Chiral and spin order in the two-dimensional $\pm J$ XY spin glass: Domain-wall scaling analysis

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This is an analytic study of the two-dimensional XY spin glass with $\pm J$ disorder. The Hamiltonian has a continuous spin symmetry and a discrete chiral symmetry, and therefore possesses, potentially, two different order parameters and correlation lengths. The cost of breaking the symmetries is probed by comparing the ground-state energy under periodic (P) boundary conditions with the one under antiperiodic (AP) and under reflecting (R) boundary conditions. Two energy differences (domain-wall energies) appear, ΔE^{AP} and ΔE^{R}, whose scaling behavior with system size is nontrivially related to the correlation length exponents. For a specific distribution of the $\pm J$ disorder we show that the chiral and spin correlation lengths diverge with the same exponent as T goes down to 0. The common exponent has a common cause, viz. the low reversal energy of domains of chiral variables. For general disorder we give a heuristic argumentation in terms of droplet excitations that leads to spin ordering on a longer, or equal, scale than chiral ordering. These results are in contrast with interpretations of Monte Carlo simulations.

I. INTRODUCTION

Due to the rotational symmetry of the XY model, any of its ground states is necessarily part of a continuum of ground states related by global spin rotations. It was first pointed out by Villain[1] that the ground state of an XY spin glass with random $\pm J$ interactions has, in addition, a twofold degeneracy. The two continua can be deduced from one another by a global spin reflection with respect to an arbitrary axis. They are characterized by opposite "chiralities," that is, by an opposite sense of rotation of the spins as one moves around a plaquette of the lattice. At finite temperature one may have domains belonging to different ground states ("chiral excitations"), so that each plaquette has to be characterized by its own chiral variable.

A number of recent theoretical papers have focused on the role of these chiralities.[2–6] The motivation for this interest is the question of what the lower critical dimension d_ℓ is of the XY spin glass, and by which mechanism this system orders just above d_ℓ. Ozeki and Nishimori[7–9] have developed arguments for the bound $d_\ell \geq 4$ (see also Schwartz and Young[10]), which is supported by substantial numerical evidence (see references in Ref. 9). However, all these authors consider the conventional Edwards-Anderson order parameter associated with the XY spins. Villain's[1] discovery of a discrete symmetry therefore naturally led to the idea, due to Kawamura and Tanemura,[2,3] that in fact d_ℓ might be less than 4. We recall briefly how this idea has been investigated in recent years.

One can define two correlation lengths ξ_c and ξ_s, associated with the chiral variables $q_{\mathbf{r}}$ (where $\mathbf{r}$ denotes the center of a plaquette) and the spin variables $\mathbf{S_i}$ (where $\mathbf{i}$ denotes a lattice site), respectively:

$$\overline{\langle q_{\mathbf{r}} q_{\mathbf{r}+\mathbf{R}}\rangle^2} \sim e^{-R/\xi_c}$$

and

$$\overline{\langle \mathbf{S_i}\cdot\mathbf{S_{i+R}}\rangle^2} \sim e^{-R/\xi_s} \quad (R\to\infty). \tag{1.1}$$

Here $\langle\cdots\rangle$ denotes the thermal and $\overline{\cdots}$ the disorder average. For the low-temperature behavior of these correlation lengths one expects for $d < d_\ell$

$$\xi_c(T) \sim T^{-\nu_c} \quad \text{and} \quad \xi_s(T) \sim T^{-\nu_s} \quad (T\to 0). \tag{1.2}$$

One method to investigate the relation between ξ_c and ξ_s is to determine the correlation length exponents ν_c and ν_s via finite-size scaling of "domain-wall" energies. In simpler cases where there is a single symmetry and a single correlation length with exponent ν, this is done by finding the ground-state energies for two appropriately chosen distinct boundary conditions. The energy difference ΔE will then generally scale with the linear system size N as $\overline{|\Delta E|} \sim N^{-1/\nu}$.

In the present case, with both a continuous and a discrete symmetry, the periodic (P) "reference" boundary conditions can be changed in two ways: to antiperiodic (AP) and to reflecting (R) ones, this latter possibility having been first suggested by Kawamura and Tanemura.[3] Hence there are two energy differences to be considered, ΔE^{AP} and ΔE^{R}. Across an antiferromagnetic seam two spins interact with each other's image under reflection in the origin (of spin space). Hence AP boundary conditions probe the spatial rigidity of spin order, and the spin correlation length exponent ν_s is given by the usual relation

$$\overline{|\Delta E^{\mathrm{AP}}|} \sim N^{-1/\nu_s}. \tag{1.3}$$

Across a reflecting seam two spins interact with each

other's image under reflection about a fixed axis (in spin space). Since the chiral variables appear only when the "spin waves" are integrated out—and notwithstanding the obvious link between R boundary conditions and chirality—it is not *a priori* clear exactly how these boundary conditions probe the chiral order. In fact the relation $\overline{|\Delta E^R|} \sim N^{-1/\nu_c}$ that one might naively expect is certainly not correct. The reason is that R boundary conditions, upon closer inspection, appear to affect also the spin-wave degrees of freedom. This was first realized with remarkable intuition by Kawamura and Tanemura,[3] who then proceeded and extracted the chiral correlation length exponent ν_c from a heuristic expression involving both ΔE^{AP} and ΔE^R.

On the basis of Monte Carlo determinations of the ground states of two- and three-dimensional systems these authors conclude that in two dimensions $\nu_s = 1.2 \pm 0.15$ and $\nu_c = 2.6 \pm 0.3$. This implies that the chiral variables have their own length scale and order in larger domains than the spin variables. Similar conclusions, namely, $\nu_s \approx 1$ and $\nu_c \approx 2$, were reached by Ray and Moore,[5] also on the basis of Monte Carlo simulations and a finite-size scaling analysis. These studies suggest that as the dimension d is increased, ν_c^{-1} may vanish before ν_s^{-1} does, so that just above d_ℓ there would be a phase with long-range chiral order but exponentially decaying spin-spin correlations. Indeed this is precisely the scenario that Kawamura and Tanemura[2,3] find in Monte Carlo simulations of the two- and three-dimensional XY spin glass. In recent work on the two- and three-dimensional Heisenberg spin glass Kawamura[11] arrives at fully analogous conclusions. If these are accepted, the XY and Heisenberg vector spin glasses have their lower critical dimensionality between $d = 2$ and $d = 3$.

In order to achieve a better understanding of the interplay between chiral and spin variables, Ney-Nifle, Hilhorst, and Moore[6] recently performed an analytic study of the random $\pm J$ XY spin glass (in its Villain[12] formulation) on the one-dimensional ladder lattice (see also Morris *et al.*[13]) Their conclusion is that the chiral correlation length ξ_c and spin correlation length ξ_s diverge for $T \downarrow 0$ with the same exponent $\nu_c = \nu_s$ (of which they determine the value 1.8999...). The purpose of this work is to extend the methods of Ref. 6 to $d = 2$ and to confront our results with those of Refs. 3 and 5.

Our starting point in Sec. II is the Villain[12] formulation of the XY model with random $\pm J$ interactions. This formulation has the advantage that the chiral variables can easily be defined and can be decoupled from the spin waves. We transform the XY Hamiltonian to a Coulomb gas Hamiltonian with charges $q_{\mathbf{r}}$ that play the role of the chiral variables. The transformation has been known[14–16] for a while, including the fact[1,17] that the charges take half-integer values on the frustrated plaquettes and integer values on the others. Here we take exactly into account all finite-size effects, essential for the finite-size scaling analysis that follows. Our result for the Coulomb gas partition function in the case of periodic (P) and antiperiodic (AP) boundary conditions, and for an arbitrary realization of the disorder, is given by Eqs. (2.38). In both cases the effective Hamiltonian contains, in addition to the Coulomb interaction, a coupling term between the total electric dipole moment and the boundary conditions. This additional term was given without derivation by Fisher, Tokuyasu, and Young who studied the gauge glass model.[18] A derivation was given very recently by Vallat and Beck.[19] They use a different approach[19] from ours, the one presented here being an extension of the standard transformation method[14–16] between the XY model and Coulomb gas. In Secs. III and IV we go further and discuss how these equations are modified in the case of reflecting (R) boundary conditions. In the limit $T \to 0$ the equations reduce to expressions for the ground-state energy, including all its finite-size corrections, *provided* the ground state itself is known or can be plausibly guessed.

Since the ground-state problem for an arbitrary realization of the disorder cannot be solved, we treat in Sec. V a more restricted two-parameter subset of realizations, in which the frustrated plaquettes are placed in a random rectangular array with infinite-range correlation along the y direction. (Disorder of the unidirectionally infinite-ranged type was also the first one considered for the random bond Ising model.[20]) We find a regime of main interest in parameter space in which the XY spin glass has a zero-temperature transition with

$$\nu_s = \nu_c. \tag{1.4}$$

There is a second regime, with a strongly anisotropic spatial distribution of the frustrated plaquettes, in which the model has a low-temperature phase with long-range chiral correlation and, very plausibly, power law decay of spin correlations. This is behavior analogous to that of fully frustrated two-dimensional XY models,[21,22] on which there exists a large literature.

