Josephson junction ladders: Ground state and relaxation phenomena

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This paper considers a Josephson junction array with the geometry of a ladder and anisotropy in the Josephson couplings. The ground-state problem for the ladder corresponds to the one for the one-dimensional chiral XY model in a twofold anisotropy field, which allows for a rigorous characterization of the ground-state phase diagram and the relevant elementary excitations for the system. The approach to equilibrium, which we study using Langevin dynamics, shows slow relaxation, typical of systems whose energy landscape in the configuration space consists of a wealth of metastable states, dynamically disconnected.

I. INTRODUCTION

Arrays of Josephson junctions (JJ) in the presence of an external magnetic field are probably one of the best examples of physical systems in which the ideas of competition between interactions (frustration), disorder effects, and complex (glassy) dynamics can be theoretically and experimentally checked in a controlled way.[1] Most of the effort devoted in the last decade to these systems has been addressed mainly to the study of two- and three-dimensional arrays as models of extreme type-II superconductors in connection with some problems in granular superconductors (as the high-T_c materials).[2] In this paper we will study a simple geometrical configuration of links, a ladder, which nevertheless shows an interesting nontrivial behavior, and present some results concerning the equilibrium (ground-state) properties as well as the dynamic approach to equilibrium using Langevin dynamics.

For two-dimensional arrays the ground state is in general unknown for arbitrary values of the frustration parameter f (which is essentially the external magnetic field in the appropriate units). Several approaches have been attempted in order to get close to the truly ground state. Halsey[3] proposed a kind of one-dimensional solution (Halsey's staircase) which gives correct configurations but only for certain values of f. He found that this solution provides a highly discontinuous function for the ground-state energy versus f, which is in contradiction with an exact result indebted to Vallat and Beck.[4] Another way is a numerical attack of the problem, by using either phase or vortex variables. Examples of such approach are the pioneering works of Teitel and Jayaprakash[5] and more recently, the "editing method" of Straley and co-workers.[6] In all these methods it is assumed that the ground-state for rational values of f, let us say $f = p/q$, is periodic with periodicity q (or $2q$ in some cases) and the density of vortices is equal to f. However, for the geometry studied here, because of the free character of the boundary conditions on the upper and lower branch of the ladder, superconducting currents on them are possible, allowing a ground-state vortex density different from f.

Different studies on superconducting networks with a ladder geometry are found in literature. Fink and co-workers and Simonin *et al.* have studied a ladder of superconducting wires in the Ginzburg-Landau approximation.[7] Current structures commensurate with the underlying lattice appear as solutions of the Ginzburg-Landau equations.

Previous to our work, Kardar[8] first and later Granato[9] have studied a ladder of JJ in the presence of a magnetic field and charging effects. Kardar, by doing various approximations to the interaction potential connects the JJ ladder and the Frenkel-Kontorova (FK) model with the "dual Coulomb gas." Granato focused his study in the effect of small deviations from a commensurate field as a function of the charging energy and in the critical behavior in the absence of external field (quantum XY model).

In the remainder of this introduction, the specific model for the Josephson junction ladder will be introduced, along with the notation used and the main approximations leading to it. In Sec. II, making a judicious choice of the gauge, is established the equivalence, regarding the ground-state problem, with the one-dimensional chiral XY model with anisotropy. Going further, an important argument due to Griffiths and Chou[10] and Sasaki and Griffiths,[11] allows the applicability of the vast amount of rigorous results on the ground states of models of spatially modulated structures with *convex* interactions[12] to our system. This provides in a rigorous way the main properties of the ground-state phase diagram, some of which were already suggested by Kardar and Granato. We also use the effective potentials method[10] for the explicit computation, with arbitrary numerical accuracy, of the ground-state configuration for any values of the model parameters.

In Sec. III we perform a linear stability analysis of representative ground-state structures which reveals its persistence as metastable states outside the domain of f values for which they are ground states. Many other metastable structures do exist in the system and the dy-

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namic approach to equilibrium, which we consider using Langevin dynamics, is characterized by a *constrained dynamics*[13] leading to slow relaxation. This is a remarkable behavior in the absence of quenched disorder; i.e., no randomness in the Hamiltonian of the system.[14]

