerically calculate the low-lying excitation spectrum, and identify the energy density, the polarization, and the magnetization operators.

II. LINE OF CONTINUOUSLY VARYING CRITICALITY IN THE REGION $\beta < 0$

A. Properties of the critical line

The critical line which divides an ordered phase and a disordered phase has the following properties.[27]

(1) The $Z_2 \times Z_2$ symmetry associated with the polarization and the magnetization operators fully breaks down on this line when we cross it from the disordered phase ($\beta > \beta_c$) to the ordered phase ($\beta < \beta_c$) where β_c is the critical point for a given λ.

(2) The Hamiltonian (1.2) has the property

$$\mathcal{H}(\beta, \lambda) = -\beta\, \mathcal{H}\left(\frac{1}{\beta}, -\lambda\right). \tag{2.1}$$

For $\beta < 0$, this maps the ground state to the ground state in the transformed system. Thus we have a duality which connects (λ, β) and $\left(-\lambda, \frac{1}{\beta}\right)$ for $\beta < 0$. Accordingly the critical lines between $\lambda = 0$ and -1 and between $\lambda = 0$ and 1 are dual.

(3) At $(\lambda, \beta) = (0, -1)$ the Hamiltonian has the self-dual property and reduces to the two decoupled antiferromagnetic Ising models at criticality.

(4) The critical exponents which were calculated by series expansions vary continuously.

B. Finite-size scaling method

In order to obtain the central charge c and the exponents to identify some of the operators on the critical line, we use the finite-size scaling method.[28–30]

The Z_4 charge operator is defined by

$$Q = \oplus_{i=1}^{N} q_i \pmod 4, \tag{2.2}$$

where

$$q_i = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}. \tag{2.3}$$

The eigenvalue of Q is 0, 1, 2, or 3. Since Q commutes with the Hamiltonian (1.2), the Hamiltonian is decomposed into the four charge sectors

$$\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \mathcal{H}_3. \tag{2.4}$$

Here $\mathcal{H}_1 \cong \mathcal{H}_3$ because the Hamiltonian is invariant under exchanges of the charge sectors 1 and 3 at every site.

The finite-size correction to the ground state energy is

$$\frac{E_G}{N} \sim e_G - c\frac{\pi v}{6}\frac{1}{N^2} + O\left(\frac{1}{N^3}\right), \tag{2.5}$$

where e_G denotes the energy per unit length in the thermodynamic limit, v is the charge velocity, and N is the number of the lattice site. The corrections to the excitation energies are classified as

$$E(q; l, \bar{l}; N) - E_G(N) \sim \frac{2\pi v}{N}\left(h_{n,m} + \bar{h}_{n,m} + l + \bar{l}\right), \tag{2.6}$$

where l and $\bar{l}$ are non-negative integers which represent the structure of the conformal tower. The indices n and m represent those of the spin-wave operator and the vortex operator in the Gaussian model,[5] respectively. Here $h_{n,m}$ denotes the scaling dimension which is associated with the primary field $\phi_{n,m}$ and $\bar{h}_{n,m}$ is the contribution from that of the antiholomorphic part. The critical exponent of the operators of the Gaussian model, $x_{n,m}$, is related to them as $x_{n,m} = h_{n,m} + \bar{h}_{n,m}$.

The excitations carry the momentum

$$P = \frac{2\pi}{N}\left(h_{n,m} - \bar{h}_{n,m} + l - \bar{l}\right). \tag{2.7}$$

C. Numerical results

Due to the duality (2.1), we only investigate the region $-1 < \lambda < 0$. The critical points were determined by series expansions up to 17th order.[27] The data used in this paper are shown in Table I. We cannot have reliable results for $\lambda < -0.8$.[31]

To estimate the central charge from (2.5), we calculate the ground state energy $E_G(N)$ and the charge velocity $v(N)$ for a finite size N. We use the Lanczös method[32] and the calculations are performed up to 12 sites under periodic boundary conditions. We first estimate v by extrapolating $v(N)$ for $N = 4$, 6, 8, 10, and 12 by the method of least squares. This and the estimates of the coefficients of $1/N^2$ in (2.5) give the central charge $c(N)$, which is shown in Fig. 1. The central charge c is obtained by extrapolating $c(N)$ for $N = 4$, 6, 8, 10, and 12 by the method of least squares. The error bars are set to include the extrapolated values from $N-2$, N, and $N+2$ where N is either 6, 8, and 10. The results are shown in Fig. 2. The numerical estimates support $c = 1$ for $\lambda > -0.8$.

