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Distribution of large currents in finite-size random resistor networks

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The distribution of large currents in random resistor networks is controlled by current enhancing defects. For randomly diluted networks the critical defect is linear and consists of two lines of insulating bonds separated by a conducting bond. For networks in which the conductances take two values ($0 < G_1 < G_2$) the critical defect for very large currents was shown previously to be a funnel shape with a 45° opening angle. We show that for an intermediate range of currents the critical defect is a funnel shape with an opening angle which differs from 45^0 and depends on the ratio G_1/G_2 and the relative concentration of the two conductances. For small currents the critical defect is linear. Finite-size corrections to the asymptotic result for the distribution of large currents are discussed.

Random resistor networks (RRN) have served to illuminate the transport and breakdown properties of disordered materials. One of the simplest models of breakdown is the random fuse model[1] where the resistors in the network change irreversibly to insulators if the current exceeds a threshold value. In this model, breakdown is initiated in resistors carrying the largest currents so it is of interest to understand the distribution of large currents in random resistor networks. More generally, RRN's provide a convenient setting for developing insights into the distribution of extreme or rare events in random systems.

In several previous studies[2–5] the authors and their collaborators have addressed the question of large currents in random resistor networks. In these papers, the asymptotic scaling behavior of the distribution of large currents was found. In the present paper we discuss "corrections to scaling" in the distribution of large currents in two-dimensional random resistor networks. The starting point for understanding the distribution of large currents in RRN's is the idea of a *critical defect.* The most probable local configuration of resistors which can produce a given current is called the critical defect for that current. By identifying the probability of finding a given current with the probability of finding the associated critical defect, the distribution of large currents can be estimated. The shape of the critical defect depends upon the distribution from which the resistors are chosen. For random dilution (i.e., a finite fraction of insulating bonds) Li and Duxbury[2] showed that the critical defect consists of two adjacent slits of insulating bonds separated by a single conductor, see Fig. 1(a). The size of the required slit grows linearly in the size of the current in the central conductor.

If all the resistances in the network are finite, Machta and Guyer[4] showed that the critical defect for large currents is a funnel-shaped region with a 45° opening angle, see Fig. 1(c). The upper and lower quadrants of the funnel are composed of good conductors while the left and right quadrants are composed of bad conductors. The size of the large current channeled into the center of the funnel scales as a power of the funnel size as discussed below.

By identifying the probability of a large current with the probability of the associated critical defect, one can obtain the expected largest current $I_{\max}$ in a finite network of size L

$$I_{\max} \sim (\ln L)^{\alpha}. \tag{1}$$

and the asymptotic behavior of the distribution, $P(I)$ of large currents in an infinite network[5]

$$P(I) \sim \exp(-cI^{1/\alpha}). \tag{2}$$

These formulas are valid for large L and I. The average current density is normalized to unity.

The exponent α depends on the critical defect and thus on the underlying distribution of resistors. For random dilution the result is[2]

$$\alpha = 1. \tag{3}$$

If the conductances take two nonzero values, $0 < G_1 < G_2$ then[4]

$$\alpha = \alpha(G) = \frac{1}{2}\left(1 - \frac{4}{\pi}\tan^{-1}(G^{1/2})\right) \tag{4}$$

where $G = G_1/G_2$. In both cases α is determined by the relation between the current in the critical defect and the number of resistors required to make the defect.

In the limit $G \to 0$, $\alpha(G) \to 1/2$, whereas for random dilution, $G = 0$, $\alpha = 1$. This discontinuity is due to the fact that the asymptotic critical defect is one dimensional for $G = 0$ and two dimensional for $G > 0$. References 4 and 5 suggested that for a given finite current there is a crossover from the 45° funnel to the two-slit configuration as G is made small. In this paper we show that the situation is somewhat more complex and that the critical defects for intermediate currents are funnel shapes with opening angles differing from 45°.

We have studied the shape of the critical defect for various values of the current and G using both analytic and numerical methods. To simplify the calculations we first consider the case where the fraction, f, of bad conductors is close to zero. In this case, the density of configurations with S bad conductors and T good conductors is nearly independent of T and well approximated by f^S. The critical defect for current I is then the configuration with the smallest number of bad conductors able to produce current I. Although the asymptotic critical defect depends only on whether $G = 0$ or $G > 0$, for finite currents the shape of the critical defect depends on both f and G.

