Suppression of the order parameter in homogeneous disordered superconductors

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We calculate the first-order correction to the order parameter Δ in a disordered superconductor, using a formalism that treats electron-electron repulsion and order parameter fluctuations on an equal footing. We find that this correction is plagued by the same low-momentum singularities from the screened Coulomb potential as the correction to the transition temperature, and that these singularities are cancelled between terms in both cases. The cancellation of leading-order terms means that we must consider all terms of the same perturbation order in the calculation of Δ and T_c if we are not to obtain qualitatively incorrect results. High momentum and frequency fluctuations then dominate, leading to a suppression of Δ proportional to $\ln^3(\Delta\tau)$. We therefore expect that the ratio Δ/T_c will be roughly constant, which is confirmed by detailed numerical evaluation. We show the general utility of our formalism in the evaluation of physical quantities in the disordered superconductor. Finally we comment on the appropriateness of the dirty-boson approach in the region where our calculations are applicable.

I. INTRODUCTION

The suppression of superconductivity in disordered thin films has been the subject of much interest[1,2] in the past few years, due to some interesting experiments[3–5] on thin homogeneous superconducting films, and to the growth of our understanding of localization phenomena in disordered metals. The majority of experiments measure the resistance per square of a family of films as a function of temperature, $R_\square(T)$, the film thickness being progressively increased (and $R_\square$ consequently decreased) by the vapor deposition of more material. The transition temperature T_c is then seen to be suppressed as the resistance per square in the normal state $R_\square$, a measure of the disorder in the system, is increased. In fact, in recent experiments[3,4] T_c goes to zero (i.e., superconductivity is completely destroyed) when $R_\square$ is of order the quantum of resistance, $R_0=h/4e^2\approx 6.45$ kΩ. Films of higher resistance show insulating behavior as $T\to 0$, and all R_T curves can be collapsed onto two curves, one for superconducting and one for insulating behavior, by the choice of appropriate reduced variables.[6] Much less data exists on the suppression of the order parameter at zero temperature, Δ_0, another measure of the superconducting state. These experiments[4,5] obtain Δ_0 by measuring the tunneling current of a normal-insulating-superconductor junction, with the thin film as the superconductor as the film thickness is increased. Δ_0 is seen to be suppressed as normal state $R_\square$ is increased in a similar manner to $T_c-\Delta_0$ and T_c appears to vanish at the same value of $R_\square$, with the ratio Δ_0/T_c remaining roughly constant[4] as this happens.

The existence of a well-defined many-body theory for the dirty metal and superconductor suggests that theorists should calculate both T_c and Δ_0 as a function of $R_\square$, and see if their results can be reconciled with experiment. These calculations will be in the perturbative (weak localization) regime, and so will not allow one to get close to the superconductor-insulator transition, but should be able to explain the weak disorder region. The appropriate disorder parameter in the perturbation theory in both the exact two-dimensional (2D) and quasi-2D cases is the ratio of $R_\square$ to the quantum of resistance, $R_0=h/4e^2$,

$$\frac{R_\square}{R_0}=\begin{cases}\dfrac{6\pi}{(k_F l)(k_F t)}, & k_F t\gg 1 \text{ (quasi-2D)}\\ \dfrac{4}{k_F l}, & k_F t\ll 1 \text{ (2D)},\end{cases} \tag{1.1}$$

where k_F is Fermi wave vector, l is elastic mean free path, and t is film thickness. Much work,[7–13] has been performed on the calculation of $T_c(R_\square)$ as this is the easier quantity to evaluate—one starts from the metallic state and looks for the superconducting instability in the electron pair propagator. The evaluation of Δ_0 is more difficult since one must now sit in the superconducting state and treat the pair correlations consistently, and consequently this problem has received less attention.[12,14] The main technical difficulty is that the extra correlation between electron pairs leads to a 2×2 matrix structure for the electron Green functions, and to greater algebraic complexity in any calculation.

Although there have been many works on the effect of disorder on T_c, it is only comparatively recently that the first-order perturbation theory has been correctly understood. To see why this is so, let us start by recalling the mean-field (BCS) formula for T_c,

$$T_c=1.13\omega_D\exp\left[-\frac{1}{N(0)(\lambda-\mu^*)}\right], \tag{1.2}$$

where λ is the effective attractive interaction between electrons mediated by phonons of frequency less than the Debye frequency ω_D, μ^* is the repulsive Coulomb pseudopotential,[15] and $N(0)$ is the single-particle density of states at the Fermi surface per spin. ω_D and λ are not ex-

pected to be strongly affected by disorder, so one focuses on the effect of disorder on $N(0)$ and μ^*. It is known[16] that $N(0)$ is strongly suppressed due to the effect of disorder on the screening of the Coulomb repulsion between electrons. At finite frequency in the dirty metal the screened Coulomb interaction has the same $1/q^{d-1}$ singularity at low momentum q in a d-dimensional system as the bare Coulomb potential due to the diffusive motion of the electrons. Most work therefore focused on the effect of the Coulomb interaction on T_c, and has ignored the effect of electron-pair fluctuations. As discussed in detail in Sec. IV, a term involving both Coulomb and pair fluctuations was not considered in most calculations.[8-10] Normally the omission of a term of the same order as those being considered would lead only to a change in numerical coefficient, but here the omitted diagram leads to cancellation of the low-momentum singularity of the screened Coulomb potential, and a qualitative change in the result. This cancellation shows us that we must include *all* diagrams of the same perturbation order in any calculation of T_c, and alerts us to the possibility of similar problems in any evaluation of Δ_0. Although the details of the first-order perturbation correction to T_c are known to theorists in this area, we will provide what we believe is the first detailed derivation in the literature together with new work on Δ_0.

Here we calculate the first-order correction to the order parameter, $\Delta(T)$, for all temperatures T, making sure that all contributions are included. In Sec. II we derive the matrix formalism used in our calculation, which treats all bosonic fluctuations on an equal footing. We derive the diagrammatic rules of this formalism, and expressions for the impurity ladder, the ladder-dressed vertices, and the effective propagators in the random-phase approximation (RPA). Density and order-parameter phase fluctuations are shown to be coupled, and this leads to the same low-momentum singularities in these propagators that occur in the normal state. In Sec. III we perform the actual calculation of the first-order correction to the BCS self-consistency equation for order parameter $\Delta(T)$. In Sec. IV we obtain the first-order correction to T_c by linearizing the equation for $\Delta(T)$, and show that this result is identical to that obtained by looking for the singularity in the pair propagator in the normal state. We show that the low-momentum singularity in the screened Coulomb potential is cancelled,[17] and that the correction to T_c can be split into a Coulomb repulsion and a pair-fluctuation term, with the former dominating. In Sec. V we analyze in detail the first-order correction to Δ_0. We show that the low-momentum singularities in density and phase propagators are cancelled except for a single term that can be shown to be due to the Goldstone mode. The similarity of behavior in the T_c and Δ_0 results means that Δ_0/T_c should be roughly independent of disorder. In Sec. VI we perform detailed numerical evaluations of $T_c(R_\square)$ and $\Delta_0(R_\square)$ to make sure we have not made any invalid approximations in Secs. IV and V. The numerical results verify the results derived analytically.

The basic conclusion of this work is that the dependence of Δ_0 and T_c on disorder is basically the same, and that this is mirrored in a similarity in possible pitfalls in calculation. We show that low-momentum singularities in screened potentials occur, and are cancelled, in both the normal and superconducting states, demonstrating the importance of keeping all terms of the same order in such a calculation. We recover not only the low-frequency, low-momentum fluctuation terms obtained by dirty-boson theories, but also the high-frequency, high-momentum fluctuations. The latter arise from the fact that we are dealing with pairs of electrons rather than single bosons, and moreover are seen to be dominant in the perturbative region. Accordingly in the regime where both approaches are valid, the dirty-boson approach misses the dominant term.

II. DERIVATION OF THE 4×4 MATRIX FORMALISM

The model Hamiltonian

We consider a system of electrons that scatter off static nonmagnetic impurities and interact with each other via the long-range Coulomb repulsion and the (phonon-mediated) BCS contact attraction. The scattering of electrons from static impurities is described by the Hamiltonian

$$H_{e-i}=\sum_\sigma \int d\mathbf{x}\,\psi_\sigma^+(\mathbf{x})\left[-\frac{\nabla_x^2}{2m}+\sum_i u(\mathbf{x}-\mathbf{x}_i)\right]\psi_\sigma(\mathbf{x}) , \tag{2.1}$$

where $\psi_\sigma^+(\mathbf{x})$, $\psi_\sigma(\mathbf{x})$ are the electron creation and annhilation operators, and $u(\mathbf{x}-\mathbf{x}_i)$ is the impurity potential at $\mathbf{x}$ due to an impurity at $\mathbf{x}_i$. The impurity scattering leads to a lifetime $\tau_\mathbf{k}$ for a state of fixed momentum $\mathbf{k}$ of

$$\frac{1}{\tau_\mathbf{k}}=2\pi N(0)n_i|u(\mathbf{k})|^2 , \tag{2.2}$$

where n_i is the impurity density, and $N(0)$ is the density of states per spin at the Fermi surface. Here we will make the usual assumption of δ-function impurity potentials so that lifetime τ is independent of momentum.

The Coulomb repulsion between electrons is described by the Hamiltonian

$$H_C=\sum_{\sigma,\sigma'}\int d\mathbf{x}\int d\mathbf{x}'\psi_\sigma^+(\mathbf{x})\psi_\sigma(\mathbf{x})\frac{e^2}{|\mathbf{x}-\mathbf{x}'|}\psi_{\sigma'}^+(\mathbf{x}')\psi_{\sigma'}(\mathbf{x}') , \tag{2.3}$$

which leads to a bare Coulomb propagator that is just the Fourier transform of the potential above. For 2D, 3D, and quasi-2D systems we obtain

$$\begin{aligned} V_{2D}^0(q)&=\frac{2\pi e^2}{q} , \\ V_{3D}^0(q)&=\frac{4\pi e^2}{q^2} , \\ V_{q-2D}^0(q_\parallel,q_\perp)&=\frac{4\pi e^2}{q_\parallel^2+q_\perp^2}[1-(-1)^n e^{-q_\parallel t/2}] , \end{aligned} \tag{2.4}$$

where in the quasi-2D case $q_\parallel$ is the continuous momen-

tum parallel to the film, and $q_\perp = 2\pi n/t$, $n = 0, \pm 1, \pm 2 \ldots$, is the discrete momentum perpendicular to the film of thickness t. Experimentally we will always have a quasi-2D system, and so should always use the quasi-2D potential V_{q-2D}. However the mathematical difficulties associated with having to both integrate over the continuous $q_\parallel$ and sum over the discrete $q_\perp$ leads us to consider the limiting 2D and 3D cases. For $0 < q_\parallel < 2\pi/t$ only $q_\perp = 0$ is important, and V_{q-2D} reduces to V_{2D}, and the sum over q is 2D; for $2\pi/t < q_\parallel < 1/l$, $q_\perp$ can be considered effectively continuous, and V_{q-2D} reduces to V_{3D}, and the sum over q is 3D. Note that for any finite thickness t the quasi-2D potential has a $1/q$ singularity at $q=0$.

The BCS attraction is described by the Hamiltonian

$$H_{\rm BCS} = -\lambda \sum_{\sigma,\sigma'} \int d\mathbf{x} \psi_\sigma^+(\mathbf{x}) \psi_\sigma(\mathbf{x}) \psi_{\sigma'}^+(\mathbf{x}) \psi_{\sigma'}(\mathbf{x}) , \quad (2.5)$$

which corresponds to an instantaneous contact interaction $-\lambda\delta(\mathbf{x}-\mathbf{x}')$. The fact that this interaction is derived from the electron-phonon interaction causes us to introduce an upper frequency cutoff at the Debye frequency ω_D. This leads to an interaction line contributing factor $-\lambda$. The contact nature of the interaction leads to the restriction $\sigma' = -\sigma$ which is due to the Pauli principle restriction that one cannot have two identical fermions at the same point in space.

Nambu-Gor'kov approach

Having introduced the model Hamiltonian, let us recall how the standard Nambu matrix field theory[18,19] of the superconductor is derived.[20] The 2×2 matrix structure is needed to include two types of correlation between electron operators—the usual particle-hole correlation $\langle \psi_\uparrow \psi_\uparrow^+ \rangle$, and the anomalous pairing correlation $\langle \psi_\uparrow \psi_\downarrow \rangle$. We introduce the vector Nambu operator

$$\Psi = \begin{bmatrix} \psi_\uparrow \\ \psi_\downarrow^+ \end{bmatrix}; \quad \Psi^+ = (\psi_\uparrow^+ \quad \psi_\downarrow) \quad (2.6)$$

whose matrix propagator now includes both types of correlation:

$$\langle \Psi \Psi^+ \rangle = \begin{bmatrix} \langle \psi_\uparrow \psi_\uparrow^+ \rangle & \langle \psi_\uparrow \psi_\downarrow \rangle \\ \langle \psi_\downarrow^+ \psi_\uparrow^+ \rangle & \langle \psi_\downarrow^+ \psi_\downarrow \rangle \end{bmatrix} . \quad (2.7)$$

In the normal state the temperature Nambu-Green function is

$$G(k, i\omega) = \begin{bmatrix} \dfrac{1}{i\omega - \xi_k} & 0 \\ 0 & \dfrac{1}{i\omega + \xi_k} \end{bmatrix} = \frac{1}{z - \xi_k \tau_3} , \quad (2.8)$$

where $z = i\omega$, $\omega = (2n+1)\pi T$ is a Fermi-Matsubara frequency, and the τ_i are Pauli matrices. The diagrammatic[20] rules are then the same as in the normal state, except for the matrix structure of the electron Green function, and the presence of Pauli matrix τ_3 at each interaction or impurity vertex due to the electron-density operator being expressed in terms of the Nambu operator by

$$\rho = \psi_\uparrow^+ \psi_\uparrow + \psi_\downarrow^+ \psi_\downarrow = \Psi^+ \tau_3 \Psi . \quad (2.9)$$

The pairing correlations in the clean superconductor can be taken into account self-consistently as shown in Fig. 1(a). With the ansatz $\Sigma = \Delta\tau_1$ for the self-energy, the Green function for the pure superconductor becomes

$$G_0(k,z) = \frac{1}{z - \xi_k \tau_3 - \Delta\tau_1} = \frac{z + \xi_k \tau_3 + \Delta\tau_1}{z^2 - \xi_k^2 - \Delta^2} , \quad (2.10)$$

and the diagram of Fig. 1(a) gives self-energy,

$$\Sigma = -\lambda T \sum_\omega N(0) \int d\xi_k \frac{\tau_3 (z + \xi_k \tau_3 + \Delta\tau_1)\tau_3}{\xi_k^2 - z^2 + \Delta^2}$$

$$= N(0)\lambda \Delta \tau_1 T \sum_\omega \frac{1}{\sqrt{\omega^2 + \Delta^2}} . \quad (2.11)$$

In the above we have linearized the momentum sum around the Fermi surface,

$$\sum_k \approx N(0) \int d\xi_k d\hat{\Omega} , \quad (2.12)$$

where $d\hat{\Omega}$ is the angular integral, and we have noted that τ_0 and τ_3 components vanish due to oddness of integrand in ω and ξ_k, respectively. We therefore obtain the BCS consistency equation for Δ

$$1 = N(0)\lambda T \sum_\omega \frac{1}{\sqrt{\omega^2 + \Delta^2}} . \quad (2.13)$$

If we also include the Coulomb repulsion in Fig. 1(a) we find that λ is replaced by $\lambda - \mu^*$, where μ^* is the Coulomb pseudopotential[15]

$$\mu^* = \frac{\mu}{1 + \mu \ln(\varepsilon_F/\omega_D)}, \quad \mu = N(0) \langle V_C(\mathbf{k}_F - \mathbf{k}_F') \rangle_{\rm FS} . \quad (2.14)$$

Henceforth λ will now include μ^*.