The result (1.4) in the first regime is different from the scenario proposed by Kawamura and Tanemura[2,3] and by Ray and Moore.[5] The basic mechanism responsible for (1.4) is that the ground state can accommodate to a changeover from P to AP boundary conditions by means of the formation of a chiral domain wall, which is energetically lower lying than the continuous spin-wave deformation that naturally comes to mind. On the basis of this example alone we cannot rule out the possibility that our conclusion, Eq. (1.4), is valid only within the restricted class of disorder realizations.

In order to treat the case of general disorder, we construct in Sec. VI a heuristic theory based on the same mechanism. It involves the lowest-lying excitations of the standard Coulomb Hamiltonian which are assumed to be collective charges reversals. We find that

$$\nu_s \geq \nu_c \,. \tag{1.5}$$

This is compatible with (1.4) but contradicts the results of Refs. 2, 3, 5 where the opposite inequality holds. Furthermore, we show that if Eq. (1.5) would hold as a strict inequality, one cannot extract the exponent ν_c from ΔE^R and ΔE^{AP}. To find ν_c, an appropriate quantity would then be the energy difference between the two ground states of the Coulomb Hamiltonian with periodic and reflecting boundary conditions, without any additional

terms.

Our conclusion is that there is no evidence that chiral order extends on a longer length scale than spin order.

II. PARTITION FUNCTION WITH PERIODIC BOUNDARY CONDITIONS

A. Villain XY model with $\pm J$ interactions

We consider an XY model on a finite square lattice, periodic in both directions, with sites $\mathbf{i} = (m, n)$, where $m = 1, \ldots, M$ and $n = 1, \ldots, N$. Each site $\mathbf{i}$ is occupied by a two-component unit vector or "spin" $\mathbf{S_i}$ whose angle $\phi_\mathbf{i}$ with a reference axis takes values in $(-\pi, \pi]$. In the ferromagnetic Villain model[12] two spins $\phi_\mathbf{i}$ and $\phi_\mathbf{j}$ linked by a nearest-neighbor bond $\langle \mathbf{i}, \mathbf{j} \rangle$ have the Boltzmann weight

$$e^{-\beta J V(\phi_\mathbf{i} - \phi_\mathbf{j})} \equiv \sum_{n=-\infty}^{\infty} e^{-\beta J (\phi_\mathbf{i} - \phi_\mathbf{j} - 2\pi n)^2} . \tag{2.1}$$

Here $J > 0$ sets the energy scale. The relation (2.1) implies that the Villain interaction $V(\phi)$ depends on βJ; when β becomes large, only one term on the right-hand side (RHS) of (2.1) will dominate, $V(\phi)$ will tend to

$$V(\phi) = \phi^2 \quad \text{for } |\phi| \leq \pi, \quad \beta = \infty, \tag{2.2}$$

and we may interpret β as the inverse temperature $1/k_B T$. For all β, the sum on n in (2.1) guarantees the periodicity property $V(\varphi) = V(\varphi + 2\pi)$ and the symmetry property $V(\varphi) = V(-\varphi)$. The antiferromagnetic Villain model has $V(\phi_\mathbf{i} - \phi_\mathbf{j} - \pi)$ instead of $V(\phi_\mathbf{i} - \phi_\mathbf{j})$.

Here we wish to consider an XY spin glass with randomly ferro- or antiferromagnetic Villain interactions, that is, with the Hamiltonian

$$\mathcal{H} = J \sum_{\langle \mathbf{i},\mathbf{j} \rangle} V(\phi_\mathbf{i} - \phi_\mathbf{j} - \pi_{\mathbf{ij}}), \tag{2.3}$$

where the sum runs over all nearest-neighbor bonds of the periodic lattice, and the $\pi_{\mathbf{ij}}$ are quenched random variables such that

$$\pi_{\mathbf{ij}} = \left\{ \begin{matrix} 0 \\ \pi \end{matrix} \right\} \text{ with probability } \frac{1}{2}. \tag{2.4}$$

The expression for the canonical partition function of this XY spin glass then is

$$Z_{M,N} = \int_{-\pi}^{\pi} \prod_\mathbf{i} d\varphi_\mathbf{i} \times \sum_{\{n_{\mathbf{ij}}\}} \exp\left(-\beta J \sum_{\langle \mathbf{i},\mathbf{j} \rangle} (\varphi_\mathbf{i} - \varphi_\mathbf{j} - \pi_{\mathbf{ij}} - 2\pi n_{\mathbf{ij}})^2 \right), \tag{2.5}$$

in which the $n_{\mathbf{ij}}$ may be seen as additional dynamical variables, and the argument of the exponential as a new effective Hamiltonian. We shall henceforth write

$$\nu_{\mathbf{ij}} \equiv n_{\mathbf{ij}}/(2\pi), \tag{2.6}$$

which is integer (half-integer) when the interaction between $\varphi_\mathbf{i}$ and $\varphi_\mathbf{j}$ is ferromagnetic (antiferromagnetic).

B. Transformation to a Coulomb gas

1. *From XY spin glass to a solid-on-solid model*

The transformation from an XY model to a solid-on-solid (SOS) model is well known[14,15] and has been extended to various types of random XY models. Here we carefully study the finite-size effects that determine the ground-state energy differences under different boundary conditions. In (2.5) we wish to carry out the integrations on the MN variables $\varphi_\mathbf{i}$. To that end we arbitrarily select a lattice site $\mathbf{i}_0$ and transform to the new variables of integration,

$$\varphi_0 = \varphi_{\mathbf{i}_0}, \tag{2.7a}$$

$$\varphi_{\mathbf{ij}} = \varphi_\mathbf{i} - \varphi_\mathbf{j}. \tag{2.7b}$$

These are $2MN + 1$ in number, and there exist $MN + 1$ relations between them. In order to formulate these, let $\mathbf{r}$, $\mathbf{s}$, ... be the position vectors of the plaquette centers. We adopt the convention that in $\langle \mathbf{i}, \mathbf{j} \rangle$ the site $\mathbf{j}$ is to the right of $\mathbf{i}$ (for a horizontal bond) or above $\mathbf{i}$ (for a vertical bond), which naturally extends across the periodic boundaries. We define furthermore a sum on the bonds $\langle \mathbf{i}, \mathbf{i}' \rangle$ surrounding a plaquette $\mathbf{r}$ by the diagram of Fig. 1 together with the formula

$$\widehat{\sum}^{\mathbf{r}} \varphi_{\mathbf{ii}'} \equiv \varphi_{12} + \varphi_{23} - \varphi_{43} - \varphi_{14}. \tag{2.8}$$

The caret on the summation sign is a reminder of the sign convention in the RHS of (2.8). The variables $\varphi_{\mathbf{ij}}$ of (2.7b) then satisfy the relations

$$\widehat{\sum}^{\mathbf{r}} \varphi_{\mathbf{ii}'} = 0 \bmod 2\pi \quad \text{for all } \mathbf{r}. \tag{2.9a}$$

These are MN relations of which only $MN - 1$ are independent. The two remaining relations correspond to loops around the torus and are

$$\sum\nolimits^{x} \varphi_{\mathbf{ij}} \equiv \sum_{m=1}^{M} \varphi_{(m,n_1),(m+1,n_1)} = 0 \bmod 2\pi, \tag{2.9b}$$

$$\sum\nolimits^{y} \varphi_{\mathbf{ij}} \equiv \sum_{n=1}^{N} \varphi_{(m_1,n),(m_1,n+1)} = 0 \bmod 2\pi, \tag{2.9c}$$

where $\mathbf{i}_1 \equiv (m_1, n_1)$ is another arbitrarily selected lattice site. Upon introducing the variables of integration (2.7)

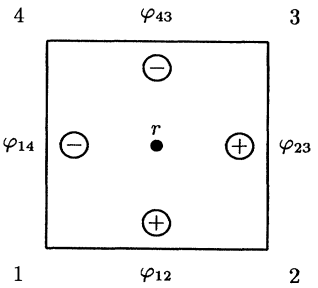

FIG. 1. Diagram indicating the sign convention in the plaquette sum of Eq. (2.8). The indices $1, \ldots, 4$ are shorthand for $\mathbf{i}_1, \ldots, \mathbf{i}_4$.

in Eq. (2.5) we must represent the conditions (2.9) by δ functions and find

$$Z_{M,N} = 2\pi \int_{-\infty}^{\infty} \prod_{\langle \mathbf{i},\mathbf{j}\rangle} d\varphi_{\mathbf{ij}} \prod_{\mathbf{r}\neq\mathbf{r}_0} \delta\left(\widehat{\sum}^{\mathbf{r}} \varphi_{\mathbf{ij}} \bmod 2\pi\right) \times \delta\left(\sum^{x} \varphi_{\mathbf{ij}} \bmod 2\pi\right) \delta\left(\sum^{y} \varphi_{\mathbf{ij}} \bmod 2\pi\right) \times \exp\left[-\beta J \sum_{\langle \mathbf{ij}\rangle} (\varphi_{\mathbf{ij}} - \pi_{\mathbf{ij}})^2\right], \tag{2.10}$$

in which $\mathbf{r}_0$ is an arbitrarily selected plaquette, and where a trivial factor 2π comes from the integration on φ_0. The δ functions may be represented by a Fourier sum with the aid of

$$\delta(x \bmod 2\pi) = \sum_{\ell=-\infty}^{\infty} \delta(x - 2\pi\,\ell) = (2\pi)^{-1} \sum_{\ell=-\infty}^{\infty} e^{i\,\ell\,x}, \tag{2.11}$$

which requires the introduction of a set $\{n_{\mathbf{r}}\}$ of summation variables for the plaquettes, and of two summation variables n_x and n_y for the loops around the torus. One finds

$$Z_{M,N} = (2\pi)^{-MN} \sum_{\{n_{\mathbf{r}}\}}{}' \sum_{n_x} \sum_{n_y} \int_{-\infty}^{\infty} \prod_{\langle \mathbf{i},\mathbf{j}\rangle} d\varphi_{\mathbf{ij}} \exp\left[i \sum_{\mathbf{r}} n_{\mathbf{r}} \widehat{\sum}^{\mathbf{r}} \varphi_{\mathbf{ij}}\right] \times \exp\left[i n_x \sum^{x} \varphi_{\mathbf{ij}} + i n_y \sum^{y} \varphi_{\mathbf{ij}}\right] \exp\left[-\beta J \sum_{\langle \mathbf{i},\mathbf{j}\rangle} (\varphi_{\mathbf{ij}} - \pi_{\mathbf{ij}})^2\right], \tag{2.12}$$

where the prime on the summation sign indicates the restriction to $n_{\mathbf{r}_0} = 0$. The integrations on the $\varphi_{\mathbf{ij}}$ can now be performed.

