Line-delocalization transitions in the presence of quenched disorder

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We classify universality classes and all possible critical singularities for line-delocalization transitions in a space of arbitrary dimensionality in the presence of quenched disorder uncorrelated along the line direction but otherwise general. The situation under consideration involves both one-line unbinding (e.g., the wetting transition in two dimensions) and many-line delocalization (e.g., the appearance of Abrikosov flux lines near the lower critical field in disordered type-II superconductors). In particular, we find that *any* unbinding transition from a short-ranged pinning potential in two dimensions is characterized by a "classical" jump in the specific heat. We also reproduce many of the results of R. Lipowsky and M. E. Fisher [Phys. Rev. Lett. **56**, 472 (1986)] and characterize their ranges of validity. We find, however, that the crossover exponent for the two-dimensional critical wetting transition in the presence of random bond disorder is 5, rather than 4. For the cases in which comparison is possible our results are in agreement with exact replica calculations.

I. INTRODUCTION

Phase transitions involving linear objects (e.g., domain walls and interfaces in two spatial dimensions, Abrikosov flux lines, polymers, and dislocations in crystals) are a subject of great importance in many branches of condensed matter physics. Apart from its intrinsic theoretical interest, the understanding of these phenomena has practical relevance. This is especially true for phase transitions in the presence of disorder, since this is a common feature of most experimental systems.

In this paper we will present a detailed description of a wide class of phase transitions involving line objects in a random medium. All the cases discussed can be regarded as line-delocalization transitions, which can be classified into two large groups.

The first group are the phase transitions involving only a single line in an external potential. The typical situation here is that the external potential is either binding or not, depending on its parameters, temperature, line stiffness, and nature of the disorder (if any). An example of such a transition is the wetting transition[1] in two spatial dimensions.

Another group involves phase transitions at which many lines appear at once as a phase transition point is passed. Examples here are the commensurate-incommensurate transition[2,1] in two spatial dimensions (the phase transition is approached by changing the free energy of the domain wall dividing commensurate domains of different registry via tuning the chemical potential), and the destruction of the Meissner state by Abrikosov vortices[3] in type-II superconductors (the transition is approached by decreasing the flux-line free energy by increasing the external magnetic field). If the line free energy is positive, the appearance of lines is costly, and the equilibrium phases have no line defects; if the line free energy becomes negative, spontaneous formation of lines takes place and only effective repulsion between them makes this transition continuous.

Our present theoretical understanding of one-line delocalization transitions in pure systems is quite complete and based mainly on the analysis of a relatively simple class of models using the solid-on-solid (SOS) approximation in which one treats the line as a structureless geometrical object placed in an external potential. An exact renormalization-group (RG) analysis of the corresponding SOS Hamiltonian has been given[4] for a symmetric pinning potential of general form and arbitrary number of space dimensionalities. However, many of the results concerning critical behavior near the one-line depinning transition can be obtained from heuristic scaling and random-walk type of arguments,[1,5] which give prominence to the concept of line wandering in the resulting physical picture.

One-line delocalization transitions in the presence of quenched disorder are less well understood. The heuristic arguments used in the pure case[1] can be extended;[1,6] in the case of the two-dimensional critical wetting transition from a short-range pinning potential in the presence of random-bond disorder, the results were supported by Kardar's exact solution[7] and by simulation studies.[8] However, Kardar's solution involves the replica trick[9] with its ill-controlled limit to zero replicas, and the numerical studies can be interpreted as supporting larger values of the exponents. Recent analytic and numerical work[10–13] concerns the issue of a bulk depinning transition, i.e., when the pinning potential is symmetric. The results for critical exponents are different from those found in Refs. 1 and 6–8, even though the universality principle[14] implies that they should be the same, as Kardar[7] already claimed. Thus the consensus opinion is not clearly yet on completely firm ground.

The weakest point of the heuristic arguments[6,1] is that

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they treat the random problem as being the same as the pure one but with a different value of the wandering exponent, seemingly ignoring the importance of the order of averagings in random systems. Kardar's scaling explanation[7] of his replica result suffers from a similar shortcoming: the replica answer can be obtained from purely scaling arguments only by *ignoring* temperature rescaling which is a general feature of random systems and the origin of a hyperscaling violation.[1,15] This does not yet mean that the answers coming from heuristic consideration are wrong, since scaling arguments by themselves do not suffice to give the correct critical behavior. However, if the heuristic arguments do give correct results, it must mean that the exact RG equations describing the one-line delocalization transition exhibit a nontrivial cancellation of temperature rescaling. We will demonstrate that this is the case for a wide class of random systems, and thus explain how it is that the heuristic arguments can work.

Many-line delocalization transitions are more complicated than the one-line transitions, just as many-body problems in quantum mechanics are more difficult than one-body problems. Nevertheless, the picture of many-line delocalization transitions in pure systems is now quite complete. The first exact solution of the commensurate-incommensurate transition in two dimensions was given by Pokrovsky and Talapov[16] and later confirmed by others.[17] A physical explanation of these results based on the concept of a long-range interline repulsion mediated by line wandering has been given by Fisher and Fisher[18] and Nattermann;[19] they also generalized the approach to the case of domain walls. The many-line delocalization transition in three dimensions has been analyzed in the context of the destruction of the Meissner state by Abrikosov vortices by Nelson[3] using random-walk arguments and by Nelson and Seung[20] through a connection to the Edwards model of interacting polymers.[21] Nelson[3] and Nelson and Seung[20] have noticed a powerful analogy between the dilute limit of the Abrikosov vortex state and the ground-state properties of a quantum many-body Bose system[22] and exploited this analogy mainly in the context of type-II superconductors. Finally, a RG theory of the ground-state properties of a dilute system of interacting bosons has been worked out for general dimensionality and interparticle interaction,[23] giving as a by-product a general treatment of the many-line delocalization problem.

The presence of quenched disorder makes the problem of many-line delocalization very complicated. Heuristic arguments[1,24,15] extending the ideas of the pure case[18,19] again produce a variety of results; however, the degree of confidence now is even less than that in the case of one-line delocalization. There is an exact replica solution of the commensurate-incommensurate transition in two dimensions in the presence of random-bond disorder,[25,7] which supports the heuristic arguments. There have been two phenomenological attempts at approaching the problem in general dimensionality. Nelson and Le Doussal[26] generalized the RG approach of Refs. 22 and 20, treating the disorder perturbatively. This, however, does not allow us to study the immediate vicinity of the phase transition where nonperturbative effects are very important. Another attempt that avoids using the replica method and goes beyond the perturbative regime was undertaken by Nattermann, Feigelman, and Lyuksyutov.[27] In the case of the commensurate-incommensurate transition in two dimensions in the presence of random-bond disorder their result differs from that of Ref. 25 by the presence of an extra logarithmic correction. They also found some limitations on the range of validity of heuristic arguments.

This review shows that the line-delocalization transitions in a random medium are still an open and controversial problem and some general approach avoiding the replica method is necessary. We will try to provide this, using a generalization of our previous analysis[4,23] of line-delocalization transitions in pure systems. For the case of many-line delocalization it is similar to that of Ref. 27. However, we will show that the RG equations of that paper miss an important contribution, which alters some of its conclusions.

This paper is organized as follows. Section II introduces essential physical notions and ideas and gives a scaling analysis of the delocalization transitions of a single line from a short-range pinning potential in the presence of point disorder. The derivation of RG equations describing the one-line delocalization transitions from a generic pinning potential in the presence of disorder uncorrelated along the average line direction (but otherwise general) is given in Sec. III. Section IV analyses these equations for a special choice of the bare parameters of the problem but including the presence of a scale-invariant tail of the pinning potential, finding all the possible critical singularities. Section V gives the form of the corrections to scaling, and in Sec. VI we find exponents associated with the presence of external fields destroying the phase transition. Section VII deals with the calculation of the critical exponents and amplitudes in the presence of a short-range pinning potential and random-bond disorder, which can be compared with the replica results. In Sec. VIII we show how the RG equations of Sec. III can be modified to describe the delocalization transitions of a system of many lines. Section IX uses these equations to analyze many-line delocalization transitions for a special choice of the bare parameters of the problem and a scale-invariant tail of the interline potential, finding all the possible critical singularities. In Sec. X we discuss the case that the long-range tail of the interline potential is not scale invariant. Section XI deals with the calculation of the critical exponents and amplitudes for the case of a short-range interline interaction and random-bond disorder, which can be compared with the replica results. The concluding section of the paper summarizes our results and comments on other approaches to the problem.

II. SCALING ANALYSIS OF THE DELOCALIZATION TRANSITIONS OF A SINGLE LINE FROM A SHORT-RANGE PINNING POTENTIAL IN THE PRESENCE OF POINT DISORDER

Let us consider a directed line object imbedded in a $(d+1)$-dimensional space and subject to both the pinning potential $V_0\delta(\mathbf{x})$ and a random potential $V_r(\mathbf{x},t)$, where $\mathbf{x}$ is the d-dimensional vector denoting the line position and t is the coordinate along the line. The corresponding SOS Hamiltonian has the form[1]

$$H=\int dt\left\{\frac{m}{2}\left[\frac{d\mathbf{x}}{dt}\right]^2+V_0\delta(\mathbf{x})+V_r(\mathbf{x},t)\right\}, \tag{2.1}$$

where m is the line stiffness. In this section we restrict ourselves to the case of the random potential uncorrelated in all directions:

$$\langle V_r(\mathbf{x},t)\rangle=0,\quad \langle V_r(\mathbf{x},t)V_r(0,0)\rangle=\Delta_0\delta(t)\delta(\mathbf{x}), \tag{2.2}$$

where $\langle\cdots\rangle$ denotes the disorder averaging and Δ_0 is the degree of disorder. In the absence of an external pinning potential that violates translational symmetry, the statistical properties of a directed polymer are described by the concept of line wandering.[1,15] When one end of a polymer is fixed, then the root-mean-square displacement x of the free end is related to the polymer length t as[1,15]

$$x\cong At^{\zeta}, \tag{2.3}$$

where A is an amplitude and ζ is the wandering exponent. In the absence of disorder they are given by the familiar random-walk expressions[1]

$$A=(T/m)^{1/2},\quad \zeta=\tfrac{1}{2}, \tag{2.4}$$

where T is the temperature. For the case of point uncorrelated disorder it is known that $\zeta=\frac{2}{3}$ for $d=1$ dimensions.[28,7,25,29] For general d there are heuristic arguments[30] leading to the conclusions

$$\zeta=\frac{2}{2+d}, \tag{2.5}$$

$$A=(\Delta_0/mT)^{1/(d+2)}, \tag{2.6}$$

valid for $d<2$; for $d>2$ and sufficiently weak disorder the wandering exponent takes on its thermal value $\zeta=\frac{1}{2}$. For the case $d=1$, Eqs. (2.5) and (2.6) reproduce the exact expressions for both the exponent ζ and the amplitude A. There is a proof[31] that (2.5) is an analytic continuation of the exact result $\zeta(d=1)=\frac{2}{3}$ for $1\le d<2$, and below we will give a series of arguments supportive to the claim that Eqs. (2.5) and (2.6) are exact.