First we report the results of the numerical work. We calculated the current in the central bond of the configurations of Fig. 1. The current as a function of the number of bad conductors is presented in Fig. 2. It is seen from this figure that it is necessary to go to large defects before the 45° funnel is the critical defect. In Fig. 2, it is seen that for $G = 1/4$ the two-slit defect produces more current for a given size until the defect has about 100 bad conductors. Thereafter the 26° funnel is dominant until the funnel size reaches nearly 500 bonds when finally the 45° funnel produces the largest current.

In Ref. 4 we used continuum methods to analyze the 45° funnel and show that it is the asymptotic critical defect. Here we use the continuum methods to estimate the shape of the critical defect for finite currents. The results are in reasonable agreement with the numerical calculations. For a funnel with a diagonal of length $2l$ and an opening angle β, see Fig. 3, the maximum current $i_{\max}$ in the central bond of the defect is given by

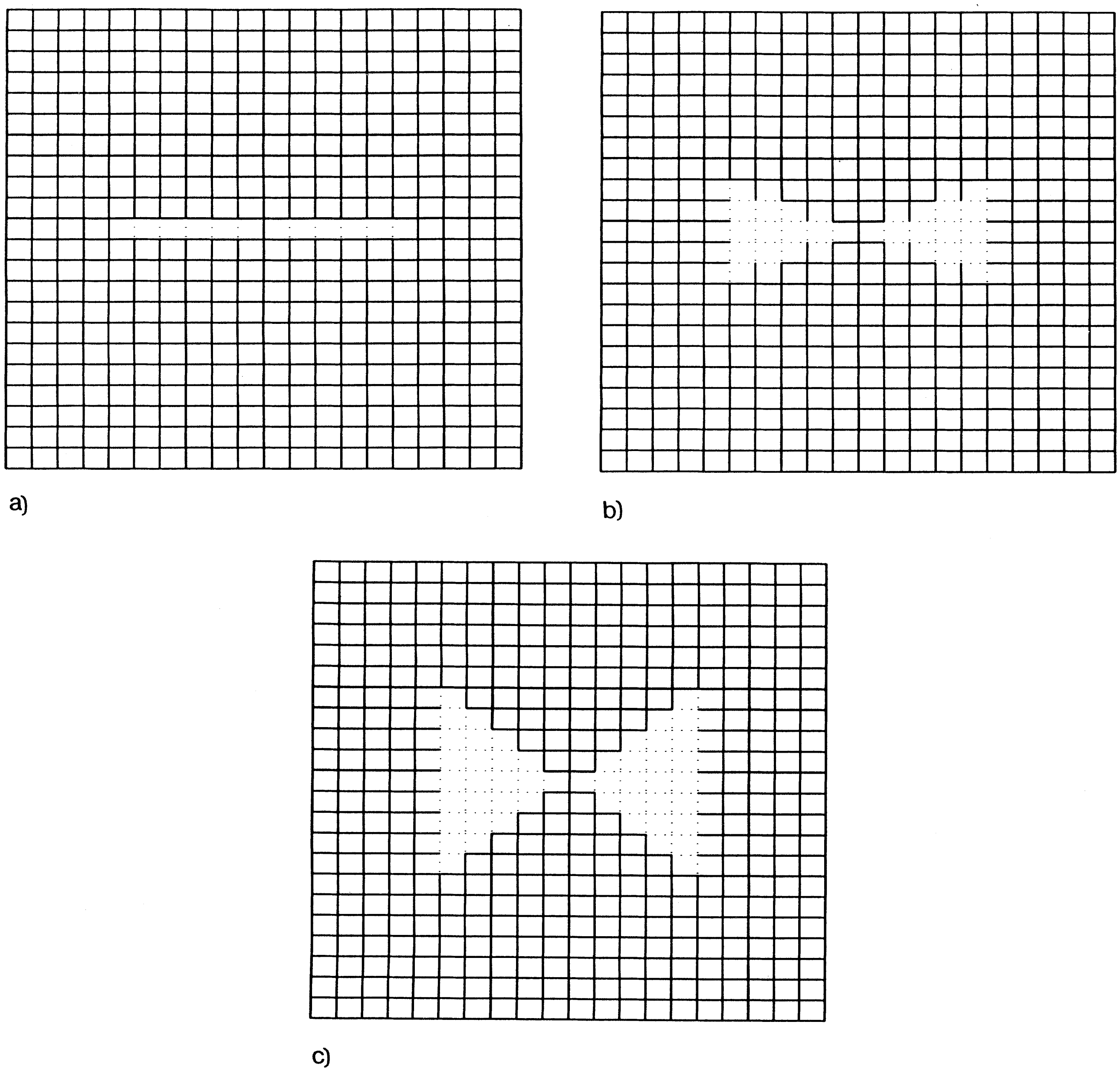

FIG. 1. Lattice funnels (a) two slits; (b) 26.6° funnel; (c) 45° funnel. The applied current is in the y direction.