Let the geometric relation between the pair of lattice sites $\langle \mathbf{i},\mathbf{j}\rangle$ and the pair of plaquette centers $\langle \mathbf{r},\mathbf{s}\rangle$ be as in Fig. 2. It is furthermore useful to define

$$\tau_{\mathbf{rs}}^{x(y)} = \begin{cases} 1 \text{ if } \langle \mathbf{i},\mathbf{j}\rangle \text{ is part of the loop (2.9b) [the loop (2.9c)],} \\ \text{otherwise.} \end{cases} \tag{2.13}$$

Henceforth we shall generally write $\pi_{\mathbf{rs}}$ instead of $\pi_{\mathbf{ij}}$ when $\langle \mathbf{i},\mathbf{j}\rangle$ and $\langle \mathbf{r},\mathbf{s}\rangle$ are related as in Fig. 2. With this notation the result of the $\varphi_{\mathbf{ij}}$ integrations in (2.12) is

$$Z_{M,N} = (2\beta J)^{-MN} \sum_{n_x} \sum_{n_y} \sum_{\{n_{\mathbf{r}}\}}{}' \exp\left[i \sum_{\langle \mathbf{r},\mathbf{s}\rangle} \pi_{\mathbf{rs}} \left(n_{\mathbf{r}} - n_{\mathbf{s}} + \sum_{\alpha=x,y} \tau_{\mathbf{rs}}^{\alpha}\, n_{\alpha}\right)\right] \times \exp\left[-(4\beta J)^{-1} \sum_{\langle \mathbf{r},\mathbf{s}\rangle} \left(n_{\mathbf{r}} - n_{\mathbf{s}} + \sum_{\alpha=x,y} \tau_{\mathbf{rs}}^{\alpha} n_{\alpha}\right)^2\right]. \tag{2.14}$$

Equation (2.14) represents the partition function of a solid-on-solid model (or "column model") with Gaussian interaction, in which, due to the randomness in the original XY model, some terms occur with negative sign. In the nonrandom case with all $\pi_{\mathbf{rs}} = 0$, we recover the partition function of the well-known discrete Gaussian model, *summed* on different step boundary conditions represented by the variables n_x and n_y.

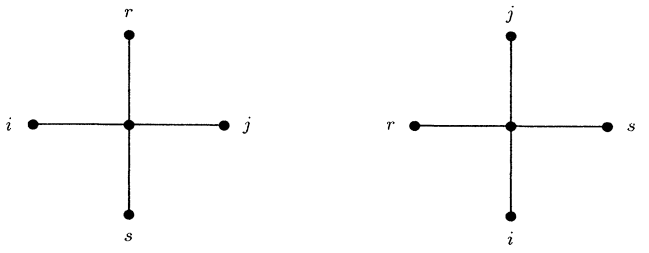

FIG. 2. Geometric relation between the pair of lattice sites $\langle \mathbf{i},\mathbf{j}\rangle$ and the pair of plaquette centers $\langle \mathbf{r},\mathbf{s}\rangle$.

2. *From the SOS model to Coulomb gas*

The transformation from a SOS model to a Coulomb gas is also well known.[16] We have to apply it here to the random SOS model of Eq. (2.14), taking properly into account again all finite-size effects. For a function $f(n)$ of an integer variable n one has

$$\sum_{n=-\infty}^{\infty} f(n) = \sum_{q=-\infty}^{\infty} \int_{-\infty}^{\infty} d\nu\; e^{2\pi i q \nu} f(\nu). \tag{2.15}$$

Applying this identity to the sum on the $\{n_{\mathbf{r}}\}$ in (2.14) we get

$$Z_{M,N} = (2\beta J)^{-MN} \int_{-\frac{1}{2}}^{\frac{1}{2}} d\lambda \sum_{n_x} \sum_{n_y} \sum_{\{q_\mathbf{r}\}} \int_{-\infty}^{\infty} \prod_{\mathbf{r}} d\nu_\mathbf{r}$$
$$\times \exp\Bigg[2\pi i \sum_{\mathbf{r}} q_\mathbf{r} \nu_\mathbf{r} + 2\pi i \lambda \nu_0$$
$$+ i \sum_{\langle \mathbf{r},\mathbf{s} \rangle} \pi_{\mathbf{rs}} \left(\nu_\mathbf{r} - \nu_\mathbf{s} + \sum_\alpha \tau^\alpha_{\mathbf{rs}}\, n_\alpha \right)$$
$$- (4\beta J)^{-1} \sum_{\langle \mathbf{r},\mathbf{s} \rangle} \left(\nu_\mathbf{r} - \nu_\mathbf{s} + \sum_\alpha \tau^\alpha_{\mathbf{rs}}\, n_\alpha \right)^2 \Bigg], \quad (2.16)$$

in which $\nu_0 \equiv \nu_{\mathbf{r}_0}$ and the integral on λ takes care of the condition $n_{\mathbf{r}_0} = 0$ in (2.14). The integrations on the $\nu_\mathbf{r}$ are Gaussian. The most convenient way to carry them out requires some preliminaries. We introduce the coordinate representation $\mathbf{r} = (x, y)$ with $x = 1, \ldots, M$ and $y = 1, \ldots, N$, and the periodicity condition that $(x+M, y)$ and $(x, y+N)$ also denote the plaquette center $\mathbf{r}$. For horizontal and vertical pairs $\langle \mathbf{r}, \mathbf{s} \rangle$ we shall have, by convention, $\mathbf{s} = (x+1, y)$ and $\mathbf{s} = (x, y+1)$, respectively. For each plaquette center $\mathbf{r}$ we define a *frustration variable* $p_\mathbf{r}$ by

$$2\pi\, p_\mathbf{r} = \widehat{\sum}^{\mathbf{r}} \pi_{\mathbf{ii}'} = \pi_{01} - \pi_{20} - \pi_{30} + \pi_{04}, \quad (2.17)$$

where the notation is as in Eq. (2.8), together with Fig. 3.

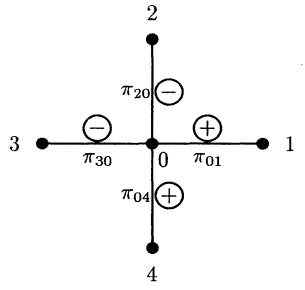

FIG. 3. Diagram indicating the sign convention in the definition (2.17) of the frustration variable $p_\mathbf{r}$. The indices $0, 1, \ldots, 4$ are shorthand for the plaquette centers $\mathbf{r}, \mathbf{r}_1, \ldots, \mathbf{r}_4$.

Let furthermore

$$\Pi_\alpha = \sum_{\langle \mathbf{r},\mathbf{s} \rangle}{}^{\alpha}\, \pi_{\mathbf{rs}}, \quad (2.18)$$

where the subscript $\alpha = x$ (or $\alpha = y$) denotes restriction to vertical (or horizontal) bonds (recall that a horizontal pair $\langle \mathbf{i}, \mathbf{j} \rangle$ corresponds to a vertical pair $\langle \mathbf{r}, \mathbf{s} \rangle$ and vice versa). At this point it is useful to fix the positions of the two loops around the torus by the specific choice

$$\tau^x_{\mathbf{rs}} = \delta_{y,N} \quad \text{for } \langle \mathbf{r}, \mathbf{s} \rangle = \langle (x, y), (x, y+1) \rangle,$$
$$\tau^y_{\mathbf{rs}} = \delta_{x,M} \quad \text{for } \langle \mathbf{r}, \mathbf{s} \rangle = \langle (x, y), (x+1, y) \rangle. \quad (2.19)$$