The effect of a weak pinning well can be studied using probabilistic arguments. A long polymer will return to the origin with accumulated probability that is proportional to $\int dt\,\delta(\mathbf{x})$; using the definition (2.3), we can estimate the integral, getting $t^{1-\zeta d}$ or, equivalently, $x^{(1/\zeta)-d}$. If $1-\zeta d<0$, the return probability goes to zero for large t, implying that a weak pinning potential will fail to localize the polymer. However, a sufficiently deep pinning potential will localize the line, thus indicating that there is a delocalization-localization transition for a symmetric pinning potential for some nonzero value of the pinning strength. When $1-\zeta d>0$, then the return probability is of order unity, implying that an arbitrarily weak pinning potential localizes the polymer. The return probability arguments set the lower critical dimensionality at $\zeta d=1$, and for the pure problem ($\zeta=\frac{1}{2}$) they give a qualitatively correct picture confirmed by the exact calculation.[4]

These arguments also give us a hint at the form of the critical singularities near the line-delocalization transition. This transition is anisotropic so that there are different correlation lengths $\xi_\parallel$ and $\xi_\perp$ for correlations along the line and transverse to it, with differing critical behavior; however, they are not independent[1] and are related by the line wandering exponent [see (2.3)]: $\xi_\perp\cong A\xi_\parallel^{\zeta}$. This implies that there is a relationship $\nu_{\perp 0}=\zeta\nu_{\parallel 0}$ between the corresponding exponents where the subscript "0" means that the pinning well is short ranged. The correlation lengths $\xi_\parallel$ and $\xi_\perp$ have the meaning that on length scales $t<\xi_\parallel$ and $x<\xi_\perp$ the polymer is essentially free and is not sufficiently altered by the presence of the localizing potential; but on scales exceeding the correlation lengths the line is localized, and the effect of the pinning well cannot be neglected. Since the accumulated return probability behaves as $t^{1-\zeta d}$ [or, equivalently, $x^{(1/\zeta)-d}$] then the correlation length exponents are given by

$$\nu_{\perp 0}^{-1}=(\zeta\nu_{\parallel 0})^{-1}=|\zeta^{-1}-d| \tag{2.7a}$$

below an upper critical dimensionality. For the pure problem ($\zeta=\frac{1}{2}$) Eq. (2.7a) gives the exact exponents[4] valid up to the upper critical dimensionality $d=4$. Setting the lower critical dimensionality by the condition $\nu_{\perp 0}^{-1}=0$ correctly gives $d=2$. For the random-bond disorder in $d=1$ (for which $\zeta=\frac{2}{3}$), Eq. (2.7a) reproduces the exact replica result.[7] For $d=1$ and arbitrary disorder (general ζ) Eq. (2.7a) coincides with the heuristic expression of Refs. 6 and 1.

The criterion setting the lower critical dimensionality at $\zeta d=1$ can also be obtained by comparing typical values of the second and third terms of (2.1). The typical value of V_r comes from (2.2) and (2.3), and is of order $(\Delta_0/A^d t^{\zeta d+1})^{1/2}$, while the typical value of the second term of (2.1) is of order $V_0/A^d t^{\zeta d}$. A weak pinning potential is irrelevant whenever

$$(\Delta_0/A^d t^{\zeta d+1})^{1/2}\gg V_0/A^d t^{\zeta d},$$

as $t\to\infty$. This will always be the case if $\zeta d-1>0$.

One more supportive argument is to say that the relevance of the pinning potential is governed by the scaling behavior of the typical value of $H_1=\int dt\,V_0\delta(\mathbf{x})$, which is proportional to the accumulated return probability.

As was mentioned by Kardar[7] the criterion we just received can be obtained in a more formal language as a result of the scaling transformation

$$x=bx',\quad t=b^{1/\zeta}t',\quad V_0=b^{d-(1/\zeta)}V_0'. \tag{2.8a}$$

Equation (2.8a) reveals the weakest (formal) point of the above arguments: the temperature rescaling has been left out even though it should be present in disordered systems.

Temperature rescaling can be incorporated into the heuristic arguments.[13] Well-separated optimal paths for a directed polymer in a random medium are characterized by a free-energy difference $\Delta F\cong mA^2t^{2\zeta-1}$, which can be obtained by estimating the elastic energy term in (2.1) with the help of (2.3). Then one may say that the right quantity to consider is the ratio to ΔF of the typical value of $H_1=\int dt\, V_0\delta(\mathbf{x})$. It behaves as $t^{2-\zeta(d+2)}$ or $x^{(2/\zeta)-d-2}$ thus implying that a weak pinning well is irrelevant for $2-\zeta(d+2)<0$, marginal for $2-\zeta(d+2)=0$ (relevant to the case $d=1$ in the presence of point disorder, where $\zeta=\frac{2}{3}$), and strongly relevant for $2-\zeta(d+2)>0$. This argument implies the following expression for the correlation length exponents:

$$\nu_{\perp 0}^{-1}=(\zeta\nu_{\parallel 0})^{-1}=\left|\frac{2}{\zeta}-d-2\right| , \tag{2.7b}$$

which can also be obtained as a consequence of the scaling transformation

$$\begin{gathered}x=bx',\quad t=b^{1/\zeta}t',\quad T=b^{2-(1/\zeta)}T' ,\\ V_0=b^{d+2-(2/\zeta)}V_0'\end{gathered} \tag{2.8b}$$

For the pure problem ($\zeta=\frac{1}{2}$) the results (2.7a) and (2.7b) as well as the transformations (2.8a) and (2.8b) are identical. References 10–13 propose that Eq. (2.7b) describes the critical singularities near a one-line delocalization transition. For the important case of random-bond disorder in $d=1$, substitution of $\zeta=\frac{2}{3}$ into (2.7b) gives $\nu_{\perp 0}^{-1}=0$, thus implying an essential singularity typical for a marginal dimensionality instead of ordinary power-law dependence.

The scaling arguments by themselves do not determine the critical behavior, and one of the goals of the present paper is to show via explicit calculation that (2.7a) is the correct answer for some range of parameters.

There is yet another heuristic argument favoring (2.7a) based on a version of the Harris criterion.[32] In the vicinity of the depinning transition for the pure problem the correlation lengths are related by $\xi_\perp\sim\xi_\parallel^{1/2}$. Weak disorder will affect the "pure" critical singularities if the integral $\int dt\, V_r(\mathbf{x},t)$ inside the correlation volume $\xi_\perp^d\xi_\parallel$ is large. Using (2.2) and (2.3) with $\zeta=\frac{1}{2}$ one finds that $\int dt\, V_r(\mathbf{x},t)$ behaves as $\xi_\parallel^{(2-d)/4}$, implying that weak disorder does not change the critical singularities of the pure depinning transition for $d>2$, changes them for $d<2$, and is marginal at $d=2$. The dimensionality $d=2$ is simultaneously the lower critical dimension for the pure problem. On the other hand, Eq. (2.7b) predicts that $d=1$ is the lower critical dimension for the random-bond disorder ($\zeta=\frac{2}{3}$). Then taking (2.7b) for granted would imply that there are two lower critical dimensionalities $d=1$ and $d=2$. This implies nonmonotonic behavior of the inverse correlation exponents as a function of space dimensionality for $1\le d\le 2$ and seems highly unlikely.

The argument based on the Harris criterion is consistent with the return probability argument since the wandering exponent as given by (2.5) takes on its thermal value in two dimensions.

III. ONE-LINE DELOCALIZATION TRANSITIONS

A. General structure of the renormalization-group equations

Here we consider the generalization of (2.1) to a more general pinning potential $V(\mathbf{x})$ and random potential $V_r(\mathbf{x},t)$:

$$H=\int dt\left\{\frac{m}{2}\left[\frac{d\mathbf{x}}{dt}\right]^2+V(\mathbf{x})+V_r(\mathbf{x},t)\right\} . \tag{3.1}$$

We will assume that the random potential has zero mean and is not correlated along the t direction:

$$\langle V_r(\mathbf{x},t)\rangle=0,\quad \langle V_r(\mathbf{x},t)V_r(\mathbf{0},0)\rangle=\Delta_0\delta(t)R(\mathbf{x}) , \tag{3.2}$$

where $R(\mathbf{x})$ is a function dependent on the kind of disorder; we assume also that $R(\mathbf{x})$ depends only on the absolute value of $\mathbf{x}$.

We will consider only the symmetric pinning potentials $V(\mathbf{x})=V(-\mathbf{x})$. According to the universality principle,[14] this will also give us the critical singularities for a wider class of pinning potentials having the same long-distance behavior; furthermore, the problem of line delocalization from a symmetric pinning potential is interesting in its own right, since no exact solution is available in two (i.e., $1+1$) spatial dimensions: Kardar's replica solution[7] corresponds to the line delocalization from an asymmetric pinning potential, and the generalization of his method to the case of a symmetric potential apparently is not possible.

The partition function $Z(\mathbf{x},t)$ for a line ending at $(\mathbf{x},t)$ satisfies the Schrödinger-like equation[28]

$$-T\frac{\partial Z}{\partial t}=\left\{-\frac{T^2}{2m}\nabla^2+V(\mathbf{x})+V_r(\mathbf{x},t)\right\}Z , \tag{3.3}$$

where ∇^2 is the d-dimensional Laplacian. The nonlinear transformation $Z=e^h$ brings Eq. (3.3) into the form introduced in Refs. 29:

$$\frac{\partial h}{\partial t}=\frac{T}{2m}\nabla^2 h+\frac{T}{2m}(\nabla h)^2-\frac{V_r(\mathbf{x},t)}{T}-\frac{V(\mathbf{x})}{T} . \tag{3.4}$$

It differs in that the last term is present. As in Refs. 29 we can make progress by going to the Fourier version of (3.4):

$$\begin{aligned}h(\mathbf{k},\omega)=&\,G(\mathbf{k},\omega)[V_r(\mathbf{k},\omega)+2\pi\delta(\omega)V(\mathbf{k})]/T\\ &-\frac{T}{2m}G(\mathbf{k},\omega)\int\frac{d\Omega\, d^dq}{(2\pi)^{d+1}}\mathbf{q}\cdot(\mathbf{k}-\mathbf{q})h(\mathbf{q},\Omega)\\ &\times h(\mathbf{k}-\mathbf{q},\omega-\Omega) ,\end{aligned} \tag{3.5}$$

where the wave-vector- and frequency-dependent quantities are the corresponding Fourier transforms defined as

$$h(\mathbf{x},t)=\int\frac{d\omega\, d^dk}{(2\pi)^{d+1}}h(\mathbf{k},\omega)\exp[i(\mathbf{k}\mathbf{x}-\omega t)] , \tag{3.6}$$

and

$$G^{-1}(\mathbf{k},\omega)=\frac{T}{2m}k^2-i\omega \tag{3.7}$$

is the inverse bare propagator. Let us define the effective propagator $\mathcal{G}(\mathbf{k},\omega)$ and the potential $V_{\text{eff}}(\mathbf{k})$ as

$$h(\mathbf{k},\omega)\equiv\mathcal{G}(\mathbf{k},\omega)[V_r(\mathbf{k},w)+2\pi\delta(\omega)V_{\text{eff}}(\mathbf{k})]/T\ . \tag{3.8}$$