The system we consider is a ladder of superconducting islands in the presence of a magnetic field perpendicular to the plane of the ladder (see Fig. 1) and we assume that each island is proximity coupled to its three nearest neighbors. The interaction Hamiltonian for the system is

$$H = -\sum_{\langle ij\rangle} J_{ij}\cos(\theta_i - \theta_j - A_{ij}) = -\sum_{\langle ij\rangle} J_{ij}\cos\gamma_{ij}, \tag{1}$$

where θ_i denotes the phase of the superconducting order parameter at the ith island or site; γ_{ij}, the gauge invariant phase difference, is restricted to the interval $(-\pi,\pi]$; and A_{ij} is proportional to the line integral of the vector potential $\vec{A}$ between the ith and the jth sites,

$$A_{ij} = \frac{2\pi}{\phi_0}\int_j^i \vec{A}\cdot d\vec{l}. \tag{2}$$

It is required that $\sum_p A_{ij} = 2\pi f$, where f is the ratio of the flux caused by the external field with the superconducting magnetic flux quantum and is a measure of the frustration. This relation expresses the discretized Maxwell equation for the vector potential and the sum is taken in a clockwise direction over the bonds surrounding the plaquette p of the lattice. Because the phases are all defined in the interval $(-\pi,\pi]$, we have $\sum_p \gamma_{ij} = 2\pi(n-f)$ where n is the integer that defines the vorticity in each plaquette. Associated to this value we define the vortex density ω as the mean value of n in the ladder.

The Hamiltonian [Eq. (1)] is the sum of the Josephson coupling energies between the neighboring islands. Here we are neglecting screening currents by assuming that A_{ij} is fully determined by the external magnetic field. This assumption is correct whenever the penetration length is much greater than the island width. In this situation, there is no flux quantization,[15] the vortex density being not a flux quanta density, but a fluxoid quanta density. Also we consider that no charging effects are present. For the coupling constants, J_{ij}, we will assume that $J_{ij} = J_x$ for horizontal links and $J_{ij} = J_y$ for vertical ones.

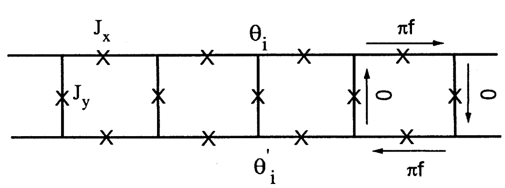

FIG. 1. Schematic picture of the JJ ladder. The gauge choice is shown in the right-most plaquette. Equation (3) gives the interaction Hamiltonian of this system.

II. GROUND STATE AND PHASE DIAGRAM

Among the different choices for the gauge one is particularly convenient: $A_{ij} = +\pi f$ for the upper links, $A_{ij} = -\pi f$ for the lower ones and $A_{ij} = 0$ for vertical links, which corresponds to a vector potential parallel to the ladder and taking opposite values on upper and lower branches. Thus

$$H = -\sum_i [J_x\cos(\theta_i - \theta_{i+1} - \pi f) + J_x\cos(\theta_i' - \theta_{i+1}' + \pi f) + J_y\cos(\theta_i - \theta_i')]. \tag{3}$$

Here θ_i (θ_i') denotes the phase on the upper (lower) branch of the ladder at the ith step. It is easy to see that the phase configurations which minimize the Hamiltonian are such that $\theta_i + \theta_i' = \text{const}$, independent of i; then, by fixing this constant to 0 and defining the anisotropy parameter, $h = J_y/2J_x$, one obtains the equivalence between the following Hamiltonian:

$$H = -2J_x\sum_i [\cos(\theta_i - \theta_{i+1} - \pi f) + h\cos(2\theta_i)] \tag{4}$$

and Eq. (3), regarding the ground-state configurations (and other local minima for the energy).

The Hamiltonian [Eq. (4)] describes a one-dimensional chiral XY model in a twofold anisotropy field.[16] It belongs to a general class of one-dimensional models of spatially modulated structures,[17] the simplest of them being the FK model, extensively studied by Aubry.[12] The equilibrium properties of these models depend crucially on the convexity of the interaction potential. In the model defined by Eq. (4), the nearest neighbor interaction potential is nonconvex. However, it can be proved[10,11] that only the convex part of the interaction term plays a role in determining the ground-state configurations, so that the ground-state properties are those of a convex model; of course, the situation may be different if one considers different aspects of the model, other than the ground state.