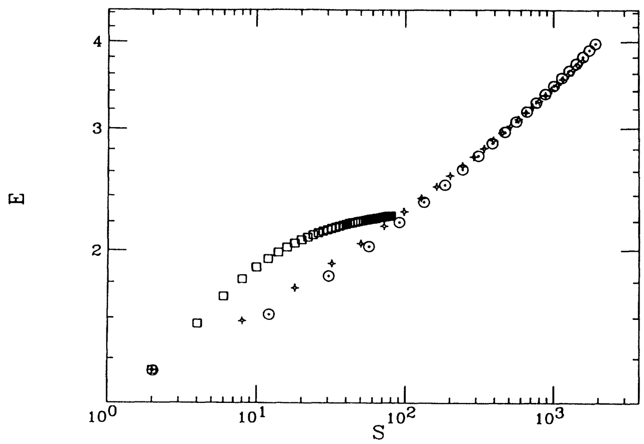

FIG. 2. The current enhancement $E = I_{\max}/I_0$ at the funnel apex (I_0 is the applied current) for the 2-slit (□); 26.6° (+) and 45° (⊙) funnels of Figure 1. S is the number of bonds in the funnel, the calculations were performed on 150×150 lattices and $G = 1/4$.

$$i_{\max} \sim \nu(l/a)^{1-\nu} , \tag{5}$$

where a is the bond length and ν is an implicit function of β:

$$\tan \nu\beta \tan \nu \left(\frac{\pi}{2} - \beta\right) = G. \tag{6}$$

The number of bad conductors in this defect is given by

$$S = (l/a)^2 \sin 2\beta. \tag{7}$$

As discussed in Ref. 4, Eq. (6) is derived by approximating the lattice funnel by continuous regions with two different conductivities as shown in Fig. 3. Ohm's law and the equation of current continuity are solved with appropriate boundary conditions along the diagonals. In the continuum limit, the current density at the center of the funnel shape diverges as $r^{\nu-1}$. When the lattice cutoff is included we arrive at Eq. (5).

For a given $i_{\max}$ we can use Eqs. (5)–(7) to solve for the β which minimizes S. Under the assumption that the critical defect is funnel shaped this yields the critical opening angle within the continuum approximation. It is straightforward to verify from these equations that for S (or equivalently $i_{\max}$) less than a crossover value $S_c(G)$, the funnel defect carrying the largest current has $\beta \neq 45°$. The optimum opening angle arises from a tradeoff between the behavior of $i_{\max}$ which diverges most strongly in l at 45° and the number of bad conductors in the lattice which is minimized when β is small. For $S \geq S_c(G)$ the divergence in $i_{\max}$ dominates and the largest current is found in the 45° funnel. In agreement with the numerical work, we find that the crossover to the 45° funnel occurs at relatively large sizes, e.g. $S_c(0.25) \approx 350$ compared to a crossover near 500 bonds for the numerical study. For $G = 0.25$ and $S = 350$, we find that $i_{\max} \approx 1.96$ in the 45° funnel. That is to say that the central bond of this funnel carries a current which is about twice the typical bond current. This is somewhat smaller than the numerical result of 2.6.

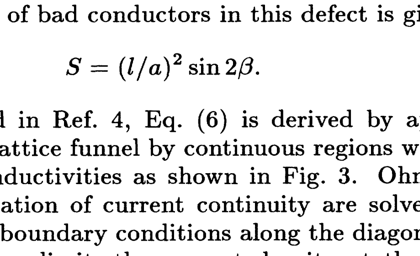

FIG. 3. Geometry of the continuum funnel showing the opening angle β and the diagonal length l. The region of bad conductance is shaded.