The substitution of Eq. (3.8) into the integral equation (3.5) leads to a complicated expression with three sorts of terms: the terms that are quadratic in V_{eff}, which have $\delta(\omega)$ frequency dependence, terms that are quadratic in V_r which clearly do not, and then there is a cross term

$$-\frac{1}{mT}G(\mathbf{k},\omega)\int\frac{d^dq}{(2\pi)^d}\mathbf{q}\cdot(\mathbf{k}-\mathbf{q})\mathcal{G}(\mathbf{q},0)V_{\text{eff}}(\mathbf{q})\times\mathcal{G}(\mathbf{k}-\mathbf{q},\omega)V_r(\mathbf{k}-\mathbf{q},\omega)\ . \tag{3.9}$$

We can show that the cross term does not contribute terms proportional to $\delta(\omega)$ by squaring Eq. (3.9) and performing the configurational averaging with the help of Eq. (3.2). This leads to something proportional to $\delta(\omega)$ [the important point here is the absence of correlations along the t direction in Eq. (3.2)], which implies that (3.9) has a singularity at $\omega=0$ which is weaker than $\delta(\omega)$. Collecting the $\delta(\omega)$ terms we get

$$\mathcal{G}(\mathbf{k},0)V_{\text{eff}}(\mathbf{k})=G(\mathbf{k},0)\left[V(\mathbf{k})+\frac{1}{2m}\int\frac{d^dq}{(2\pi)^d}\mathbf{q}\cdot(\mathbf{k}-\mathbf{q})\mathcal{G}(\mathbf{q},0)\mathcal{G}(\mathbf{k}-\mathbf{q},0)V_{\text{eff}}(\mathbf{q})V_{\text{eff}}(\mathbf{k}-\mathbf{q})\right]\ . \tag{3.10}$$

Let us seek $\mathcal{G}(\mathbf{k},\omega)$ is the limit $\mathbf{k},\omega\to 0$ in the form

$$\mathcal{G}^{-1}(\mathbf{k},\omega)=\frac{\mathcal{T}}{2m}k^2-i\omega\ , \tag{3.11}$$

with some effective diffusion constant parametrized by an effective temperature $\mathcal{T}$. The value of $\mathcal{T}$ is determined by the finite-frequency equation for $\mathcal{G}(\mathbf{k},\omega)$ which was not written down. We argue that its behavior is determined by the discussion given in Refs. 29. This involves the assumption that in the low-$(\mathbf{k},\omega)$ limit the presence of the cross term (3.9) in the equation for $\mathcal{G}(\mathbf{k},\omega)$ does not affect the effective temperature and thus can be ignored. Physically, this means that the pinning potential in (3.1) does not change the statistical properties of the medium, which are set entirely by the interplay between the thermal and disorder fluctuations. On the other hand, the properties of the medium do affect the pinning potential and this is the effect to be studied in this paper. We do not have decisive formal arguments in favor of this assumption and will check it later by comparison with available results.

Substituting (3.11) into (3.10), and going to the limit $\mathbf{k}\to\mathbf{0}$ we get an equation for the effective potential,

$$V_{\text{eff}}(\mathbf{k})=\frac{\mathcal{T}}{T}\left[V(\mathbf{k})-\frac{2m}{\mathcal{T}^2}\int\frac{d^dq}{(2\pi)^d}\frac{V_{\text{eff}}^2(\mathbf{q})}{q^2}\right]\ . \tag{3.12}$$

Without disorder, the relationship $\mathcal{T}=T$ holds, and Eq. (3.12) reduces to the starting point of Ref. 4.

The attempt to solve (3.12) iteratively immediately leads to the conclusion[4] that the lowest-order corrections will be divergent at the small-q limit of the corresponding integrals at low space dimensions. To lowest order in the bare potential we will have $V_{\text{eff}}(\mathbf{k})=\mathcal{T}V(\mathbf{k})/T$. Substituting this back into Eq. (3.12) we will have to second order in the pinning potential

$$V_{\text{eff}}(\mathbf{k})=\frac{\mathcal{T}}{T}\left[V(\mathbf{k})-\frac{2m}{T^2}\int\frac{d^dq}{(2\pi)^d}\frac{V^2(\mathbf{q})}{q^2}\right]\ . \tag{3.13}$$

Whenever $V(\mathbf{q}\to\mathbf{0})$ is nonzero the integral in (3.13) diverges for $d\le 2$. Then we would have to include an infinite set of terms to go beyond the perturbative regime. Similar divergences occur in the consideration of the equation for $\mathcal{G}(\mathbf{k},\omega)$ which determines the effective temperature $\mathcal{T}$.

These divergences can be treated in a systematic way by using renormalization-group methods.[14] Let us integrate over the short-wavelength degrees of freedom having wave vectors in the interval from $\Lambda(1-dl)$ to Λ, where Λ is a momentum cutoff set either by the short-range part of the potential $V(\mathbf{x})$ or by the function $R(\mathbf{x})$ (3.2) (for the sake of simplicity we assume that they have the same correlation length), and dl is the infinitesimal. For the effective temperature the results must have the form

$$\mathcal{T}=T[1+f_d(\Delta,\Lambda,m,T)dl]\ , \tag{3.14}$$

where $f_d(\Delta,\Lambda,m,T)$ is some function dependent on the space dimensionality, the type and the degree of disorder, short-distance cutoff, the line stiffness, and the original temperature acting in the system of modes with wave vectors between zero and Λ. Although the function f_d is known only in the one-loop approximation,[29] the critical exponents governing the line-delocalization transition do not depend on its specific form. Substituting (3.14) in (3.13) and performing the integration over a shell of thickness $\Lambda\,dl$ we get to the lowest order in dl

$$V_{\text{eff}}(k)=V(k)[1+f_d(\Delta,\Lambda,m,T)dl]-\frac{2mK_d\Lambda^{d-2}V^2(\Lambda)dl}{T^2}-\frac{2mK_d}{T^2}\int_0^{\Lambda(1-dl)}dq\,q^{d-3}V^2(q)\ , \tag{3.15}$$

where $K_d=S_d/(2\pi)^d$ and $S_d=2\pi^{d/2}/\Gamma(d/2)$ is the surface area of a d-dimensional unit sphere; upon arriving at (3.15) we also assumed that the pinning potential depends only on the absolute value of the wave vector. The presence of the temperature renormalization factor f_d in this equation is an important difference between our work and that of others.[27,10,11]

Equation (3.15) can be used to define the renormalized potential

$$V^R(k)=V(k)+\left[V(k)f_d(\Delta,\Lambda,m,T)-\frac{2mK_d\Lambda^{2-d}}{T^2}V^2(\Lambda)\right]dl\ . \qquad (3.16)$$

This demonstrates that disorder leads to the multiplicative renormalization of the external potential (it changes the properties of the medium where the line is placed) in addition to the generation of a short-range contribution characteristic of the pure system.[4]

Let us consider a pinning potential comprised of both a short-range part proportional to $\delta_a(x)$ [here $\delta_a(x)$ corresponds to any well-localized function having width $a=\Lambda^{-1}$ that transforms into the mathematical δ function as $a\to 0$] and a long-range tail of the form $V_s/|\mathbf{x}|^s$, where V_s is the amplitude and s is some exponent. The Fourier transform of this potential in the low-k limit takes the form[4]

$$V(k)=V_0+\frac{V_sAk^{s-d}}{d-s}\ ,$$
$$A=2^{d-s+1}\pi^{d/2}\Gamma\left[\frac{d-s}{2}+1\right]\Big/\Gamma(s/2)\ . \qquad (3.17)$$

Substitution of the expression (3.17) into (3.16) gives the renormalized values for V_0^R and V_s^R:

$$V_0^R=V_0+V_0f_d(\Delta,\Lambda,m,T)dl-\frac{2mK_d\Lambda^{d-2}}{T^2}\left[V_0+\frac{V_sA\Lambda^{s-d}}{d-s}\right]^2dl\ , \qquad (3.18)$$

$$V_s^R=V_s+V_sf_d(\Delta,\Lambda,m,T)dl\ . \qquad (3.19)$$

To complete the description of the effect of exclusion of the short-wavelength degrees of freedom we have to write down the analog of Eqs. (3.14), (3.18), and (3.19) for the parameters of the disorder function $R(\mathbf{x})$ [(3.2)]. In contrast to the cases of the temperature and the pinning potential, however, this cannot be given in general form and we will have to go into the details. In Sec. VII we will give the discussion for random-bond disorder. However, many general results already follow from the way the pinning potential is coupled to the temperature.

The original form of the Hamiltonian (3.1) is recovered by the scaling transformation (2.8b) along with $V_s=b^{s+2-(2/\zeta)}V_s'$, where now $b=1+dl$ is the scaling factor and the wandering exponent ζ is to be selected to find a fixed point of the resulting RG equations. We also assume the presence of a corresponding scaling rule for disorder which we do not specify. The combination of Eqs. (3.14), (3.18), (3.19), and (2.8b) leads to the set of RG equations

$$\frac{dT(l)}{dl}=\{\zeta^{-1}-2+f_d[\Delta(l),\Lambda,m,T(l)]\}T(l)\ , \qquad (3.20)$$

$$\frac{dV_0(l)}{dl}=\{2\zeta^{-1}-d-2+f_d[\Delta(l),\Lambda,m,T(l)]\}V_0(l)-\frac{2mK_d\Lambda^{d-2}}{T^2(l)}\left[V_0(l)+\frac{V_s(l)A\Lambda^{s-d}}{d-s}\right]^2\ , \qquad (3.21)$$

$$\frac{dV_s(l)}{dl}=\{2\zeta^{-1}-s-2+f_d[\Delta(l),\Lambda,m,T(l)]\}V_s(l)\ , \qquad (3.22)$$

$$\frac{d\Delta(l)}{dl}=y(\zeta,d)\Delta(l)+z_d[\Delta(l),\Lambda,m,T(l)]\ , \qquad (3.23)$$

where the last equation symbolizes the expectable form of the RG equation(s) for disorder. Δ here can be considered as a vector having several components: we do know from the one-loop calculation[29] that any type of noise under renormalization generates a white-noise contribution similar to Eq. (3.16) that always generates a short-range potential. The function(s) $y(\zeta,d)$ is due to a scaling transformation like (2.8b): it depends only on ζ (type of disorder) and the space dimensionality d. The function(s) $z_d(\Delta,\Lambda,m,T)$ is due to the integration over the high-momentum degrees of freedom.