We can treat the presence of nonmagnetic impurities by including an extra self-energy diagram to describe the dressing of the electron line by impurities as shown in Fig. 1(b). We then make the ansatz that the pairing self-

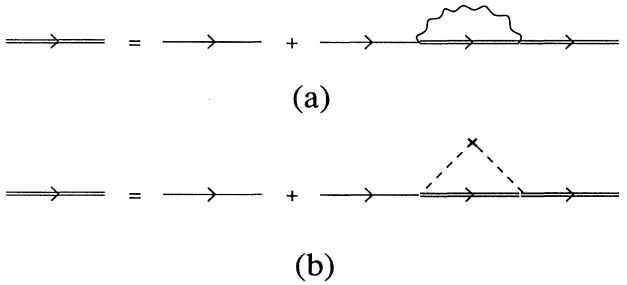

FIG. 1. Diagrammatic equation for mean-field electron Green function in (a) pure superconductor and (b) disordered superconductor. The double solid line is the mean-field electron Green function, the single straight line is the electron Green function for the pure metal, the wavy line is the BCS interaction, and the dashed line is the impurity interaction.

energy has the form $\Sigma_p=\Delta\tau_1$, and the impurity self-energy the form $\Sigma_{\rm imp}=-(\bar{z}-z)+(\bar{\Delta}-\Delta)\tau_1$, so that the Green function for the dirty superconductor is

$$G_0(k,z)=\frac{\bar{z}+\xi_k\tau_3+\bar{\Delta}\tau_1}{\bar{z}^2-\xi_k^2-\bar{\Delta}^2}\ , \tag{2.15}$$

which is just the Green function for the clean superconductor with z, Δ, replaced by $\bar{z}$, $\bar{\Delta}$, respectively. The diagrammatic definition of $\Sigma_{\rm imp}$ then allows us to write $\bar{z},\bar{\Delta}$ in terms of z,Δ,

$$\begin{aligned}\bar{z}&=z\left[1+\frac{i}{2\tau\sqrt{z^2-\Delta^2}}\right]\\&=i\omega\left[1+\frac{1}{2\tau\sqrt{\omega^2+\Delta^2}}\right];\quad \bar{z}/z=\bar{\Delta}/\Delta\ .\end{aligned} \tag{2.16}$$

The diagrammatic definition of the pairing energy Σ_p leads to the same self-consistency equation or Δ except that z,Δ are replaced by $\bar{z}$, and $\bar{\Delta}$. The relation $\bar{z}/\bar{\Delta}=z/\Delta$ means that this equation is exactly the same as before, no superconductivity is unaffected by nonmagnetic impurities at the mean-field level—a result known as Anderson's theorem.[21,22]

We note that the z and $\bar{z}$ defined above are purely imaginary—we make this choice as a way of keeping track of factors of i. Other notations we will use extensively are $W=\sqrt{\omega^2+\Delta^2}$, $\varepsilon=\sqrt{\bar{z}^2-\bar{\Delta}^2}$, $z'=i\omega'=i(\omega+\Omega)$, where Ω is a bosonic Matsubara frequency from an external potential. For the new frequency z' we define $\bar{z}'$, W', $\bar{\Delta}'$, and ε' in the same manner as for $\bar{z}$, W, $\bar{\Delta}$, and ε, with $z=i\omega$ replaced by $z'=i\omega'$. We summarize all the notations used in Table I for ease of reference.

The 4×4 matrix formalism

The Nambu field theory above is a complete theory of the pure or disordered superconductor, but suffers from the problem that the electron-electron interaction, which is due to electron-density fluctuations, is treated differently than the other fluctuations in the system—the order-parameter amplitude and phase fluctuations. To include the latter fluctuations in the Nambu theory, one needs to include diagrams in which a pair of electrons repeatedly interact with each other as shown in Fig. 2(a). The electron-electron interactions shown in the ladder include both BCS and Coulomb parts so that λ includes μ^* as previously discussed. In calculations of T_c such pairs of electron lines can be rewritten as an electron pair propagator, leading to the Coulomb interaction and pair fluctuations being treated identically. In the superconductor the matrix structure makes such a rewriting less easy, but it can be done in the case of contact interaction

$$\begin{aligned}&-\lambda[\Psi^+(\mathbf{x})\tau_3\Psi(\mathbf{x})][\Psi^+(\mathbf{x})\tau_3\Psi(\mathbf{x})]\\&\quad=-\tfrac{1}{2}\lambda[\Psi^+(\mathbf{x})\tau_1\Psi(\mathbf{x})][\Psi^+(\mathbf{x})\tau_1\Psi(\mathbf{x})]\\&\qquad-\tfrac{1}{2}\lambda[\Psi^+(\mathbf{x})\tau_2\Psi(\mathbf{x})][\Psi^+(\mathbf{x})\tau_2\Psi(\mathbf{x})]\ ,\end{aligned} \tag{2.17}$$

which can be written diagrammatically as in Fig. 2(b). The diagrams in Fig. 2(a) are then equivalent to the usual screening diagrams for the electron-electron interaction except that they have τ_1 or τ_2 at their vertices, as shown in Fig. 2(c). We are therefore led to a matrix theory in which τ_1, τ_2, and τ_3 are also allowed at vertices, and the bare matrix potential is

TABLE I. A summary of the notation used in the calculations of Secs. II and III. $\omega=(2n+1)\pi T$ is a Fermi-Matsubara frequency, while $\Omega=2m\pi T$ is a Bose-Matsubara frequency. We define the imaginary quantities $z=i\omega$ and $z'=i\omega'$ to make it easier to keep track of factors of i in calculations. The overlined quantities are those appearing in the electron Green function after it has been renormalized by disorder. W and W' are purely real, and the square root in their definition has positive real part; ε and ε' are purely imaginary and the square root in their definition has positive imaginary part. The $\alpha_\pm$, $\bar{\alpha}_\pm$, and $\beta_\pm$ are coherence factors.

$z=i\omega$	$z'=i\omega'=i(\omega+\Omega)$
$\bar{z}/z=\bar{\Delta}/\Delta=1+\dfrac{i}{2\tau\sqrt{z^2-\Delta^2}}$	$\bar{z}'/z'=\bar{\Delta}'/\Delta=1+\dfrac{i}{2\tau\sqrt{z'^2-\Delta^2}}$
$W=\sqrt{\omega^2+\Delta^2}$	$W'=\sqrt{\omega'^2+\Delta^2}$
$\varepsilon=\sqrt{\bar{z}^2-\bar{\Delta}^2}$	$\varepsilon'=\sqrt{\bar{z}'^2-\bar{\Delta}'^2}$
$\alpha_\pm=\bar{\alpha}_\pm-2=1-\dfrac{\bar{z}\bar{z}'\pm\bar{\Delta}\bar{\Delta}'}{\varepsilon\varepsilon'}$	$\beta_\pm=\dfrac{\bar{z}'\bar{\Delta}\pm\bar{z}\,\bar{\Delta}'}{\varepsilon\varepsilon'}$

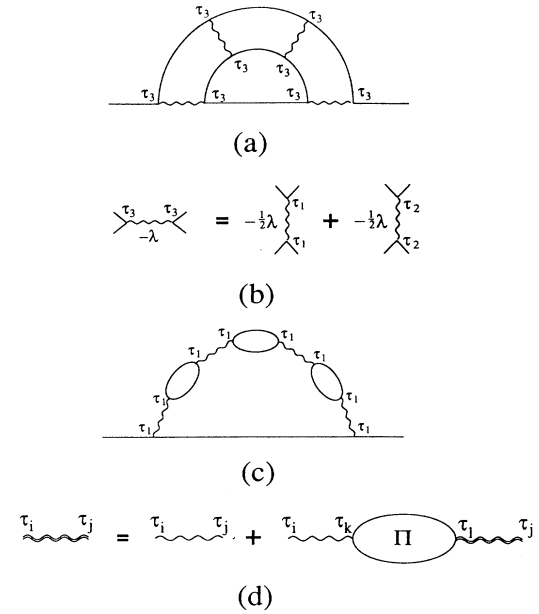

FIG. 2. (a) A typical correction to the self-energy of a superconductor due to order-parameter fluctuations. (b) Diagrammatic rewriting of the BCS contact interaction in terms of order parameter amplitude and phase operators. (c) Order-parameter fluctuation diagram (a) after the rewriting in (b). We see that the fluctuation diagram now has the same form as the screening diagram for the electron-electron interaction. (d) Screening of electron-electron interactions in a superconductor by density and order parameter fluctuations. All fluctuation terms can be now be treated in a single screening diagram in which all Pauli matrices τ_1, τ_2, and τ_3 may occur at vertices. This defines the screened interaction V_{ij}.

$$V^0=\begin{bmatrix} -\frac{1}{2}\lambda & 0 & 0 \\ 0 & -\frac{1}{2}\lambda & 0 \\ 0 & 0 & V_C(q) \end{bmatrix}, \tag{2.18}$$

where the rows and columns of the matrix correspond to the τ_1, τ_2, and τ_3 vertices. The order parameter fluctuations are then included in the screening of the matrix potentials

$$V=V^0+V^0\Pi V\ , \tag{2.19}$$

as shown in Fig. 2(d). We find it more convenient to work with $2V^0$ since then we have potential $-\lambda$ in the 11 and 22 channels, and $2V_C$ in the 33 channel, which can be regarded as the Coulomb repulsion for both spins. We therefore shuffle factors of 2 in the screening equation to get

$$(2V)=(2V^0)+(2V^0)(\tfrac{1}{2}\Pi)(2V)\ , \tag{2.20}$$

so we will need to include an extra factor of $\frac{1}{2}$ in the definition of Π, and overall factor $\frac{1}{2}$ to reduce $2V$ to V. We note that essentially similar techniques have previously been used[23,24,13] in work on dirty superconductors.

So far all of the above has been purely algebraic manipulation to enable us to perform perturbation theory in a more systematic manner. Physically the matrix formalism derived above corresponds to viewing the electron-electron interactions as the exchange of bosonic collective modes between electrons. There are four possible bosonic operators in the system that are made by forming bilinear combinations of two Nambu electron operators. A good physical choice is to use the four Pauli matrices, τ_1, τ_2, τ_3, and $\mathbf{k}\tau_0$ which lead to the operators for order-parameter amplitude Δ, order parameter phase ϕ, electronic density ρ, and electronic current j, respectively, as shown in Table II. The current fluctuations lead to electromagnetic interactions, and including these leads to the so-called relativistic formulation,[25] so the final potential will be a 4×4 matrix. We shall ignore these fluctuations here as they are less important than the other terms. We note that our formalism is equivalent to the path-integral description[26] of superconductivity, where the BCS contact interaction and Coulomb interactions are split up using the Hubbard-Stratonovich transformation.

Ladders, vertices, and polarization bubbles

Having derived our matrix formalism, we proceed to evaluate the impurity ladder for the superconductor shown in Fig. 3(a), the impurity dressed vertices shown in Fig. 3(b), and the RPA polarization bubble shown in Fig. 3(c), all of which are needed for the calculations of Sec. III.

The impurity ladder is given by the geometric series

$$\Gamma=\Gamma_0+\Gamma_0S\Gamma_0+\Gamma_0S\Gamma_0S\Gamma_0+\cdots\ , \tag{2.21}$$

where Γ_0 is the impurity line

$$\Gamma_0=\frac{1}{2\pi N(0)\tau}\tau_3\otimes\tau_3 \tag{2.22}$$

and S is the momentum sum of a direct product of Green functions

$$S=\sum_k G(k,i\omega)\otimes G(k+q,i\omega+i\Omega)\ . \tag{2.23}$$

Performing the sum over momentum leads to the expression for S

$$S=\pi N(0)\tau I\left[\tau_3\otimes\tau_3-\frac{(\bar{z}-\bar{\Delta}\tau_1)\otimes(\bar{z}'-\bar{\Delta}'\tau_1)}{\varepsilon\varepsilon'}\right], \tag{2.24}$$

where I is the integral

$$I=\frac{1}{\pi\tau}\int d\xi_k d\hat{\Omega}\frac{\xi_k(\xi_k-\mathbf{q}\cdot\mathbf{v}_F)}{(\xi_k^2-\varepsilon^2)[(\xi_k-\mathbf{q}\cdot\mathbf{v}_F)^2-\varepsilon'^2]}$$
$$\approx 1-(Dq^2+W+W')\tau\ , \tag{2.25}$$

where the approximation above is valid in the limit $\Omega\tau,ql,\Delta\tau\ll1$, which we call the dirty-limit approximation. The geometric series for Γ can then be summed to yield

$$\Gamma=\frac{1}{2\pi N(0)\tau}\tau_3\otimes\tau_3+\frac{1}{4\pi N(0)\tau^2(Dq^2+W+W')}\left[\tau_3\otimes\tau_3-\frac{(\bar{z}-\bar{\Delta}\tau_1)\otimes(\bar{z}'-\bar{\Delta}'\tau_1)}{\varepsilon\varepsilon'}\right], \tag{2.26}$$

TABLE II. The bosonic propagators in a disordered superconductor, and the Pauli matrices that occur at their interaction vertices. One can make four bosonic operators from bilinear combinations of the Nambu operator with the Pauli matrices. Each of these has a physical meaning as shown, and possesses a coupling in the model Hamiltonian. Since the electronic current is coupled to the vector electromagnetic interaction, which is relativistically weaker than the scalar interaction of the electronic density, we will ignore its effect.

Physical meaning	Symbol	Operator, $O_i=\Psi^+\tau_i\Psi$	Pauli matrix, τ_i
Order parameter amplitude	Δ	$\psi^+_{\mathbf{k}\uparrow}\psi^+_{-\mathbf{k}\downarrow}+\psi_{-\mathbf{k}\downarrow}\psi_{\mathbf{k}\uparrow}$	τ_1
Order parameter phase	ϕ	$-i(\psi^+_{\mathbf{k}\uparrow}\psi^+_{-\mathbf{k}\downarrow}-\psi_{-\mathbf{k}\downarrow}\psi_{\mathbf{k}\uparrow})$	τ_2
Electron density	ρ	$\psi^+_{\mathbf{k}\uparrow}\psi_{\mathbf{k}\uparrow}+\psi^+_{-\mathbf{k}\downarrow}\psi_{-\mathbf{k}\downarrow}$	τ_3
Electron current	j	$\mathbf{k}(\psi^+_{\mathbf{k}\uparrow}\psi_{\mathbf{k}\uparrow}-\psi^+_{-\mathbf{k}\downarrow}\psi_{-\mathbf{k}\downarrow})$	$\mathbf{k}\tau_0$

where we note that although the first term is $O(\Delta\tau)$ smaller than the second, it cannot be ignored as it has a different matrix structure, and so may yield a nonzero result when the larger term gives nothing.