We pass in Eq. (2.16) from the $\nu_\mathbf{r}$ to new variables of integration $\nu'_\mathbf{r}$ defined by

$$\nu'_\mathbf{r} = \nu_\mathbf{r} + M^{-1} x n_y + N^{-1} y n_x, \quad (2.20)$$

where as before $\mathbf{r}$ runs through (x, y) with $x = 1, \ldots, M$ and $y = 1, \ldots, N$. After rearranging terms in the exponential one finds for the partition function (2.16)

$$Z_{M,N} = (2\beta J)^{-MN} \int_{-\frac{1}{2}}^{\frac{1}{2}} d\lambda \sum_{n_x} \sum_{n_y} \exp\Big[-2\pi i \lambda (n_x + n_y) + i \left(M^{-1} \Pi_y n_y + N^{-1} \Pi_x n_x \right)$$
$$- (4\beta J)^{-1} \left(N^{-1} M n_x^2 + M^{-1} N n_y^2 \right) \Big]$$
$$\times \sum_{\{q_\mathbf{r}\}} \exp\left[-2\pi i \sum_\mathbf{r} q_\mathbf{r} (M^{-1} x n_y + N^{-1} y n_x) \right] \times Z'_{M,N}, \quad (2.21a)$$

with

$$Z'_{M,N} = \int_{-\infty}^{\infty} \prod_\mathbf{r} d\nu'_\mathbf{r}\, \exp\left[2\pi i \sum_\mathbf{r} \nu'_\mathbf{r} (q_\mathbf{r} + p_\mathbf{r} + \lambda \delta_{\mathbf{r},\mathbf{r}_0}) - (4\beta J)^{-1} \sum_{\langle \mathbf{r},\mathbf{s} \rangle} (\nu'_\mathbf{r} - \nu'_\mathbf{s})^2 \right]. \quad (2.21b)$$

We shall abbreviate

$$Q_\mathbf{r} = q_\mathbf{r} + p_\mathbf{r} + \lambda \delta_{\mathbf{r},\mathbf{r}_0}. \quad (2.22)$$

The integrals on the $\nu'_\mathbf{r}$ in Eq. (2.21b) are now easily carried out with the aid of the Fourier variables

$$\widehat{\nu}'_\mathbf{k} = (MN)^{-\frac{1}{2}} \sum_\mathbf{r} e^{-i\mathbf{k}\cdot\mathbf{r}} \nu'_\mathbf{r} \quad (2.23)$$

and analogously defined $\widehat{Q}_\mathbf{k}$, where

$$\mathbf{k} = (k_x, k_y) = 2\pi \left(M^{-1} \kappa_x, N^{-1} \kappa_y \right), \quad (2.24a)$$

$$\kappa_x = 0, 1, \ldots, M-1\,, \quad \kappa_y = 0, 1, \ldots, N-1. \quad (2.24b)$$

It is useful to make the specific choice $\mathbf{r}_0 = (M, N)$. Taking proper care of the integration on $\widehat{\nu}'_\mathbf{0}$, which is excep-

tional and gives a factor $\delta(\widehat{Q}_0)$, one finds from (2.21b)

$$Z'_{M,N} = (MN)^{\frac{1}{2}} \delta\left(\sum_{\mathbf{r}} q_{\mathbf{r}} + \lambda\right) C_{M,N} \times \exp\left[-4\pi^2 \beta J \sum_{\mathbf{k}\neq\mathbf{0}} \lambda_{\mathbf{k}}^{-1} |Q_{\mathbf{k}}|^2\right], \tag{2.25}$$

with

$$\lambda_{\mathbf{k}} = 4\left(\sin^2\frac{k_x}{2} + \sin^2\frac{k_y}{2}\right), \tag{2.26}$$

$$C_{M,N} = \prod_{\mathbf{k}\neq\mathbf{0}} (4\pi\beta J/\lambda_{\mathbf{k}})^{\frac{1}{2}}. \tag{2.27}$$

Upon inserting (2.25) in (2.21a) one may carry out the λ integration with the result

$$Z_{M,N} = (2\beta J)^{-MN} (MN)^{\frac{1}{2}} C_{M,N} \sum_{n_x}\sum_{n_y} \exp\left[i\left(M^{-1}\Pi_y n_y + N^{-1}\Pi_x n_x\right) - (4\beta J)^{-1}\left(N^{-1}Mn_x^2 + M^{-1}Nn_y^2\right)\right] \times {\sum_{\{q_{\mathbf{r}}\}}}' \exp\left[-2\pi i \sum_{\mathbf{r}} \left(M^{-1}x n_y + N^{-1} y n_x\right) - \beta\mathcal{H}_C\left(\{q_{\mathbf{r}} + p_{\mathbf{r}}\}\right)\right]. \tag{2.28}$$

Here the prime restricts the summation to neutral "charge" configurations $\{q_{\mathbf{r}}\}$, i.e., satisfying

$$\sum_{\mathbf{r}} q_{\mathbf{r}} = 0, \tag{2.29}$$

and $\mathcal{H}_C$ is the Coulomb Hamiltonian,

$$\mathcal{H}_C\left(\{q_{\mathbf{r}} + p_{\mathbf{r}}\}\right) = 8\pi^2 J \sum_{\mathbf{r}}\sum_{\mathbf{r}'} U_{M,N}(\mathbf{r} - \mathbf{r}') \times (q_{\mathbf{r}} + p_{\mathbf{r}})(q_{\mathbf{r}'} + p_{\mathbf{r}'}), \tag{2.30}$$

where $U_{M,N}$ is the Coulomb potential on a periodic lattice,

$$U_{M,N}(\mathbf{r}) = \frac{1}{2MN} \sum_{\mathbf{k}\neq\mathbf{0}} \frac{e^{-i\mathbf{k}\cdot\mathbf{r}} - 1}{\lambda_{\mathbf{k}}}. \tag{2.31}$$

This function has the periodicity properties

$$U_{M,N}(\mathbf{r}) = U_{M,N}\left(\mathbf{r} + M\mathbf{e}_1\right) = U_{M,N}\left(\mathbf{r} + N\mathbf{e}_2\right), \tag{2.32}$$

where $\mathbf{e}_1 \equiv (1,0)$ and $\mathbf{e}_2 \equiv (0,1)$.

The last step needed is to transform the sums on n_x and n_y in (2.28) according to Eq. (2.15), by introducing the continuous variables ν_x and ν_y, and new summation variables q_x and q_y. The integrations on ν_x and ν_y are again Gaussian and easily carried out. The result can be slightly rewritten with the aid of the relations

$$M^{-1}\left(\Pi_y - 2\pi\sum_{\mathbf{r}} x q_{\mathbf{r}}\right) = -2\pi M^{-1} P_x + \pi_y, \tag{2.33a}$$

$$N^{-1}\left(\Pi_x - 2\pi\sum_{\mathbf{r}} y q_{\mathbf{r}}\right) = -2\pi N^{-1} P_y + \pi_x, \tag{2.33b}$$

where π_y and π_x are sums along loops around the torus,

$$\pi_y = \sum_{y=1}^{N} \pi_{(M,y),(1,y)}, \quad \pi_x = \sum_{x=1}^{M} \pi_{(x,N),(x,1)}, \tag{2.34}$$

and $\mathbf{P}$ is the electric dipole moment,

$$\mathbf{P} = (P_x, P_y) = \sum_{\mathbf{r}} \mathbf{r}(q_{\mathbf{r}} + p_{\mathbf{r}}). \tag{2.35}$$

The final result for $Z_{M,N}$ then becomes

$$Z_{M,N} = (2\beta J)^{-MN} (MN)^{\frac{1}{2}} C_{M,N} \times {\sum_{\{q_{\mathbf{r}}\}}}' \Theta_{N/M}\left[N^{-1}P_y - \pi_x/(2\pi)\right] \Theta_{M/N} \times \left[M^{-1}P_x - \pi_y/(2\pi)\right] e^{-\beta\mathcal{H}_C}, \tag{2.36}$$

where

$$\Theta_a(u) = (4\pi\beta J a)^{\frac{1}{2}} \sum_{q=-\infty}^{\infty} \exp\left[-4\pi^2\beta J a (q-u)^2\right] \tag{2.37}$$

has the periodicity property $\Theta_a(u) = \Theta_a(u+1)$, and the prime refers to the charge neutrality condition (2.29).

C. Summary

In view of the relatively technical character of the above transformation, and for easy later reference, we summarize here the result. The partition function $Z_{M,N}$ [see Eq. (2.5)] of the $\pm J$ Villain XY model [Eqs. (2.3) and (2.4)] can be expressed as the partition function of a system of Coulomb charges $\{q_{\mathbf{r}}\}$ on the dual lattice,

$$Z_{M,N} = 2\pi(2\beta J)^{-MN+1} (MN)^{\frac{1}{2}} C_{M,N} \times \sum_{q_x,q_y=-\infty}^{\infty} {\sum_{\{q_{\mathbf{r}}\}}}' e^{-\beta\hat{\mathcal{H}}(q_x,q_y,\{q_{\mathbf{r}}\})}, \tag{2.38a}$$

where

$$\hat{\mathcal{H}}(q_x,q_y,\{q_{\mathbf{r}}\}) = 4\pi^2 JNM^{-1}\left[q_y + N^{-1}\sum_{\mathbf{r}} y(q_{\mathbf{r}}+p_{\mathbf{r}}) + (2\pi)^{-1}\sum_{x=1}^{M}\pi_{(x,N),(x,1)}\right]^2$$
$$+4\pi^2 JN^{-1}M\left[q_x + M^{-1}\sum_{\mathbf{r}} x(q_{\mathbf{r}}+p_{\mathbf{r}}) + (2\pi)^{-1}\sum_{y=1}^{N}\pi_{(M,y),(1,y)}\right]^2$$
$$+8\pi^2 J\sum_{\mathbf{r}}\sum_{\mathbf{r}'} U_{M,N}(\mathbf{r}-\mathbf{r}')(q_{\mathbf{r}}+p_{\mathbf{r}})(q_{\mathbf{r}'}+p_{\mathbf{r}'})\ . \tag{2.38b}$$

The Coulomb potential $U_{M,N}$ and the constant $C_{M,N}$ are given by (2.31) and (2.27), respectively, together with (2.26) and (2.24). The $q_{\mathbf{r}}$ run through all integer values subject to the charge neutrality condition (2.29). The sums on $\mathbf{r}$ and $\mathbf{r}'$ run through (x,y) with $x = 1,\ldots,M$ and $y = 1,\ldots,N$. Each nearest-neighbor bond $\langle \mathbf{r},\mathbf{s}\rangle$ on the charge lattice is dual to a nearest neighbor bond $\langle \mathbf{i},\mathbf{j}\rangle$ of the original spin lattice, and the disorder variables $\pi_{\mathbf{rs}}$ that appear in (2.38) are equal to the corresponding $\pi_{\mathbf{ij}}$. The frustration variables $p_{\mathbf{r}}$ are defined in terms of the $\pi_{\mathbf{rs}}$ by the diagram of Fig. 3; they are half-integer for a frustrated plaquette, and integer for an unfrustrated plaquette. Even though the first two terms in (2.38b) seem to favor a definite coordinate representation of the lattice, one may verify with the aid of some algebra that $Z_{M,N}$ is invariant under translation of the origin, as of course it should be.