To estimate the critical exponents from (2.6), we numerically calculate the low-lying excitation spectrum $E_{q,j}(N)$ where j label the energy levels. Since the numerical results strongly support that $c = 1$ on the critical line, we determine v from (2.5) under the assumption $c = 1$. The estimated critical exponents from (2.6) are shown in Fig. 3. The exponents obtained by extrapolation are shown in Fig. 4. Here and hereafter the er-

TABLE I. Critical points β_c estimated by series expansions of the operators $O_\pm$ and O_2 (see Ref. 16 for definitions). Column (a) represents values of β_c obtained by magnetizations and column (b) represents those obtained by susceptibilities. The estimates for the quantities are obtained by averaging three or four highest-order diagonal elements and near-diagonal elements, $[n-1,n]$, $[n,n]$, and $[n+1,n]$, of the Padé tables. Error bars are set to include these three or four values. At $\lambda=0$, the exact value of β_c is -1.

λ	β_c			
	O_2		$O_\pm$	
	(a)	(b)	(a)	(b)
0.0	———	−1.0002(9)	−1.0016(3)	−1.001(2)
−0.3	−0.674283(9)	−0.6734(5)	−0.675(2)	−0.672(2)
−0.5	−0.4962(2)	−0.4965(2)	−0.4980(7)	−0.4971(5)
−0.6	−0.4124(2)	−0.4146(5)	−0.4135(5)	−0.4157(2)
−0.7	−0.3287(2)	−0.3346(1)	−0.3298(6)	———
−0.75	−0.2858(4)	−0.2943(4)	−0.2870(8)	−0.2962(2)
−0.8	−0.2533(4)	−0.2532(3)	−0.243(1)	−0.2556(7)
−0.82	−0.2108(5)	−0.192(6)	−0.224(1)	−0.239(1)
−0.85	−0.20523(1)	−0.164(6)	−0.201(5)	−0.154(8)
−0.9	−0.152(4)	−0.12(2)	−0.14(1)	−0.11(1)

ror bars are set from the same method as that used for the central charge. At $\lambda=0$ the numerically obtained values of $h_{n,m}+\bar{h}_{n,m}$ are 0.990(1) (sector 0), 0.1238(5) (sector 1), and 0.2477(6) (sector 2). The exact values of the exponents (the two decoupled antiferromagnetic Ising models) are, for the thermal exponent, $x_T=1$, for the exponent of the polarization operator, $x_P=\frac{1}{4}$, and for the magnetization operator, $x_H=\frac{1}{8}$. Thus we identify the energy density operator as $\phi_{2,0}$, the polarization operator as $\phi_{1,0}$, and the two magnetization operators as $\left(\frac{1}{16},\frac{1}{16}\right)$ (see Fig. 4). The magnetization operator is in the sector $\mathcal{H}_1$; thus the sector $\mathcal{H}_3$ has another one. The data suggest that the magnetic exponent x_H is independent of λ. They also suggest that x_T and x_P are continuously varying. The numerical estimates, however, suggest that the ratio x_T/x_P is constant and equals 4 for $\lambda\geq-0.5$. See Fig. 5.

III. CLASSIFICATION OF THE UNIVERSALITY CLASS

The numerical results support that $c=1$ on the critical line. By the numerical analysis, we identified the thermal and the polarization operators which have the continuously varying critical exponents. The results suggest that their ratio equals 4. These are responsible for the classification to the Gaussian universality class. In the Gaussian universality class, there are two kinds of continuously varying criticality. One is the original Gaussian line (or circle line),[33] and the other is the S^1/Z_2 orbifold line. We describe the difference between these lines.

We can derive the operator content of the theory from the requirement of the modular invariance of the partition function.[34–36] Given a modular-invariant model, we can construct another modular-invariant model pro-

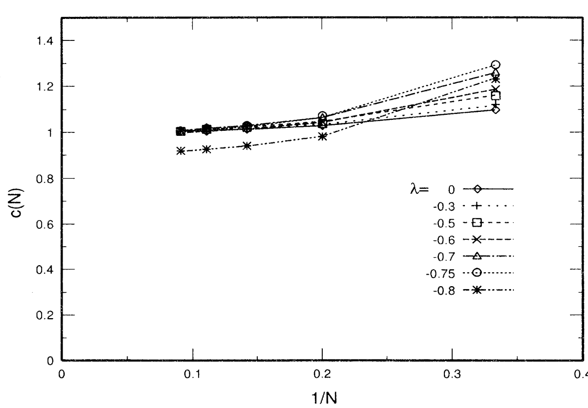

FIG. 1. Estimates of the central charges $c(N)$ for a finite size N. The parameter λ are 0, -0.3, -0.5, -0.6, -0.7, -0.75, and -0.8.