The impurity ladder dressed vertices Λ_i can then easily be evaluated from the impurity ladder Γ. For these vertices we need to calculate the geometric series

$$\Lambda=\Gamma_0 S+\Gamma_0 S\Gamma_0 S+\cdots=(\Gamma-\Gamma_0)\Gamma_0^{-1} \tag{2.27}$$

from which we can obtain the dressed vertex Λ_i by inserting τ_i between the two terms of the direct product,

$$\Lambda_i=\frac{1}{2\tau(Dq^2+W+W')}\times\left[\tau_3\tau_3\tau_i\tau_3\tau_3-\frac{(\bar{z}-\bar{\Delta}\tau_1)\tau_3\tau_i\tau_3(\bar{z}'-\bar{\Delta}'\tau_1)}{\varepsilon\varepsilon'}\right]. \tag{2.28}$$

This leads to the expressions for the three dressed vertices,

$$\begin{aligned}
&2\tau(Dq^2+W+W')\Lambda_\Delta(\omega,\omega')\\
&\quad=\left[1+\frac{(\bar{z}-\bar{\Delta}\tau_1)(\bar{z}'-\bar{\Delta}'\tau_1)}{\varepsilon\varepsilon'}\right]\tau_1\\
&\quad=-(\bar{\alpha}_+ +\beta_+\tau_1)\tau_1\ ,\\
&2\tau(Dq^2+W+W')\Lambda_\phi(\omega,\omega')\\
&\quad=\left[1+\frac{(\bar{z}-\bar{\Delta}\tau_1)(\bar{z}'+\bar{\Delta}'\tau_1)}{\varepsilon\varepsilon'}\right]\tau_2\\
&\quad=-(\bar{\alpha}_- +\beta_-\tau_1)\tau_2\ ,
\end{aligned} \tag{2.29}$$

$$\begin{aligned}
&2\tau(Dq^2+W+W')\Lambda_\rho(\omega,\omega')\\
&\quad=\left[1-\frac{(\bar{z}-\bar{\Delta}\tau_1)(\bar{z}'+\bar{\Delta}'\tau_1)}{\varepsilon\varepsilon'}\right]\tau_3\\
&\quad=(\alpha_- +\beta_-\tau_1)\tau_3\ ,
\end{aligned}$$

where from now on we replace the labels $i=1,2,3$ by the physical quantities Δ, ϕ, and ρ, respectively, that they represent, as described in Table II. The α and β terms that occur in these vertices are coherence factors

$$\alpha_\pm=1-\frac{\bar{z}\bar{z}'\pm\bar{\Delta}\bar{\Delta}'}{\varepsilon\varepsilon'};\quad \bar{\alpha}_\pm=\alpha_\pm-2;\beta_\pm=\frac{\bar{z}'\bar{\Delta}\pm\bar{z}\,\bar{\Delta}'}{\varepsilon\varepsilon'}\ . \tag{2.30}$$

We note that the vertices $\Lambda_i(\omega',\omega)$, which are obtained by swapping primed and unprimed quantities, are the same as above except that the order of the matrix product is reversed, for example,

$$2\tau(Dq^2+W+W')\Lambda_\phi(\omega',\omega)=-\tau_2(\bar{\alpha}_- +\beta_-\tau_1)\ . \tag{2.31}$$

To evaluate the polarization bubbles Π_{ij} shown in Fig. 3(c) we need to evaluate the geometric series

$$\Pi=S+S\Gamma_0 S+S\Gamma_0 S\Gamma_0 S+\cdots=\Gamma_0^{-1}(\Gamma-\Gamma_0)\Gamma_0^{-1} \tag{2.32}$$

and then insert the vertices τ_i and τ_j between the two factors of the direct product and take the trace. Finally we need the factor -1 for a closed fermion loop, the extra factor $\frac{1}{2}$ discussed in Sec. II, and the sum over ω to get

$$\Pi_{ij}(q,\Omega)=-\frac{1}{2}\pi N(0)T\sum_\omega\frac{1}{Dq^2+W+W'}\mathrm{Tr}\left[\tau_3\tau_3\tau_3\tau_i\tau_3\tau_3\tau_3\tau_j-\frac{\tau_3(\bar{z}-\bar{\Delta}\tau_1)\tau_3\tau_i\tau_3(\bar{z}'-\bar{\Delta}'\tau_1)\tau_3\tau_j}{\varepsilon\varepsilon'}\right]. \tag{2.33}$$

In the above derivation we have made the assumption that we can change the order of summation over frequency and momentum and perform the momentum sum first. This is not true for the first term in the diagrammatic series of Fig. 3(c) when we evaluate $\Pi_{\rho\rho}$. When we treat this term with more care we find that we obtain an extra term $+N(0)$ upon interchanging the order of the sums. The nonzero Π_{ij} are then given by

$$\begin{aligned}
\Pi_{\Delta\Delta}(q,\Omega)&=\pi N(0)T\sum_\omega\left[1+\frac{\omega\omega'-\Delta^2}{WW'}\right]\frac{1}{Dq^2+W+W'}\ ,\\
\Pi_{\phi\phi}(q,\Omega)&=\pi N(0)T\sum_\omega\left[1+\frac{\omega\omega'+\Delta^2}{WW'}\right]\frac{1}{Dq^2+W+W'}\ ,\\
\Pi_{\rho\rho}(q,\Omega)&=-\pi N(0)T\sum_\omega\left[1-\frac{\omega\omega'+\Delta^2}{WW'}\right]\frac{1}{Dq^2+W+W'}+N(0)\ ,\\
\Pi_{\phi\rho}(q,\Omega)&=-\pi N(0)T\sum_\omega\frac{\Delta\Omega}{WW'}\frac{1}{Dq^2+W+W'}=-\Pi_{\rho\phi}(q,\Omega)\ .
\end{aligned} \tag{2.34}$$

We can then derive the screened potentials V_{ij},

$$V=\begin{bmatrix}(-\lambda^{-1}+\Pi_{\Delta\Delta})^{-1} & 0 & 0\\ 0 & [(2V_C(q))^{-1}+\Pi_{\rho\rho}]/\mathcal{D} & -\Pi_{\phi\rho}/\mathcal{D}\\ 0 & \Pi_{\phi\rho}/\mathcal{D} & (-\lambda^{-1}+\Pi_{\phi\phi})/\mathcal{D}\end{bmatrix}, \tag{2.35}$$

where

$$\mathcal{D}\equiv(-\lambda^{-1}+\Pi_{\phi\phi})[(2V_C(q))^{-1}+\Pi_{\rho\rho}]+\Pi_{\phi\rho}^2 . \tag{2.36}$$

The coupling between the phase and density fluctuations is a manifestation of gauge invariance.[27]

Properties of the screened potentials

The first thing we can check is that setting $\Delta=0$ reproduces the normal-state results. $\Pi_{\phi\rho}$ is zero as it is proportional to Δ, so the Π matrix becomes diagonal. $\Pi_{\rho\rho}$ reduces to

$$\begin{aligned}\Pi_{\rho\rho}&=N(0)-2\pi N(0)T\sum_{\Omega}\frac{\theta(-\omega(\omega+\Omega))}{Dq^2+|\Omega|}\\&=N(0)\frac{Dq^2}{Dq^2+|\Omega|}\end{aligned} \tag{2.37}$$

the familiar polarization bubble for the dirty metal. We note that $\Pi_{\rho\rho}$ satisfies the sum rules[28]

$$\lim_{q\to0}\lim_{\Omega\to0}\Pi_{\rho\rho}(q,\Omega)=N(0);\quad\lim_{\Omega\to0}\lim_{q\to0}\Pi_{\rho\rho}(q,\Omega)=0 . \tag{2.38}$$

Both $\Pi_{\Delta\Delta}$ and $\Pi_{\phi\phi}$ reduce to the same function

$$\begin{aligned}\Pi_{\Delta\Delta}&=2\pi N(0)T\sum_{\omega}\frac{\theta(\omega(\omega+\Omega))}{Dq^2+|2\omega+\Omega|}\\&=4\pi N(0)T\sum_{\omega>0}\left[\frac{1}{Dq^2+2\omega+|\Omega|}-\frac{1}{2\omega}\right]+2\pi N(0)T\sum_{\omega>0}\frac{1}{\omega}\\&=\ln\left[\frac{1.13\omega_D}{T}\right]+\psi\left[\frac{1}{2}\right]-\psi\left[\frac{1}{2}+\frac{Dq^2+|\Omega|}{4\pi T}\right],\end{aligned} \tag{2.39}$$

so that $V_{\Delta\Delta}$ and $V_{\phi\phi}$ are given by

$$\begin{aligned}V_{\Delta\Delta}(q,\Omega)&=V_{\phi\phi}(q,\Omega)\\&=-N(0)^{-1}\left[\ln\left[\frac{T}{T_{c0}}\right]+\psi\left[\frac{1}{2}+\frac{Dq^2+|\Omega|}{4\pi T}\right]-\psi\left[\frac{1}{2}\right]\right]^{-1}\\&=-L_0(q,\Omega),\end{aligned} \tag{2.40}$$

where $L_0(q,\Omega)$ is the BCS electron-pair propagator, as discussed in Sec. IV.

Having shown that the effective potentials for the dirty superconductor reduce to those for the dirty metal, let us now look in detail at their properties in the superconductor. We note that we can use the BCS self-consistency equation (2.13) to replace λ^{-1} by $\pi N(0)T\sum_\omega 1/W$ in the RPA results for V_{ij}. *In the formulas below we absorb the* λ^{-1} *into the* $\Pi_{\Delta\Delta}$ *and* $\Pi_{\phi\phi}$ *terms.* We first show that there is no singularity in the order-parameter amplitude propagator $V_{\Delta\Delta}$ at zero frequency and momentum for any nonzero Δ,

$$\begin{aligned}V_{\Delta\Delta}(0,0)^{-1}&=\pi N(0)T\sum_{\omega}\left[\left[1+\frac{\omega^2-\Delta^2}{\omega^2+\Delta^2}\right]\frac{1}{2\sqrt{\omega^2+\Delta^2}}-\frac{1}{\sqrt{\omega^2+\Delta^2}}\right]\\&=-\pi N(0)T\sum_{\omega}\frac{\Delta^2}{(\omega^2+\Delta^2)^{3/2}}\neq0 .\end{aligned} \tag{2.41}$$

The presence of the nonzero order parameter has led to fluctuations along this order-parameter direction becoming massive.

We next show that the propagators $V_{\phi\phi}$, $V_{\phi\rho}$, and $V_{\rho\rho}$ all have a singularity at zero momentum for all nonzero frequency and all temperatures. This is the analog of the singularity of the screened potential in the normal metal, which singularity is known to strongly affect that system's properties. To show the existence of this singularity we need only show that the denominator $\mathcal{D}$ vanishes at $q=0$ for all $\Omega\neq0$ and T. Since $V_C(q)\sim q^{d-1}$, we need only prove that

$$\Pi_{\phi\phi}(0,\Omega)\Pi_{\rho\rho}(0,\Omega)+\Pi_{\phi\rho}(0,\Omega)^2=0 , \tag{2.42}$$

since then we know that

$$\Pi_{\phi\phi}(q,\Omega)\Pi_{\rho\rho}(q,\Omega)+\Pi_{\phi\rho}(q,\Omega)^2=O(q^2) . \tag{2.43}$$

This is shown in Appendix A, in fact that

$$-x^2\Pi_{\rho\rho}(0,\Omega)=x\Pi_{\phi\rho}(0,\Omega)$$
$$=\Pi_{\phi\phi}(0,\Omega)$$
$$=2\Omega^2\pi N(0)T\sum_{\omega}\frac{1}{(4\omega^2-\Omega^2)W}\ . \qquad (2.44)$$

where $x=\Omega/2\Delta$ in Table I. The existence of this low-momentum singularity means that in any calculation we perform we must be careful to keep all the singular terms in case cancellations occur. If we next expand the $\Pi_{ij}(q,\Omega)$ around $q=0$ to $O(q^2)$,

$$\Pi_{ij}(q,\Omega)\approx\Pi_{ij}(0,\Omega)+Dq^2\Pi'_{ij}(0,\Omega) \qquad (2.45)$$

we see that Eq. (2.36) for $\mathcal{D}(q,\Omega)$ can be written using Eq. (2.45) to relate $\Pi_{\phi\phi}$, $\Pi_{\phi\rho}$, and $\Pi_{\rho\rho}$,

$$\mathcal{D}\approx-\Pi_{\rho\rho}(0,\Omega)\{V_C^{-1}(q)x^2+Dq^2[x^2\Pi'_{\rho\rho}(0,\Omega)+2x\Pi'_{\phi\rho}(0,\Omega)-\Pi'_{\phi\phi}(0,\Omega)]\}\ . \qquad (2.46)$$

The singular parts of the potentials $V_{\phi\phi}$, $V_{\phi\rho}$, and $V_{\rho\rho}$ in Eq. (2.35) are then related by $V_{\rho\rho}(q,\Omega)=-x^2V_{\phi\phi}(q,\Omega)$, $V_{\phi\rho}(q,\Omega)=xV_{\phi\phi}(q,\Omega)$, and $V_{\phi\phi}(q,\Omega)$ is given by

$$V_{\phi\phi}(q,\Omega)=-\{V_C^{-1}(q)x^2+Dq^2[x^2\Pi'_{\rho\rho}(0,\Omega)+2x\Pi'_{\phi\rho}(0,\Omega)-\Pi'_{\phi\phi}(0,\Omega)]\}\ . \qquad (2.47)$$

From Eq. (2.35) we see that although $V_{\phi\phi}$, $V_{\phi\rho}$, and $V_{\rho\rho}$ are all singular at $q=0$ for $\Omega\neq0$, only $V_{\phi\phi}$ is singular at $q=0$ when $\Omega=0-$ the other two potentials have factors of Ω that send the singular part to zero at $\Omega=0$. This is to be expected since $V_{\phi\phi}$ represents the Goldstone mode of the system which must have a $1/q^2$ singularity at $\Omega=0$.

Another thing we may note is that if we rewrite $V_{\rho\rho}$ in the form,

$$V_{\rho\rho}(q,\Omega)^{-1}=(2V_C(q))^{-1}+\Pi^{\rm eff}_{\rho\rho}(q,\Omega) \qquad (2.48)$$

so that the effective polarization operator for the electron-electron interaction, $\Pi^{\rm eff}_{\rho\rho}$, is given by

$$\Pi^{\rm eff}_{\rho\rho}(q,\Omega)=\Pi_{\rho\rho}(q,\Omega)+\frac{\Pi_{\phi\rho}(q,\Omega)^2}{[-\lambda^{-1}+\Pi_{\phi\phi}(q,\Omega)]} \qquad (2.49)$$

the usual sum rules[28] for this polarization operator are obeyed

$$\lim_{q\to0}\lim_{\Omega\to0}\Pi^{\rm eff}_{\rho\rho}(q,\Omega)=N(0);\quad \lim_{\Omega\to0}\lim_{q\to0}\Pi^{\rm eff}_{\rho\rho}(q,\Omega)=0\ . \qquad (2.50)$$

This explains the apparent paradox that $\Pi_{\rho\rho}$ does not satisfy the usual sum rule—the mixing of phase and density fluctuations means that it is the effective polarization operator $\Pi^{\rm eff}_{\rho\rho}$, defined directly from the physical potential, that satisfies these relations.

We finally note that the vanishing of the denominator $\mathcal{D}$ at $q=0$ has been demonstrated using the V_{ij} we obtained by making the dirty-limit approximation. If we keep the full dependence of the V_{ij} on $\Delta\tau$ will the denominator still vanish, or will it be finite but of size $O(\Delta\tau)$? To get the exact expressions for the Π_{ij} it can be shown that we need simply make the replacement

$$\frac{1}{Dq^2+W+W'}\to\frac{\tau}{\sqrt{[1+\tau(W+W')]^2+2Dq^2\tau}-1} \qquad (2.51)$$

and we see that putting $q=0$ yields the same result in both cases. The vanishing of the denominator is therefore not just an artifice of the dirty-limit approximation.

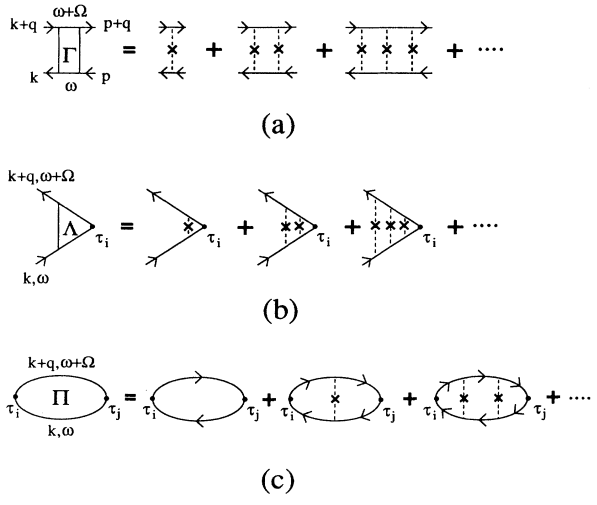

FIG. 3. Diagrammatic definition of (a) the impurity ladder Γ, (b) the impurity dressed vertices Λ_i, and (c) the RPA polarization functions Π_{ij}.

III. DIAGRAMMATIC CALCULATION OF FIRST-ORDER CORRECTION TO ORDER-PARAMETER SELF-CONSISTENCY EQUATION

We evaluate the self-energy diagrams shown in Fig. 4. The self-consistency equation for the order parameter is

$$\Delta=\lambda\sum_k T\sum_\omega\frac{1}{2}\mathrm{Tr}[\tau_1G(k,\omega)]\ , \qquad (3.1)$$

which when G is expanded to first order becomes

$$\Delta=\lambda\sum_k T\sum_\omega\frac{1}{2}\mathrm{Tr}\{\tau_1[G_0(k,\omega)+G_0(k,\omega)\Sigma(k,\omega)G_0(k,\omega)]\}\ . \qquad (3.2)$$

In what follows we will evaluate $\sum_k G_0\Sigma G_0$ for each self-energy contribution, from which the corrected self-consistency equation can then be derived.