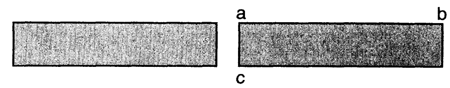

FIG. 4. Geometry of the continuum two-slit defect. The region of bad conductance is shaded.

It is also possible to study the two-slit defect in the continuum approximation. As the size of the two-slit defect increases the central current reaches a maximum value, i_{2sl}. This value can be estimated using the continuum version shown in Fig. 4. The analysis of the current in the central region of the defect proceeds by imposing current continuity across the horizontal boundary *a*-*b* and electric field continuity across the vertical boundary *a*-*c*. The result[4] is $i_{2sl} = 1/G$. This result is an overestimate, for example when $G = 0.25$, the numerical results shown in Fig. 2 suggest that $i_{2sl} \approx 2.5$, but is expected to yield the correct scaling of i_{2sl} with $1/G$.

How does the critical defect depend upon f, the fraction of bad conductors? The numerical work applies to the small f regime since the background lattice consists entirely of good conductors. The continuum approximation, Eqs. (5)–(7), is explicitly independent of f and yields the same result for either β or $\pi/2 - \beta$. However, the assumption that f is either near 0 or 1 is implied in Eq. (7). If f is near zero or one then only the number of minority conductors enters into the expression for the density of the defect. If both kinds of conductors must be specified as is the case when $f \approx 0.5$ then the effective area of the defect does not shrink according to Eq. (7) and the advantage gained by having the angle deviate from 45° is lost. If f is near 1 rather than near zero defects consist of a configuration of good conductors in a background of bad conductors. In this case, it is the good conductors whose area must be minimized and the solution of Eqs. (5)–(7) with $\beta > \pi/4$ is the appropriate one. Thus, if good conductors predominate, the critical defect for intermediate current values will be a funnel with angle greater than 45°. For small currents, the appropriate linear defect is a line of good conductors parallel to the current flow. The maximum current enhancement due to a line of good conductors can also be estimated by the continuum method as $1/G$.

As is clear from either the numerical work or the above analysis, the current in a funnel defect is a very weak function of angle for small defects however as the defect size increases beyond $S_c(G)$ the difference between the current found in the 45° funnel and funnels with other angles increases with S. This can be seen by considering derivatives of $i_{\max}$ with respect to β,

$$\left[\frac{di_{\max}}{d\beta}\right]_{S,\beta=\frac{\pi}{4}} = 0, \quad -\left[\frac{d^2 i_{\max}}{d\beta^2}\right]_{S,\beta=\frac{\pi}{4}} \sim S^{\alpha(G)}\ln(S) \tag{8}$$

with $\alpha(G)$ defined in Eq. (4). This result supplements the arguments given in Ref. 4 that the 45° funnel controls the very large current tail of the distribution of currents in a random resistor network. The conclusion is that currents in a random resistor network which are many times larger than average are almost surely found in 45° funnels.

On the other hand, for the range of currents (and associated defect sizes) accessible to numerical simulation many types of configurations will contribute to the current distribution. For example, the largest expected defect size in a RRN of size 200 × 200 with $f \approx 0.25$ is roughly $10 - 20$ bonds. It is therefore surprising that numerical studies[5,6] for $P(I)$ are in reasonably good agreement with the asymptotic predictions of Eqs. (2) and (4). An indication of why this is so is found in the fact that $i_{\max}$ is a weak function of β until very large values of S are reached. This suggests that a range of favorable defect shapes all contribute to the current distribution in a way which is similar to the 45° funnel. The numerical studies yield effective values of α which lie above $\alpha(G)$ and which deviate toward $\alpha = 1$ as G becomes small. This is explained by the crossover from one-dimensional (two-slit or line) defects with $\alpha = 1$ to the two-dimensional funnel defect. For large but finite sample sizes, the crossover occurs at a current which scales as $1/G$.

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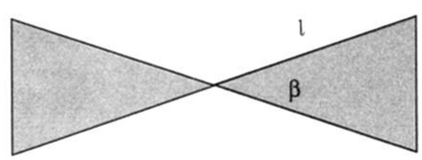

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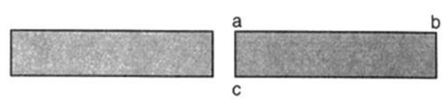

FIG. 4. Geometry of the continuum two-slit defect. The region of bad conductance is shaded.