Hydrodynamic model for the degenerate free-electron gas: Generalization to arbitrary frequencies

P. Halevi

Instituto de Física de la Universidad Autónoma de Puebla, Apartado Postal J-48, Puebla, Puebla 72570, Mexico

(Received 6 September 1994)

The hydrodynamic model for the degenerate free-electron gas has been generalized for arbitrary frequencies. The result for the longitudinal dielectric function is $\epsilon(\omega,q)=1-\omega_P^2[\omega(\omega+i\nu)-\beta^2q^2]^{-1}$, where ω is the circular frequency, q is the wave vector, ω_P is the plasma frequency, ν is the collision frequency, and $\beta^2=v_F^2(\frac{3}{5}\omega+\frac{1}{3}i\nu)/(\omega+i\nu)$, where v_F is the Fermi velocity. This interpolation formula for β^2 reduces to the well-known low- and high-frequency limits. The derivation is based on comparison, for small q, with the Boltzmann model for $\epsilon(\omega,q)$; however, modified to include the Mermin correction for relaxation to the local equilibrium. The bulk plasmon dispersion relation for this model is also found, and it includes collision-modified Landau damping (that is absent for $\omega\ll\nu$ and $\omega\gg\nu$).

I. INTRODUCTION

Since its inception 60 years ago,[1] the hydrodynamic model has proved to be very useful in describing electrical transport and optical properties of conductors. The main advantage of this model lies in the simplicity of accounting for nonlocality or spatial dispersion, as manifested in the wave-vector ($\vec{q}$) dependence of the dielectric function $\epsilon(\omega,\vec{q})$. Of course, precisely because of this simplicity, the model fails when sophistications such as Landau damping and band-structure effects are present. Then one has to resort to more advanced treatments, such as the Boltzmann model or the Lindhard, random-phase approximation (RPA) model.[2] Very recently the hydrodynamic model was applied to two-dimensional electron gases,[3] to small spheres,[4] to one-dimensional quantum wires,[5] and to the derivation of additonal boundary conditions at the interface between two conductors.[6]

The hydrodynamic model, as applied to the degenerate free-electron gas, has a serious shortcoming, namely, it is valid only for frequencies ω that are either very small or very large in comparison to the collisional frequency ν.[7] For $\omega\ll\nu$ collisions predominate and thus a conduction electron possesses three degrees of freedom. In the opposite limit, $\omega\gg\nu$, the influence of collisions is negligible and the particle motion is essentially limited to the direction of the electric field. This then corresponds to one degree of freedom, rather than three. One wonders, how many degrees of freedom does an electron have for an arbitrary ratio ω/ν? Is the hydrodynamic model applicable at all if ω/ν is neither very small nor very large? In the past such queries were ignored and the model was frequently used, on the basis of some phenomenological argument, as reasonable for any value of the frequency. In particular, the well-known β parameter (see below) was considered as a fitting parameter, which in principle can be determined from comparison with experiments for any value of ω. In this communication I wish to actually calculate $\beta(\omega)$, retrieving the low-frequency limit $\beta(0)=(1/3)^{1/2}v_F$ and the high-frequency limit $\beta(\infty)=(3/5)^{1/2}v_F$, where v_F is the Fermi velocity, as special cases. This will also lead to expressions for the adiabatic law constant $\kappa(\omega)$ and for the effective number of degrees of freedom $f(\omega)$. In short, I present the generalization of the hydrodynamic model to arbitrary frequencies.

The approach taken here is based on a straightforward comparison of the hydrodynamic model with a more sophisticated one, namely, the Boltzmann model *with the Mermin correction*[8] that takes into account the relaxation of the charge carriers to the local equilibrium (modulated by the wave). The Mermin correction is normally applied to the Lindhard or RPA model of the dielectric function; for the present purpose it suffices to consider the much simpler Boltzmann dielectric function.