First let us consider the self-energy diagrams with two impurity dressed vertices shown in Fig. 4(a), and start by computing the $V_{\Delta\Delta}$ term, which we denote as $\Sigma^{(2)}_{\Delta\Delta}$, the superscript (2) indicating two dressed vertices. The diagram rules enable us to write down the expression

$$\sum_k G_0\Sigma^{(2)}_{\Delta\Delta}G_0=-\frac{1}{2}T\sum_\Omega\sum_q\sum_k G_0(k,\omega)\Lambda_\Delta(\omega,\omega+\Omega)G_0(k+q,\omega+\Omega)\Lambda_\Delta(\omega+\Omega,\omega)G_0(k,\omega)V_{\Delta\Delta}(q,\Omega)\ . \tag{3.3}$$

An idea of the complexity of this term can be obtained by writing out the full matrix structure of each component to give

$$\sum_k G_0\Sigma^{(2)}_{\Delta\Delta}G_0=\frac{N(0)}{8\tau^2}T\sum_\Omega\sum_q\frac{1}{(Dq^2+W+W')^2}V_{\Delta\Delta}(q,\Omega)$$
$$\times\int d\xi_k\frac{(\bar z+\bar\Delta\tau_1+\xi_k\tau_3)(\bar\alpha_++\beta_+\tau_-)\tau_1(\bar z'+\bar\Delta'\tau_1+\xi_k\tau_3)\tau_1(\bar\alpha_++\beta_+\tau_1)(\bar z+\bar\Delta\tau_1+\xi_k\tau_3)}{(\xi_k^2-\varepsilon^2)^2(\xi_k^2-\varepsilon'^2)}\ , \tag{3.4}$$

where we have set $q=0$ in the second Green function as we expect the major contribution to come from small q due to the form of the vertex functions. The major difficulty comes in performing the matrix product in the numerator of the above expression. We have developed a general technique for simplifying such expressions, which we demonstrate in Appendix A. The final expression is relatively simple,

$$\sum_k G_0\Sigma^{(2)}_{\Delta\Delta}G_0=\frac{N(0)}{4\tau^2}T\sum_\Omega\sum_q\bar\alpha+\frac{\bar z+\bar\Delta\tau_1}{\varepsilon}\frac{1}{(Dq^2+W+W')^2}V_{\Delta\Delta}(q,\Omega)\int d\xi_k\frac{\varepsilon^2\varepsilon'-2\varepsilon\xi_k^2+\varepsilon'\xi_k^2}{(\xi_k^2-\varepsilon^2)^2(\xi_k^2-\varepsilon'^2)}\ . \tag{3.5}$$

We can make the approximation $\varepsilon\approx\varepsilon'\approx i/2\tau$, valid in the dirty limit, to evaluate the latter integral. After some algebraic manipulation we obtain

$$\sum_k G_0\Sigma^{(2)}_{\Delta\Delta}G_0=\frac{1}{2}\pi N(0)T\sum_\Omega\sum_q\frac{i\omega+\Delta\tau_1}{W}\left[1+\frac{\omega\omega'-\Delta^2}{WW'}\right]V_{\Delta\Delta}(q,\Omega)\ . \tag{3.6}$$

The other two-vertex terms corresponding to different effective propagators V_{ij} have the same matrix structure $i\omega+\Delta\tau_1$, and differ from Eq. (3.6) only in having the appropriate V_{ij}, and a new coherence factor in front of it.

We next show that the three-ladder diagram of Fig. 4(c) is identically zero. This diagram can be obtained from the two-vertex diagrams of Fig. 4(a) which were evaluated above. The final ladder can be made up impurity by impurity, the nth term T_n being obtained by induction

$$T_n=\frac{1}{2\pi\tau}\int d\xi_k G_0\tau_3T_{n-1}\tau_3G_0 \tag{3.7}$$

with T_0 being the two-vertex result. Noting that T_0 has matrix structure $\bar z+\bar\Delta\tau_1$, T_1 has the matrix structure

$$\int d\xi_k\frac{(\bar z+\bar\Delta\tau_1+\xi_k\tau_3)(\bar z-\bar\Delta\tau_1)(\bar z+\bar\Delta\tau_1+\xi_k\tau_3)}{(\xi_k^2-\varepsilon^2)^2}=(\bar z+\bar\Delta\tau_1)\int d\xi_k\frac{\xi_k^2+\varepsilon^2}{(\xi_k^2-\varepsilon^2)^2}=0 \tag{3.8}$$

where the final integral equals zero. Therefore T_1 and all other T_n are zero, and the three-ladder diagram is thus zero. Put another way, the matrix structure forbids this diagram, just as the requirement that the two lines of a ladder must have opposite frequency forbids it in the normal metal.

Let us now evaluate the one-vertex diagram shown in Fig. 4(b). We will first evaluate the diagram without the external ladder, which we know will be $O(\Delta\tau)$ smaller than the two-vertex diagrams of Fig. 4(a), and we will then add the external ladder impurity by impurity as above. The diagram rules give

$$\sum_k G_0\Sigma^{(1)}_{\Delta\Delta}G_0=-2T\sum_\Omega\sum_q\sum_k G_0(k,\omega)\Lambda_\Delta(\omega,\omega+\Omega)G_0(k+q,\omega+\Omega)\tau_1G_0(k,\omega)V_{\Delta\Delta}(q,\omega) \tag{3.9}$$

the factor 2 arising as there are two places to put the dressed vertex, each yielding the same result. The matrix product is simplified in Appendix B to give

$$\sum_k G_0\Sigma^{(1)}_{\Delta\Delta}G_0=-\frac{N(0)}{\tau}T\sum_\Omega\sum_q\frac{1}{Dq^2+W+W'}V_{\Delta\Delta}(q,\Omega)$$
$$\times\left\{\bar\alpha_+\frac{\bar z+\bar\Delta\tau_1}{\varepsilon}\int d\xi_k\frac{\varepsilon^2\varepsilon'-2\varepsilon\xi_k^2+\varepsilon'\xi_k^2}{(\xi_k^2-\varepsilon^2)^2(\xi_k^2-\varepsilon'^2)}-\beta_+\frac{(\bar z+\bar\Delta\tau_1)\tau_1}{\varepsilon}\int d\xi_k\frac{\varepsilon^2\varepsilon'-\varepsilon'\xi_k^2}{(\xi_k^2-\varepsilon^2)^2(\xi_k^2-\varepsilon'^2)}\right\}\ . \tag{3.10}$$

We see that there are two types of matrix structure present — the $\bar z+\bar\Delta\tau_1$ also found in the two-vertex diagrams, and a

new form $(\bar{z}+\bar{\Delta}\tau_1)\tau_1$. We know that the term with matrix structure $\bar{z}+\bar{\Delta}\tau_1$ yields zero upon putting the external ladder back, so we can ignore this term as the only nonzero result it yields is $O(\Delta\tau)$ smaller than the two-vertex terms. We put back the external ladder around the term proportional to $(\bar{z}+\bar{\Delta}\tau_1)\tau_1$ one impurity line at a time, as in Eq. (3.7). Here since T_0 has the matrix structure $(\bar{z}+\bar{\Delta}\tau_1)\tau_1$, T_1 has structure

$$\begin{aligned} T_1 &= \frac{1}{2\pi\tau}\int d\xi_k \frac{(\bar{z}+\bar{\Delta}\tau_1+\xi_k\tau_3)\tau_3(\bar{z}+\bar{\Delta}\tau_1)\tau_1\tau_3(\bar{z}+\bar{\Delta}\tau_1+\xi_k\tau_3)}{(\xi_k^2-\varepsilon^2)^2} \\ &= C(\bar{z}+\bar{\Delta}\tau_1)\tau_1\frac{1}{2\pi\tau}\int d\xi_k \frac{1}{\xi_k^2-\varepsilon^2} \\ &= \frac{i}{2\tau\varepsilon}C(\bar{z}+\bar{\Delta}\tau_1)\tau_1 , \end{aligned} \tag{3.11}$$

so that repeating this process leads to "amplification" by a geometric series

$$1+\left[\frac{i}{2\tau\varepsilon}\right]+\left[\frac{i}{2\tau\varepsilon}\right]^2+\cdots=\frac{\varepsilon}{\varepsilon-i/2\tau}\approx\frac{1}{2W\tau} . \tag{3.12}$$

Including this fact, which makes the one-vertex terms have the same order of magnitude as the two-vertex term, and making the dirty-limit approximation $\varepsilon\approx\varepsilon'\approx i/2\tau$ leads to the expression

$$\sum_k G_0\Sigma^{(1)}_{\Delta\Delta}G_0=-\frac{1}{2}\pi N(0)T\sum_\Omega\sum_q\frac{1}{Dq^2+W+W'}\frac{\omega\tau_1-i\Delta}{W^2}\left[\frac{(\omega+\omega')\Delta}{WW'}\right]V_{\Delta\Delta}(q,\Omega) . \tag{3.13}$$

The other one-vertex diagrams corresponding to different effective propagators V_{ij} have the same matrix structure $\omega\tau_1-i\Delta$, and differ from Eq. (3.13) in having the appropriate propagator V_{ij}, and a new coherence factor in front of it.

If we add up all the two-vertex and one-vertex terms, and substitute them into the order parameter self-consistency equation, we get

$$\begin{aligned} \frac{1}{N(0)\lambda}-T\sum_\omega\frac{1}{W} &= -\pi T\sum_\omega T\sum_\Omega\sum_q\frac{1}{(Dq^2+W+W')^2W} \\ &\times\left[\frac{1}{2}\left[1-\frac{\omega\omega'+\Delta^2}{WW'}\right]V_{\rho\rho}(q,\Omega)-\frac{1}{2}\left[1+\frac{\omega\omega'+\Delta^2}{WW'}\right]V_{\phi\phi}(q,\Omega)-\frac{(\omega'-\omega)\Delta}{WW'}V_{\phi\rho}(q,\Omega)\right. \\ &\left.-\frac{1}{2}\left[1+\frac{\omega\omega'-\Delta^2}{WW'}\right]V_{\Delta\Delta}(q,\Omega)\right] \\ &+\pi T\sum_\omega T\sum_\Omega\sum_q\frac{\omega}{\Delta(Dq^2+W+W')W^2}\left[\frac{1}{2}\frac{(\omega'-\omega)\Delta}{WW'}V_{\rho\rho}(q,\Omega)+\frac{1}{2}\frac{(\omega'-\omega)\Delta}{WW'}V_{\phi\phi}(q,\Omega)\right. \\ &\left.-\frac{\omega\omega'+\Delta^2}{WW'}V_{\phi\rho}(q,\Omega)+\frac{1}{2}\frac{(\omega'+\omega)\Delta}{WW'}V_{\Delta\Delta}(q,\Omega)\right] , \end{aligned} \tag{3.14}$$

which, together with the definition of the V_{ij}, is the final result of the calculation.

If we keep just the left side in Eq. (3.14), we recover the BCS result, which can be written in the form

$$\frac{1}{N(0)\lambda}=\ln\left[\frac{\omega_D}{\Delta_0(T)}\right]+F\left[\frac{\Delta_0(T)}{T}\right] , \tag{3.15}$$

which means the Eq. (3.14) can be rewritten in the form

$$\ln\left[\frac{\Delta(T)}{\Delta_0(T)}\right]=-\pi T\sum_\Omega\sum_q T\sum_\omega\cdots , \tag{3.16}$$

where the $\cdots$ just denotes the first-order correction terms on the right-hand side (RHS) of Eq. (3.14). We can obtain a very similar result for T_c by setting $\Delta=0$ in the RHS of Eq. (3.14) to yield

$$\ln\left[\frac{T_c}{T_{c0}}\right]=-\pi T\sum_\Omega\sum_q T\sum_\omega\cdots , \tag{3.17}$$

where the $\cdots$ are the first-order correction terms with $\Delta=0$.

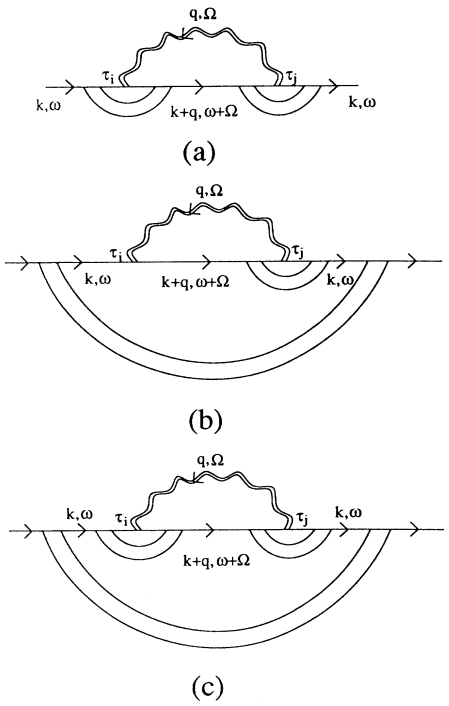

FIG. 4. The first-order corrections to the self-energy of a disordered superconductor. (a) is the two-vertex diagram, whilst (b) is the one-vertex diagram. Both of these diagrams have two impurity ladders Γ defined in Fig. 3(a). (c) is the three-ladder diagram which is shown to be identically zero in the text. The double wavy line is the screened interaction line defined by Fig. 2(d) with the RPA polarization function defined in Fig. 3(c).

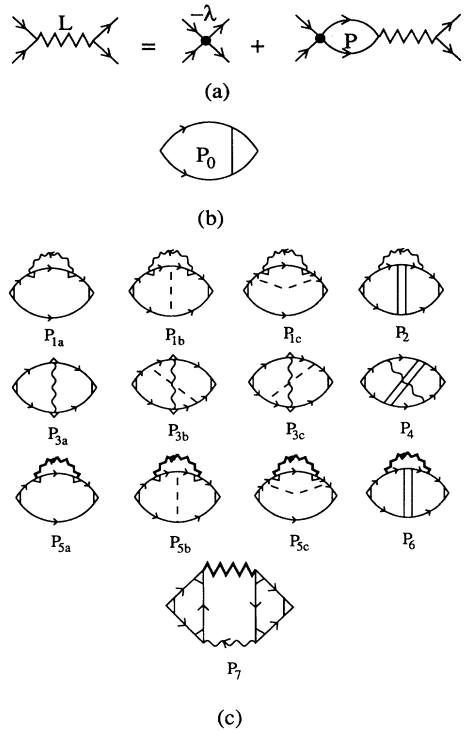

FIG. 5. (a) Diagrammatic definition of the electron-pair propagator L (zig-zag line), and the pair-polarization operator P. (b) Zeroth-order (BCS) contribution to the pair-polarization operator. (c) First-order contributions to the pair-polarization operator. The wavy line is the Coulomb interaction between electrons, while the zig-zag line is the BCS electron-pair propagator L_0.

