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Critical behavior of random transverse-field Ising spin chains

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A real-space renormalization-group treatment of random transverse-field Ising spin chains that was introduced previously is developed and extensively analyzed. It yields results that are asymptotically exact in the critical region near the zero-temperature para-to-ferromagnetic quantum phase transition. In particular, the exact scaling function is obtained for the magnetization as a function of a uniform applied magnetic field and the distance to the critical point, and up to the solution of a linear ordinary differential equation whose solution can be exhaustively analyzed, the scaling function of the average spin-spin correlation function is also obtained. Thus *more* exact information is obtainable about the critical behavior for this random model than is known for the pure version which is equivalent to the two-dimensional Ising model. The basic reason for this is the extremely broad distribution of energy scales that occurs at low energies near the critical point of the random system. For the random chain the distribution of the magnetization of the first spin in a semi-infinite system is also studied and the results found to agree in the scaling limit with results of McCoy obtained from the exact solution of the closely related McCoy-Wu Ising model; this provides strong justification for the validity of the present approach. The singular properties of the weakly ordered and weakly disordered "Griffiths' phases" that occur at zero temperature near the critical point are also studied, as well as the behavior at low but nonzero temperature. Possible extensions of the results and general lessons drawn from them for other random systems are briefly discussed.

I. INTRODUCTION

Understanding of the equilibrium statistical mechanics of random systems has been greatly hampered by the almost complete lack of solvable models with randomness coupled with a severe shortage of reliable nonperturbative methods. The exceptions are mostly infinite range models which for random systems tend to be either somewhat trivial, like the mean-field limit of random exchange ferromagnets, or rather pathological and difficult to understand, like the Sherrington-Kirkpatrick model of spin glasses.[1] Almost the only exception to this is the strip-random two-dimensional (2D) Ising model introduced and partially solved by McCoy and Wu a quarter century ago.[2,3] It consists of a nearest neighbor Ising model on a rectangular lattice, on which all the vertical exchanges K are the same, while the horizontal bonds J_i are identical to each other within each column but differ from column to column, in the simplest case being drawn independently from a distribution $\pi(J)dJ$. From a statistical-mechanical point of view, this appears to be very artificial; this is perhaps one of the reasons the McCoy-Wu model has received remarkably little attention.

But the transfer matrix of the McCoy-Wu (MW) model in the vertical direction is essentially equivalent to a much more natural system: the quantum-mechanical random transverse-field Ising spin chain with Hamiltonian

$$\mathcal{H} = -\sum_i J_i \sigma_i^z \sigma_{i+1}^z - \sum_i h_i \sigma_i^x - H \sum_i \sigma_i^z \ , \qquad (1.1)$$

where the $\{\sigma_i^\alpha\}$ are Pauli matrices. The transverse fields h_i are nonrandom in the MW model, $h_i = h$, and are related to the original vertical bonds K; we will, following Shankar and Murthy,[4] later generalize to random h_i's drawn independently from a distribution $\rho(h)dh$. A uniform magnetic field H in the z direction can also be added as in Eq. (1.1), although this destroys the solvability of the model. Note that with $H = 0$ a gauge transformation can always be performed on the variables to make *all* J_i and h_i *positive.* We will thus only consider this case.

Some of the properties of the ground state of the random transverse-field Ising chain are known from the exact solution;[2–4] these are summarized in the next subsection. The behavior is found to be rather remarkable. In particular, as the quantum fluctuations, controlled by h, are reduced, the susceptibility diverges below some h_χ, but the system remains paramagnetic with ferromagnetism only occurring below a *smaller* critical value of h, h_c. In addition, it is known that the distributions of various physical quantities are extremely broad with rare anomalous values dominating the averages of them, but just such averages are what would be measured in macroscopic experiments. This seemingly pathological behavior has perhaps been another reason for the neglect of the MW model; however, as we shall see, related behavior is in fact generic for random quantum systems. But perhaps the dominant reason for the ignoring of the MW results by most of the community working on random systems is that they do not *seem* to fit naturally into the conventional field-theoretic or renormalization-group descriptions of phase transitions. One of the main points of this paper is to show otherwise: Not only can the rich behavior of the random transverse-field Ising chain be understood within a renormalization-group framework,

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but the simple renormalization-group transformation introduced here can be used to derive new *exact* results in the scaling limit of low temperatures, small applied field H, and long length scales near the critical point, including expressions for some quantities which are not known exactly even for the pure Ising model. Nevertheless, the phase transition in this random system is rather strange and does not seem to be forceable into a conventional field-theoretic or replica description. In the last section, we will see that, in spite of its strange behavior, many of the features of the MW model will obtain in other random quantum systems.

This paper is organized as follows: In the remainder of the Introduction, the previously known results are summarized; the renormalization-group (RG) transformation from which we will obtain virtually all our results is introduced; the resulting qualitative RG flows are discussed; and the main results of the paper are summarized. In Sec. II, the renormalization-group transformation is analyzed in detail and a special solution of the flow equations that includes a scaling limit from which information about the critical region can be extracted is found.

The consequences of the RG flows, the special solution, and its generalization to enable information on spin correlations to be analyzed are studied in the next section. Exact critical scaling functions for the magnetization as a function of an applied field H and the average two-point spin correlation function are thereby derived. Sections II and III contain the main technical analysis found in the text, but demonstration of the convergence of general initial Hamiltonians to the special solution of the flow equations and details of some of the needed analysis are relegated to Appendixes A–C. In Sec. IV, the properties of the weakly ordered and disordered phases that exist near the critical point are analyzed and the general structure of the phase diagram is discussed.