The essential physics of the model is the competition between the anisotropy term (coming from the vertical Josephson couplings) which tends to pin the phases value to 0 or π, and the interaction term (coming from the horizontal Josephson couplings) which tends to fix $\theta_i - \theta_{i+1} = \pi f$, that is, it tries to keep the value of the vortex density at the frustration value, $\omega = f$. The ground-state phase configuration at given values of the parameters (h, f) is the result of the compromise between both competing tendencies. In order to compute them we use the effective potentials method[10] which has become the standard method to obtain the phase diagram for these type of models. When used in combination with Newton and relaxation techniques it gives the ground-state configuration with arbitrary accuracy.[18] The computed phase diagram is shown in Fig. 2 where, for clarity, only a few transition lines are represented. Characterizing a ground state by the value of the vortex density (ω), we see in Fig. 3, as predicted by the rigorous arguments mentioned above,[10,11] that $\omega(f)$, for fixed h, is a devil's

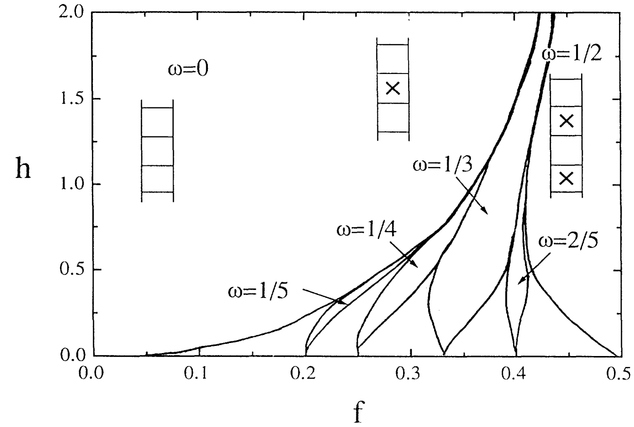

FIG. 2. The phase diagram of the JJ ladder obtained using the method of effective potentials. Each phase is defined by the value of ω and, for clarity, only a few of the transition lines are represented. The sketches of the ladders show the vortex configurations of the simplest commensurate states.

staircase: a continuous function but such that for each rational value of the vortex density there is an interval of values of the field for which ω remains constant.

This phase diagram is quite different from the one expected for the isotropic ($h = 0.5$) two-dimensional JJ arrays. For the 2D system, though there is no rigorous proof for it, it is assumed that there are no intervals of stability for rational values of ω, that is, $\omega = f$ everywhere.[3,5] In the case of the ladder, however, the vortex density is not equal to the field. If we move along inside a step, the ground-state configuration for the gauge invariant phase differences do change with f, while the vortex configuration remains unchanged; correspondingly, supercurrents along the links of the ladder keep varying to compensate for the increase of the field

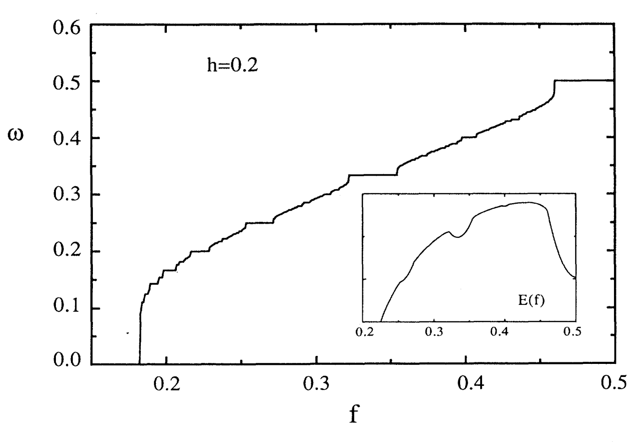

FIG. 3. $\omega(f)$ for the ground-state configurations when $h = 0.2$. This function is a devil's staircase: a continuous function with an step for each commensurate value of ω. The inset shows the continuity of the ground-state energy as a function of the frustration.

with no change in the vortex (fluxoid) density. In the case of the ground-state for zero vortex density, it is tempting to speak of this effect as a sort of Meissner effect, but one should notice that no flux is expelled from the ladder, and one cannot interpret the value of f for which the ground state changes as an analog of the critical field for superconductors: it is the fluxoid density what changes from zero at that value, not the flux density.

Although $\omega(f)$ shows such complex aspect the ground-state energy is a continuous function of the frustration (Fig. 3). This is also the case for a 2D JJ array, as proved analytically in Ref. 4.

The vortex configuration n_i corresponding to a ground state of vortex density $\omega(< 1)$ is explicitly given by

$$n_i = \chi_\omega(i\omega + \alpha), \tag{5}$$

where α is an arbitrary constant and $\chi_\omega(x) = \chi_\omega(x+1)$ is the characteristic function of the interval $[0,\omega)$:

$$\chi_\omega(x) = \begin{cases} 1 & \text{if } 0 \le x < \omega, \\ 0 & \text{if } \omega \le x < 1. \end{cases} \tag{6}$$

Then, n_i is a periodic sequence (with minimal period) for rational values of ω and a quasiperiodic sequence for irrational values of the vortex density; it is traditional to speak of commensurate and incommensurate ground states, respectively.