Introducing the dimensionless variables

$$u(l)=\frac{2mK_d}{T^2(l)a^{d-2}}\left[V_0(l)+\frac{V_s(l)Aa^{d-s}}{d-s}\right]=\frac{2mK_d}{T^2(l)a^{d-2}}U(l)\ , \qquad (3.24)$$

which measures the amplitude of the short-range interaction, and

$$g(l)=\frac{2mK_dA}{T^2(l)a^{s-2}}V_s(l)\ , \qquad (3.25)$$

which is the amplitude of the long-range (r^{-s}) tail, puts Eqs. (3.21) and (3.22) into a form more convenient for analysis:

$$\frac{du}{dl}=[2-d-f_d(\overline{\Delta})]u-u^2+g\ , \qquad (3.26)$$

$$\frac{dg}{dl}=[2-s-f_d(\overline{\Delta})]g\ , \qquad (3.27)$$

where we assume (relying on the one-loop calculation of Ref. 29) that the parameters Δ, Λ, m, and T can be combined into a single dimensionless parameter, the disorder degree $\overline{\Delta}$. These equations together with Eqs. (3.20) and (3.23) are to be solved subject to the initial conditions[4]

$$u_0=u(l=0)=\frac{2mK_d}{T^2a^{d-2}}\left[V_0+\frac{V_sAa^{d-s}}{d-s}\right]\ , \qquad (3.28)$$

$$g_0=g(l=0)=\frac{2mK_dA}{T^2a^{s-2}}V_s\ , \qquad (3.29)$$

while the physical temperature and disorder degree give the initial conditions for Eqs. (3.20) and (3.23).

The RG equations (3.20), (3.26), (3.27), and (3.23) together with the initial conditions (3.28) and (3.29) are the main results of the present section.

B. Properties of the RG equations and critical singularities

Before actually using the RG equations, let us discuss the range of validity of the RG equations (3.26) and (3.27) for the short- and long-range parts of the pinning potential. The derivation came from perturbative RG; however, we claim that they can be used for exact determination of critical singularities. In the absence of disorder (which is the case $f_d=0$) they describe a delocalization transition in a pure system and lead to the correct critical singularities:[4] in fact, these equations with initial conditions (3.28) and (3.29) are equivalent to the exact solution of the radial Schrödinger equation over the large-distance region, with the short-range behavior replaced by a boundary condition. The special character of the coupling between the temperature and potential (3.12) allows us to claim that the only way that disorder influences the pinning potential is through the wandering exponent ζ and the unknown function $f_d(\overline{\Delta})$ in the parts of Eqs. (3.26) and (3.27) that are linear in potential.

In the pure case, the connection between the Schrödinger equation and Eqs. (3.26) and (3.27) enabled us to extract the critical singularities:[4] the function u in (3.26) is related to the spatial behavior of the zero-energy radial wave function $R(r)$ (here r is the radial coordinate) in the following fashion:

$$R=\text{const}\times\exp\int\frac{u(r)dr}{r}\ , \tag{3.30}$$

so that in any case that there is a finite scale l^* at which $u(l\to l^*)=-\infty$ (i.e., $R=0$), the oscillation theorem[33] leads to a localization length

$$\xi_\perp=a\ \exp l^* \tag{3.31}$$

of a negative-energy wave function. Simultaneously, this gives us the behavior of a *transverse* correlation length in the context of the one-line delocalization transition. The explicit connection between one-line delocalization and one-particle quantum mechanics[4] also shows that the initial condition (3.28) is sometimes insufficiently accurate to give the correct critical singularities, but we know how to fix this.[4]

For the pure problem, the connection with the Schrödinger equation gave us a valuable way to check our ideas. For the random problem, the connection with the Schrödinger equation is lost (and just as well, since the Schrödinger equation is now "time" dependent); however, the renormalization ideas are sufficiently well founded in physical ideas that we may confidently assume that the renormalization equations (3.26) and (3.27) together with the initial conditions (3.28) and (3.29) continue to give the spatial behavior of the properly averaged zero-energy wave function.

Now let us look at the RG equations describing the temperature (3.20) and the disorder renormalization (3.23). Their right-hand sides depend on the as yet undetermined wandering exponent, which is determined by a stable finite fixed point of Eqs. (3.20) and (3.23). This can be found from the coupled system of equations

$$\zeta^{-1}-2+f_d(\overline{\Delta}^*)=0\ , \tag{3.32}$$

$$y(\zeta,d)\Delta^*+z_d(\Delta^*,\Lambda,m,T^*)=0\ , \tag{3.33}$$

where $\overline{\Delta}^*$, Δ^*, and T^* are finite, and the asterisks correspond to the fixed-point values.

Finally we have to connect the critical behavior of the free energy per unit length F with the singularity of the transverse correlation length (3.31). In the pure case[4] we used the condition $F=-DT^2/m\xi_\perp^2$ (where D is a constant), having the clear quantum-mechanical meaning of an energy level. Now in the presence of disorder we have a temperature renormalization, so that we are going to use the same expression with the effective temperature $\mathcal{T}$ instead of T. The effective temperature $\mathcal{T}$ is related to the fixed-point value T^* via the inverse scaling transformation [compare with (2.8b)] as follows:

$$\mathcal{T}=T^*(\xi_\perp/a)^{2-(1/\zeta)}\ . \tag{3.34}$$

Therefore the critical singularity of the free energy is given by the formula

$$F=-\frac{DT^{*2}}{ma^2}(\xi_\perp/a)^{2[1-(1/\zeta)]}\ . \tag{3.35}$$

For a pure system ($\mathcal{T}=T^*$, $\zeta=\frac{1}{2}$) this reduces to a known hyperscaling relation.[1] However, for $\zeta\neq\frac{1}{2}$, hyperscaling is violated due to the temperature renormalization. The singularities predicted by Eq. (3.35) are in agreement with both heuristic arguments[6,1] and replica results.[7] The former give the free-energy singularity in the form

$$F=-\text{const}\times mA^{2/\zeta}\xi_\perp^{2[1-(1/\zeta)]}\ . \tag{3.36}$$

The comparison with (3.35) allows us to relate the amplitude A introduced in (2.3) with the fixed-point value T^* as follows:

$$T^*=\text{const}\times mA^{1/\zeta}a^{2-(1/\zeta)}\ . \tag{3.37}$$

For the pure problem when $T^*=T$ and A and ζ are given by (2.4), Eq. (3.37) holds automatically.

C. Critical singularities: specific choice of initial conditions

The solution of Eqs. (3.32) and (3.33) can be presented in the form

$$\overline{\Delta}^*=A_d\ , \tag{3.38}$$

where A_d is some dimension-dependent numerical constant. If one selects the initial dimensionless degree of disorder exactly at the stable fixed point (3.32) and (3.33),

$$\overline{\Delta}_0=\overline{\Delta}^*\ , \tag{3.39}$$

that lies on some curve (manifold) in the disorder-temperature (hyper)plane Δ_0-T, then the right-hand sides of Eqs. (3.20) and (3.23) will always stay at zero values, the fixed-point value of the temperature T^* will always be equal to the physical temperature $T^*=T$, and the physics of the delocalization transition will be determined just by Eqs. (3.26) and (3.27). For any other initial conditions this will still be true only on the largest length scales, where the dimensionless disorder degree is close to its fixed-point value. Clearly, the critical singularities for the special initial conditions (3.39) and general initial conditions will be the same, and only prefactors will be different. As will be seen, the technical advantage of the special choice (3.39) is that it enables us to solve the problem without going into the nature of the disorder, allowing us to parametrize it by the wandering exponent ζ. Substituting the expression for f_d from Eq. (3.32) into Eqs. (3.26) and (3.27), we get

$$\frac{du}{dl}=(\zeta^{-1}-d)u-u^2+g \ , \tag{3.40}$$

$$\frac{dg}{dl}=(\zeta^{-1}-s)g \ . \tag{3.41}$$

The terms of Eqs. (3.40) and (3.41) that are linear in the potential demonstrate the important feature of the problem mentioned in the Introduction: they look like the result of the scaling transformation (2.8a) along with $V_s=b^{s-(1/\zeta)}V_s'$, not involving any temperature rescaling at all: the substitution of f_d from (3.32) into (3.26) and (3.27) canceled out the temperature rescaling. This is why heuristic[1] and purely scaling[7] arguments can still work.

The discussion of the one-line delocalization transition for the special initial condition (3.39) and a general pinning potential is very similar to that in the pure system,[4] so that we will be following Ref. 4 and omitting details which are common for both problems.

IV. DELOCALIZATION TRANSITIONS IN THE PRESENCE OF A SCALE-INVARIANT LONG-RANGE TAIL OF THE PINNING POTENTIAL

Let us start with the marginal case $1/\zeta=s$ for which the amplitude of the long-range part of the pinning potential is scale invariant [according to Eq. (3.41)] and thus plays only the role of a constant parameter $g_0=g$ in Eq. (3.40). The fixed points at which the right-hand side of (3.40) vanishes are given by the roots of the quadratic equation

$$(\zeta^{-1}-d)u-u^2+g=0 \ . \tag{4.1}$$

This has real solutions

$$u_1=\tfrac{1}{2}\{\zeta^{-1}-d+[(\zeta^{-1}-d)^2+4g]^{1/2}\} \ , \tag{4.2}$$

$$u_2=\tfrac{1}{2}\{\zeta^{-1}-d-[(\zeta^{-1}-d)^2+4g]^{1/2}\} \ , \tag{4.3}$$

whenever g satisfies the inequality

$$g\geq-\frac{(\zeta^{-1}-d)^2}{4} \ . \tag{4.4}$$

The solution of Eq. (3.40) expressed in terms of u_1 and u_2 is

$$u(l)=u_1+\frac{(u_1-u_2)(u_0-u_1)}{(u_0-u_2)\exp(u_1-u_2)l-u_0+u_1} \ , \tag{4.5}$$

while for $u_1=u_2=(\zeta^{-1}-d)/2$ [which occurs for $g=-(\zeta^{-1}-d)^2/4$] this reduces to

$$u(l)=\frac{\zeta^{-1}-d}{2}+\frac{u_0-(\zeta^{-1}-d)/2}{1+[u_0-(\zeta^{-1}-d)/2]l} \ . \tag{4.6}$$

The delocalization transition occurs when the initial value u_0 coincides with the unstable fixed point u_2 (4.3), which divides the unbound states [for which $u(l\to\infty)$ tends to the stable fixed point u_1 (4.2)] from the bound states [for which $u(l\to l^*)=-\infty$]:

$$u_0=\tfrac{1}{2}\{\zeta^{-1}-d-[(\zeta^{-1}-d)^2+4g]^{1/2}\} \ . \tag{4.7}$$

For the bound states, the denominator of Eq. (4.5) vanishes for

$$\exp(u_1-u_2)l^*=(\xi_\perp/a)^{u_1-u_2}=\frac{u_0-u_1}{u_0-u_2} \ , \tag{4.8a}$$

where we have used the definition of $\xi_\perp$ (3.31). The variable u has here the same physical meaning as in the pure case;[4] as in that case, the condition (4.8a) is not sufficiently accurate: we have to subtract from each u_0 the dimensionless value of the "energy level"

$$E_0=-Fma^2/DT^2=-(\xi_\perp/a)^{2[1-(1/\zeta)]}$$

[see Eq. (3.35)], because this is the way the energy appears in the original Schrödinger equation (see Ref. 4 for a detailed explanation). After the subtraction, we get instead of (4.8a) (omitting a numerical constant)

$$(\xi_\perp/a)^{u_1-u_2}=\frac{u_0-u_1+(a/\xi_\perp)^{2[(1/\zeta)-1]}}{u_0-u_2+(a/\xi_\perp)^{2[(1/\zeta)-1]}} \ . \tag{4.8b}$$

The critical singularities are to be extracted from the solution of (4.8b) for $u_0\to u_2$. Several different cases are possible here.