The hydrodynamic model is based on Newton's second law for an electron of effective mass m, charge $-e$, and average velocity $\vec{v}$,

$$m\frac{d\vec{v}}{dt}=-e\left(\vec{E}+\frac{\vec{v}}{c}\times\vec{B}\right)-m\nu\vec{v}-\frac{\nabla p}{n}. \quad (1)$$

As we see, the particle is subject to the Lorentz force of the electromagnetic wave (there are no applied fields), to a phenomenological damping force (ν being the collision frequency), and to a pressure force (n is the local electron density). This last contribution in Eq. (1) is proportional to the pressure gradient and derives, hence, entirely from the inhomogeniety of n — the result of the modulation by the wave. Denoting by "0" equilibrium quantities (which are homogeneous for our bulk electron gas) and by "1" small out-of-equilibrium quantities (which are inhomogeneous),

$$\vec{\nabla}p=\vec{\nabla}p^{(1)}=\frac{\partial p^{(1)}}{\partial n^{(1)}}\vec{\nabla}n^{(1)}. \quad (2)$$

The parameter β is defined as

$$\beta^2 = \frac{1}{m}\frac{\partial p^{(1)}}{\partial n^{(1)}}. \quad (3)$$

If one also uses the continuity equation

$$\frac{\partial n}{\partial t} + \vec{\nabla}\cdot(n\vec{v}) = 0, \quad (4)$$

then it is easy to show that the dielectric function for longitudinal response is

$$\epsilon_H(\omega, q) = 1 - \frac{\omega_P^2}{\omega(\omega + i\nu) - \beta^2 q^2}, \quad (5)$$

where $\omega_P = (4\pi n^{(0)} e^2/m)^{1/2}$ is the plasma frequency. So this is the well-known hydrodynamic result.

For an adiabatic process p is proportional to n^κ, where κ is the adiabatic constant. Then

$$\partial p^{(1)}/\partial n^{(1)} = \kappa p^{(0)}/n^{(0)}. \quad (6)$$

For a degenerate free-electron gas

$$p^{(0)} = \frac{2}{3}u = \frac{2}{5}n^{(0)}\epsilon_F, \quad (7)$$

where u is the particle energy per unit volume and ϵ_F is the Fermi energy. Because $\epsilon_F = \frac{1}{2}mv_F^2$, from Eqs. (3), (6), and (7),

$$\beta^2 = \frac{1}{5}\kappa v_F^2. \quad (8)$$

It is well known that the value of κ depends on the number of degrees of freedom f, according to the formula

$$\kappa = (f+2)/f. \quad (9)$$

In this paper we are concerned with a three-dimensional free-electron gas. However, $f = 3$ only for very low frequencies, where the randomness of the collisions indeed permits motion in all three dimensions. For very high frequencies the motion is deterministic, with the velocity parallel to the direction of the electric field; thus $f = 1$ is appropriate. Therefore Eq. (9) gives

$$\kappa = \begin{cases} 5/3, & \omega \ll \nu \\ 3, & \omega \gg \nu, \end{cases} \quad (10)$$

and from Eq. (8)

$$\beta = \begin{cases} (1/3)^{1/2} v_F, & \omega \ll \nu \\ (3/5)^{1/2} v_F, & \omega \gg \nu. \end{cases} \quad (11)$$

By now it is quite obvious that, for a three-dimensional, degenerate free-electron gas, f, κ, and β are functions of the frequency. So we shall proceed to determine the functions $f(\omega)$, $\kappa(\omega)$, and $\beta(\omega)$.

II. DETERMINATION OF $\beta(\omega)$

On the basis of the Boltzmann equation the following expression is found for the longitudinal dielectric function:[2]

$$\epsilon_B(\omega, q) = 1 + \frac{3\omega_P^2}{\omega q v_F}\left[\frac{\omega}{q v_F} + \frac{1}{2}\left(\frac{\omega}{q v_F}\right)^2 \ln\frac{1 - q v_F/\omega}{1 + q v_F/\omega}\right]. \quad (12)$$

This result is obtained in the limit $\omega \ll \nu$. Often the effect of collisions is included by replacing all ω in the square brackets by $\tilde{\omega} = \omega + i\nu$. However, in the presence of charge-density oscillations this procedure is incorrect. It ignores the fact that the accelarated charges do *not* relax to a state of uniform density n_0, but to a local density $n(\vec{x}) = n^{(0)} + n^{(1)}(\vec{x})$. This dependence of the perturbation $n^{(1)}(\vec{x})$ on position is a consequence of the modulation by the electric field of the wave which, in our situation, is longitudinal ($\vec{E}\cdot\vec{q} \neq 0$). The correct procedure was indicated by Mermin,[8] who actually applied it to the RPA model of $\epsilon(\omega, q)$. Here we apply this approach to the Boltzmann model, Eq. (12). Then the Boltzmann model with the Mermin correction reads

$$\epsilon_{BM}(\omega, q) = 1 + \frac{\tilde{\omega}[\epsilon_B(\tilde{\omega}, q) - 1]}{\omega + i\nu[\epsilon_B(\tilde{\omega}, q) - 1]/[\epsilon_B(0, q) - 1]}. \quad (13)$$