Commensurate ground states are, for any value of the parameter h, pinned and defectible. A configuration is pinned when there exists a finite value I_d (depinning current) such that if a current $I < I_d$ is injected into each island on the upper branch and extracted from each island on the lower branch, the vortex configuration remains unchanged. In this case the phases change to a new equilibrium configuration and no voltage appears on the links (the ladder remaining superconducting). For values of the external current greater than the depinning current, the phases configuration varies with time and a voltage can be measured. A defectible configuration admits discommensurations (defects), that is, there exist minimum energy configurations of the same vortex density which are not ground states.[12] An elementary discommensuration in a commensurate configuration corresponds to a domain wall separating equivalent ground-state vortex configurations which are shifted relative to each other, with minimum increase (or decrease) of the local vortex density (see Fig. 4 for an example). Notice that the vortex configuration of an elementary discommensuration, though only locally different from the underlying commensurate vortex configuration, cannot be obtained from this through a finite number of local changes, but entails the whole rearrangement of a semi-infinite part of the system. It is important to keep this point in mind when dynamic approaches to equilibrium are studied. The creation energy of an elementary discommensuration goes to zero at the value of the frustration parameter where the ground-state vortex density changes, and a C-IC transition takes place.

Incommensurate ground states show two different regimes, separated by an Aubry transition (transition by

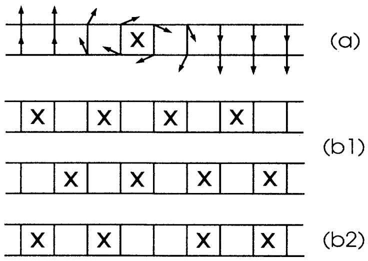

FIG. 4. (a) An elementary discommensuration (DC) in a $\omega = 0$ state. We show the DC from both phase and vortex points of view. (b1) The two possible vortex sequences for the $\omega = 1/2$ ground state. (b2) Elementary DC in the $\omega = 1/2$ commensurate configuration.

breaking of analyticity) (Ref. 12) at a critical value h_c of the parameter h, which depends on the irrational vortex density, ω. Below this critical value, the ground-state configuration is unpinned (any external current produces the appearance of voltage on the links) and no defects can be sustained. In this regime the sequence of gauge invariant phase differences γ_{ij} can be expressed in terms of an analytical hull function. Thus, in the case of a vertical link $\gamma_i = \theta_i - \theta'_i = g(-i2\pi\omega + \beta)$, with β an arbitrary constant (Fig. 5). The situation changes when h grows above the critical value h_c: the hull function develops infinitely many discontinuities and the incommensurate ground-state there becomes pinned and defectible. Our estimate of h_c for a golden incommensurability ratio, $\omega = (3 - \sqrt{5})/2$, is $h_c = 0.245\ldots$. This estimate should certainly be improved, for we have used rather poor rational approximants of ω. On the basis of the plausible irrelevance of the deviations from quadratic of the interaction potential, one can conjecture that the main gap of the hull function for golden irrational vortex density behaves as $\Delta \simeq (h - h_c)^{\xi}$, with $\xi = 0.712$, the critical exponent obtained by Mackay[19] for the Aubry transition in the standard FK model. The depinning current, I_d, for the golden incommensurate structure has been estimated using simulations in the RSJ approximation.[20] It has been shown that I_d scales as $(h - h_c)^{\nu}$ with $\nu = 2.75$ close to the estimation, $\nu = 3.011$, of MacKay for the standard FK model.

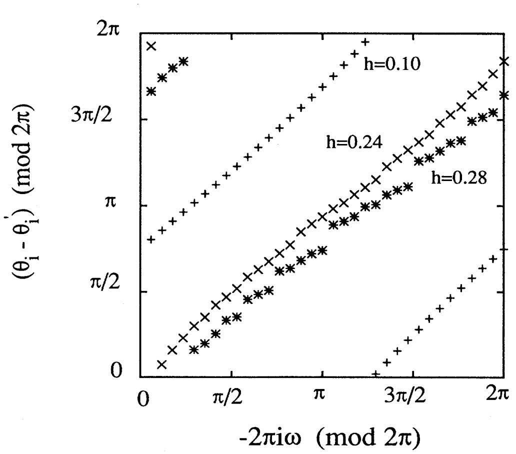

FIG. 5. 2π module representation of the hull function for the gauge invariant difference of phase in the vertical links for the $\omega = 13/34$ ground state. This state reflects the behavior of true incommensurate (irrational ω) phases. Above a critical value of h the function develops infinite discontinuities being analytic for values of h lower than the critical one. In this case $h_c \simeq 0.24$.