A. $2(\zeta^{-1}-1)<u_1-u_2$

In this case, g is bounded by

$$g>(\zeta^{-1}-1)^2-(\zeta^{-1}-d)^2/4 \ . \tag{4.9}$$

The leading terms of the localization length and the free-energy (3.35) expansions are given by

$$\xi_\perp=a(u_2-u_0)^{-\zeta/2(1-\zeta)} \ , \tag{4.10}$$

$$F=-\frac{DT^2}{ma^2}(u_2-u_0) \ , \tag{4.11}$$

where we can drop the distinction between T and T^* because they are identical for the initial condition (3.39). The phase transition characterized by Eqs. (4.10) and (4.11) is very unusual, since the divergence of the correlation length $\xi_\perp$ (4.10) implies that this is a second-order phase transition while the behavior of the free energy F (4.11) is typical for a first-order phase transition. It is clear from the condition (4.9) that the critical singularities (4.10) and (4.11) cannot be realized for a two-dimensional ($d=1$) delocalization transition from a purely short-range ($g=0$) pinning potential for any form of the disorder potential (i.e., for no value of ζ). On the border of the regime (4.9), where

$$g=(\zeta^{-1}-1)^2-(\zeta^{-1}-d)^2/4\ , \tag{4.12}$$

we expect the presence of extra logarithmic corrections to Eqs. (4.10) and (4.11).

B. $0<u_1-u_2<2(\zeta^{-1}-1)$

In this case, g is bounded by

$$-(\zeta^{-1}-d)^2/4<g<(\zeta^{-1}-1)^2-(\zeta^{-1}-d)^2/4\ . \tag{4.13}$$

This case occurs in all of the physical situations ($\zeta<1$) as well as dimensions $2-\zeta^{-1}<d<3\zeta^{-1}-2$; it includes the important special case $g=0$, which corresponds to a short-range pinning potential (but excluding the case $g=0$, $\zeta^{-1}=d$, which will be analyzed separately). The correction term in Eq. (4.8b) is irrelevant in leading order for $u_0\to u_2$, and the localization length and the free-energy singularities are as follows:

$$\xi_\perp=a\left[\frac{u_2-u_0}{u_1-u_0}\right]^{-\nu_\perp}, \tag{4.14}$$

$$\nu_\perp^{-1}=u_1-u_2=[(\zeta^{-1}-d)^2+4g]^{1/2}\ , \tag{4.15}$$

$$F=-\frac{DT^2}{ma^2}\left[\frac{u_2-u_0}{u_1-u_0}\right]^{2\nu_\perp(\zeta^{-1}-1)}, \tag{4.16}$$

where we have introduced the correlation length exponent $\nu_\perp$ according to the standard definition.[14]

For $\zeta^{-1}>d$ and $g=0$ the condition $u_0=u_2$ for a depinning transition reduces to $u_0=0$. This means that a depinning transition from a short-range pinning potential is impossible along the curve $\overline{\Delta}_0=\overline{\Delta}^*$ and the line is localized by an arbitrarily weak attractive well. An important special case satisfying $\zeta^{-1}>d$ is the two-dimensional ($d=1$) unbinding transition in the presence of random-bond disorder, for which the exact value $\zeta=\frac{2}{3}$ is available.[28,7,25,29] For $g=0$, Eq. (4.15) reduces to (2.7a). We can rewrite Eq. (4.16) in a form appropriate for both $\zeta^{-1}>d$ and $\zeta^{-1}<d$:

$$F=-\frac{DT^2}{ma^2}\left[\frac{|u_0|}{\zeta^{-1}-d+|u_0|}\right]^{2(\zeta^{-1}-1)/(\zeta^{-1}-d)}, \tag{4.17}$$

where we have written $u_0=-|u_0|$ to be explicit on signs (the line can only be pinned for $u_0<0$). For $d=1$, Eq. (4.17) gives the "classical"[14] quadratic dependence corresponding to a jump in the specific heat, and the *actual value of ζ (or type of disorder) does not play any role.* It does determine the localization length exponent (2.7a) which is $\nu_{\perp 0}=2$ for $d=1$ and $\zeta=\frac{2}{3}$ (random-bond disorder). The physical reason why we can compare critical singularities (2.7a) and (4.17) for unbinding from a symmetric pinning potential with those for the asymmetric one (two-dimensional wetting transition[7]) is because in the vicinity of the wetting transition the distance between the line (interface) and the substrate goes to infinity, thus implying that asymmetry of the effective pinning potential becomes less and less important in the critical region. In other words, both problems belong to the same universality class even though their phase transition points do not coincide.

For $\zeta^{-1}<d$ and $g=0$, the condition $u_0=u_2$ for the depinning transition reduces to $u_0=\zeta^{-1}-d$: the line will be localized only by a sufficiently deep pinning potential. The critical singularities from the localized side of the transition are given again by Eqs. (2.7a) and (4.17).

For the case of nonzero g, Eqs. (4.15) and (4.16) tell us that the critical exponents depend upon g and thus are *nonuniversal.* In the presence of random-bond disorder in two spatial dimensions ($d=1$) the nonuniversal regime (4.14)–(4.16) is realized when the long-range tail of the pinning potential $V_s/|\mathbf{x}|^s$ has the exponent $s=\zeta^{-1}=\frac{3}{2}$.

C. Multicritical point $g=-(\zeta^{-1}-d)^2/4$

This is the marginal case $u_1=u_2=(\zeta^{-1}-d)/2$; we have to use Eq. (4.6) to find the scale l^* at which $u(l\to l^*)=-\infty$:

$$l^*=\frac{1}{(\zeta^{-1}-d)/2-u_0}\ . \tag{4.18}$$

The expressions for the localization length and the free energy follow from (3.31) and (3.35):

$$\xi_\perp=a\exp\left[\frac{1}{(\zeta^{-1}-d)/2-u_0}\right], \tag{4.19}$$

$$F=-\frac{DT^2}{ma^2}\exp\left[-\frac{2(\zeta^{-1}-1)}{(\zeta^{-1}-d)/2-u_0}\right]. \tag{4.20}$$

The condition for the phase transition reduces now to $u_0=(\zeta^{-1}-d)/2$, and the localized region is given by $u_0<(\zeta^{-1}-d)/2$.

In the special case $\zeta^{-1}=d$ ($g=0$) Eqs. (4.19) and (4.20) reduce to

$$\xi_\perp=a\exp(1/|u_0|)\ , \tag{4.21}$$

$$F=-\frac{DT^2}{ma^2}\exp(-2(d-1)/|u_0|)\ . \tag{4.22}$$

Here an arbitrarily weak short-range pinning potential localizes the line.

D. Kosterlitz-Thouless-like transition

Hitherto we have looked at cases in which the right-hand side of the RG equation (3.40) has real zeros, which are the fixed points. However, when

$$g<-(\zeta^{-1}-d)^2/4 \tag{4.23}$$

there are no real solutions, and the solution of the RG equation is

$$u(l)=\frac{\zeta^{-1}-d}{2}+\sqrt{\lambda}\tan\left[\arctan\frac{u_0-(\zeta^{-1}-d)/2}{\sqrt{\lambda}}-\sqrt{\lambda}l\right], \tag{4.24}$$

$$\lambda=-(\zeta^{-1}-d)^2/4-g\ , \tag{4.25}$$

instead of Eqs. (4.5) and (4.6). Equation (4.24) has an infinite periodic sequence of values of l^* at which $u(l \to l^*) = -\infty$, implying the presence of an infinite number of zeros of the radial wave function. Following the analysis of Ref. 4, we find the following Kosterlitz-Thouless-like[34] singularities as $\lambda \to 0$:

$$\xi_\perp = a \exp(\pi/\sqrt{\lambda}) , \tag{4.26}$$

$$F = -\frac{DT^2}{ma^2} \exp[-2\pi(\zeta^{-1}-1)/\sqrt{\lambda}] , \tag{4.27}$$

for $u_0 > (\zeta^{-1}-d)/2$, and

$$\xi_\perp = a \exp(\pi/2\sqrt{\lambda}) , \tag{4.28}$$

$$F = -\frac{DT^2}{ma^2} \exp[-\pi(\zeta^{-1}-1)/\sqrt{\lambda}] , \tag{4.29}$$

for $u_0 = (\zeta^{-1}-d)/2$. The latter equations reflect the thermodynamic singularities for passage through the multicritical point (see the previous subsection) at fixed $u_0 = (\zeta^{-1}-d)/2$. Previously [see Eqs. (4.19) and (4.20)] we have found the critical behavior if one passes through the multicritical point for fixed $g = -(\zeta^{-1}-d)^2/4$. Therefore we can rewrite Eqs. (4.19) and (4.28) in more general form in terms of $\Delta u = (\zeta^{-1}-d)/2 - u_0$ and λ (4.25) as follows:

$$\xi_\perp = a \exp\left\{ \frac{\Omega(\lambda/\Delta u^2)}{\Delta u} \right\} , \tag{4.30}$$

where the shape function Ω has the properties

$$\Omega(0) = 1 \tag{4.31}$$

and

$$\Omega(y \to \infty) \approx \pi/2\sqrt{y} . \tag{4.32}$$

V. DELOCALIZATION TRANSITION IN THE PRESENCE OF A LONG-RANGE TAIL $V_s/|\mathbf{x}|^s$ WITH $\zeta^{-1} < s$

When the long-range part of the pinning potential has a tail $V_s/|\mathbf{x}|^s$ with $\zeta^{-1} < s$, the variable g (3.41) is irrelevant in the RG sense.[14] Then the critical singularities in leading order are given by the results of the previous section for $g=0$; the next-order corrections are calculated below. Even when $V(x)$ has a "long-range" tail that is irrelevant in the RG sense,[14] the loci of the phase transitions found for $g=0$ are shifted, as discussed in Ref. 4. We will restrict ourselves to the case $2-\zeta^{-1} < d < 3\zeta^{-1}-2$, and use the general ideas for calculating corrections to scaling.[14] To simplify the formulas we will use a thermal scaling field τ to denote the dimensionless proximity to the phase transition point, and omit all dimensional and irrelevant numerical factors.