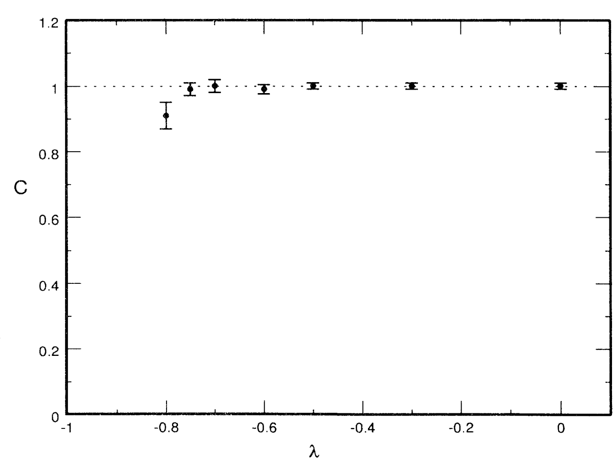

FIG. 2. Central charges obtained by the extrapolations. See the text for the error bars. The dashed line is $c=1$.

jecting onto states which are invariant under a discrete group.[24,25,37,38] Using the modular-invariant partition function of the Gaussian model, we can construct the S^1/Z_2 orbifold model by projecting onto the states which are invariant under the Z_2 group. It is crucial that the S^1/Z_2 orbifold model consists of two sectors. One of the sectors is identical to the original Gaussian model. It describes the continuously varying critical exponents. The other sector consists of the operators whose scaling dimension is independent of the coupling constant. On the critical line ($\beta = 1, -1 \le \lambda \le 1$), Yang[23] constructed a partition function and found that the magnetic exponent is associated with the two primary fields $(\frac{1}{16}, \frac{1}{16})$ in the

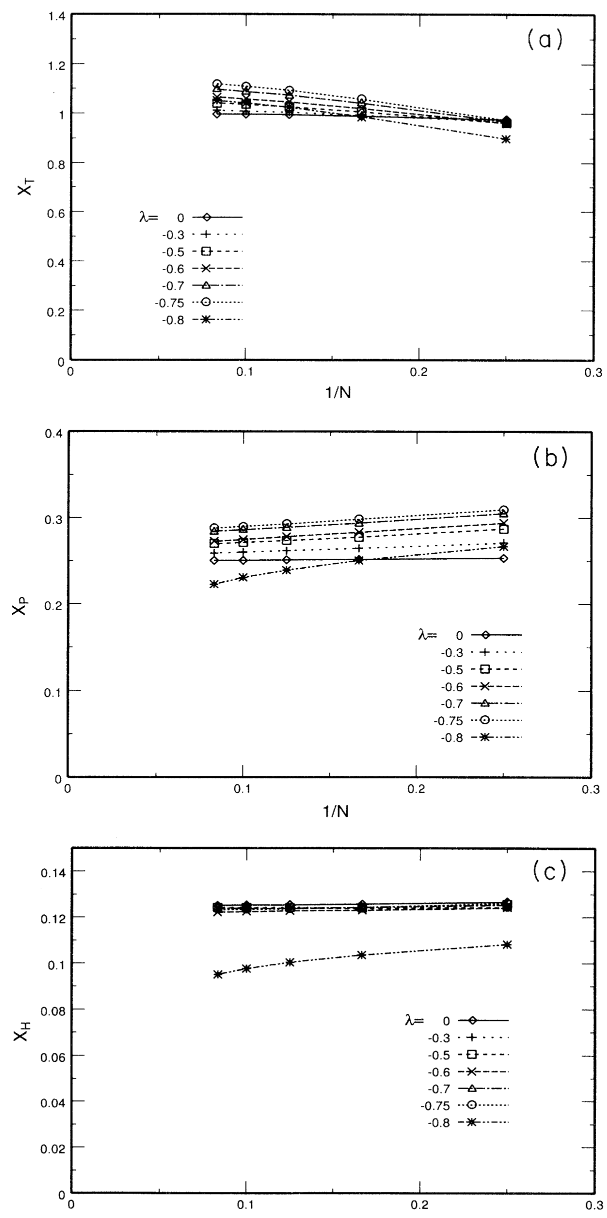

FIG. 3. Estimates of the critical exponents of the operators (a) x_T, (b) x_P, and (c) x_H for a finite size N. The parameter λ are 0, -0.3, -0.5, -0.6, -0.7, -0.75, and -0.8.