IV. FIRST-ORDER CORRECTION TO THE TRANSITION TEMPERATURE

The first thing we can do with the order-parameter self-consistency equation derived above is to linearize with respect to Δ to obtain the equation for the first-order correction to the transition temperature. The latter can also be obtained directly from the normal state by calculating the pair propagator $L(q,\Omega)$ as shown in Fig. 5(a), and looking for the superconducting instability at $q=\Omega=0$. L is given by

$$L^{-1}(q,\Omega)=\lambda^{-1}+P(q,\Omega)\ , \tag{4.1}$$

where $P(q,\Omega)$ is the pair polarization bubble. The zeroth-order polarization bubble $P_0(q,\Omega)$ is shown in Fig. 5(b), and leads to the mean-field result

$$L_0^{-1}(q,\Omega)=N(0)\left[\ln\left[\frac{T}{T_{c0}}\right]+\psi\left[\frac{1}{2}+\frac{Dq^2+|\Omega|}{4\pi T}\right]-\psi\left[\frac{1}{2}\right]\right]\ . \tag{4.2}$$

In Sec. II we showed that the propagators for order-parameter amplitude and phase, $V_{\Delta\Delta}$ and $V_{\phi\phi}$, both reduce to $-L_0(q,\Omega)$ when Δ is set to zero. We note that the extra factor of -1 that is associated with $L(q,\Omega)$ is due to the fact that we can interchange the two particle lines in any diagram containing the pair propagator to obtain both a direct and an exchange term, giving the extra factor $-2+1=-1$. A correction to the polarization operator $\delta P(0,0)$ leads to a change in T_c, which is defined as the temperature at which the denominator becomes zero, given by

$$\ln\left[\frac{T_c}{T_{c0}}\right]=\frac{\delta P(0,0)}{N(0)}\ . \tag{4.3}$$

The first-order diagrams for the correction to $P(0,0)$ are shown in Fig. 5(c), and can be evaluated using the many-body theory of the normal metal. Diagrams P_1-P_4 contain only the Coulomb repulsion, whilst P_5 and P_6 contain only pair fluctuations. The final diagram, P_7 contains both Coulomb and pair propagator, and must be kept if the singularity in the Coulomb potential is to be removed. As it was already mentioned[11] this is the diagram that has often been overlooked in work on the correction to the transition temperature. We note its superficial resemblance to the Aslamazov-Larkin[29] contribution to fluctuation conductivity. The results of the evaluation are

$$P_1=-\pi N(0)T\sum_\omega T\sum_\Omega\sum_q\left[\frac{1}{|\omega|^2}\frac{1}{Dq^2+|\Omega|}+\frac{2}{|\omega|}\frac{1}{Dq^2+|\Omega|}\right]\times V_C(q,\Omega)\theta(-\omega(\omega+\Omega)),$$

$$P_2=\pi N(0)T\sum_\omega T\sum_\Omega\sum_q\frac{1}{|\omega|^2}\frac{1}{Dq^2+|2\omega+\Omega|}\times V_C(q,\Omega)\theta(\omega(\omega+\Omega)),$$

$$P_3=-\pi N(0)T\sum_\omega T\sum_\Omega\sum_q\frac{1}{|\omega||\omega+\Omega|}\frac{1}{Dq^2+|\Omega|}\times V_C(q,\Omega)\theta(-\omega(\omega+\Omega)),$$

$$P_4=-\pi N(0)T\sum_\omega T\sum_\Omega\sum_q\frac{1}{|\omega||\omega+\Omega|}\frac{1}{Dq^2+|2\omega+\Omega|}\times V_C(q,\Omega)\theta(\omega(\omega+\Omega)), \quad (4.4)$$

$$P_5=-\pi N(0)T\sum_\omega T\sum_\Omega\sum_q\left[\frac{1}{|\omega|^2}\frac{1}{Dq^2+|2\omega+\Omega|}+\frac{2}{|\omega|}\frac{1}{(Dq^2+|2\omega+\Omega|)^2}\right]\times L(q,\Omega)\theta(\omega(\omega+\Omega)),$$

$$P_6=\pi N(0)T\sum_\omega T\sum_\Omega\sum_q\frac{1}{|\omega|^2}\frac{1}{Dq^2+|\Omega|}\times L(q,\Omega)\theta(-\omega(\omega+\Omega)),$$

$$P_7=4\pi^2N(0)^2T\sum_\Omega\sum_q\left[T\sum_\omega\frac{\Omega}{|\omega||\omega+\Omega|}\times\frac{1}{Dq^2+|\omega|+|\omega+\Omega|}\right]^2\times V_C(q,\Omega)L(q,\Omega).$$

We can now perform our consistency check by setting $\Delta\to 0$ in the order-parameter self-consistency equation derived above. The potentials then become $V_{\rho\rho}\to 2V_C$, $V_{\Delta\Delta}$, $V_{\phi\phi}\to -L$, $V_{\phi\rho}\to 2\Pi_{\phi\rho}LV_C$. Since $W\to|\omega|$, $W'\to|\omega+\Omega|$, the coherence factors become Heaviside functions that set the relative signs of frequencies

$$1-\frac{\omega\omega'\pm\Delta^2}{WW'}\to 2\theta(-\omega(\omega+\Omega));\qquad 1+\frac{\omega\omega'\pm\Delta^2}{WW'}\to 2\theta(\omega(\omega+\Omega)). \quad (4.5)$$

The denominator Dq^2+W+W' becomes $Dq^2+|\Omega|$ for $\omega,\omega+\Omega$ of opposite sign, $Dq^2+|2\omega+\Omega|$ for $\omega,\omega+\Omega$ of the same sign. Making all these substitutions we find that we can reproduce the equation for T_c suppression term by term. Diagram P_7 derives from the one-vertex $V_{\phi\rho}$ term, which shows that its origin is due to coupling between order-parameter and density fluctuations. The $1/(Dq^2+|\Omega|)^2$ term in P_1 and $1/(Dq^2+|2\omega+\Omega|)^2$ term in P_5 are the only one remaining from the two-vertex part of Eq. (3.14). All the other terms come from the one-vertex part by using the substitutions

$$\frac{\omega\omega'}{W^3W'}\to\frac{1}{|\omega|^2}[\theta(\omega(\omega+\Omega))-\theta(-\omega(\omega+\Omega))],\qquad \frac{\omega^2}{W^3W'}\to\frac{1}{|\omega||\omega+\Omega|}[\theta(\omega(\omega+\Omega))+\theta(-\omega(\omega+\Omega))]. \quad (4.6)$$

Having shown that the two methods of calculation produce the same expression for the correction to T_c, we now proceed to evaluate it. We will split the result into two parts: the Coulomb part consisting of those terms that contain a Coulomb propagator, (P_1-P_4,P_7), and consequently require special attention at $q=0$, and the fluctuation part consisting of the terms with only a fluctuation propagator (P_5,P_6). If we perform the ω sum first we get the Coulomb part

$$\ln\left[\frac{T_c}{T_{c0}}\right]_C=-T\sum_\Omega\sum_q\left\{-\frac{1}{2\pi T}\frac{Dq^2}{\Omega^2-(Dq^2)^2}\psi'\left[\frac{1}{2}+\frac{|\Omega|}{2\pi T}\right]+\frac{2Dq^2[\Omega^2+(Dq^2)^2]}{\Omega[\Omega^2-(Dq^2)^2]^2}\left[\psi\left[\frac{1}{2}+\frac{|\Omega|}{2\pi T}\right]-\psi\left[\frac{1}{2}\right]\right]-\frac{4(Dq^2)^2}{[\Omega^2-(Dq^2)^2]^2}\frac{[\psi(1/2+|\Omega|/2\pi T)-\psi(1/2)]^2}{\{\psi[1/2+(Dq^2+|\Omega|)/4\pi T]-\psi(1/2)\}}\right\}V_C(q,\Omega). \quad (4.7)$$

Since the worst singularity possible in $V_C(q,\Omega)$ goes as $1/q^2$, we see that the overall q^2 factor multiplying V_C in the Coulomb part means that this singularity is removed.[17] The removal of such a singularity seems to be a general feature when we evaluate a physically measurable quantity such as T_c as opposed to a property such as the density of states which is not measurable. We note also that there is no pole in the expression at $Dq^2=|\Omega|$, as may be seen by expanding out the last term around this point to second order.

As was stressed in Ref. 17, the appearance of the overall factor q^2 in Eq. (4.7) means that the correction to the transition temperature is universal in the sense that it is not sensitive to the low-momentum structure of the Coulomb potential, therefore one can use 3D or 2D form of the Coulomb potential.

Equation (4.7) is still an exact expression, from which we will now extract the largest term by approximation. If we use the 3D screened Coulomb potential and define $s=Dq^2$, noting that $\sum_q=\int ds/4\pi D$, we note that the largest contributions to the s integrals come from the first and second terms, which fall off as $1/s$ for large s. Since these $1/s$ terms are cut off by $1/\tau$ at the upper limit and $|\Omega|$ at the lower limit this gives

$$\ln\left[\frac{T_c}{T_{c0}}\right]_C \approx -\frac{1}{8\pi N(0)D} T\sum_{\Omega}\left\{-\frac{1}{2\pi T}\psi'\left[\frac{1}{2}+\frac{|\Omega|}{2\pi T}\right]+\frac{2}{|\Omega|}\left[\psi\left[\frac{1}{2}+\frac{|\Omega|}{2\pi T}\right]-\psi\left[\frac{1}{2}\right]\right]\right\}\ln\left[\frac{1}{|\Omega|\tau}\right]. \tag{4.8}$$

The main contribution to the Ω sum comes from large ω where $\psi(x)\approx\ln(x)$. We can ignore the $\psi'(x)$ term as it is less singular at large x $-\psi'(x)\approx 1/x$. Putting $\Omega=2\pi Tm$ gives

$$\ln\left[\frac{T_c}{T_{c0}}\right]_C \approx -\frac{1}{4\pi^2 N(0)D}\sum_{m>0}\frac{\ln(m)}{m}\left[\ln\left[\frac{1}{T\tau}\right]-\ln(m)\right]. \tag{4.9}$$

Approximating this sum by an integral and cutting off at $\Omega\sim 1/\tau$ gives the leading order term,

$$\ln\left[\frac{T_c}{T_{c0}}\right]_C \approx -\frac{1}{24\pi^2 N(0)D}\ln^3\left[\frac{1}{T_c\tau}\right]. \tag{4.10}$$

Changing the numerical factor of the cutoff in the log-cubed term does not affect the leading order log-cubed term, but alters any log-squared terms. In the asymptotic limit $T_c\tau\ll 1$ this does not matter, but when we put in actual experimental numbers, the log-squared terms can be important. We can therefore identity two sources of error in the term written above—those due to the approximations used to get the leading order term from the original expression, and those due to the arbitrariness of the upper cutoff.

If we next consider the suppression of T_c due to pair fluctuations, after performing the ω sum we obtain,

$$\ln\left[\frac{T_c}{T_{c0}}\right]_f = \frac{1}{2\pi N(0)}\sum_{\Omega}\sum_{q}\frac{Dq^2}{\Omega^2-(Dq^2)^2}\frac{\{\psi'[1/2+(Dq^2+|\Omega|)/4\pi T]-\psi'(1/2+|\Omega|/2\pi T)\}}{\{\psi[1/2+(Dq^2+|\Omega|)/4\pi T]-\psi(1/2)\}}. \tag{4.11}$$

We note that the zero-frequency (classical) fluctuations diverge as $\sum_q 1/q^2$ at $q=0$, corresponding to fluctuations of the Goldstone mode, and that this is the dominant term,

$$\ln\left[\frac{T_c}{T_{c0}}\right]_f = -\frac{7\zeta(3)}{2\pi^4 N(0)D}\int\frac{ds}{s}. \tag{4.12}$$

The upper cutoff in this integral will be $s=Dq^2\sim T_c$, at which point the low-q expansion used breaks down.

The main problem is deciding what to use for the lower cutoff in q. The most physical choice appears to be the inverse length of the system in the transverse direction. We then see that for an infinite system T_c is suppressed to zero, reflecting the fact that order-parameter fluctuations destroy long-range broken-symmetry order in two or less dimensions. This is just a statement of the Mermin-Wagner-Hohenberg theorem.[30,31] In this sense the problem of the suppression of T_c in a system of two or less dimensions is ill-defined.[32] The "idealistic" result we then get is

$$\ln\left[\frac{T_c}{T_{c0}}\right] = -\frac{1}{24\pi}\frac{R_\square}{R_0}\ln^3\left[\frac{1}{2\pi T_c\tau}\right] - \frac{7\zeta(3)}{2\pi^3}\frac{R_\square}{R_0}\ln\left[\frac{L^2\Delta}{D}\right]. \tag{4.13}$$

However since there are experiments on two-dimensional systems that give perfectly well-defined T_c, we would like to have some way to deal with the fluctuation term. If we insert the actual size of the sample for L we find the suppression is much stronger than that found experimentally. We could just conveniently ignore the term, preferring to consider only "Coulombic" suppression, but there is obviously no justification to this. The only other approach in the literature has been to use the zeroth-order fluctuation propagator rather than the exact fluctuation propagator, which immediately yields a lower cutoff of $|\ln(T_c/T_{c0})|$ for s. If one then solves the resulting implicit equation by iteration, the first term in the iterated expression has the $R_\square/R_0$ factor under the logarithm,

$$\ln\left[\frac{T_c}{T_{c0}}\right] = -\frac{1}{24\pi}\frac{R_\square}{R_0}\ln^3\left[\frac{1}{2\pi T_c\tau}\right] + \frac{7\zeta(3)}{2\pi^3}\frac{R_\square}{R_0}\ln\left[\frac{7\zeta(3)}{2\pi^3}\frac{R_\square}{R_0}\right]. \tag{4.14}$$

This method was first used by Strongin *et al.*[33] who evaluated the phase-fluctuation contribution using a Ginburg-Landau approach. Ovchinnikov[7] noted that the above equation could be rewritten in terms of the smearing of the phase transition by the fluctuation contribution to the conductivity above T_c, which has smearing parameter $\tau_0=R_\square/R_0$, so that the smearing τ_0 appears as the lower cutoff in s. Eckern and Pelzer[12] in essence reversed this argument saying that the smearing of the transition means that one should cut off at the smearing parameter due to the uncertainty in knowing exactly where T_c is. We believe that the above arguments are incorrect, and that the actual lower cut-off should be the inverse length o the system, $1/L$. But in order to compare the numerical T_c and T_0 results to each other in Sec. V we will however use this lower cutoff procedure for Eq. (4.12), noting that similar reasoning (using the mean-field rather than exact propagators) leads to the same cutoff for Δ_0.

V. FIRST-ORDER CORRECTION TO THE ORDER PARAMETER

Let us now consider the equation for the correction to the order parameter, Eq. (3.14). The first thing we have to do is to calculate the singular low-momentum contribution from the potentials $V_{\phi\phi}$, $V_{\phi\rho}$, and $V_{\rho\rho}$. We can then subtract this singular term from the complete expression to obtain a term that is well-behaved at low momentum which we will also analyze. Since the singular potentials satisfy $V_{\rho\rho}(q,\Omega)=-x^3V_{\phi\phi}(q,\Omega)$, $V_{\phi\rho}(q,\Omega)=xV_{\phi\phi}(q,\Omega)$, the singular contribution can be written ("lm" for low-momentum)

$$\ln\left[\frac{\Delta(T)}{\Delta_0(T)}\right]_{\rm lm}=\frac{1}{2}\pi T\sum_{\omega}T\sum_{\Omega}\left\{\frac{\Delta}{(W+W')^2W}\left[x^2\left[1-\frac{\omega\omega'+\Delta^2}{WW'}\right]+\left[1+\frac{\omega\omega'+\Delta^2}{WW'}\right]+2x\frac{2\Delta^2x}{WW'}\right]\right.$$
$$\left.+\frac{\omega}{W^2(W+W')}\left[-x^2\frac{2\Delta^2x}{WW'}+\frac{2\Delta^2x}{WW'}-2x\frac{\omega\omega'+\Delta^2}{WW'}\right]\right\}\sum_q V_{\phi\phi}(q,\Omega)\ . \tag{5.1}$$

This expression is simplified in Appendix C to yield after much algebra

$$\ln\left[\frac{\Delta(T)}{\Delta_0(T)}\right]_{\rm lm}=\frac{1}{4}\pi T\sum_{\omega}\frac{1}{W^3}T\sum_{\Omega}\sum_q V_{\phi\phi}(q,\Omega)\ . \tag{5.2}$$

We see that the singular terms do not cancel each other completely, but reduce to a very simple form. To understand the physical significance of this term we can set $T=T_c$. We then find that this term yields half the fluctuation contribution to T_c given in Eq. (4.12), and thus interpret it as being due to the order-parameter phase fluctuations. (The other half of the fluctuation correction to T_c comes from the order-parameter amplitude fluctuations which are cutoff below T_c by the order parameter). Since the Coulomb interaction and phase fluctuations are coupled below T_c, the singular correction to T_c due to pair fluctuations has to emerge from the same denominator as the nonsingular correction to T_c from the Coulomb interaction. Let us now evaluate this low-momentum correction to the order parameter.