Consider first the case $\zeta^{-1} \neq d$. Here the leading term of the free-energy expansion is given by $F \sim \tau^{2\nu_{10}(\zeta^{-1}-1)}$ with ν_{10} from Eq. (2.7a). The value of $g(l) = g_0 \exp[(\zeta^{-1}-s)l]$ [see Eq. (3.41)] evaluated at the correlation length scale $e^{l*} = \tau^{-\nu_{10}}$ is equal to $g^* = g_0 \tau^{\nu_{10}(s-\zeta^{-1})}$ and is small compared with unity for $\tau \ll 1$. Therefore we can seek the free-energy singularity in the form

$$F \sim \tau^{2\nu_{10}(\zeta^{-1}-1)} f[g_0 \tau^{\nu_{10}(s-\zeta^{-1})}] ,$$

where $f(x)$ is an analytic function behaving for $x \ll 1$ as $f(x) = 1-x$. This leads to the expansion of the free energy

$$F \sim \tau^{2\nu_{10}(\zeta^{-1}-1)} - g_0 \tau^{\nu_{10}(\zeta^{-1}-2+s)} . \tag{5.1}$$

For $\zeta^{-1} = d$ in the absence of a long-range perturbation the correlation length and free energy have singularities $\exp(1/\tau)$ and $\exp[-2(d-1)/\tau]$, respectively [see (4.21) and (4.22)]. The value of $g(l) = g_0 \exp(\zeta^{-1}-s)l$ evaluated at the correlation length scale $e^{l*} = \exp(1/\tau)$ is given by $g^* = g_0 \exp[(\zeta^{-1}-s)/\tau]$. The scaling argument now leads to the expression

$$F \sim \exp[-2(d-1)/\tau]\{1 - g_0 \exp[(\zeta^{-1}-s)/\tau]\} . \tag{5.2}$$

VI. LONG-RANGE TAIL $V_s/|\mathbf{x}|^s$ WITH $\zeta^{-1} > s$

If the long-range part of the pinning potential falls off as $1/|\mathbf{x}|^s$ with $\zeta^{-1} > s$, it is relevant in the RG sense and grows under rescaling as $g(l) = g_0 \exp(\zeta^{-1}-s)l$ [see Eq. (3.41)], leading to a different physical picture. The outcome depends on the sign of the long-range tail of the pinning potential. For the case of a repulsive tail ($g_0 > 0$) and an attractive short-range well, a first-order depinning transition occurs.

For the case of an attractive tail ($g_0 < 0$) the line is always pinned. We can calculate the free-energy density in the limit $|g_0| \ll 1$, because our picture of the delocalization transition with $g_0 = 0$ holds for intermediate scales less than the spatial scale e^{l^*} imposed by the presence of the long-range tail. For small $|g(l)|$ and $\zeta^{-1} \neq d$ the system can be described on scale l by the exponent (4.15), where now $g = g(l)$ is scale dependent. The maximal scale at which such a description is still meaningful is given by the zero of the expression in the square brackets of Eq. (4.15):

$$(\zeta^{-1}-d)^2 = 4|g_0| \exp(\zeta^{-1}-s)l^* .$$

Therefore the localization length and the free energy are given by [see Eqs. (3.31) and (3.35)]

$$\xi_\perp \simeq a[(\zeta^{-1}-d)^2/|g_0|]^{1/(\zeta^{-1}-s)} , \tag{6.1}$$

$$F \sim [|g_0|/(\zeta^{-1}-d)^2]^{2(\zeta^{-1}-1)/(\zeta^{-1}-s)} . \tag{6.2}$$

The last expression has an important special case. Let us imagine the presence of an additional term in the Hamiltonian (3.1) imitating an external field term which localizes the line and destroys the phase transition. For the edge-depinning (e.g., two-dimensional wetting) transition this term is proportional to $|x|$,[1] which corresponds to a relevant long-range tail with $s=-1$. Then we obtain from (6.2) the free-energy density above the depinning transition in an external field $|g_0| = h$ for $h \to +0$ in the form

$$F \sim h^{2(\zeta^{-1}-1)/(\zeta^{-1}+1)} . \quad (6.3)$$

We can combine (6.3) with $F \sim \tau^{2\nu_{10}(\zeta^{-1}-1)}$ [ν_{10} is from (2.7a)] for $h=0$ and rewrite F for both nonzero τ and h in the form

$$F = \tau^{2\nu_{10}(\zeta^{-1}-1)}\Omega[h/\tau^{\nu_{10}(\zeta^{-1}+1)}] , \quad (6.4)$$

where the scaling function Ω has the properties $\Omega(0)=\text{const}$ and $\Omega(y) \to y^{2(\zeta^{-1}-1)/(\zeta^{-1}+1)}$ as $y \to \infty$. The analogs of Eqs. (6.1) and (6.2) for $\zeta^{-1}=d$ have extra logarithmic corrections instead of a singularity at $\zeta^{-1}=d$. The free energy in this case can be obtained by an extension of the corresponding calculation of Ref. 4.

In two spatial dimensions ($d=1$) (6.4) reduces to

$$F = \tau^2 \Omega(h/\tau^\delta) , \quad (6.5)$$

$$\delta = \frac{1+\zeta}{1-\zeta} , \quad (6.6)$$

where we have introduced the crossover exponent δ according to the standard definition,[1] and used (2.7a). No exact calculations of δ are available in two spatial dimensions. Heuristic arguments[6,1] give

$$\delta_{\text{heur}} = \frac{2-\zeta}{1-\zeta} . \quad (6.7)$$

In a pure system, $\zeta=\frac{1}{2}$, and these give the same result: $\delta=3$. However, in the presence of random-bond disorder ($\zeta=\frac{2}{3}$) Eq. (6.6) predicts $\delta=5$, while (6.7) gives $\delta_{\text{heur}}=4$. Although in general our approach reproduces the heuristic results, in this one place there is a disagreement.

VII. CALCULATION OF CRITICAL SINGULARITIES FROM PERTURBATIVE RG: RANDOM-BOND DISORDER

Up to now we have been considering the case of the special initial conditions (3.39), which enabled us to find the critical exponents in terms of the wandering exponent ζ without needing an explicit model for the disorder. This kept out of sight some important features of general initial conditions, such as temperature renormalization (3.20), the behavior of the critical amplitudes, and their dependence on disorder degree. Moreover, the expressions for the correlation length (3.31) and the singular part of the free energy (3.35) involve the short-range cutoff a, while the problem Kardar solved via the Bethe ansatz[7] had $a=0$; thus it is desirable to verify that the limit $a=0$ is accessible to our approach, which requires that we start from general initial conditions. These issues will be considered in this section.

We consider the case when the pinning potential in (3.1) is purely short ranged and the dependence $R(\mathbf{x})$ in Eq. (3.2) corresponds to a well-localized function of the size a: $R(\mathbf{x})=\delta_a(\mathbf{x})$, i.e., it describes a short-ranged or random-bond disorder. Then the amplitude Δ_0 in (3.2) will be the parameter of the problem that quantifies the disorder. The function $f_d(\bar{\Delta})$ in Eq. (3.14) is known perturbatively[29] in disorder degree:

$$f_d(\bar{\Delta}) = \frac{2-d}{d}\bar{\Delta} , \quad (7.1)$$

where the dimensionless parameter $\bar{\Delta}$ is defined as[29]

$$\bar{\Delta} = mK_d a^{2-d}\Delta/T^3 . \quad (7.2)$$

Taking the equation for disorder degree from Refs. 29 we will have instead of Eqs. (3.20)–(3.23)

$$\frac{dT}{dl} = \left[\zeta^{-1} - 2 + \frac{2-d}{d}\bar{\Delta}\right] T , \quad (7.3)$$

$$\frac{du}{dl} = \left[2-d-\frac{2-d}{d}\bar{\Delta}\right] u - u^2 , \quad (7.4)$$

$$\frac{d\bar{\Delta}}{dl} = \left[2-d+\frac{2(2d-3)}{d}\bar{\Delta}\right]\bar{\Delta} . \quad (7.5)$$

For a finite amount of disorder and $d<\frac{3}{2}$, Eqs. (7.3) and (7.5) have a stable fixed point

$$\bar{\Delta}^* = \frac{d(2-d)}{2(3-2d)} , \quad (7.6)$$

which leads to the result[29]

$$\zeta^{-1} = 2 - \frac{(2-d)^2}{2(3-2d)} . \quad (7.7)$$

For $d \to \frac{3}{2}$, $\bar{\Delta}^*$ diverges, while for $\frac{3}{2}<d<2$ no stable fixed points are found. These properties are undoubtedly artifacts of the one-loop calculation.[29] For $d>2$, $\bar{\Delta}^*$ becomes unstable, implying a depinning transition caused by randomness.[29] The strong-coupling regimes for $d \geq 2$ are not accessible in the one-loop approximation. In what follows we assume that $d<2$, $\zeta^{-1}>d$, and in a theory more accurate than that leading to (7.6) a stable positive fixed-point value $\bar{\Delta}^*$ exists. Equation (7.4) has a stable fixed point u_1 [see Eq. (4.2)]:

$$u_1 = (2-d)\left[1-\frac{\bar{\Delta}^*}{d}\right] = \zeta^{-1} - d . \quad (7.8)$$

Even though the first representation of u_1 in terms of d and $\bar{\Delta}^*$ is perturbative, the second one in terms of ζ and d is exact. We note that the depinning transition is governed by the unstable fixed point u_2 of Equation (7.4) [see Eq. (4.3)] which is now located at $u_2=0$, which implies that the line is always pinned by a symmetric short-ranged attractive pinning potential in the presence of random-bond disorder. To find the critical singularities near the phase transition point $u_0=0$, we have to solve the system (7.3)–(7.5) with the initial conditions $u(l=0)=u_0$ [see (3.28) for $V_s=0$], $T(l=0)=T$, and $\bar{\Delta}(l=0)=\bar{\Delta}_0$ [see (7.2)]. The solutions are

$$u^{-1}(l)=\frac{\exp(d-2)l}{\zeta^{-1}-d}[\exp(2-d)l+\overline{\Delta}^*/\overline{\Delta}_0-1]$$
$$\times\left[1+\left[\frac{u_1}{u_0}-\frac{\overline{\Delta}^*}{\overline{\Delta}_0}\right](\overline{\Delta}_0/\overline{\Delta}^*)^{(2-\zeta^{-1})/(2-d)}[\exp(2-d)l+\overline{\Delta}^*/\overline{\Delta}_0-1]^{-(\zeta^{-1}-d)/(2-d)}\right], \tag{7.9}$$

$$T(l)=T[(1-\overline{\Delta}_0/\overline{\Delta}^*)\exp(d-2)l+\overline{\Delta}_0/\overline{\Delta}^*]^{(2-\zeta^{-1})/(2-d)}. \tag{7.10}$$