The transformation from the ferromagnetic two-dimensional (2D) XY model to the Coulomb gas Hamiltonian $\mathcal{H}_C$, including the charge neutrality condition, has been known for a while.[15] Similarly the replacement of the charges $q_{\mathbf{r}}$ by $q_{\mathbf{r}} + p_{\mathbf{r}}$ in the random case has been known since Villain[1] and was treated in a more general context in Ref. 17. Equation (2.38) show how on a *finite lattice* the charge configurations receive a supplementary weight, represented by the first two terms in (2.38b), which depend on the electric dipole moment, in agreement with Refs. 18, 19. Whereas this weight plays no role for a bulk system in the thermodynamic limit, it is essential for finite-size scaling considerations, and therefore [see Eqs. (1.2) and (1.3)] for the determination of the low-temperature behavior of the correlation lengths. This fact will be exploited in the example of Sec. V and in the general heuristic theory of Sec. VI.

III. ANTIPERIODIC BOUNDARY CONDITIONS

In view of the remark preceding Eq. (2.3), it is very simple to change the periodic boundary conditions in the system discussed above into antiperiodic boundary conditions (we shall always take these along the seam joining the Mth and the first column). It amounts to changing $\pi_{\mathbf{ij}}$ into $\pi_{\mathbf{ij}} + \pi$ on the antiferromagnetic seam. This just means drawing another member of the class of random systems under consideration. In particular, frustrated (unfrustrated) plaquettes remain frustrated (unfrustrated). The corresponding antiperiodic partition function $Z^{\mathrm{AP}}_{M,N}$ differs from Eq. (2.36) only in that $\pi_x/(2\pi)$ is replaced by $\pi_x/(2\pi) + \frac{1}{2}$. This difference is at the origin of the finite-size effect of interest.

The following example, although trivial, shows how ground-state energy differences are extracted from the final equations of Sec. II. We compare the ground states of an XY ferromagnet with periodic and antiperiodic boundary conditions. The energy difference ΔE^{AP} is that of a spin wave of wavelength $2M$, which can be written down immediately:

$$\Delta E^{\mathrm{AP}} = NMJ\left(\frac{2\pi}{2M}\right)^2 = \pi^2\frac{N}{M}J. \tag{3.1}$$

To see how the same result can be obtained from Eq. (2.36) we use that for both types of boundary conditions all $p_{\mathbf{r}}$ vanish, and in the ground state all $q_{\mathbf{r}}$ as well, and furthermore that the periodic boundary conditions have $\pi_x = 0$ and the antiperiodic ones $\pi_x = \frac{1}{2}$. This gives

$$\Delta E^{\mathrm{AP}} = -\lim_{\beta\to\infty}\frac{1}{\beta}\ln\Theta_{N/M}\left(\frac{1}{2}\right), \tag{3.2}$$

which in view of (2.37) exactly coincides with (3.1).

The Θ functions in Eq. (2.36) or, equivalently, the first two terms in Eq. (2.38b) can be traced back mathematically to the global constraint on the spin variable differences $\varphi_{\mathbf{ij}}$. This example shows that they represent the energy of a continuous spin-wave deformation forced into the system by the boundary conditons. We shall extend this interpretation to the case of general disorder, and speak of a *global spin wave.*

Ground-state energy differences between P and AP boundary conditions for more complicated situations can also be obtained from (2.36) or (2.38), at least in those cases where the ground state of the charge system is known or can be plausibly guessed. A nontrivial example is discussed in Sec. V.

IV. REFLECTING BOUNDARY CONDITIONS

Reflecting boundary conditions are introduced into the XY Hamiltonian by letting the spins $\mathbf{S}_{(1,n)}$ in the first lattice column interact with the images of the $\mathbf{S}_{(M,n)}$ in the last column under reflection about an axis in spin space. The counterpart of the periodic Hamiltonian (2.3) is

$$\mathcal{H}^R = J\sum_{\langle \mathbf{i},\mathbf{j}\rangle}^{\mathrm{reg}} V(\varphi_{\mathbf{i}} - \varphi_{\mathbf{j}} - \pi_{\mathbf{ij}})$$
$$+J\sum_{\langle \mathbf{i},\mathbf{j}\rangle}^{\mathrm{exc}} V(\varphi_{\mathbf{i}} + \varphi_{\mathbf{j}} - \pi_{\mathbf{ij}})\,. \tag{4.1}$$

Its partition function will be denoted by $Z^R_{M,N}$, the upper index R indicating, here and henceforth, reflecting boundary conditions. The subscript "reg" in (4.1) refers to the regular terms and the subscript "exc" to the exceptional ones, modified by the reflecting boundary conditions.

A. Transformation to a Coulomb gas

The conversion of the Hamiltonian (4.1) into a Coulomb gas Hamiltonian proceeds via the same succession of transformations as in the case of periodic boundary conditions. However, several differences occur, and we shall indicate the main modifications below.

(i) When transforming to the variables of Eq. (2.7), it is convenient to choose $\mathbf{i}_0 = (1,1)$. The expressions for $\varphi_{\mathbf{i}} + \varphi_{\mathbf{j}}$ in the exceptional terms in (4.1) then become

$$\varphi_{(M,n)} + \varphi_{(1,n)} = \varphi_{(M,n),(1,n)} - 2\sum_{\ell=2}^{n} \varphi_{(1,\ell-1),(1,\ell)} + 2\varphi_0 \qquad (n = 1,2,\ldots,N) \quad . \quad (4.2)$$

(ii) The integration on φ_0 can be carried out only after the introduction of the plaquette and loop variables $\{n_{\mathbf{r}}\}$, n_x, and n_y, and leads to the result $2\pi\delta_{n_x,0}$. The integrations on the $\varphi_{(M,n),(1,n)}$ and $\varphi_{(1,n-1),(1,n)}$ are exceptional.

(iii) The discrete Gaussian partition function, obtained as an intermediate result, now becomes

$$Z^R_{M,N} = (2\beta J)^{-MN} \sum_{n_y} \sum_{\{n_{\mathbf{r}}\}}{}' \exp\left[i \sum_{\langle \mathbf{r},\mathbf{s}\rangle}^{\text{reg}} \pi_{\mathbf{rs}}(n_{\mathbf{r}} - n_{\mathbf{s}}) - (4\beta J)^{-1} \sum_{\langle \mathbf{r},\mathbf{s}\rangle}^{\text{reg}} (n_{\mathbf{r}} - n_{\mathbf{s}})^2 \right] \times \exp\left[i \sum_{\langle \mathbf{r},\mathbf{s}\rangle}^{\text{exc}} \pi_{\mathbf{r},\mathbf{s}}(-n_{\mathbf{r}} - n_{\mathbf{s}} + n_y) - (4\beta J)^{-1} \sum_{\langle \mathbf{r},\mathbf{s}\rangle}^{\text{exc}} (n_{\mathbf{r}} - n_{\mathbf{s}} - n_y)^2 \right], \quad (4.3)$$

in which, as before, $\mathbf{r} = (x,y)$ with $x = 1,\ldots,M$ and $y = 1,\ldots,N$, and the exceptional terms are the horizontal bonds linking the Mth to the first column; we have chosen $\tau^y_{\mathbf{rs}} = 1$ if $\langle \mathbf{r},\mathbf{s}\rangle$ is exceptional (and hence $\tau^y_{\mathbf{rs}} = 0$ if $\langle \mathbf{r},\mathbf{s}\rangle$ is regular); and the prime restricts the summation to configurations $\{n_{\mathbf{r}}\}$ with $n_{(M,N)} = 0$. The exceptional terms show that between the columns M and 1, in addition to the step boundary condition represented by the variable n_y, a reflection is imposed with respect to the zero level of the column heights.

(iv) Upon continuing the succession of transformations, one finds that the integral on ν_y gives $2\delta(2q_y + \sum_{bfr} q_{\mathbf{r}} + \lambda)$. After integration over λ this becomes $2\delta(2q_y - \sum_{\mathbf{r}} q_{\mathbf{r}}, 0)$ where $\delta(a,b) \equiv \delta_{a,b}$, and after summation on q_y one gets $2\delta(\sum_{\mathbf{r}} q_{\mathbf{r}} \bmod 2, 0)$; i.e., the sum of the charges $q_{\mathbf{r}}$ should be even.