At $T=0$ we know[12] the exact form of $V_{\phi\phi}(q,\Omega)$ up to $O(q^2)$ in the denominator,

$$V_{\phi\phi}(q,\Omega)^{-1}=-\left[(2V_C(q))^{-1}x^2+\frac{Dq^2}{2\Delta}\sqrt{1+x^2}E\left[\frac{x}{\sqrt{1+x^2}}\right]\right]\ . \tag{5.3}$$

If we use a 3D Coulomb potential this diverges as $1/q^2$ at $q=0$, and ignoring the q^2 term from $(2V_C(q))^{-1}$ as it is so small, the low-momentum contribution is ("lmqf" for low-momentum quantum fluctuations)

$$\ln\left[\frac{\Delta(0)}{\Delta_0(0)}\right]_{\rm lmqf}=\frac{1}{4\pi^2N(0)D}\int_0^{1/2\Delta\tau}\frac{dx}{\sqrt{1+x^2}E(x/\sqrt{1+x^2})}\int\frac{ds}{s}$$
$$=-\frac{1}{4\pi}\frac{R_\Box}{R_0}\ln\left[\frac{1}{2\Delta\tau}\right]\ln\left[\frac{L^2\Delta}{D}\right], \tag{5.4}$$

where we replaced the frequency sums by integrals, and noted that the main contribution to the x integral comes from the upper limit, which is $1/2\Delta\tau$. If we were to use the 2D Coulomb potential there is no longer a divergence at $q=0$, and the $\ln(L^2\Delta/D)$ term is replaced by $\ln(D\kappa_2^2/\Delta)$, where κ_2 is the 2D screening wave vector. Note that although this term is now finite it is rather large in value.

We will now consider the nonsingular part of Eq. (3.14) which is what is left over after the singular part has been subtracted off. Since this has no singular behavior at low q, it follows that a large contribution can come only form large q and Ω. We can easily calculate the contribution from large q and Ω since the large Dq^2 and Ω are so much greater than Δ, that we can set $\Delta=0$ here. The leading terms are the same as in the normal metal, so that we get a log-cubed term. The coefficient is the same as for the normal metal and is again universal. This contribution can thus be written ("hmqf" for high-momentum quantum fluctuations)

$$\ln\left[\frac{\Delta(0)}{\Delta_0(0)}\right]_{\rm hmqf}=-\frac{1}{24\pi}\frac{R_\Box}{R_0}\ln^3\left[\frac{1}{2\Delta\tau}\right]. \tag{5.5}$$

The "idealistic" result for the suppression of $\Delta(0)$ for a 3D Coulomb potential is then

$$\ln\left[\frac{\Delta(0)}{\Delta_0(0)}\right]=-\frac{1}{24\pi}\frac{R_\Box}{R_0}\ln^3\left[\frac{1}{2\Delta\tau}\right]-\frac{1}{4\pi}\frac{R_\Box}{R_0}\ln\left[\frac{1}{2\Delta\tau}\right]\ln\left[\frac{L^2\Delta}{D}\right] \tag{5.6}$$

and for a 2D Coulomb potential is

$$\ln\left[\frac{\Delta(0)}{\Delta_0(0)}\right]=-\frac{1}{24\pi}\frac{R_\Box}{R_0}\ln^3\left[\frac{1}{2\Delta\tau}\right]-\frac{1}{4\pi}\frac{R_\Box}{R_0}\ln\left[\frac{1}{2\Delta\tau}\right]\ln\left[\frac{D\kappa_2^2}{\Delta}\right]. \tag{5.7}$$

If we cut off the low-momentum divergence by using the zeroth-order propagators that have value of the order pa-

rameter, this gives a cutoff $|\ln(\Delta(0)/\Delta_0(0))|$. This leads to the "pragmatic" result

$$\ln\left[\frac{\Delta(0)}{\Delta_0(0)}\right]=-\frac{1}{24\pi}\frac{R_\square}{R_0}\ln^3\left[\frac{1}{2\Delta\tau}\right]+\frac{1}{4\pi}\frac{R_\square}{R_0}\ln\left[\frac{1}{2\Delta\tau}\right]\ln\left[\frac{1}{4\pi}\frac{R_\square}{R_0}\right]. \quad (5.8)$$

This pragmatic cutoff is exactly equivalent to that used in the T_c case to obtain Eq. (4.14).

Let us compare Eq. (5.8) for Δ and Eq. (4.14) for T_c. We see that they have essentially identical high-momentum quantum fluctuation terms that are universal in that they are independent of the nature of the screened potential. The low-momentum fluctuation term for Δ_0 differs from that for T_c in that it has an extra factor $\ln(\frac{1}{2}\Delta\tau)$ and its exact form depends on the nature of the electron-electron interaction. We note that the "pragmatic" results for T_c [Eq. (4.14)] and Δ_0 [Eq. (5.8)] are dominated by the high-momentum quantum fluctuation term and so Δ_0/T_c should be asymptotically constant as a function of $R_\square/R_0$. We investigate this numerically in the next section.

To see how we go continuously from the low-momentum classical fluctuation correction to T_c to the low-momentum quantum fluctuation correction to Δ_0, we will approximate the singular low-momentum correction to $\Delta(T)$ for general T. For $T\neq 0$ we can split this into two terms—the classical fluctuations from $\Omega=0$ only, and the quantum fluctuations from the rest of the Ω sum, which we expect to be dominated by large Ω. For the classical fluctuations one has the potential

$$V_{\phi\phi}(q,0)^{-1}=-\frac{1}{2}\pi N(0)Dq^2T\sum_\omega\frac{1}{W^2}. \quad (5.9)$$

Now we can approximate

$$T\sum_\omega\frac{1}{W^2}\propto\frac{1}{\max[\Delta(T),T]};\quad T\sum_\omega\frac{1}{W^3}\propto\frac{1}{\max[\Delta(T),T]^2} \quad (5.10)$$

so that the low-momentum classical fluctuation (mcf) term is

$$\ln\left[\frac{\Delta(T)}{\Delta_0(T)}\right]_{\rm lmcf}\approx -C(T)\frac{T}{\max[\Delta(T),T]}\frac{R_\square}{R_0}\times\ln\left[\frac{L^2\max[\Delta(T),T]}{D}\right], \quad (5.11)$$

where $C(T)$ is an $O(1)$ constant that varies slowly with T. We immediately see that there is no classical fluctuation term at $T=0$. To evaluate the quantum fluctuation term, we need the asymptotic behavior of $V_{\phi\phi}(q,\Omega)$ at large Ω, which is given by

$$V_{\phi\phi}(q,\Omega)^{-1}\sim-\frac{N(0)Dq^2\Omega}{4\Delta^2}, \quad (5.12)$$

which leads to

$$\ln\left[\frac{\Delta(T)}{\Delta_0(T)}\right]_{\rm lmqf}\approx -D(T)\frac{R_\square}{R_0}\frac{\Delta^2}{\max[\Delta(T),T]^2}\times\ln\left[\frac{1}{\max[\Delta(T),T]\tau}\right]\times\ln\left[\frac{L^2\max[\Delta(T),T]}{D}\right] \quad (5.13)$$

where again $D(T)$ is an $O(1)$ constant that varies slowly with T. We see that there is no quantum fluctuation contribution at $T=T_c$ since $\Delta=0$ there. Note that the classical fluctuation terms cannot be cut off by the 2D Coulomb potentials whilst the quantum fluctuations can. We note also that the high-momentum contribution is simply obtained by replacing Δ by $\max[\Delta(T),T]$ in Eq. (5.5) to give

$$\ln\left[\frac{\Delta(T)}{\Delta_0(T)}\right]_{\rm hmqf}=-\frac{1}{24\pi}\frac{R_\square}{R_0}\ln^3\left[\frac{1}{\max[\Delta(T),T]\tau}\right]. \quad (5.14)$$

VI. NUMERICAL RESULTS

In this section we will numerically evaluate the first-order corrections to T_c, and Δ_0, and compare these to the asymptotic forms given in Eqs. (4.14) and (5.8), respectively. The complete T_c expression is given by the sum of Eqs. (4.7) and (4.11). To evaluate this we introduce the dimensionless variables $m=\Omega/2\pi T_c$ and $y=Dq^2/2\pi T_c$ to give

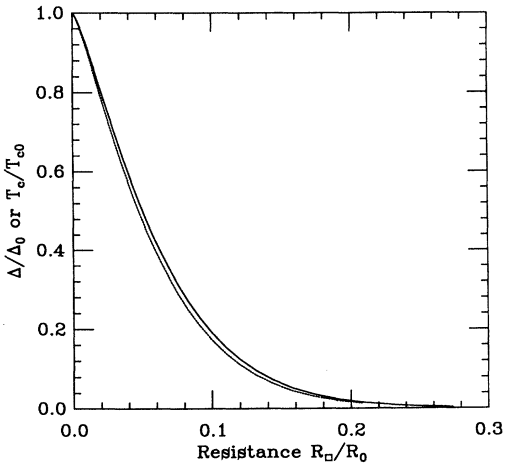

FIG. 6. Numerical results for the first-order correction to the order parameter, Δ/Δ_0 (dotted line), and to the transition temperature, T_c/T_{c0} (solid line), as a function of resistance $R_\square$. In each case the singular logarithm in the low-momentum fluctuation terms has been cut off in the "pragmatic" way by the square resistance. The two curves are very similar, showing that Δ/T_c is roughly independent of resistance. We have used the value of T_c/ε_F for Sn.

$$\ln\left[\frac{T_c}{T_{c0}}\right]=-\frac{1}{4\pi^2}\frac{R_\square}{R_0}\left\{\sum_{m=1}^{m_0}\ln\left[\frac{m_0+m}{m}\right]\left[\frac{2}{m}[\psi(\tfrac{1}{2}+m)-\psi(\tfrac{1}{2})]+\psi'(\tfrac{1}{2}+m)\right]\right.$$
$$-\sum_{m=1}^{m_0}\int_0^{m_0}dy\frac{4y}{(m-y)(m^2-y^2)}$$
$$\times\left[\frac{[\psi(1/2+m)-\psi(1/2)]^2}{[\psi(1/2+y/2+m/2)-\psi(1/2)]}-[\psi(\tfrac{1}{2}+m)-\psi(\tfrac{1}{2})]+\frac{y-m}{2}\psi'(\tfrac{1}{2}+m)\right]$$
$$+\frac{28\zeta(3)}{\pi^2}\int_{y_c}^1\frac{dy}{y}-\int_0^{m_0}\frac{dy}{y}\left[\frac{\psi'(1/2+y/2)-\psi'(1/2)}{\psi(1/2+y/2)-\psi(1/2)}-\frac{\psi''(1/2)}{\psi'(1/2)}\theta(1-y)\right]$$
$$\left.+\sum_{m=1}^{m_0}\int_0^{m_0}dy\frac{2y}{m^2-y^2}\frac{\psi'(1/2+y/2+m/2)-\psi'(1/2+m)}{\psi(1/2+y/2+m/2)-\psi(1/2)}\right\},\tag{6.1}$$

where $m_0=1/2\pi\tau T_c$ is the upper cutoff for m and y, and $y_c=7\zeta(3)R_\square/2\pi^3R_0$ is the lower cutoff for y in the singular pair fluctuation term. We see that the disorder strength $1/\tau$ occurs only in the factor $R_\square/R_0$ and the upper cutoff m_0 (we have cut off Dq^2 and Ω at $1/\tau$). In our calculations we vary the cutoff m_0 which can then be related to $R_\square/R_0$ for a given superconductor if we know the ratio T_c/ε_F,

$$\frac{R_\square}{R_0}=\frac{4\pi T_c m_0}{\varepsilon_F}\ .\tag{6.2}$$

The first term in Eq. (6.1) is the leading term from the Coulombic contribution, Eq. (4.8), and has log-cubed asymptotic behavior. The second term is the remainder of the Coulombic contribution, the difference between the exact Eq. (4.7) and its approximant Eq. (4.8). The third term is the leading term from the pair-fluctuation contribution, Eq. (4.12). The fourth and fifth terms form the remainder of the pair-fluctuation contribution, the difference between the exact Eq. (4.11) and its approximant Eq. (4.12). The fourth term is the zero-frequency part of the remainder, whilst the fifth term is the finite-frequency part. Although the leading asymptotic behavior is the log-cubed term, the asymptotic regime is reached so slowly that we need to consider subdominant terms to get reasonable agreement between numerical results from Eq. (6.1) and asymptotic approximants. After much algebra one obtains

$$\ln\left[\frac{T_c}{T_{c0}}\right]=-\frac{1}{8\pi}\frac{R_\square}{R_0}\left\{-\frac{28\zeta(3)}{\pi^2}\ln\left[\frac{7\zeta(3)}{2\pi^3}\frac{R_\square}{R_0}\right]+\frac{1}{3}\ln^3(m_0)-\left[\psi(1/2)+\frac{3}{2}\right]\ln^2(m_0)\right.$$
$$\left.+\left[\frac{\pi^2}{4}+\ln(8)\right]\ln(m_0)+\frac{1}{2}\ln(m_0)\ln(\ln(m_0))\right\}.\tag{6.3}$$

Numerical solution of Eq. (6.3) is presented in Fig. 6. A comparison of the numerical and asymptotic results for the T_c case is given in Fig. 7.

The complete Δ_0 expression is given by Eq. (3.14). To evaluate this we introduce the dimensionless variables $x=\Omega/2\Delta$ and $y=Dq^2/2\Delta$. As for the T_c a case the disorder strength $1/\tau$ occurs only in the term $R_\square/R_0$ and the cutoff for x and y, $x_0=1/2\Delta\tau$. The singular term Eq. (5.2) is subtracted from the full expression Eq. (3.14) to give an integral that is well-behaved as $y=0$. Equation (5.2) is then separately evaluated and is cut off at $y_c=R_\square/4\pi R_0$. There are three levels of integration to perform—the outer x and y integrals, and the inner ω integral over the coherence factors (and a similar integral needed to produce the V_{ij}). The x and y integrals are split into intervals [0,1], and $[1,x_0]$, the latter region being integrated logarithmically. In our calculation we vary the upper cutoff x_0, which is related to $R_\square/R_0$ by

$$x_0=\frac{4\Delta x_0}{\varepsilon_F}\ .\tag{6.4}$$

We plot the numerical results for $\ln(T_c/T_{c0})$ and $\ln(\Delta/\Delta_0)$ for the value of T_c/ε_F corresponding to Sn in Fig. 6 for $R_\square/R_0$ up to about 0.2. We see that the curves are very similar so that Δ/T_c is roughly constant as a function of $R_\square/R_0$.

VII. CONCLUSIONS

In this paper we have evaluated the equation for the first-order perturbation correction to the order parameter, $\Delta(T)$, due to the electron-electron interaction in a disordered thin-film superconductor using a self-energy approach. From this we evaluated the dependence of the transition temperature, T_c, and the zero-temperature order parameter, Δ, as a function of the resistance $R_\square/R_0$ both analytically (in the asymptotic limit) and numerically. The analytic results are summarized in Table III, and the numerical results are plotted in Fig. 6. We find that T_c and Δ are suppressed in a similar manner, and that Δ/T_c is roughly constant for $R_\square/R_0$ up to 0.2. The analytic results for T_c have been evaluated previously,[7,11,12] providing a check upon our calculation, while those for Δ are new.

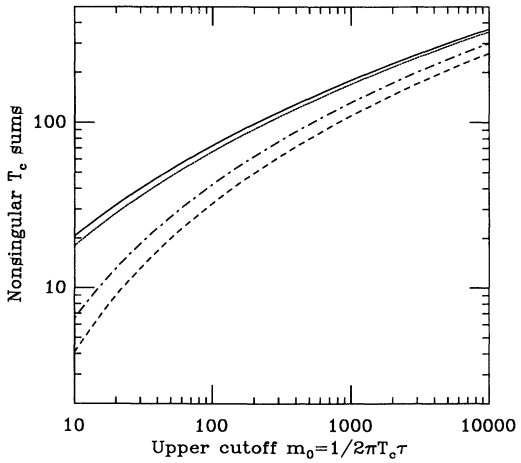

FIG. 7. Comparison of the nonsingular sums (i.e., all but the third term) of Eq. (6.1) for $\ln(T_c/T_{c0})$ and the asymptotic approximation to these terms of Eq. (6.3). The solid line is the numerical result; the dashed line is the leading asymptotic term $\frac{1}{3}\ln^3(m_0)$; the dot-dashed line contains in addition the next term $-[\psi(\frac{1}{2})+\frac{3}{2}]\ln^2(m_0)$; the dotted line contains in addition the terms $[\frac{1}{4}\pi^2+\ln(8)]\ln(m_0)+\frac{1}{2}\ln(m_0)\ln(\ln(m_0))$. We see that the asymptotic approach is slow, and in the physically important range of m_0 we must include nonleading asymptotic terms to get good agreement with the numerics.