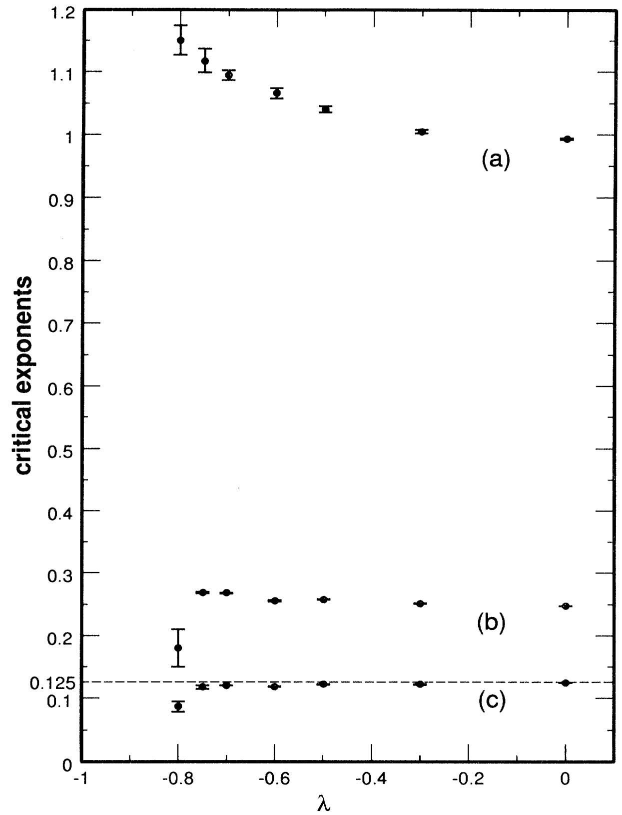

FIG. 4. Critical exponents obtained by the extrapolations. See the text for the error bars.

constant sector. In Fig. 6, we show a survey of $c = 1$ models.[37,38] It is worth noticing that each line has the corresponding special points where the enhancement of the symmetry occurs. The correspondence is one to one. We use also these relations to identify the points on the critical line.

To make identification of the critical line, the most important result obtained numerically is that critical expo-

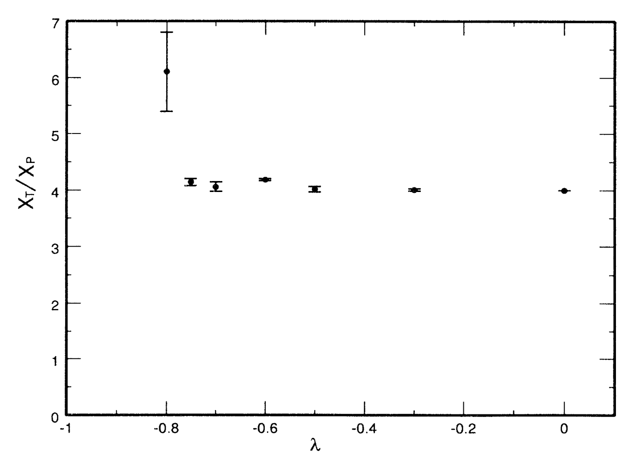

FIG. 5. Ratios between the critical exponents x_T and x_P obtained by the extrapolations. See the text for the error bars.

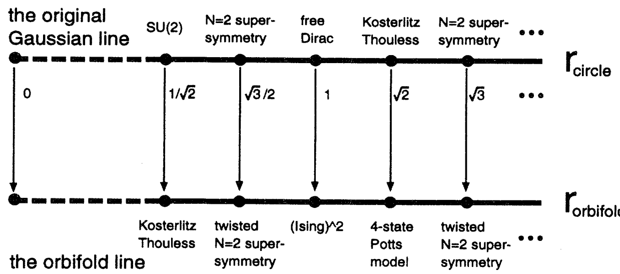

FIG. 6. Two lines of the continuously varying criticality in the Gaussian universality class. (For example, see Ref. 36.) The values along these lines are the compactification radius r. The arrows represent the corresponding models with the same compactification radius.