III. METASTABILITY AND RELAXATION PHENOMENA

One of the characteristics of frustrated models is the existence of a large number of metastable states, a feature which influences dramatically the dynamic approach to equilibrium. Those states are local minima of the energy (i.e., stable solutions of the equilibrium equation, $\frac{\partial H}{\partial \theta_i} = 0$). For the model we are considering, it seems plausible that the existence of truly chaotic metastable states could be justified, as an extension of the results obtained in Ref. 21, for the FK model. In order to illustrate the nature of metastability in the JJ ladder, we will consider the linear stability of a very restricted class of configurations, namely those vortex configurations which are ground state for some values of the model parameters. To analyze the stability of such states we have worked out the spectrum of small linear perturbations. The procedure has been the following

(a) First, by fixing the model parameters at values inside the tongue (see phase diagram) corresponding to the selected particular value of ω, and using the method of effective potentials, the ground-state phase (and vortex) configuration is obtained.

(b) Now we vary finely the parameter f (typically $\Delta f \simeq 10^{-3}$) and using a Newton method to solve the system of (nonlinear) equilibrium equations $\frac{\partial H}{\partial \theta_i} = 0$, we determine the evolution of the equilibrium vortex configuration under quasistatic changes in f, along with the energy variation.

(c) At each value of f, the spectrum of the small perturbations matrix, $\{\frac{\partial^2 H}{\partial \theta_i \partial \theta_j}\}$, around the corresponding equilibrium phase configuration, is computed.

In all the cases, a zero eigenvalue is found, which corresponds to the (continuous symmetry) invariance of the Hamiltonian (3) under uniform rotation of all the phases. If the rest of the eigenvalues are all positive, the configuration is linearly stable, and the state is a local minimum of the energy for the fixed value of ω under consideration; when the lowest eigenvalue takes on a negative value, the configuration is linearly unstable, usually corresponding to a local maximum of the energy.

In Fig. 6 we show the evolution of the (nonzero) lowest eigenvalue of the stability matrix for some simple values of the commensurability ratio $\omega = 1/2, 1/3, 1/4, 1/5, 2/5$ when f is varied between 0 and 0.5 and $h = 0.5$ ($J_x = J_y$). Not surprisingly, the range of stability is much wider than the interval of f values for which a given state is a

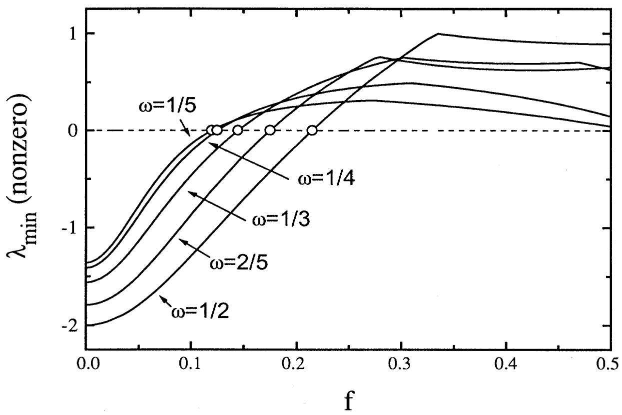

FIG. 6. Lowest of the nonzero eigenvalues of the matrix of stability of a state when f is varied. The picture shows the cases of some simple commensurate states when $h = 0.5$. Open circles show transition from a minimum to a maximum of the energy when f is decreased.

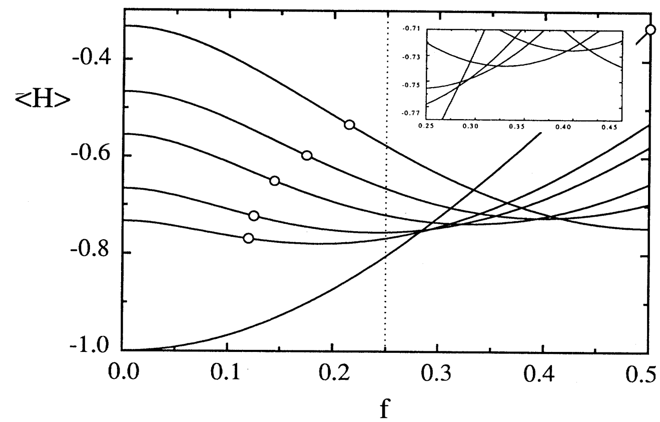

FIG. 7. Energy diagram of some commensurate states as a function of f. The lowest envelope is the ground-state energy. Open circles mark the limit of stability of each state, see Fig. 6. The dotted line is for $f = 0.25$, the value of the field we choose in the relaxation calculations we present in Sec. III. $h = 0.5$.