The result for T_c can be written as the sum of a high-momentum quantum fluctuation term and a low-momentum classical fluctuation term. The high-momentum quantum fluctuation term consists of all contributions that contain the screened Coulomb interaction, which is singular at low momentum. This low-momentum singularity is cancelled between the various contributions, leading to a result that is dominated by high-momentum and high-frequency fluctuations. This term is proportional to $\ln^3(1/2\pi T_c\tau)$ with a coefficient that is independent of the exact nature of the screened electron-electron interaction. The low-momentum term consists of those contributions that contain only the electron-pair propagator, and is divergent for an infinite system in two or less dimensions. This is consistent with the Mermin-Wagner-Hohenberg theorem[30,31] that there is no long-range broken-symmetry order at finite temperature in two or less dimensions.

The result for $\Delta(T)$ can be written as the sum of a high-momentum quantum fluctuation term and a low-momentum term. This occurs in the following way. The electronic density fluctuations (Coulomb interaction) and order-parameter phase fluctuations in the superconductor are coupled, and as a result their propagators have the same low-momentum singularity as the bare Coulomb interaction for any T less than T_c. The low-momentum term is the part of the equation for $\Delta(T)$ that retains this singularity; the high-momentum part is the remainder after the singular term is subtracted off. The high-momentum quantum fluctuation term is proportional to $\ln^3(1/2\Delta\tau)$ with the same coefficient as the equivalent term in the T_c result. This is because at high momentum and high frequency the superconducting gap is not visible, and one gets the same results as in the normal metal. At $T=0$ the low-momentum term in the equation for Δ is dominated by high-frequency quantum fluctuations to give a result proportional to $\ln(1/2\Delta\tau)\ln(L^2\Delta/D)$ for a 3D Coulomb potential and $\ln(1/2\Delta\tau)\ln(D\kappa_2^2/\Delta)$ for a 2D Coulomb potential. The low-momentum quantum-fluctuation term is thus divergent for a 3D Coulomb potential, and convergent (but very large in value) for a 2D Coulomb potential. For any finite temperature T the low-momentum term in the equation for $\Delta(T)$ has both a classical and a quantum fluctuation contribution, and the classical fluctuation part is always divergent. At $T=T_c$ this low-momentum term has only a classical fluctuation part, and is equal to half of the classical fluctuation correction to T_c.

From the above discussion we see that the low-momentum terms in both the T_c and $\Delta(T)$ equations are divergent for infinite 2D systems except in the special case of zero temperature and a 2D Coulomb interaction between electrons. To be able to compare the results for T_c and Δ as a function of $R_\square/R_0$ we must deal with these divergences. We do not know of any physically justifiable way of doing this, but content that the treatment must be the same for both T_c and Δ. One possible approach, for example, would be to ignore the low-momentum terms. We choose to follow the practice used in work on T_c suppression[33,7,12] and introduce a lower momentum cut-off by using the uncorrected BCS value of T_c or Δ in interaction propagators. This leads to a lower cutoff of Dq^2/T_c or Dq^2/Δ proportional to $R_\square/R_0$. With this cutoff we find that the corrections to T_c and Δ are dominated by the high-momentum quantum fluctuation terms and predict that Δ/T_c is asymptotically constant as a function of $R_\square/R_0$. This is confirmed by the numerical results, and consistent with experiment. We do not believe previous work[12] suggesting that Δ is suppressed more strongly than T_c since this work appears to cut off the divergent low-momentum classical fluctuations in the T_c case as described above, but does not cut off the low-momentum quantum fluctuation correction to Δ (which converges in the special case of a 2D Coulomb potential at $T=0$).

The result that the first-order correction to $\Delta(T)$ can be written as the sum of a high-momentum and a low-momentum term enables us to make a connection to dirty-boson theories[34,35] of the disordered superconductor. These theories treat the Cooper pairs as effective bosons, ignoring their fermionic composition. We might expect this approximation to be valid at low momentum, q, and low frequency Ω. Indeed we see that the phase propagator in both the dirty-boson model[34] and our superconductor model [see Eq. (2.47)] has the form $1/(q^2+\Omega^2 q^{d-1})$. If we perform first-order perturbation theory for any quantity using the dirty-boson theory, we would get contributions of the form

$$T\sum_{\Omega}\sum_{q}\frac{1}{q^2+\Omega^2 q^{d-1}}\ . \tag{7.1}$$

TABLE III. Summary of results for the first-order perturbation correction to transition temperature, T_c, zero-temperature order parameter, Δ_0, and order parameter at finite temperature, $\Delta(T)$. The results are split into three terms: high-momentum quantum fluctuations which come from high momentum and high frequency, and are usually called Coulomb terms in papers on T_c; low-momentum classical fluctuations which come from low-momentum and zero frequency, and are usually called classical or pair fluctuations in papers on T_c; and low-momentum quantum fluctuations which come from low-momentum and high frequency and are a new feature of the Δ calculation. The 2D, and 3D mean that we are approximating the quasi-2D Coulomb interaction by a 2D, or a 3D Coulomb interaction, respectively. The terms "idealistic" and "pragmatic" refer to how we choose to cut off the singular low-momentum fluctuation terms. In the "idealistic" version we must cut off both classical and quantum terms by the inverse system size, $q_0=1/L$ (we note that in the special case of $T=0$ for a 2D Coulomb potential there is no low-momentum singularity). Since this idealism gives results that disagree with experiment, the "pramagnetic" version involves a cutoff of $Dq^2/T_c \sim \ln(T_c/T_{c0})$ or $Dq^2/\Delta \sim \ln(\Delta/\Delta_0)$, obtained by using the zeroth-order propagators—solution by iteration then puts $R_\square/R_0$ under the logarithm. The $\Delta(T)$ results are included to show how the low-momentum contribution changes smoothly from all quantum at $T=0$ to all classical at $T=T_c$.

Property	High momentum Quantum	Low momentum Classical	Low momentum Quantum
Idealistic $\ln\left[\frac{T_c}{T_{c0}}\right]$	$-\frac{1}{24\pi}\frac{R_\square}{R_0}\ln^3\left[\frac{1}{2\pi T_c\tau}\right]$	$-\frac{7\zeta(3)}{2\pi^3}\frac{R_\square}{R_0}\ln\left[\frac{L^2T_c}{D}\right]$	
Pragmatic $\ln\left[\frac{T_c}{T_{c0}}\right]$	$-\frac{1}{24\pi}\frac{R_\square}{R_0}\ln^3\left[\frac{1}{2\pi T_c\tau}\right]$	$+\frac{7\zeta(3)}{2\pi^3}\frac{R_\square}{R_0}\ln\left[\frac{7\zeta(3)}{2\pi^3}\frac{R_\square}{R_0}\right]$	
3D Idealistic $\ln\left[\frac{\Delta}{\Delta_0}\right]$	$-\frac{1}{24\pi}\frac{R_\square}{R_0}\ln^3\left[\frac{1}{2\Delta\tau}\right]$		$-\frac{1}{4\pi}\frac{R_\square}{R_0}\ln\left[\frac{1}{2\Delta\tau}\right]\ln\left[\frac{L^2\Delta}{D}\right]$
2D Idealistic $\ln\left[\frac{\Delta}{\Delta_0}\right]$	$-\frac{1}{24\pi}\frac{R_\square}{R_0}\ln^3\left[\frac{1}{2\Delta\tau}\right]$		$-\frac{1}{4\pi}\frac{R_\square}{R_0}\ln\left[\frac{1}{2\Delta\tau}\right]\ln\left[\frac{D\kappa_2^2}{\Delta}\right]$
Pragmatic $\ln\left[\frac{\Delta}{\Delta_0}\right]$	$-\frac{1}{24\pi}\frac{R_\square}{R_0}\ln^3\left[\frac{1}{2\Delta\tau}\right]$		$+\frac{1}{4\pi}\frac{R_\square}{R_0}\ln\left[\frac{1}{2\Delta\tau}\right]\ln\left[\frac{1}{4\pi}\frac{R_\square}{R_0}\right]$
Idealistic $\ln\left[\frac{\Delta(T)}{\Delta_0(T)}\right]$	$-\frac{1}{24\pi}\frac{R_\square}{R_0}\ln^3\left[\frac{1}{M\tau}\right]$[a]	$-C(T)\frac{T}{M}\frac{R_\square}{R_0}\ln\left[\frac{L^2}{MD}\right]$[a]	$-D(T)\frac{\Delta^2(T)}{M^2}\ln\left[\frac{1}{M\tau}\right]\ln\left[\frac{L^2M}{D}\right]$[a]

[a]Where $M=\max[\Delta(T),T]$.

We get a similar result for first-order perturbation theory within our superconductor model, and we also know the form of the propagator for any frequency and momentum. We find that the contribution from the phase propagator is dominated by high frequency, leading to a prefactor $\ln(1/\Delta\tau)$ for any low-momentum singularity. This feature cannot be derived from the bosonic models as they are valid only at low frequency, $\Omega \ll \Delta \ll 1/\tau$. The boson models therefore cannot obtain the correct form for the singular low-momentum contribution to a physical quantity (such as T_c or Δ) to first order in perturbation theory. We also note that these theories also cannot obtain the high-momentum quantum fluctuation contributions which are important in explaining the dependence of T_c and Δ upon disorder.

ACKNOWLEDGMENT

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APPENDIX A: LOW-MOMENTUM SINGULARITIES IN DENSITY AND PHASE PROPAGATORS

In this appendix we prove the relation

$$\Pi_{\phi\phi}(0,\Omega)\Pi_{\rho\rho}(0,\Omega)+\Pi_{\phi\rho}(0,\Omega)^2=0 \tag{A1}$$

for any temperature $0 \le T \le T_c$, which leads to the potentials $V_{\phi\phi}$, $V_{\phi\rho}$, and $V_{\rho\rho}$ having $1/q^{d-1}$ singularities at low momentum q. As discussed in Sec. III these singularities and their possible cancellation are technical details necessary for the calculation of any quantity in the disordered superconductor. This result has been proved[12] at $T=0$ where the $\Pi_{ij}(0,\Omega)$ are known elementary functions, but has not been shown for general T. For this reason we provide a detailed algebraic proof for all T.

We start with the equations for $\Pi_{\phi\phi}$, $\Pi_{\phi\rho}$, and $\Pi_{\rho\rho}$,

$$\Pi_{\phi\phi}(0,\Omega)=\pi N(0)T\sum_{\omega}\left[\frac{WW'+\omega\omega'+\Delta^2}{WW'(W+W')}-\frac{1}{W}\right],$$
$$\Pi_{\phi\rho}(0,\Omega)=-\pi N(0)T\sum_{\omega}\frac{\Omega\Delta}{WW'(W+W')}, \tag{A2}$$
$$\Pi_{\rho\rho}(0,\Omega)=N(0)-\pi N(0)T\sum_{\omega}\frac{WW'-\omega\omega'-\Delta^2}{WW'(W+W')}.$$

To simplify these formulas we note that $\omega\omega'$ can be rewritten as

$$\omega\omega'=\tfrac{1}{2}[\omega^2+\omega'^2-(\omega'-\omega)^2]$$
$$=\tfrac{1}{2}[W^2+W'^2-2\Delta^2-\Omega^2] \tag{A3}$$

and the coherence factors for $\Pi_{\phi\phi}$ and $\Pi_{\rho\rho}$ can be rewritten

$$WW'+\omega\omega'+\Delta^2=\tfrac{1}{2}[(W+W')^2-\Omega^2];$$
$$WW'-\omega\omega'-\Delta^2=-\tfrac{1}{2}[(W-W')^2-\Omega^2]. \tag{A4}$$

Substituting these expressions in the equations for $\Pi_{\phi\phi}$, $\Pi_{\rho\rho}$ yields

$$\Pi_{\phi\phi}=\pi N(0)T\sum_{\omega}\left[\frac{1}{2W}+\frac{1}{2W'}-\frac{1}{W}\right]$$
$$-\frac{1}{2}\Omega^2\pi N(0)T\sum_{\omega}\frac{1}{WW'(W+W')}$$
$$=-\frac{1}{2}\Omega^2\pi N(0)T\sum_{\omega}\frac{1}{WW'(W+W')},$$
$$\Pi_{\rho\rho}=N(0)+\frac{1}{2}\pi N(0)T\sum_{\omega}\left[\frac{1}{W}-\frac{1}{W'}\right]\frac{W'-W}{W+W'}$$
$$-\frac{1}{2}\Omega^2\pi N(0)T\sum_{\omega}\frac{1}{WW'(W+W')}$$
$$=N(0)+\pi N(0)T\sum_{\omega}\frac{W'-W}{W(W+W')}$$
$$-\frac{1}{2}\Omega^2\pi N(0)T\sum_{\omega}\frac{1}{WW'(W+W')}, \tag{A5}$$

where in the last step in each case we used the substitution $\omega\to-\omega'$ which maps $W\to W'$ and vice versa. We immediately see that $\Pi_{\phi\phi}$ and $\Pi_{\phi\rho}$ are proportional to each other. To show that $\Pi_{\rho\rho}$ is proportional to $\Pi_{\phi\phi}$ and $\Pi_{\phi\rho}$ we must relate the two sums

$$S_1=T\sum_{\omega}\frac{1}{WW'(W+W')};$$
$$S_2=T\sum_{\omega}\frac{W'-W}{W(W+W')}=T\sum_{\omega}\left[\frac{1}{W}-\frac{2}{W+W'}\right]. \tag{A6}$$

To do this we will multiply out top and bottom by $W'-W$ and use the identity

$$W'^2-W^2=(\omega+\Omega)^2-\omega^2=\Omega(2\omega+\Omega) \tag{A7}$$

and the transformation $\omega\to-\omega'$ to get

$$S_1=\frac{T}{\Omega}\sum_{\omega}\frac{W'-W}{WW'(2\omega+\Omega)}=\frac{2T}{\Omega}\sum_{\omega}\frac{1}{W(2\omega+\Omega)},$$
$$S_2=T\sum_{\omega}\left[\frac{1}{W}--\frac{2(W'-W)}{\Omega(2\omega+\Omega)}\right] \tag{A8}$$
$$=T\sum_{\omega}\left[\frac{1}{W}+\frac{4W}{\Omega(2\omega+\Omega)}\right].$$

If we combine the two terms in S_2 over a common denominator, and expand out W^2 in powers of $2\omega+\Omega$ we get

$$S_2=T\sum_{\omega}\left[\frac{2\omega}{\Omega W}+\frac{4\Delta^2+\Omega^2}{(2\omega+\Omega)W}\right]. \tag{A9}$$

The second term in this expression is proportional to S_1. The first term is a sum over an odd function of ω, and therefore appears at first glance to be zero. However, since the upper and lower limits of this sum must be invariant under $\omega\to-(\omega+\Omega)$, there are $\Omega/2\pi T$ more negative ω than positive ω summed over. For large ω the summand is simply $2\,\mathrm{sgn}(\omega)/\Omega$, so this sum gives

$$T\sum_{\omega}\frac{2\omega}{\Omega W}=-T\left[\frac{\Omega}{2\pi T}\right]\left[\frac{2}{\Omega}\right]=-\frac{1}{\pi} \tag{A10}$$

and thus

$$S_2=-\frac{1}{\pi}+T\sum_{\omega}\frac{4\Delta^2+\Omega^2}{(2\omega+\Omega)W}. \tag{A11}$$

Substituting the expressions for S_1 and S_2 back into that for $\Pi_{\rho\rho}$, we see that the constant term $N(0)$ is cancelled out, and the resulting formula for $\Pi_{\rho\rho}$ is proportional to those for $\Pi_{\phi\phi}$ and $\Pi_{\phi\rho}$,

$$-x^2\Pi_{\rho\rho}(0,\Omega)=x\Pi_{\phi\rho}(0,\Omega)$$
$$=\Pi_{\phi\phi}(0,\Omega)=-\Omega\pi N(0)T\sum_{\omega}\frac{1}{(2\omega+\Omega)W}, \tag{A12}$$

where $x=\Omega/2\Delta$, which proves our original assertion.

APPENDIX B: DETAILS OF CALCULATION OF FIRST-ORDER CORRECTION TO GREEN FUNCTION

In this appendix we will demonstrate how the complex matrix products that occur in the calculation of Sec. III may be simplified. This is of course crucial to the utility of our method, for if all the many-body theory yields is long expressions containing matrix products that we cannot simplify, then although the method is formally correct it is useless practically. We present a method that is found empirically to work for any diagrammatic calculation within our approach, not just those of Sec. III. This cannot be an accident, although we do not yet have a detailed proof that the method will always work. The reader is encouraged to plough through the algebra below to get a feel for the detailed mechanics.