(v) The relevant wave vectors are now

$$\mathbf{k} = (k_x, k_y) = 2\pi\left(M^{-1}\left(\kappa_x + \frac{1}{2}\right), N^{-1}\kappa_y \right), \quad (4.4)$$

with κ_x and κ_y as in (2.24b); this set of wave vectors will be referred to by the upper index R on summation and product signs. We define in particular

$$C^R_{M,N} = \prod_{\mathbf{k}}{}^R (4\pi\beta J/\lambda_{\mathbf{k}})^{\frac{1}{2}} . \quad (4.5)$$

(vi) The variable $q_{\mathbf{r}} - \delta_{x,M}\pi_{\mathbf{rs}}/\pi$, which appears in the Coulomb Hamiltonian, may be renamed $q_{\mathbf{r}}$ by a shift of the variables $q_{(M,y)}$.

The final result is

$$Z^R_{M,N} = 2(2\beta J)^{-MN} C^R_{M,N} \sum_{\{q_{\mathbf{r}}\}}{}^{\text{par}} e^{-\beta \mathcal{H}^R_C}, \quad (4.6)$$

where the subscript "par" refers to the parity condition

$$\left(\sum_{\mathbf{r}} q_{\mathbf{r}} + \pi_y/\pi \right) \bmod 2 = 0 \quad (4.7)$$

and where

$$\mathcal{H}^R_C(\{q_{\mathbf{r}} + p_{\mathbf{r}}\}) = 8\pi^2 J \sum_{\mathbf{r}} \sum_{\mathbf{r}'} U^R_{M,N}(\mathbf{r} - \mathbf{r}') \times (q_{\mathbf{r}} + p_{\mathbf{r}})(q_{\mathbf{r}'} + p_{\mathbf{r}'}) \quad (4.8)$$

with $\mathbf{r}$ and $\mathbf{r}'$ in the range $(x,y), x = 1,\ldots,M$, $y = 1,\ldots,N$, and with

$$U^R_{M,N}(\mathbf{r}) = \frac{1}{2MN} \sum_{\mathbf{k}}{}^R \frac{e^{i\mathbf{k}\cdot\mathbf{r}}}{\lambda_{\mathbf{k}}}. \quad (4.9)$$

This function has the (anti)periodicity properties

$$U^R_{M,N}(\mathbf{r}) = U^R_{M,N}(\mathbf{r} + N\mathbf{e}_2) = -U^R_{M,N}(\mathbf{r} + M\mathbf{e}_1) . \quad (4.10)$$

Hence two charges at a fixed finite distance on opposite sides of the reflecting boundary interact with each other's charge-conjugated image: Reflecting boundary conditions for the XY spins lead to charge-conjugating boundary conditions for the Coulomb charges.

We now comment on this result and compare it to its counterpart for P and AP boundary conditions, Eq. (2.36). The fact that in the partition function (4.6) no charge neutrality is imposed becomes understandable if one writes the interaction U^R as

$$U^R_{M,N}(\mathbf{r}) = U_{2M,N}(\mathbf{r}) - U_{2M,N}(\mathbf{r} + M\mathbf{e}_1), \quad (4.11)$$

which is easily checked. It is as though we have a double system, of size $2M \times N$ and with periodic boundary conditions, in which every charge at a site $\mathbf{r}$ of the original system is paired up with an image charge, equal but of opposite sign, at the corresponding site $\mathbf{r} + M\mathbf{e}_1$. Using Eq. (4.11) and obvious symmetry properties one can make this explicit by writing the Hamiltonian (4.8) as

$$\mathcal{H}_C^R(\{q_\rho + p_\rho\}) = 4\pi^2 J \sum_\rho \sum_{\rho'} U_{2M,N}(\boldsymbol{\rho} - \boldsymbol{\rho}') \times (q_\rho + p_\rho)(q_{\rho'} + p_{\rho'}), \tag{4.12}$$

where $\boldsymbol{\rho}$ and $\boldsymbol{\rho}'$ run through the $2M \times N$ lattice and it is understood that corresponding sites carry opposite charges. The extra prefactor $\frac{1}{2}$ that (4.12) has with respect to (4.8) indicates that the energy density has been smeared out over twice the volume. The self-interaction present in (4.8) has become the interaction energy between a charge and its image in (4.12).

V. EXAMPLE

A. Specific expressions for the disorder: Ground state and ground-state energy with P boundary conditions

As an example we consider an $M \times N$ lattice containing a rectangular array of $M' \times N'$ frustrated plaquettes, consisting of the points $\mathbf{r}_{a,b} \equiv (x_a, y_b)$ with $a = 1, 2, \ldots, M'$ and $b = 1, 2, \ldots, N'$, where $y_b = bN/N' \equiv b\ell_y$ with ℓ_y an even integer, and the x_b are drawn randomly and independently with a density $M'/M \equiv 1/\ell_x$. The density of frustrated plaquettes is therefore $1/\ell_x\ell_y$. (Note that drawing the frustrated plaquettes randomly is not the same thing as drawing the negative bonds randomly;[24,25] the latter procedure creates pairs of frustrated plaquettes, leading to a Coulomb gas with quenched dipoles rather than quenched charges.) We let $Q^0_{\mathbf{r}}$ denote the ground-state value of $q_{\mathbf{r}} + p_{\mathbf{r}}$. The arguments of this section rest on the hypothesis that we can find the true ground state. For M' and N' even we may safely assume that the ground state has the checkerboard array of charges

$$Q^0_{(x_a,y_b)} = \pm\frac{1}{2}(-1)^{a+b} \tag{5.1}$$

and zero charges on all the other plaquettes. The ground state value of $\mathcal{H}_C$ therefore is, from (5.1) and (2.30),

$$\mathcal{H}_C(\{Q^0_{\mathbf{r}}\}) = 2\pi^2 J \sum_{a,b} \sum_{a',b'} U_{M,N}(\mathbf{r}_{a,b} - \mathbf{r}_{a',b'}) \times (-1)^{a+a'+b+b'}. \tag{5.2}$$

Upon using the explicit form of the lattice Coulomb potential, Eqs. (2.31) and (2.26), we can rewrite (5.2) as a sum of interaction energies between the charge-carrying lattice columns,

$$\mathcal{H}_C(\{Q^0_{\mathbf{r}}\}) = \pi^2 J \ell_y^{-2} N \sum_{a,a'} (-1)^{a+a'} \sum_\nu V_\nu(x_a - x'_a), \tag{5.3}$$

where ν runs through the values $\pm\frac{1}{2}, \pm\frac{3}{2}, \ldots, \pm(\frac{1}{2}\ell_y - \frac{1}{2})$, and in which, with $c_\nu \equiv \cos 2\pi\nu\ell_y^{-1}$,

$$V_\nu(x) = \frac{1}{M}\sum_{k_x} \frac{e^{ik_x x}}{4 - 2\cos k_x - 2c_\nu} \simeq \frac{1}{2}\left[(2-c_\nu)^2 - 1\right]^{-1/2} \times \{2 - c_\nu - [(2-c_\nu)^2 - 1]^{1/2}\}^{|x|}, \tag{5.4}$$

the last step involving the limit $M \to \infty$ at finite $|x|$. This expression takes the much simpler form

$$V_\nu(x) \sim (2\pi|\nu|)^{-1}\ell_y \exp(-2\pi|\nu x|\ell_y^{-1}) \tag{5.5}$$

in the limit $\ell_x, \ell_y \to \infty$ with ℓ_x/ℓ_y fixed and x of order ℓ_x. In this limit, also, the sum on ν in (5.3) can be evaluated exactly and one finds

$$\mathcal{H}_C(\{Q^0_{\mathbf{r}}\}) \simeq -\pi J \ell_y^{-1} N \sum_{a,a'} (-1)^{a+a'} \times \ln\tanh(\pi|x_a - x_{a'}|\ell_y^{-1}). \tag{5.6}$$

This shows that the interaction energy is negative between two neighboring columns (a and $a+1$), that it diverges logarithmically at small distances $|x_a - x_{a'}|$, as expected, and tends to zero as $\exp(-\pi|x_a - x_{a'}|\ell_y^{-1})$ for large $|x_a - x_{a'}|$, so that it is short ranged.

One remark is in place. The limit $\ell_x, \ell_y \to \infty$ corresponds to a *dilute* array of frustrated plaquettes. These must be joined pairwise by long ladders of negative bonds—for which there is no physical reason. The analysis that follows will use Eq. (5.6); however, qualitatively similar results can be obtained from Eqs. (5.3) and (5.4), where ℓ_x and ℓ_y are arbitrary, the key point being that the large distance behavior of the column-column interaction is still determined by the $\nu = \pm\frac{1}{2}$ terms in Eq. (5.3). We shall come back to the case of general ℓ_x and ℓ_y in the discussion in Sec. VI.

B. Comparison of P, R, and AP boundary conditions

We now wish to consider the ground state energies and their differences for the three different boundary conditions P, AP, and R.

1. Periodic boundary conditions

For the frustrated plaquettes located on the array defined in the beginning of this section, we can always choose the negative bonds such that $\pi_x = \pi_y = 0$. Since the ground state $\{Q^0_{\mathbf{r}}\}$ has $P_x = P_y = 0$ [see Eq. (2.35)], the first two terms in (2.38b) are minimized by $q_x = q_y = 0$ and so do not contribute to the ground-state energy, which is therefore entirely given by (5.6).