Obviously, Eq. (13) corresponds to a more sophisticated physical situation than Eq. (5). In fact, if $\omega/q < v_F$, then the logarithm in Eq. (12) gives rise to an imaginary part, which describes collisionless or Landau damping that is absent in the hydrodynamic model. Nevertheless, ϵ_{BM} and ϵ_H have the same limiting *form* for weak spatial dispersion. Then expanding Eq. (12) in powers of $q v_F/\omega$ one obtains

$$\epsilon_B(\tilde{\omega}, q) - 1 \simeq -\frac{\omega_P^2}{\tilde{\omega}^2}\left(1 + \frac{3}{5}\frac{q^2 v_F^2}{\tilde{\omega}^2}\right). \quad (14)$$

Also, with no approximations, Eq. (12) gives

$$\epsilon_B(0, q) - 1 = 3(\omega_P/q v_F)^2. \quad (15)$$

Substituting Eqs. (14) and (15) in Eq. (13) one finds

$$\epsilon_{BM}(\omega, q) \simeq 1 - \frac{\omega_P^2}{\omega\tilde{\omega}}\left[1 + \left(\frac{3}{5} + \frac{i}{3}\frac{\nu}{\omega}\right)\frac{q^2 v_F^2}{\tilde{\omega}^2}\right]. \quad (16)$$

A similar expansion of Eq. (5) gives

$$\epsilon_H(\omega, q) \simeq 1 - \frac{\omega_P^2}{\omega\tilde{\omega}}\left[1 + \frac{q^2\beta^2}{\omega\tilde{\omega}}\right]. \quad (17)$$

Comparison of the coefficients of q^2 in the last two equations determines that

$$\beta^2(\omega) = \frac{\frac{3}{5}\omega + \frac{1}{3}i\nu}{\omega + i\nu}v_F^2 \quad . \quad (18)$$

This "interpolation formula" reproduces the low- and high-frequency limits given by Eq. (11); however, notice that, in general, β^2 is complex.

Further, if Eq. (8) is now taken to be the definition of

$\kappa(\omega)$, then

$$\kappa(\omega) = \frac{3\omega + \frac{5}{3}i\nu}{\omega + i\nu}. \qquad (19)$$

This, again, gives the correct limits Eq. (10). Finally, an effective number of degrees of freedom is obtained from Eq. (9):

$$f(\omega) = \frac{\omega + i\nu}{\omega + \frac{1}{3}i\nu}. \qquad (20)$$

This complex number reduces to 3 and to 1 in the low- and high-frequency cases, respectively.

The behavior of the real and imaginary parts of $f(\omega)$ is quite interesting. Re $f(\omega)$ decreases monotonically from 3 to 1, and has the value 2 for $\omega/\nu = 1/3$. On the other hand, Im $f(\omega)$ vanishes for $\omega = 0$ and for $\omega = \infty$, as expected, and for $\omega/\nu = 1/3$ attains the maximum value of 1.

III. BULK PLASMONS

As an interesting application we can now derive the bulk plasmon dispersion relation within the hydrodynamic approximation. We impose the condition $\epsilon_H(\omega, q) = 0$ on Eq. (5) and substitute the value (18) for β^2. This yields

$$\omega(\omega + i\nu)^2 - \omega_P^2(\omega + i\nu) - \left(\frac{3}{5}\omega + \frac{i}{3}\nu\right) v_F^2 q^2 = 0. \qquad (21)$$

This is an implicit dispersion relation from which $\omega(q)$ may be determined numerically, even for large damping.