Consider the matrix product that occurs in the

numerator of Eq. (3.4),

$$N_2=(\bar{z}+\bar{\Delta}\tau_1+\xi_k\tau_3)(\bar{\alpha}_++\beta_+\tau_1)\tau_1(\bar{z}'+\bar{\Delta}'\tau_1+\xi_k\tau_3)\times\tau_1(\bar{\alpha}_++\beta_+\tau_1)(\bar{z}+\bar{\Delta}\tau_1+\xi_k\tau_3)\ . \tag{B1}$$

To expand out this product we split each Green function into piece that are even (E), and odd (O), with respect to $\xi_k-\bar{z}+\bar{\Delta}\tau_1$, and $\xi_k\tau_3$, respectively. We then consider the $2^{3-1}=4$ product terms that are even in ξ_k—EEE, EOO, OEO, and OOE—since the terms that are odd in ξ_k integrate to zero. These terms are simplified using the easily verified identities

$$(\bar{z}+\bar{\Delta}\tau_1)(\bar{\alpha}_++\beta_+\tau_1)(\bar{z}'+\bar{\Delta}'\tau_1)=\varepsilon\varepsilon'(\bar{\alpha}_+-\beta_+\tau_1);\quad \bar{\alpha}_+^2-\beta_+^2=-2\bar{\alpha}_+\ . \tag{B2}$$

Multiplying out the four terms using these identities gives

$$\text{EEE: } (\bar{z}+\bar{\Delta}\tau_1)(\bar{\alpha}_++\beta_+\tau_1)\tau_1(\bar{z}'+\bar{\Delta}'\tau_1)(\bar{\alpha}_++\beta_+\tau_1)\tau_1(\bar{z}+\bar{\Delta}\tau_1)=\varepsilon\varepsilon'(\bar{\alpha}_+-\beta_+\tau_1)(\bar{\alpha}_++\beta_+\tau_1)(\bar{z}+\bar{\Delta}\tau_1)=-2\varepsilon\varepsilon'\bar{\alpha}_+(\bar{z}+\bar{\Delta}\tau_1)\ ,$$

$$\text{EOO: } (\bar{z}+\bar{\Delta}\tau_1)(\bar{\alpha}_++\beta_+\tau_1)\tau_1\xi_k\tau_3(\bar{\alpha}_++\beta_+\tau_1)\tau_1\xi_k\tau_3=-\xi_k^2(\bar{\alpha}_++\beta_+\tau_1)(\bar{\alpha}_+-\beta_+\tau_1)(\bar{z}+\bar{\Delta}\tau_1)=+2\xi_k^2\bar{\alpha}_+(\bar{z}+\bar{\Delta}\tau_1)\ ,$$

$$\text{OEO: } \xi_k\tau_3(\bar{\alpha}_++\beta_+\tau_1)\tau_1(\bar{z}'+\bar{\Delta}'\tau_1)(\bar{\alpha}_++\beta_+\tau_1)\tau_1\xi_k\tau_3=\frac{\varepsilon'}{\varepsilon}\xi_k^2\tau_3(\bar{\alpha}_++\beta_+\tau_1)(\bar{\alpha}_+-\beta_+\tau_1)(\bar{z}-\bar{\Delta}\tau_1)\tau_3=-2\frac{\varepsilon'}{\varepsilon}\xi_k^2\bar{\alpha}_+(\bar{z}+\bar{\Delta}\tau_1)\ ,$$

$$\text{OOE: } \xi_k\tau_3(\bar{\alpha}_++\beta_+\tau_1)\tau_1\xi_k\tau_3(\bar{\alpha}_++\beta_+\tau_1)\tau_1(\bar{z}+\bar{\Delta}\tau_1)=-\xi_k^2(\bar{\alpha}_+-\beta_+\tau_1)(\bar{\alpha}_++\beta_+\tau_1)(\bar{z}+\bar{\Delta}\tau_1)=+2\xi_k^2\bar{\alpha}_+(\bar{z}+\bar{\Delta}\tau_1)\ .$$

We see that all terms have the same matrix structure $\bar{z}+\bar{\Delta}\tau_1$, and that they all have the factor $\bar{\alpha}_+$ which is the coherence factor for the $V_{\Delta\Delta}$ term. The empirical result that all the terms above give the product of the same matrix structure and coherence factor is the key to the success of the method. All one has to do is to sum up the appropriate coefficients of this product. The final result for this numerator is

$$N_2=2\bar{\alpha}_+\frac{\bar{z}+\bar{\Delta}\tau_1}{\varepsilon}[\varepsilon^2\varepsilon'-2\varepsilon\xi_k^2+\varepsilon'\xi_k^2]\ . \tag{B3}$$

Similar reasoning and identities can be used for the numerators corresponding to the other potentials V_{ij}, giving the same result except for a change in the coherence factor.

Next consider the numerator from Eq. (3.9)

$$N_1=(\bar{z}+\bar{\Delta}\tau_1+\xi_k\tau_3)(\bar{\alpha}_++\beta_+\tau_1)\times\tau_1(\bar{z}'+\bar{\Delta}'\tau_1+\xi_k\tau_3)\tau_1(\bar{z}+\bar{\Delta}\tau_1+\xi_k\tau_3)\ . \tag{B4}$$

We evaluate this term by term as discussed above

$$\text{EEE: } (\bar{z}+\bar{\Delta}\tau_1)(\bar{\alpha}_++\beta_+\tau_1)\tau_1(\bar{z}'+\bar{\Delta}'\tau_1)\tau_1(\bar{z}+\bar{\Delta}\tau_1)=\varepsilon\varepsilon'(\bar{\alpha}_+-\beta_+\tau_1)(\bar{z}+\bar{\Delta}\tau_1)\ ,$$

$$\text{EOO: } (\bar{z}+\bar{\Delta}\tau_1)(\bar{\alpha}_++\beta_+\tau_1)\tau_1\xi_k\tau_3\tau_1\xi_k\tau_3=-\xi_k^2(\bar{\alpha}_++\beta_+\tau_1)(\bar{z}+\bar{\Delta}\tau_1)\ , \tag{B5}$$

$$\text{OEO: } \xi_k\tau_3(\bar{\alpha}_++\beta_+\tau_1)\tau_1(\bar{z}'+\bar{\Delta}'\tau_1)\tau_1\xi_k\tau_3=\frac{\varepsilon'}{\varepsilon}\xi_k^2\tau_3(\alpha_+-\beta_+\tau_1)(\bar{z}-\bar{\Delta}\tau_1)\tau_3=\frac{\varepsilon'}{\varepsilon}\xi_k^2(\bar{\alpha}_++\beta_+\tau_1)(\bar{z}+\bar{\Delta}\tau_1)\ ,$$

$$\text{OOE: } \xi_k\tau_3(\bar{\alpha}_++\beta_+\tau_1)\tau_1\xi_k\tau_3\tau_1(\bar{z}+\bar{\Delta}\tau_1)=-\xi_k^2\xi_k(\bar{\alpha}_+-\beta_+\tau_1)(\bar{z}+\bar{\Delta}\tau_1)\ .$$

In this case there are two types of matrix structure $-\bar{z}+\bar{\Delta}\tau_1$ and $(\bar{z}+\bar{\Delta}\tau_1)\tau_1$, which are multiplied by the coherence factors $\bar{\alpha}_+$ and β_+, respectively. The final result for this numerator is

$$N_1=\bar{\alpha}_+\frac{\bar{z}+\bar{\Delta}\tau_1}{\varepsilon}[\varepsilon^2\varepsilon'-2\varepsilon\xi_k^2+\varepsilon'\xi_k^2]-\beta_+\frac{(\bar{z}+\bar{\Delta}\tau_1)\tau_1}{\varepsilon}[\varepsilon^2\varepsilon'-\varepsilon'\xi_k^2]\ . \tag{B6}$$

The numerators corresponding to other potentials V_{ij} can be similarly evaluated, giving the same result except for changes in coherence factors.

The method described above can evaluate the matrix products that result from any diagrammatic expression our formalism can produce. In other words we can evaluate any diagram we may care to write down. This makes our formalism a very powerful tool for the perturbative analysis of the disorder superconductor.

APPENDIX C: CANCELLATION OF LOW-MOMENTUM SINGULARITIES IN THE ORDER-PARAMETER EQUATION

In this appendix we will show that the low-momentum singularities that occur in Eq. (3.14) due to corresponding singularities in the potentials $V_{\phi\phi}$, $V_{\rho\rho}$, and $V_{\phi\rho}$ cancel amongst themselves to leave a single term that can be interpreted as a phase fluctuation term.

From Sec. V we know that the singular contribution to the order parameter is

$$\ln\left[\frac{\Delta}{\Delta_0}\right]=\frac{\pi}{2}T\sum_{\Omega}T\sum_{\omega}\left[\frac{\Delta}{W(W+W')^2}\left\{-x^2\left[1-\frac{\omega\omega'+\Delta^2}{WW'}\right]+\left[1+\frac{\omega\omega'+\Delta^2}{WW'}\right]+x\frac{2\Delta\Omega}{WW'}\right\}\right.$$
$$\left.+\frac{\omega}{W^2(W+W')}\left\{-x^2\frac{\Delta\Omega}{WW'}+\frac{\Delta\Omega}{WW'}-2x\frac{(\omega\omega'+\Delta^2)}{WW'}\right\}\right]\sum_q V_{\phi\phi}(q,\Omega)\ . \tag{5.1'}$$

This leads us to define the sums A_1-A_5,

$$A_{1,2,3}=T\sum_{\omega}\frac{1}{W(W+W')^2}\left[1,\frac{1}{WW'},\frac{\omega\omega'+\Delta^2}{WW'}\right],$$
$$A_{4,5}=T\sum_{\omega}\frac{\omega}{W^2(W+W')}\left[\frac{1}{WW'},\frac{\omega\omega'+\Delta^2}{WW'}\right]. \tag{C1}$$

The singular contribution is then given by

$$\ln\left[\frac{\Delta}{\Delta_0}\right]=\frac{\pi}{2}T\sum_{\Omega}V_{\phi\phi}(0,\Omega)[\Delta(1+x^2)A_1+4\Delta^3x^2A_2+\Delta(1-x^2)A_3+\Delta^2x(1-x^2)A_4-2xA_5]\ . \tag{C2}$$

We can expand out the sums A_1-A_3 by multiplying numerator and denominator by $W'-W$, noting that $W'^2-W^2=\Omega(2\omega+\Omega)$, and using the symmetry of each term in square brackets under $\omega\rightarrow-\omega'$ to get

$$T\sum_{\omega}\frac{1}{W(W+W')^2}[\cdots]=\frac{1}{2}T\sum_{\omega}\frac{1}{(W+W')^2}\left[\frac{1}{W}+\frac{1}{W'}\right][\cdots]=\frac{1}{2}T\sum_{\omega}\frac{1}{WW'(W+W')}[\cdots]$$
$$=\frac{1}{2\Omega}T\sum_{\omega}\frac{W'-W}{(2\omega+\Omega)WW'}[\cdots]$$
$$=\frac{1}{\Omega}T\sum_{\omega}\frac{1}{(2\omega+\Omega)W}[\cdots]\ . \tag{C3}$$

In the case of A_3 we further expand

$$\frac{1}{\Omega}T\sum_{\omega}\frac{\omega(\omega+\Omega)+\Delta^2}{(2\omega+\Omega)W^2W'}=\frac{1}{\Omega}T\sum_{\omega}\left[\frac{1}{(2\omega+\Omega)W'}+\frac{\Omega}{2}\left[1-\frac{\Omega}{2\omega+\Omega}\right]\frac{1}{W^2W'}\right]$$
$$=-\frac{1}{\Omega}T\sum_{\omega}\frac{1}{(2\omega+\Omega)W}+\frac{1}{2}T\sum_{\omega}\frac{1}{W^2W'}-\frac{\Omega}{2}T\sum_{\omega}\frac{1}{(2\omega+\Omega)W^2W'}\ , \tag{C4}$$

so that we can finally write the A_1-A_3 in terms of the simpler B_1-B_3,

$$A_1=\frac{1}{2\Delta x}B_1;\quad A_2=\frac{1}{2\Delta x}B_2;\quad A_3=\frac{1}{2\Delta x}(-B_1-2\Delta^2x^2B_2+\Delta xB_3)\ , \tag{C5}$$

where

$$B_1=T\sum_{\omega}\frac{1}{(2\omega+\Omega)W};\quad B_2=T\sum_{\omega}\frac{1}{(2\omega+\Omega)W^2W'};\quad B_3=T\sum_{\omega}\frac{1}{W^2W'}\ . \tag{C6}$$

Expanding A_4 and A_5 in the same manner yields

$$A_4=\frac{1}{\Omega}T\sum_{\omega}\frac{\omega(W'-W)}{(2\omega+\Omega)W^3W'}=\frac{1}{2\Omega}T\sum_{\omega}\left[1-\frac{\Omega}{2\omega+\Omega}\right]\left[\frac{1}{W^3}-\frac{1}{W^2W'}\right]$$
$$=\frac{1}{2\Omega}T\sum_{\omega}\frac{1}{W^3}-\frac{1}{2\Omega}T\sum_{\omega}\frac{1}{W^2W'}-\frac{1}{2}T\sum_{\omega}\frac{1}{(2\omega+\Omega)W^3}+\frac{1}{2}T\sum_{\omega}\frac{1}{(2\omega+\Omega)W^2W'} \tag{C7}$$

and

$$A_5=\frac{1}{\Omega}T\sum_{\omega}\frac{\omega(W^2+\Omega\omega)(W'-W)}{(2\omega+\Omega)W^3W'}=\frac{1}{2\Omega}T\sum_{\omega}\left[1-\frac{\Omega}{2\omega+\Omega}\right]\frac{W'-W}{WW'}+T\sum_{\omega}\frac{W^2-\Delta^2}{W^3W'}\frac{W^2-W}{2\omega+\Omega}$$

$$=T\sum_{\omega}\frac{1}{(2\omega+\Omega)W}-\Delta^2T\sum_{\omega}\frac{1}{(2\omega+\Omega)W^3}+\Delta^2T\sum_{\omega}\frac{1}{(2\omega+\Omega)W^2W'}\ . \tag{C8}$$

This allows us to write A_4 and A_5 in the form

$$A_4=\frac{1}{2}B_2-\frac{1}{4\Delta x}B_3-\frac{1}{2}B_4+\frac{1}{4\Delta x}B_5;\quad A_5=B_1+\Delta^2B_2-\Delta^2B_4\ , \tag{C9}$$

where B_4 and B_5 are defined by

$$B_4=T\sum_{\omega}\frac{1}{(2\omega+\Omega)W^3};\quad B_5=T\sum_{\omega}\frac{1}{W^3}\ . \tag{C10}$$

Substituting this into Eq. (C2) gives us the result for the singular term

$$\ln\left[\frac{\Delta}{\Delta_0}\right]=\frac{\pi}{2}T\sum_{\Omega}V_{\phi\phi}(0,\Omega)[-xB_1+\Delta^2x(1+x^2)B_4+\tfrac{1}{2}\Delta(1-x^2)B_5]\ . \tag{C11}$$

It turns out that B_1, B_4, and B_5 are linearly related since

$$B_1=T\sum_{\omega}\frac{1}{(2\omega+\Omega)W}=\frac{1}{4}T\sum_{\omega}\frac{4(\omega^2+\Delta^2)}{(2\omega+\Omega)W^3}=\frac{1}{4}T\sum_{\omega}\frac{(2\omega+\Omega)^2-2\Omega(2\omega+\Omega)+4\Delta^2+\Omega^2}{(2\omega+\Omega)W^3}$$

$$=-\frac{1}{2}\Delta xB_5+\Delta^2(1+x^2)B_4\ , \tag{C12}$$

so the final result is

$$\ln\left[\frac{\Delta}{\Delta_0}\right]=\frac{1}{4}\pi T\sum_{\Omega}\sum_{q}V_{\phi\phi}(q,\Omega)T\sum_{\omega}\frac{1}{W^3}\ . \tag{5.2'}$$

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