nents of the magnetic operator be constant on the critical line. It is identified with the two primary fields $(\frac{1}{16}, \frac{1}{16})$. Thus the numerical results suggest that the critical line is the orbifold line. In order to support this identification, let us examine the points $(\lambda, \beta) = (0, -1)$ and $(1, -\infty)$. Apparently the point $(\lambda, \beta) = (0, -1)$ is the $(\text{Ising})^2$ point. Next, we investigate the point $(\lambda, \beta) = (1, -\infty)$. By a nonlocal unitary transformation, the Hamiltonian (1.2) is mapped to the staggered XXZ model,[16]

$$\mathcal{H} = \sum_j \left(S^x_{2j}S^x_{2j+1} + S^y_{2j}S^y_{2j+1} + \lambda S^z_{2j}S^z_{2j+1}\right) + \beta \sum_j \left(S^x_{2j-1}S^x_{2j} + S^y_{2j-1}S^y_{2j} + \lambda S^z_{2j-1}S^z_{2j}\right), \quad (3.1)$$

where S^x_j, S^y_j, and S^z_j are the $S = 1/2$ spin operators. At $\lambda = 1$ the first-order degenerate perturbation in $1/\beta$ from the limit $\beta \to -\infty$ gives the $S = 1$ antiferromagnetic Heisenberg model. It was studied[39–41] in relation to the Haldane gap problem.[42] The Hamiltonian (3.1) reduces to decoupled $S = 1$ spins at $\lambda = 1$ and has an SU(2) symmetry. Therefore, it is reasonable to identify it with the SU(2) point on the original Gaussian line. Since the Hamiltonian (3.1) is obtained by a nonlocal transformation, it is possible that the point $(\lambda, \beta) = (1, -\infty)$ is the Kosterlitz-Thouless point on the orbifold line which corresponds to SU(2) point on the original Gaussian line (see Fig. 6). Moreover, the point $(\lambda, \beta) = (1, -\infty)$ is dual to the point $(\lambda, \beta) = (-1, 0)$ which is expected to be the end point of the critical line of the Kosterlitz-Thouless transition.

For the variations of the critical exponents, the numerical results show that the thermal exponent increases from $\lambda = 0$ to 1 along the critical line. It is consistent with the above identification since the thermal exponent is related to the compactification radius as $x_T = 1/4r^2$.

TABLE II. The correspondence between the eigenvalues of θ and the Ising variables s and t.

θ	s	t	st	$\cos\theta = \frac{s+t}{2}$	$\cos 2\theta = st$
0	1	1	1	1	1
$\pi/2$	1	-1	-1	0	-1
π	-1	-1	1	-1	1
$3\pi/2$	-1	1	-1	0	-1

TABLE III. The correspondence between the eigenvalues of p and σ.

p	σ	$\cos p = \frac{\sigma+\sigma^\dagger}{2}$	$\cos 2p = \sigma^2$
0	1	1	1
$\pi/2$	i	0	-1
π	-1	-1	1
$3\pi/2$	$-i$	0	-1

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APPENDIX

We describe the correspondence among the Hamiltonians (1.1), (1.2), and that in Ref. 16. The Hamiltonian of the classical Ashkin-Teller model in two dimensions is

$$H = -\sum_{\langle i,j\rangle}[K_2(s_is_j + t_it_j) + K_4 s_is_jt_it_j], \quad (A1)$$

where $s_i = \pm 1$ and $t_i = \pm 1$ are two kinds of Ising variables. The Ashkin-Teller quantum chain is obtained by a highly anisotropic limit of (A1). For details, see Sec. II of Ref. 16. The quantum Hamiltonian is

$$\mathcal{H} = \sum_j [2(1 - \cos p_j) + \lambda(1 - \cos 2p_j)] - \beta \sum_j [2\cos(\theta_j - \theta_{j+1}) + \lambda \cos(2\theta_j - 2\theta_{j+1})]. \quad (A2)$$

The operator θ_j acts on a site j and has four eigenvalues: 0, $\pi/2$, π, and $3\pi/2$. Its conjugate operator p_j changes the eigenstates of the operator θ_j as $e^{inp_j}|\theta_j\rangle = |\theta_j + \pi n/2\rangle$, where n is an integer and θ_j is defined by mod 2π. They obey the relation $e^{in\theta_k}e^{inp_j} = e^{inp_j}e^{in\theta_k}e^{in\frac{\pi}{2}\delta_{jk}}$. The combination of the classical variables s_i and t_i makes four possible states at each site, which corresponds to the four values of θ in (A2). The correspondence is listed in Table II.

Using the operators defined in (1.3), the Hamiltonian (A2) can be written [Eq. (1.2)]. The operator σ_j acts on a site j and has four eigenvalues: 1, i, -1, and $-i$. Its conjugate operator Γ_j changes the eigenstate of the operator σ_j as $\Gamma_j|\sigma_j\rangle = |i \times \sigma_j\rangle$. The four eigenstates of the operator p_j corresponds to those of σ_j. We list the correspondence in Table III.

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