The latter expression gives us the fixed-point value [see Eq. (3.34)]

$$T^*=T(l=\infty)=T(\overline{\Delta}_0/\overline{\Delta}^*)^{(2-\zeta^{-1})/(2-d)}. \tag{7.11}$$

The behavior of the correlation length is set by the zero of the right-hand side of (7.9). Using Eqs. (7.9), (7.11), and (3.35), we find the following critical singularities in the limit $u_0\to 0$:

$$\xi_\perp\cong a(u_1/|u_0|)^{1/(\zeta^{-1}-d)}(\overline{\Delta}_0/\overline{\Delta}^*)^{(2-\zeta^{-1})/[(2-d)(\zeta^{-1}-d)]}, \tag{7.12}$$

$$F=-\frac{DT^2}{ma^2}(u_1/|u_0|)^{2(1-\zeta^{-1})/(\zeta^{-1}-d)}$$
$$\times(\overline{\Delta}_0/\overline{\Delta}^*)^{2(2-\zeta^{-1})(1-d)/[(2-d)(\zeta^{-1}-d)]}. \tag{7.13}$$

The critical exponents governing the phase transition are surely the same as we found before but the behavior of the critical amplitudes is new. Counting the powers of a in (7.12) [including those built into the definitions (3.28) and (7.2)] shows that the expression for the localization length (7.12) is independent of the cutoff for any value of the wandering exponent ζ. This allows a comparison of (7.12) with the replica result of Kardar.[7] We remind the reader that the present discussion is for the symmetric line-depinning problem, whereas Kardar treats the unsymmetric case, so that the phase transition points are different; however, they belong to the same universality class, and therefore will exhibit the same exponents and the same behavior of the critical amplitudes. Substituting into (7.12) the exact value[28] of $\zeta=\frac{2}{3}$ (for $d=1$), we find that the localization length is *proportional* to the disorder degree Δ_0, in agreement with Refs. 7; perturbative RG reproduces this exact dependence because it becomes asymptotically exact in the limit $a\to 0$, where $\overline{\Delta}$ is small for any finite Δ, according to Eq. (7.2).

Another count of powers of a shows that the free-energy singularity (7.13) is cutoff independent for any value of ζ. Furthermore, for $d=1$, F in (7.13) is independent of the disorder degree Δ_0, again regardless of the actual value of ζ. We conclude that the critical amplitude of the free-energy singularity is proportional to T^2/m in agreement with the replica results[7] (our parameter T^2/m corresponds to the parameter γ used by Kardar[7]). Even though for $d=1$ the form of the singularity (7.13) does not depend upon the disorder degree, the width of the critical region where the singularities (7.12) and (7.13) are observed does depend on Δ_0. The condition of validity of Eqs. (7.12) and (7.13) coming from (7.9) and (7.10) is

$$u_1/|u_0|\gg\overline{\Delta}^*/\overline{\Delta}_0, \tag{7.14}$$

and again cutoff independent.

The expression for the fixed-point value of the temperature (7.11) can be used to verify the predictions (2.5) and (2.6). Combining (7.11) with Eqs. (2.5), (2.6), and (7.2) we see that they satisfy Eq. (3.37) identically.

There is one more problem that needs further study. In Sec. V we calculated corrections to scaling due to the presence of a long-range tail of the pinning potential that was irrelevant in the RG sense, using ideas developed for "pure" systems.[14] That was possible because for the special choice (3.39) the problem looks like a pure problem and the effect of disorder is parametrized by the wandering exponent ζ. For general initial conditions there are other sources of corrections to scaling, for instance, correction to the fixed-point value (7.11) coming from Eq. (7.10) and giving a more accurate value of T^*:

$$T^*=T(\overline{\Delta}_0/\overline{\Delta}^*)^{(2-\zeta^{-1})/(2-d)}$$
$$\times\left[1+\frac{2-\zeta^{-1}}{2-d}(\overline{\Delta}^*/\overline{\Delta}_0-1)(\xi_\perp/a)^{d-2}\right], \tag{7.15}$$

and probably some others. We will not speculate in this direction since we do not know in general how to calculate corrections to scaling in disordered systems. However, the implication is that the corrections to scaling for general initial conditions may be significantly different from those calculated in Sec. V.

VIII. MANY-LINE DELOCALIZATION TRANSITIONS

The continuum SOS Hamiltonian for a collection of N directed interacting line objects placed in a random medium of d transverse dimensions has the form

$$H=\int dt\sum_{i=1}^{N}\left\{\left[\frac{m}{2}\left[\frac{d\mathbf{x}_i}{dt}\right]^2-\mu\right]+V_r(\mathbf{x}_i,t)\right\}$$
$$+\int dt\sum_{i<j}^{N}V(|\mathbf{x}_i-\mathbf{x}_j|), \tag{8.1}$$

where μ is the chemical potential that together with the two-body interline interaction $V(x)$ sets an average line density n. Without a random potential in (8.1), this looks like an imaginary-time classical action describing a many-body quantum problem in the path integral formulation:[3,20,22] temperature plays the role of Planck's constant, m is the particle mass, and t is the imaginary-time variable. The ground-state properties of the corresponding quantum Bose system can be classified for arbitrary space dimensionality and general interparticle interaction in the dilute limit by means of a RG method.[22,20,23] In the language of the many-line problem the dilute limit corresponds to the vicinity of the many-line delocalization transition, which is our ultimate goal.

The ground-state properties of a dilute Bose system can be found following the prescription of Refs. 23, which will now be briefly outlined; we will follow a similar program for the Hamiltonian (8.1), looking at it as a kind of "quantum" many-body problem. First note that the most important physical process in the dilute limit is the interaction between pairs of particles. Two-body scattering of particles of masses m interacting via an interparticle potential $V(|\mathbf{x}_1-\mathbf{x}_2|)$ is the same as that of a single particle of reduced mass $m/2$ in an external potential $V(x=|\mathbf{x}_1-\mathbf{x}_2|)$; this is why we are using the same notation $V(x)$ for the interline interaction as was used for the external pinning potential in Eq. (3.1). Multiple scattering events convert the bare interaction $V(x)$ into a pseudopotential $V_{\text{eff}}(x)$ determined by Eq. (3.12) with $\mathcal{T}=T$ and m instead of $2m$. The pseudopotential can be related to the chemical potential by means of a "renormalized" Hartree condition.

To describe the many-line delocalization transitions we will use the one-line RG equations of the previous sections, neglecting contributions to their right-hand sides due to the many-body nature of the problem, because their effect goes away in the dilute limit. Then we can adopt Eqs. (3.24), (3.28), and (3.29), after replacing $2m$ by m everywhere. We will also need scaling equations for the chemical potential and particle density:

$$\frac{d\mu(l)}{dl}=2(\zeta^{-1}-1)\mu(l)\ , \tag{8.2}$$

which comes from the scaling transformation

$$\mu'=b^{2(\zeta^{-1}-1)}\mu \tag{8.3}$$

set by rescaling of the longitudinal coordinate t and the temperature in H/T [H is from (8.1)], and

$$\frac{dn(l)}{dl}=dn(l)\ , \tag{8.4}$$

which follows from simple dimensional grounds. Equations (8.2) and (8.4) are to be solved subject to initial conditions $\mu(l=0)=\mu$ (the measurable chemical potential), and $n(l=0)=n$ (the measurable density). The solutions to (8.2) and (8.4) are immediately found:

$$\mu(l)=\mu\exp[2(\zeta^{-1}-1)l],\quad n(l)=n\exp(dl)\ . \tag{8.5}$$

Our starting point for looking at the vicinity of the many-line delocalization transition is a Hartree-like relationship among the renormalized chemical potential, line density, and the full interaction potential $U(l)$ in (3.24) (Refs. 20, 23)

$$\mu(l)=n(l)U(l)\ . \tag{8.6}$$

Substituting the solutions for $\mu(l)$ and $n(l)$ into (8.6), and expressing $U(l)$ in terms of the variables $T(l)$ and $u(l)$ as given by (3.24), we obtain

$$\mu=\frac{T^2(l)a^{d-2}n\exp(d+2-2\zeta^{-1})l}{mK_d}u(l)\ . \tag{8.7}$$

The renormalization process is valid in the dilute approximation, which is defined by

$$\mu ma^2/T^2\ll 1\ . \tag{8.8}$$

If we replace the physical μ and T in (8.8) by their "renormalized" counterparts $\mu(l)$ and $T(l)$ from (8.2) and (3.20), and assume (as always) that there is a finite fixed-point value $T^*=T(l=\infty)$, we will find that there will be a scale l^* above which the inequality (8.8) will no longer be valid. The RG flow will be interrupted on this scale, which is defined as

$$\mu(l^*)=C_dT^{*2}/ma^2\ , \tag{8.9}$$

where C_d is a constant of order unity (in the interest of simplicity, this factor will be frequently omitted in what follows). This determines l^* in terms of T^* and the measurable chemical potential [given by Eq. (8.5)],

$$\exp 2(\zeta^{-1}-1)l^*=T^{*2}/ma^2\mu\ , \tag{8.10}$$

which can be combined with (8.7) to give

$$(\mu ma^2/T^{*2})^{d/2(\zeta^{-1}-1)}=\frac{na^d}{K_d}u\left[\frac{\ln(T^{*2}/\mu ma^2)}{2(\zeta^{-1}-1)}\right]\ . \tag{8.11}$$

This gives implicitly the dependence of the chemical potential μ on the line density n in the dilute limit (8.8), and provides us with the critical singularities in the vicinity of the many-line delocalization transition.

In the analysis below, we adopt the route we followed while studying one-line unbinding transitions. First we look at the case of the special initial condition (3.39) and general interline potential—this will allow us to pick up all the possible critical singularities without going into the details of the disorder. Then we will consider the case of random-bond disorder and a short-ranged interline potential, which will enable us to make comparison with the exact replica calculations.[25,7]

IX. MANY-LINE DELOCALIZATION TRANSITIONS IN THE PRESENCE OF A SCALE-INVARIANT LONG-RANGE TAIL OF THE INTERLINE POTENTIAL: SPECIAL INITIAL CONDITION (3.39)

As in the case of one-line unbinding transitions (Sec. IV), the dependence $u(l)$ in (8.11) is determined solely by Eq. (3.40) for the initial condition (3.39) and $\zeta^{-1}=s$, and we have $T^*=T$. However, now the possibility of a continuous many-line delocalization transition depends on whether or not, the root u_1 (4.2) that gives us the *stable* fixed point of Eq. (3.40) is non-negative, and then whether for the given initial conditions u is eventually carried by the RG into the region $u(l\to\infty)>0$.