One possible type of excitations from the ground state are those consisting of reversing all charges $Q^0_{\mathbf{r}}$ in one column a. The excitation energy is of the order of

$$\Delta E(\xi_a) \equiv 2\pi J \ell_y^{-1} N \ln\tanh(\pi\xi_a\ell_y^{-1}), \tag{5.7}$$

where ξ_a is the distance between x_a and the closer one of x_{a-1} and x_{a+1}. Because of its exponential decay with $|x_a - x_{a'}|$, the interaction between a and more distant columns does not change this estimate.

2. *Reflecting boundary conditions*

We first of all have to find the *a priori* unknown ground state. To this end we consider again the checkerboard configuration of charges of Eq. (5.1) and calculate its energy $\mathcal{H}_C^R(\{Q_{\mathbf{r}}^0\})$ under reflecting boundary conditions, starting from Eqs. (4.8) and (4.9). The result is again Eq. (5.6), but with charge-conjugating boundary conditions, the interaction terms between columns a and a' on different sides of the reflecting boundary have an extra minus sign. Since the number M' of columns was supposed even, this means that there is somewhere a mismatch consisting of two neighboring columns a and $a+1$ that are identically instead of oppositely charged. The energy cost is $\sim \Delta E(\xi)$, with ξ the distance between these columns, and can be minimized by shifting the mismatch to the largest interneighbor distance. The typical maximum distance that can be found is

$$\xi_{\max} \simeq \ell_x \ln\left(\frac{M}{\ell_x}\right) + \mathcal{O}(1) \quad (M \to \infty), \tag{5.8}$$

and therefore the ground-state energy difference ΔE^R between reflecting and periodic boundary conditions is of the order of $\Delta E(\xi_{\max})$, which leads to

$$\Delta E^R \sim \left(\frac{N}{\ell_y}\right)\left(\frac{M}{\ell_x}\right)^{-\pi\ell_x/\ell_y}. \tag{5.9}$$

We can now link the finite-size scaling of the ground state to the low-temperature behavior of the correlation length as indicated in the Introduction. Since the mechanism leading to Eq. (5.9) is the reversal of large domains of chiral variables, one deduces that in the case $\pi\ell_x/\ell_y > 1$, there is in the thermodynamic limit $M \sim N \to \infty$ no chiral order at any finite temperature T, and that for $T \downarrow 0$ the chiral correlation length ξ_c diverges with an exponent $\nu_c = 1/(1 - \pi\ell_x/\ell_y)$. In the case $\pi\ell_x/\ell_y < 1$ (which corresponds to relatively strong spatial anisotropy), one deduces similarly that the system has a low-temperature phase with long-range chiral order. We shall come back to both cases after discussing AP boundary conditions.

3. *Antiperiodic boundary conditions*

Under AP boundary conditions the energy of a charge configuration is again given by (2.38b). In this case, $\mathcal{H}_C(\{Q_{\mathbf{r}}^0\})$ is as given by (5.6), but since now $\pi_y = 0$ and $\pi_x = \pi$, the energy of the charge configuration $\{Q_{\mathbf{r}}^0\}$ has an extra contribution $\pi^2 JNM^{-1}$ from the first two terms in (2.38b), which is exactly the spin-wave energy encountered in the discussion of the ferromagnet in Sec. III. One cannot, however, without further inspection, identify this contribution as the ground-state energy difference ΔE^{AP}, since $\{Q_{\mathbf{r}}^0\}$ is not necessarily the ground state any more. The remainder of the discussion depends on the value of ℓ_x/ℓ_y. We discuss first the case $\pi\ell_x/\ell_y > 1$. Curiously, in this case the mechanism to construct a lower-lying state is the same as it was for R boundary conditions, viz., columnwise charge reversal.

Reversing a single column leaves $M^{-1}P_x = 0$ but changes $N^{-1}P_y$ from 0 to $\frac{1}{2}$, which annihilates the effect of $\pi_x = \pi$ and therefore cancels the spin-wave contribution $\pi^2 JNM^{-1}$. The same cancellation can be obtained by reversing the charges in an odd number of otherwise arbitrary columns. The energy cost is the excitation energy of the columns. It can be minimized by choosing an odd number of consecutive columns $a, a+1, \ldots, a'$, such that the distances $|x_{a-1} - x_a|$ and $|x_{a'+1} - x_{a'}|$ are as large as possible. Typically, they will again be of order $\ell_x \ln(M/\ell_x)$. Hence we arrive for ΔE^{AP} at the estimate

$$\Delta E^{\mathrm{AP}} \sim \left(\frac{N}{\ell_y}\right)\left(\frac{M}{\ell_x}\right)^{-\pi\ell_x/\ell_y} \quad (\pi\ell_x/\ell_y > 1), \tag{5.10}$$

identical to ΔE^R. This energy difference is less than the spin-wave energy $\pi^2 JNM^{-1}$, and therefore the ground state will adjust to AP boundary conditions by a chiral excitation, and not by a spin-wave excitation. In view of Eq. (1.3) and our result for ν_c found above we now conclude that $\nu_s = \nu_c = 1/(1 - \pi\ell_x/\ell_y)$. Hence at low temperature the spin and chiral correlation lengths, ξ_s and ξ_c, behave as

$$\xi_s \sim \xi_c \sim T^{-1/(1-\pi\ell_x/\ell_y)} \quad (\pi\ell_x/\ell_y > 1). \tag{5.11}$$

In the case $0 < \pi\ell_x/\ell_y < 1$ (large spatial anisotropy and, as seen above, a low-temperature phase with long-range chiral order), the situation is different. The minimum excitation energy of the columns is still $(N/\ell_y)(M/\ell_x)^{-\pi\ell_x/\ell_y}$, but this is higher than the spin-wave contribution which it cancels. Therefore in this case the ground state $\{Q_{\mathbf{r}}^0\}$ remains unchanged (and in the spin representation acquires only an additional global spin wave), and

$$\Delta E^{\mathrm{AP}} = \pi^2 JNM^{-1} \quad (0 < \pi\ell_x/\ell_y < 1). \tag{5.12}$$

In the thermodynamic limit $M \sim N \to \infty$ it follows [see Eqs. (1.2) and (1.3)] that $\nu_s = \infty$, which is most naturally (although not strictly necessarily) associated with a low-temperature phase with power law decay of the spin-spin correlation. In this case the energy ΔE^{AP} is not the result of a domain wall, but smeared out across the system.

All of the above discussion is for an even number M' of charge-carrying columns. We now comment on the differences that intervene when M' is odd. In that case the value of $N^{-1}P_y$ in (2.36) is $\pm\frac{1}{4}$ and is changed to $\mp\frac{1}{4}$ (modulo 1) under replacement of P by AP boundary conditions. Hence the value of $\Theta_{M/N}$ does not change and (since $M^{-1}P_x = 0$ both before and after) we have $\Delta E^{\mathrm{AP}} = 0$. The interpretation is that a global spin wave is "caught" in the system and cannot release its energy. When we replace P by R boundary conditions, the global spin-wave terms disappear from the Hamiltonian and release their energy so that ΔE^R is negative. The phenomenon of spin-wave energy release was first observed

by Kawamura and Tanemura[3] in their Monte Carlo simulations of an XY model with general disorder. It appears here analytically, but seemingly as a consequence of the distinction between even and odd M' and hence closely tied to the present example. But we shall see in the next section that it also appears in the general case.

VI. GENERAL CASE

On the basis of the experience gained we now attempt a theory, admittedly heuristic, for the general case. Let, as before, $\mathcal{H}_C$ denote the standard lattice Coulomb Hamiltonian (2.30), and $\mathcal{H}_C^R$ its counterpart (4.8) with R boundary conditions. Let in this section $\hat{\mathcal{H}}^P$ denote the Hamiltinian $\hat{\mathcal{H}}$ of Eq. (2.38b), and $\hat{\mathcal{H}}^{\rm AP}$ its counterpart with AP boundary conditions. We shall take the unknown ground state $\mathbf{Q}^0$ of $\mathcal{H}_C$ as our reference state and denote its energy by E_0. Our strategy will be to try to determine how the ground states of $\mathcal{H}_C^R$ and $\hat{\mathcal{H}}^{P,\rm AP}$ differ from $\mathbf{Q}^0$.

Since the remaining discussion concerns the low-temperature regime, it is convenient to work with charges $\pm\frac{1}{2}$ on the frustrated plaquettes and charges zero on all other plaquettes. This simplification, which was also made by Villain,[12] leads to the problem of an Ising model on a random lattice (namely, the one composed of centers of the frustrated plaquettes), with logarithmically increasing antiferromagnetic interactions and zero magnetization. The logarithmic interaction will be screened in a way that we cannot precisely describe (it has to be if the energy per charge is to be finite) and it is reasonable to imagine that there exist some effective interaction decaying as a power law between spatially separated neutral sets of charges.