ground state: a C-IC transition does not have associated a lack of stability of the commensurate state (which remains a local minimum of the energy), but it corresponds to the vanishing of the creation energy of elementary discommensurations, a feature that cannot manifest itself through a linear stability analysis. At the edge of the stability intervals, marked by the negative sign of the lowest eigenvalue, the extremal character of the configuration changes from a minimum to a maximum of the energy. An interesting feature is that, at the border of the stability interval, the gauge invariant phase configuration is always such that there is a link where the supercurrent reaches its maximum (critical current) value, which in turn coincides with the interaction potential leaving its domain of convexity. At this point any small change in the field cannot be sustained by an increase of the currents. The vortex structure becomes unstable and the nearer (in configuration space) stable phases possess a different value of ω.

In Fig. 7 we represent, as a function of the frustration, the energy of some simple commensurate states. The ground-state energy corresponds to the lower envelope of these (and infinitely many other) curves. The stability transition points are marked by open circles. We can observe that near the borders $f = 0$ or $f = 1/2$ the energies of these commensurate states are well separated while at intermediate values of f they are very close. The energies of other (commensurate and incommensurate) stable states, not included in the figure, also lie around. Besides all those stable states corresponding to minimum energy configurations which exist as ground states for some f values, one can construct other metastable states, corresponding to almost arbitrary rearrangement of vortices, with energies also lying about. Then, for intermediate values of f, the energy landscape consists of an extremely complex set of local minima, with comparable energies, corresponding to phase configurations which are, generically, rather separated in configuration space and, from a dynamic perspective, almost disconnected, a situation which is sometimes referred to as *constrained dynamics*.[13] We remind the reader that most of the states are unreachable from a given one. The dynamically reachable states are those which come from the annihilation (or creation) of a finite density of vortices and not from the rearrangement of part of the lattice. This fact introduces a hierarchy of states in the relaxation dynamics which is relevant in the glassy properties of the model.[22] We can conclude that this model presents ingredients of systems with glass behavior: a complex structure of metastable states and constrained dynamics. It is worthwhile to emphasize that here there is no quenched disorder in the Hamiltonian. As we will see such a scenario is strongly confirmed by numerical simulations of the dynamics in the presence of noise.

The existence of strong dynamic constraints in configuration space is a microscopic feature which leads to the macroscopic phenomenon known as *slow relaxation*. Such behavior has been observed in many systems like spin glass compounds, polymers, granular superconductors, etc. In order to check this phenomenon in the JJ ladder we study the Langevin relaxational dynamics[23]

$$\dot{\theta}_i(t) = -\Gamma \frac{\partial H}{\partial \theta_i} + \lambda_i(t), \tag{7}$$

where H is defined by Eq. (3) and we use $\Gamma = 1$ and $J_x = 1$; λ_i is an additive thermal noise in the phases and satisfies $\langle \lambda_i(t) \rangle = 0$, $\langle \lambda_i(t) \lambda_j(t') \rangle = 2T\delta_{ij}\delta(t - t')$.

We look for the relaxation of commensurate as well as random vortex configurations for different values of the temperature T. For each one we have computed the density of vortices as a function of time and followed the time evolution of the corresponding vortex spatial configuration. In these simulations we use ladders with 400, 2000, and 5000 plaquettes to avoid finite size effects and periodic boundary conditions are used in the longitudinal direction. The results presented here have been obtained for $f = 0.25$ and $h = 0.5$ ($J_y = J_x$). The ground state for those parameter values has zero vortex density, $\omega = 0$, and the configuration space shows there an extremely complex structure of metastable states with close energy values.

In the case of an initially ordered configuration, three temperature regimes are found (Fig. 8). At very low values of temperature the state, of course, remains as a metastable configuration. From some higher value of T, which depends on the particular initial state, and until $T \simeq 0.05$ one observes the decay from the initial ordered state to a disordered metastable vortex state. Such state is basically of the same type obtained after the $T = 0$ relaxation of a random phase configuration. For values of temperature above $T = 0.05$ the states relax slowly to the zero-vortex ground-state configuration. Such relaxation for the values of the parameters chosen is observable in the range of temperature from $T \simeq 0.05$ to $T \simeq 0.15$. About this last value of T, thermal activation of vortices and antivortices appears superposed to the purely relaxational effects.