In the absence of nonlocality (negligible q), the last term in Eq. (21) can be neglected. Then the solution is

$$\omega(q \to 0) = (\omega_P^2 - \nu^2/4)^{1/2} - i\nu/2. \qquad (22)$$

For weak spatial dispersion, ω in the last term of Eq. (21) can be replaced by this limit. If we also assume that $\nu^2/4 \ll \omega_P^2$ (as usually satisfied) then we obtain the explicit dispersion

$$\omega \simeq \omega_P + \frac{3}{10}\frac{q^2 v_F^2}{\omega_P} - i\frac{\nu}{2}\left(1 + \frac{4}{15}\frac{q^2 v_F^2}{\omega_P^2}\right). \qquad (23)$$

The real part is the ubiquitous result for bulk plasmons in the long-wavelength limit. The usual phenomenological damping is given by the term $-i\nu/2$. The interesting aspect of Eq. (23) is the q-dependent part of Im ω. Certainly this is not "collisionless damping"— after all this term is proportional to ν. Nevertheless, this term comes from the collision-modified Landau damping that is incorporated in the Boltzmann-Mermin model. For $\nu = 0$ Landau damping exists only for large wave vectors, such that $q > \omega/v_F$. Collisions ($\nu \neq 0$) soften this requirement, so that collision-modified Landau damping extends even to small wave vectors. Years ago a similar situation was encountered for Landau damping of helicon waves.[9]

IV. DISCUSSION AND CONCLUSION

The hydrodynamic model applied to the degenerate free-electron gas has been used incorrectly for many years when applied to arbitrary ratios ω/ν. This is because the β^2 coefficient of q^2 in $\epsilon(\omega, q)$ was known only in the low- and high-frequency limits. Here we have determined β^2 for arbitrary ω by comparing the hydrodynamic model with a more sophisticated one, namely the Boltzmann model including the Mermin correction. The result for β^2 is a complex expression, Eq. (18), that is consistent with the well-known low- and high-frequency limits. The complex β^2 leads, in turn, to a complex adiabatic constant $\kappa(\omega)$, Eq. (19). There is no need for alarm: considering the definition of β^2, Eq. (3), the complex $\kappa(\omega)$ only implies that the pressure fluctuations are *not* in phase with the density fluctuations. The adiabatic law is formally satisfied with this $\kappa(\omega)$. We have also defined an effective number of degrees of freedom, a complex number, Eq. (20).

With β^2 as given by Eq. (18) substituted in Eq. (5), the hydrodynamic model becomes identical with the Boltzmann-Mermin (and, as well, with the Lindhard-Mermin) model *for small wave vectors*. For finite values of q, obviously it is preferable to use the Boltzmann-Mermin model. Nevertheless, for the sake of simplicity, it is quite reasonable to use the hydrodynamic model, with β^2 determined from the small-q limit, even for finite q.

Finally, we have applied the above surveyed results to the derivation of the bulk plasmon dispersion relation. This leads to collision-modified Landau damping even for small wave vectors.

ACKNOWLEDGMENTS

I wish to acknowledge the hospitality of the Soreq Nuclear Research Center in Israel, where this paper was written. Also I am grateful to Dr. Rafael Ruppin for a careful reading of the manuscript. This research was supported by CONACyT Project No. 3923-E.

[1] F. Bloch, Helv. Phys. Acta **7**, 385 (1934); H. Jensen, Z. Phys. **106**, 620 (1937).
[2] See, for instance, the review by Ronald Fuchs and P. Halevi, in *Spatial Dispersion in Solids and Plasmas*, edited by P. Halevi, Electromagnetic Waves—Recent Developments in Research Vol. 1 (North-Holland, Amsterdam, 1992), p. 4.
[3] V. Fessatidis and H.L. Cui, Phys. Rev. B **43**, 11 725 (1991).
[4] R. Ruppin, Phys. Rev. B **45**, 11 209 (1992).
[5] B.S. Mendoza and W.L. Schaich, Phys. Rev. B **43**, 6590 (1991).
[6] Marcelo del Castillo-Mussot, W. Luis Mochán, and Bernardo S. Mendoza, J. Phys. Condens. Matter **5**, A393 (1993).
[7] A.D. Boardman, in *Electromagnetic Surface Modes*, edited by A.D. Boardman (Wiley, Chichester, 1982), p. 1.
[8] N.D. Mermin, Phys. Rev. B **1**, 2362 (1970).
[9] P. Halevi, Solid State Commun. **10**, 1189 (1972).