A. Positive stable fixed point: $u_1>0$

The fixed point u_1 in (4.2) will be strictly positive when either

$$\zeta^{-1}>d \quad \text{and } g \geq -(\zeta^{-1}-d)^2/4 \tag{9.1}$$

or

$$\zeta^{-1}\leq d \quad \text{and } g>0 . \tag{9.2}$$

Using Eqs. (8.11) and (3.40) we find to leading order in the dilute limit (8.8)

$$\mu=C_d^{(d+2-2\zeta^{-1})/d}\frac{T^2}{ma^2}(na^d u_1/K_d)^{2(\zeta^{-1}-1)/d} , \tag{9.3}$$

which leads to the scaling dependence

$$n\sim\mu^{\beta} , \tag{9.4}$$

where

$$\beta=\frac{d\zeta}{2(1-\zeta)} . \tag{9.5}$$

This result was already found by heuristic arguments for the less general case of a short-ranged ($g=0$) interline interaction potential,[24,15,1] which is included in the condition (9.1). Substituting (9.3) back into (8.8) we find that now the condition of the dilute limit reduces quite naturally to $na^d \ll 1$.

B. Zero-value stable fixed point: $u_1=0$

There are two cases in which the character of the many-line delocalization transition is governed by the stable zero-value fixed point of Eq. (3.40): in both cases we must have $u_0>0$.

The first case is $u_2<u_1=0$; according to (4.2) and (4.3), this holds if

$$\zeta^{-1}<d \quad \text{and } g=0 , \tag{9.6}$$

that is, for a purely short-ranged interline potential. Using Eq. (8.11) and the solution for $u(l)$ [Eq. (4.5)] we find to lowest order in the dilute limit (8.8)

$$\mu=C_d^{2\zeta-1}\frac{T^2}{ma^2}\left[\frac{(d-\zeta^{-1})u_0}{d-\zeta^{-1}+u_0}\frac{na^d}{K_d}\right]^{2(1-\zeta)} , \tag{9.7}$$

which leads to the scaling dependence (9.4) with

$$\beta=\frac{1}{2(1-\zeta)} . \tag{9.8}$$

Note that Eqs. (9.5) and (9.8) agree for $\zeta^{-1}=d$, which is the borderline of their ranges of validity. The range of validity of (9.7) can be obtained by putting it back into (8.8).

The second case of relevance is when the fixed point $u_1=0$ is marginal, i.e., [see Eqs. (4.2) and (4.3)] $u_1=u_2=0$, and this holds if $\zeta^{-1}=d$ and $g=0$. Using Eq. (8.11) and the solution for $u(l)$ we find to lowest order in the dilute limit (8.8)

$$\frac{1}{2(d-1)}(\mu ma^2/T^2)^{d/2(d-1)}\ln(C_d T^2/\mu ma^2) = C_d^{(2-d)/2(d-1)} na^d/K_d , \tag{9.9}$$

which is the result intermediate between Eqs. (9.3) and (9.7), and the extra logarithmic factor in (9.9) is to be expected in this marginal case. This regime is possible only for $d>1$ since the physical values of ζ are always less than unity.

X. VALUES OF s OTHER THAN $\zeta^{-1}=s$

If the interline potential $V(x)$ in (8.1) has a long-range tail of the form $V_s/|x|^s$ with $\zeta^{-1}\neq s$ the physical outcome depends on the relationship between ζ^{-1} and s. For $\zeta^{-1}<s$ the long-range tail is irrelevant in the RG sense [see Eq. (3.41)] and the results of the previous section for $g=0$ are valid.

For $\zeta^{-1}>s$ the long-range tail is relevant [see (3.41)], and the result depends on its sign. If it is attractive, a continuous many-line delocalization transition is impossible. If it is repulsive, both disorder-induced and thermal fluctuations are irrelevant in contrast to the tail, and one may expect the dependence (9.4) with

$$\beta=d/s , \tag{10.1}$$

which comes from comparing [in (8.1)] the orders of magnitude of the chemical potential and the value of the long-distance part of the interline potential evaluated at the mean interline distance $1/n^{1/d}$.

XI. MANY-LINE UNBINDING IN THE PRESENCE OF SHORT-RANGED INTERLINE INTERACTION AND RANDOM-BOND DISORDER

Here, as in Sec. VII, we look at the case of general initial conditions for random-bond disorder, and assume that the interline interaction is purely short ranged. Furthermore, we restrict ourselves to the situation $\zeta^{-1}>d$ and $d<2$ which puts us in the regime given by Eqs. (9.1) and (9.3)–(9.5). Using the expression for the fixed-point value of T^*, the condition (7.11), and the value $u_1=\zeta^{-1}-d$ [see Eq. (4.2) for $g=0$], we obtain

$$\mu=C_d^{(d+2-2\zeta^{-1})/d}\frac{T^2}{ma^2}(\overline{\Delta}_0/\overline{\Delta}^*)^{2(2-\zeta^{-1})/(2-d)} \times[na^d(\zeta^{-1}-d)/K_d]^{2(\zeta^{-1}-1)/d} , \tag{11.1}$$

which for $\overline{\Delta}_0=\overline{\Delta}^*$ comes back to (9.3). Counting the powers of a in (11.1), and noting the definition of the dimensionless $\overline{\Delta}_0=mK_d a^{2-d}\Delta_0/T^3$ [see (7.2)] shows that the result (11.1) is *independent of the short-range cutoff a*. Substituting (11.1) back into (8.8) shows that it holds automatically in the limit $a\to 0$. To find the range of validity of Eq. (11.1) we must either calculate the next-order term, or compare it with the purely thermal result. This is a special case of (11.1) with $\zeta=\frac{1}{2}$. Hence we find that the result (11.1) is valid in the limit

$$m\Delta_0/T^3 n^{(2-d)/d}\gg 1 , \tag{11.2}$$

which is both cutoff and wandering exponent (ζ) independent. Therefore Eqs. (11.1) and (11.2) can be directly compared with the replica results.[25,7] Substituting into (11.1) $d=1$, $\zeta=\frac{2}{3}$, we find the dependence

$$\mu\cong\Delta_0 n/T \tag{11.3}$$

valid in the limit $m\Delta_0/T^3 n\gg 1$, in agreement with Refs.

25 and 7. We note that for $d=1$ and $\zeta=\frac{2}{3}$ the power of C_d in (11.1) is identically zero. The dependence on the line density n in (11.1) is in agreement with heuristic arguments,[24,15,1] and we can combine them with Eq. (11.1) to write down an expression replacing (11.1) for $d<2$, and having only a single implicit parameter.

The heuristic arguments[24,15,1] give instead of (11.1) the following relationship between the chemical potential and the line density:

$$\mu \cong m A^{2/\zeta} n^{2(\zeta^{-1}-1)/d} . \tag{11.4}$$

Comparing Eqs. (11.1) and (11.4) we conclude that the amplitude A in (11.4) must both be cutoff independent and depend upon the disorder degree Δ_0 and the temperature. The expression for the amplitude A (2.6) is consistent with these properties. The substitution of (2.5) and (2.6) into (11.1) and (11.4) leads to consistent expressions up to an undetermined dimensionless factor in the less accurate Eq. (11.4). Then combining (11.1) and (2.5) we find

$$\mu = \frac{2-d}{2\overline{\Delta}^*} \frac{\Delta_0 n}{T} . \tag{11.5}$$

For the choice (2.5) the exponent of the parameter C_d in (11.1) is identically zero, and the only unknown quantity in (11.5) is the fixed-point value $\overline{\Delta}^*$. We see from Eq. (11.5) that the exponent β defined in (9.4) is *superuniversal* for $1\le d<2$ and equals unity. The fixed-point value $\overline{\Delta}^*$ is known exactly only for $d=1$, when the perturbative RG of Refs. 29 gives the exact value $\zeta=\frac{2}{3}$. We find from Eq. (7.6) that correspondingly $\overline{\Delta}^*=\frac{1}{2}$ and that (11.5) becomes

$$\mu = \Delta_0 n / T . \tag{11.6}$$

For $d\to 2-0$ the fixed-point value Δ^* should vanish as $2-d$ since for $d=2$ Eq. (2.5) reproduces the thermal wandering exponent $\zeta=\frac{1}{2}$.

XII. CONCLUSIONS

In this paper we have presented a RG theory of line-delocalization transitions in the presence of quenched disorder. Both one-line and many-line delocalization were studied using a generalization of previously developed ideas that proved to be successful in analysis of defect-free systems.[4,23] For the case of two spatial dimensions and short-range interactions we find agreement with the exact replica calculations.[7,25] We also reproduce many of the heuristic results,[6,1,15] while making clearer their range of validity. All of the critical exponents are expressed in terms of the single-line wandering exponent ζ and the space dimensionality d excepting a marginal case considered in Sec. IV. For short-ranged disorder, the kind of disorder most likely in practice, the results of Refs. 30 and 31 (see also the previous section) imply that for the case of three spatial dimensions ($d=2$ transverse dimensions) the value of ζ is the same ($\zeta=\frac{1}{2}$) as that for a pure system, so that the critical singularities of line-delocalization transitions, at least to leading order, are the same in both cases.

Other attempts have been made at RG analysis of this problem. One-line unbinding transitions were studied by Tang and Lyuksyutov[10] and by Balents and Kardar.[11] For the case of short-ranged interactions they used the following phenomenological RG equation for the renormalized potential:

$$\frac{du}{dl} = \left[\frac{2}{\zeta} - d - 2\right] u + c u^2 , \tag{12.1}$$

where c is some phenomenological constant. Equation (12.1) differs from our Eq. (3.40) for $g=0$ in the part linear in the potential. The combination $2/\zeta-d-2$ comes from the scaling transformation (2.8b) only and the difference between (12.1) and (3.40) is due to the fact that the function f_d [see Eq. (3.21)] was not taken into account. We also note that (12.1) was not actually derived so that there was a relevant question about the sign of c. For $c<0$, Eq. (12.1) leads to the localization length exponent given by (2.7b) and valid at least in some range of parameters. We note that the result (2.7b) as well as (4.10) valid at high dimensionalities were also found by Balents and Kardar[12] using different phenomenological arguments. Substituting in (2.7b) the values $\zeta=\frac{2}{3}$ and $d=1$ one finds $\nu_{10}=\infty$, implying that random-bond disorder makes the unbinding transition in two spatial dimensions marginal.

Recently Hwa and Nattermann[13] reported that they derived Eq. (12.1), but we do not understand the method of this work.

The critical exponents for many-line delocalization transitions coming from Eq. (12.1) were derived by Nattermann, Feigelman, and Lyuksyutov;[27] they were unable to reproduce the $(1+1)$-dimensional replica answer[25,7] (11.6), because the contribution f_d [see Eq. (3.21)] was missing. For the same reason they found a linear density dependence instead of our Eq. (9.7) replacing heuristic results (9.3)–(9.5); they give a different range of applicability for the heuristic arguments even though they handle the problem in essentially the same way.

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