Many classes of functions have been proposed to fit a slow relaxation curve. One of the most general is the KWW (Ref. 24) (Kohlrausch, Williams, and Watts) or two-parameter stretched exponential law, defined by $\bar{n}(t) \sim \exp[-(t/\tau)^{\beta}]$. Slow relaxation corresponds to values of the parameter $\beta < 1$. For $\beta = 0$ logarithmic relaxation is seen. A value of β from 0.5 to 0.7 is common

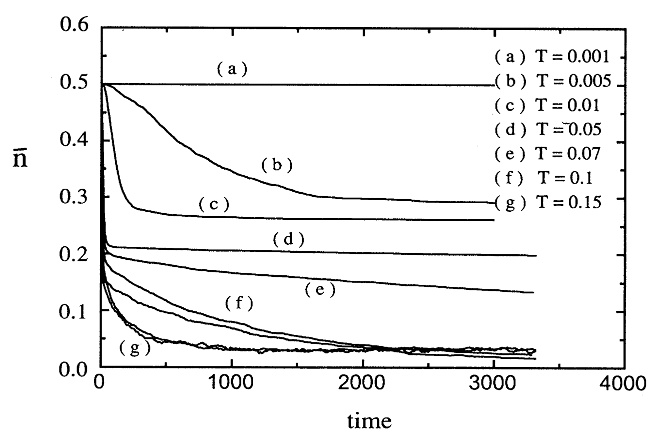

FIG. 8. Time evolution of the mean number of vortices in the ladder, $\bar{n}$, in the relaxation curves at different temperatures. The initial state is the $\omega = 1/2$ ordered state. This state is stable at this value of f and h ($f = 0.25, h = 0.5$). At low values of T no relaxation is seen. For values of T below $T \simeq 0.05$ the state decays to a new metastable one. At temperatures above this value it decays slowly to the $\omega = 0$ ground state. At highest temperatures the thermal generation of vortices is dominant, being $\langle \bar{n}(t) \rangle_t = f$. In the cases of $T = 0.1$ and $T = 0.15$ we also show the relaxation curves of random initial configuration. Those curves are nearly close to the $\omega = 1/2$ relaxation curves.

in glasses. The fittings of our simulation data give values of β between 0.5 and 0.9, depending on temperature (see Fig. 9). We have not been able to find a functional dependence of the exponent β with the temperature. However $\tau \simeq \exp(\alpha/T)$ as could be expected.[13]

Finally, we have also investigated the microscopic characteristics of the vortex dynamics in the different regimes of the relaxation. At low temperatures the relaxation of the commensurate phases is dominated by the nucleation of new structures compatible with the initial one. The long time state is a metastable one formed by different commensurate structures separated by domain walls, see Fig. 10. The metastable states reached from random phase configurations are essentially the same. At higher temperatures these intermediate metastable states decay to the ground-state configuration. As a result of the characteristics of configuration space in this range of temperature we find slow relaxation. Vortices are slowly (thermally) expelled from the array, see Fig. 11.

At very high temperatures the dynamics is dominated by the thermal generation of vortices and antivortices. Phases are randomly distributed between 0 and 2π and $\bar{n}$ approaches to f.

IV. SUMMARY AND CONCLUDING REMARKS

In this paper we have analyzed a Josephson junction ladder in the presence of a perpendicular magnetic field. The ground-state problem of this system is equivalent to the one of a FK model with convex interparticle interaction, which allows to apply the Aubry theory for this class of models. We have calculated exactly the ground-state phase diagram which shows tongues of stability for rational values of the vortex density and a devil's staircase structure. Incommensurate structures exist and can be described by a hull function. Below a critical value h_c the hull function is continuous and the structure is sliding (zero depinning current). Above such value the

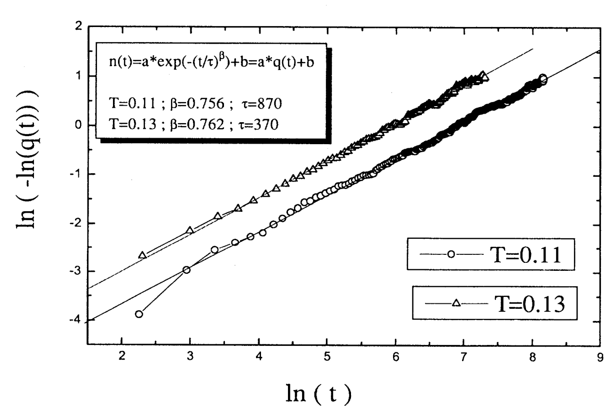

FIG. 9. Slow relaxation curves with random initial conditions have been adjusted using the two-parameter stretched exponential law in the range of temperatures between 0.05 and 0.15. Different exponents between 0.5 and 0.9 are found. ($h = 0.5, f = 0.25$.)