Assuming that we know the ground state $\mathbf{Q}^0$, the next relevant question is what the lowest-lying excitations are. Clearly all excitations can be described in terms of charge reversals with respect to the ground state $\mathbf{Q}^0$ or, alternatively, in terms of contours on the dual lattice (given any reasonable planar graph representing the nearest-neighbor relations on the lattice of charges). In the spirit of Fisher and Huse[27] we shall consider the lowest-lying excitations involving the reversal of order $\sim L^2$ charges and localized in an area of linear size $\sim 2L$ around a preassigned point in space. Such excitations will be called droplets. Let their excitation energy scale as L^{-1/ν_c}, where, since we are below d_ℓ, the exponent ν_c is positive. Due to the long-range forces, the droplets so defined need not constitute single domains. (For short-range forces they should and reduce to the droplets of Fisher-Huse[27] theory.) A typical scale-L droplet may have to be represented by a set of disconnected contours. We shall assume, nevertheless, that it is compact enough so that there is always, typically, one main contour enclosing $\sim L^2$ charges. This amounts to assuming that a domain wall can be defined, whatever its width, around the reversed area. Excitations that would not fall in this category are, for example, those that consist of many small-size domains dispersed in the volume L^2, or fractal excitations extending throughout the volume L^2 but having fractal dimension less than 2. Our hypothesis is that such reversals have excitation energies at least as high as those of the scale-L droplets.

The ground states $\mathbf{Q}^{0,R}$ and $\mathbf{Q}^0$, of $\mathcal{H}_C^R$ and $\mathcal{H}_C$, respectively, must differ at least by a contour going around the torus in the y direction, that is, by a scale-N droplet. For the energy difference between the two ground states we therefore have

$$E_0^R - E_0 \simeq c\, N^{-1/\nu_c}, \tag{6.1}$$

where c is a (positive or negative) random constant, and ν_c is as before the exponent of the chirality-chirality correlation function; see Eq. (1.2). This gives a way to determine this exponent.

We now turn to the boundary conditions P and AP. Let $\mathbf{Q}^{0,P}$ ($\mathbf{Q}^{0,\rm AP}$) be the ground state of $\hat{\mathcal{H}}^P$ ($\hat{\mathcal{H}}^{\rm AP}$) and let E_0^P ($E_0^{\rm AP}$) be its energy. Since $\hat{\mathcal{H}}^{P,\rm AP}$ differ from $\mathcal{H}_C$ by the *addition* of two quadratic terms, we have necessarily $E_0^{P,\rm AP} - E_0 \geq 0$. In the ground state $\mathbf{Q}^0$ these additional terms take a random positive value of order 1 (at least, if we assume that $M^{-1}P_x$ and $N^{-1}P_y$ do so in the ground state). This value can be reduced by reversing domains of charges in $\mathbf{Q}^0$. The ground state $\mathbf{Q}^{0,P}$ ($\mathbf{Q}^{0,\rm AP}$) is now determined by a compromise between the minimization of $\mathcal{H}_C$ and the minimization of the additional terms, which drive the total electric dipole moment to a specific value. Since this is a global biasing force, and since the excitation energies go down with increasing length scale, we expect that the compromise leads to a $\mathbf{Q}^{0,P}$ ($\mathbf{Q}^{0,\rm AP}$) that differs from $\mathbf{Q}^0$ by an excitation on the scale N of the system. This excitation might not be of the same type as the droplet excitation discussed above, and therefore we put

$$E_0^{P,\rm AP} - E_0 \simeq c^{P,\rm AP}\, N^{-1/\nu_s}, \tag{6.2}$$

where c^P and $c^{\rm AP}$ are positive random constants and the exponent ν_s is also positive; it is the spin correlation length exponent of Eq. (1.2). But since, by hypothesis, the droplet excitations are the lowest-lying ones that exist at each given scale, the excitation that leads to (6.2) is, at best, also a scale-N droplet excitation in which case $\nu_s = \nu_c$, or is possibly a combination of droplet excitations on smaller scales in which case one might have $\nu_s^{-1} \leq \nu_c^{-1}$. Hence

$$\nu_s^{-1} < \nu_c^{-1}. \tag{6.3}$$

Upon combining these considerations we arrive at the conclusion

$$\begin{aligned} \Delta E^{\rm AP} &\simeq (c^{\rm AP} - c^P)\, N^{-1/\nu_s}, \\ \Delta E^R &\simeq -c^P\, N^{-1/\nu_s} - c\, N^{-1/\nu_c}, \end{aligned} \tag{6.4}$$

in which $c^{P,\rm AP} \geq 0$ and c may be of either sign with $\overline{c} = 0$. It follows that $\overline{\Delta E^R} \simeq -\overline{c^P} N^{-1/\nu_s}$, which is negative. This is the phenomenon of the release of spin-wave energy as one passes from P to R boundary conditions. One sees furthermore that

$$\overline{(\Delta E^{\mathrm{AP}})^2}^{1/2} \simeq \overline{(\Delta E^{R})^2}^{1/2} \sim N^{-1/\nu_s}; \tag{6.5}$$

i.e., both energy differences scale with the same power of N.

We now discuss the two possibilities implied by Eq. (6.3). If it holds with the equality sign, we are in the case of a single correlation length for the spin and the chiral variables, as found in the example studied. This case needs no further comment. If Eq. (6.3) were to hold as a strict inequality, then there are two different exponents ν_s and ν_c. This would imply a longer correlation length for the spin variables than for the chiral variables. Moreover, ΔE^R is then not the appropriate quantity to determine ν_c. In fact, in this case ν_c cannot be found from a comparison of P, AP, and R boundary conditions, but should result from a comparison of $\mathcal{H}_C$ and $\mathcal{H}_C^R$ according to Eq. (6.1). Kawamura and Tanemura[3] extracted the exponent ν_s from a numerical determination of ΔE^{AP} and ν_c from a postulated expression involving both ΔE^{AP} and ΔE^R. In this work there is a strict inequality that goes in the sense opposite to (6.3).

The conclusion is that we find no evidence for chiral order extending on a longer length scale than spin order.

VII. FURTHER COMMENTS AND CONCLUSIONS

We have considered the two-dimensional XY spin glass with $\pm J$ interactions. In Secs. II–IV general formulas are presented that exhibit the interplay between chiral and spin variables in determining domain-wall energies on finite $M \times N$ lattices with various types of boundary conditions. In Sec. V we have considered, as an example, a specific type of disorder with infinite-ranged correlations in the y direction. The spatial distribution of the frustrated plaquettes depends on two parameters ℓ_x and ℓ_y, whose ratio is an anisotropy parameter. The scaling properties of the domain-wall energies with system size lead us to the following conclusions.

(i) *XY spin glasses with critical temperature $T = 0$.* For ℓ_x, $\ell_y \to \infty$ and for $\pi\ell_x/\ell_y > 1$, the system of Sec. V has only a $T = 0$ critical point at which the chiral and spin correlation lengths diverge with the same exponent $\nu_s = \nu_c = 1/(1 - \pi\ell_x/\ell_y)$. We add here without proof that the phenomenon of a $T = 0$ transition with $\nu_s = \nu_c$ holds in the entire region of the $\ell_x\ell_y$ plane determined by

$$4\ell_x(\ell_x - 1)\sin^2\frac{\pi}{2\ell_y} > 1 \quad (\ell_x > 1 \text{ and } \ell_y = 1, 2, \ldots). \tag{7.1}$$

This phenomenon is analogous to what we found on the one-dimensional ladder[6] and tube[26] lattices. It is distinct, however, from the scenario proposed by Kawamura and Tanemura[2,3] and by Ray and Moore[5] for the two-dimensional uncorrelated random $\pm J$ XY spin glass, according to which one would have $0 < \nu_c^{-1} < \nu_s^{-1}$. In order to rule out the possibility that our conclusions are restricted to a specific class of correlated disorder, we considered in Sec. VI the general case of random disorder. The heuristic theory presented yields a different inequality, namely, $0 \le \nu_s^{-1} \le \nu_c^{-1}$. We therefore find no evidence for chiral order extending on a longer scale than spin order in two dimensions.

One may now go one step further and speculate that since we had $\nu_s = \nu_c$ for $d = 1$, the mechanism uncovered above, which enables the ground state to accommodate to a change from P to AP boundary conditions by a low-energy chiral excitation, is general for uncorrelated random $\pm J$ XY model, and that $\nu_s = \nu_c$ for all $d < d_\ell$.

(ii) *Random XY models with nonzero critical temperature.* This class encompasses the model of Sec. V when Eq. (7.1) is not satisfied. In this case there is a low-temperature phase with long-range chiral order and, very probably, power law decay of the spin-spin correlations. This is the same scenario that is believed to hold for a fully frustrated two-dimensional XY model,[21–23] which is, as a matter of fact, recovered in the special limit $\ell_x = \ell_y = 1$, and which is characterized by $\nu_c^{-1} < 0 = \nu_s^{-1}$. It cannot, therefore, be identified with the scenario $\nu_c^{-1} < 0 < \nu_s^{-1}$ that Kawamura and Tanemura[3] propose for the *three*-dimensional random $\pm J$ XY model.

Finally we comment on the transition line $4\ell_x(\ell_x - 1)\sin^2(\pi/2\ell_y) = 1$ between the regions (i) and (ii). Clearly the distinction between these two types of ground-state behavior is not due to the density of frustrated plaquettes, which is $1/\ell_x\ell_y$, but to the nature of correlations in their spatial arrangement. The different behaviors can be distinguished by the ground-state response to a change between P and AP boundary conditions. In region (ii), the ground state is elastic (it deforms continuously), whereas in region (ii) it is "brittle" (it responds by the formation of a chiral domain wall). This transition appears in this work somewhat as a by-product of the analysis, but it is interesting in itself and merits a separate study.

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