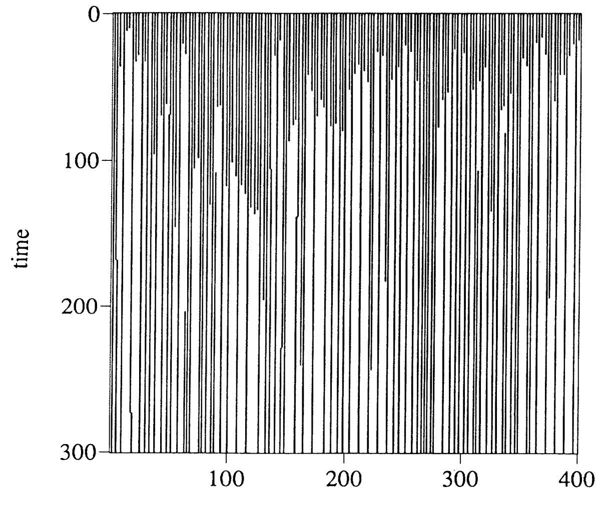

FIG. 10. Time evolution of a vortex configuration in the ladder, $n_j(t)$, showing nucleation processes. Such processes dominate the "low temperature" relaxations of ordered structures. Here the $\bar{n}(t = 0) = 1/2$ ordered state decays to a disordered metastable state, see Fig. 8. In the picture a black mark represents a vortex. ($h = 0.5, f = 0.25, T = 0.01$.)

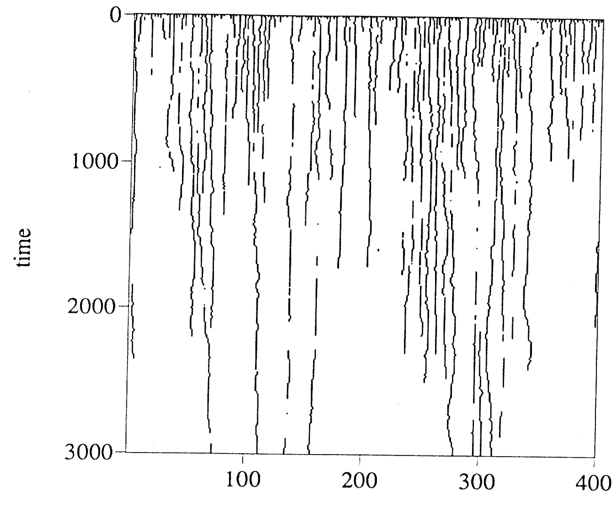

FIG. 11. Time evolution of a vortex configuration in the ladder, $n_j(t)$, showing slow relaxation. Such processes dominate the "mean temperature" relaxations. Here the initial state is a $\bar{n}(t = 0) = 1/2$ ordered state which decays quickly to a disordered metastable state, see Fig. 8, in the manner shown in Fig. 10. Then, such states decay slowly towards the corresponding ground state, in this case ($h = 0.5, f = 0.25, T = 0.1$) and $\omega_{gs} = 0$, not one vortex in the ladder.

hull function develops discontinuities and a nonzero pinning force appears. Modern microlithographic techniques along with methods to detect vortices[25] may check the above results in large JJ ladders.

Relaxation of arbitrary vortex configurations fits to a "slow dynamics" function although no structural disorder is present in the model. Disorder is introduced via initial random configuration and thermal fluctuations. The existence of a complex structure of multiple metastable states and a strongly constrained dynamics in the configuration space are the essential ingredients for this "glassy" dynamics.

The results summarized above have several consequences. Due to commensurability effects, finite size in the ladder direction can produce changes in the dynamic response under dc and/or ac currents.[26] Mismatch in the boundary condition generates defective vortex configurations with a peculiar dynamics.[27] This is an important effect in order to interpret correctly dynamic results in the ladder. Though the equilibrium properties of the JJ ladder are those of a FK model with convex interaction, this equivalence does not hold when dealing with the dynamics of the model. For instance, the dc driven dynamics in the ladder does not, in general, show a unique $V(I)$ as it should occur for a convex model.[28] Finally, if screening currents are considered, the anisotropic JJ ladder can be a good model for long Josephson junctions and stacked JJ.[29] In this way, an extensive study of the Josephson junction dynamics under dc + ac driving currents is in progress.

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