The properties of the first spin in a semi-infinite chain are analyzed in Sec. V. This enables comparisons to be made with exact results of McCoy.[3] Further justification of the results are discussed in Sec. VI, with a study of the effects of correlations between the random couplings and a preliminary analysis of the transfer matrix approach[4] contained in, respectively, Appendixes D and E. Finally, in Sec. VII, possible extensions to other random quantum systems and parallels to more general random problems are discussed.

A. Previously known results

The random transverse-field Ising chain in zero applied field H is partially exactly solvable. McCoy and Wu[2] analyzed the free energy of the strip-random two-dimensional Ising model, and McCoy[3] computed the properties of the surface of a half-plane cut parallel to the nonrandom ("time") direction with which we will later compare our results. We will work, however, solely in the quantum spin chain representation which corresponds to an anisotropic continuum limit of the MW model in the uniform (time) direction. Shankar and Murthy (SM) have studied the (zero-temperature) random transverse-field chain by a method which will be more familiar to many readers. It is discussed in some detail in Appendix E, in which we also develop a variant of the methods of the present paper that can be justified by the exact solvability.

The essential ingredient of the exact solutions is the diagonalization in the time direction using the free fermion nature of the Ising spin chain. This enables each frequency ω to be treated *independently*, resulting in, effectively, independent random *classical* chains with ω as a parameter.[4] Some properties can then be obtained by taking products of the resulting random transfer matrices $T_j(\omega)$ (Appendix E). This is equivalent to a factorization into frequency components of the full transfer matrix in the *random* direction—rather unnatural from the point of view of a quantum-mechanical system, but like that used for studying localization of free electrons in random one-dimensional wires.[5]

The critical point of the random transverse-field chain is obtained quite straightforwardly by SM. Defining

$$\Delta_h \equiv \overline{\ln h} \ , \tag{1.2}$$

which is the control parameter that we will use, they find

$$\Delta_c = \Delta_J \equiv \overline{\ln J} \ . \tag{1.3}$$

A special case of this result can be guessed by duality: A transformation from site variables $\{\sigma_i^z\}$ to bond variables,

$$\tau_i^x = \sigma_i^z \sigma_{i+1}^z \ ,$$

$$\tau_i^z = \prod_{j \le i} \sigma_j^x \ , \tag{1.4}$$

yields a transverse-field chain with the role of h's and J's interchanged. Thus if the distributions of these are identical, i.e., $\pi = \rho$, we would expect to be at the critical point.

For $\Delta_h > \Delta_c$, the system is paramagnetic, while for $\Delta_h < \Delta_c$, there is a nonzero, but unknown, spontaneous magnetization density M_0. MW showed that there is only an essential singularity in the ground state energy density at Δ_c, in contrast to the behavior in the pure system.

Some information on the behavior of correlations was also obtained by SM,[4] although they confused the rather important distinction between mean correlations and typical correlations. What they actually showed is that for operators O that are local in the fermion representation,

$$C_O^t(x,y) \equiv \langle O(x)O(y)\rangle - \langle O(x)\rangle\langle O(y)\rangle \tag{1.5}$$

has for $\Delta_h \neq \Delta_J$ the *typical* behavior

$$-\ln C_O^t(x,y) \approx \frac{|x-y|}{\tilde{\xi}} \ , \tag{1.6}$$

with probability one as $|x-y| \to \infty$, with the length

$$\tilde{\xi} \sim \frac{1}{|\delta|^{\tilde{\nu}}} , \tag{1.7}$$

when δ, the distance from criticality

$$\delta \propto \Delta_h - \Delta_J , \tag{1.8}$$

is small. The exponent

$$\tilde{\nu} = 1 \tag{1.9}$$

is identical to the correlation length exponent of the pure model (for which, of course, no probabilistic statement is needed).

Thus, in some sense, for almost all widely separated pairs (x,y), $C^t_O(x,y)$ decays exponentially with the *typical correlation length* $\tilde{\xi}$. As we shall see, this is in fact also true for the typical (truncated) spin correlations

$$C^t_{ij} \equiv \langle \sigma^z_i \sigma^z_j \rangle - \langle \sigma^z_i \rangle \langle \sigma^z_j \rangle \tag{1.10}$$

at long distances away from criticality; nevertheless, the *mean correlations* $\overline{C^t_{ij}}$ will decay quite differently, and it is *these* that are measurable by, for example, neutron scattering.

Some indication of the wide variations of the correlations at a fixed distance can already be seen from SM's calculations. At Δ_c, they find that

$$-\ln C^t_O(x,y) \sim \sqrt{|x-y|} , \tag{1.11}$$

with the proportionality *coefficient* in Eq. (1.11) having a distribution with width of order 1, indicating that the mean $\overline{C_0}$ will behave quite differently from the typical correlations. This will be made more precise later.

At this point, no exact results are available about either the spontaneous magnetization or the mean correlations. However, McCoy's[3] calculations on the properties of the first spin σ_1, in a half-line, do provide lower bounds on the former. McCoy calculates the end-point magnetization

$$M_1 \equiv \langle \sigma_1 \rangle \tag{1.12}$$

of the first spin on a semi-infinite chain as a function of a field H_1 applied only to this spin. He computes the mean $\overline{M_1}(H_1)$ and other moments of the distribution. Since the mean end-point spontaneous magnetization $\overline{M}_{1,0} \equiv \lim_{H_1 \to 0^+} \overline{M}_1(H_1)$ can be nonzero if and only if the bulk spontaneous magnetization is nonzero, and is furthermore a lower bound for the bulk spontaneous magnetization, McCoy's result that

$$\overline{M}_{1,0} \sim |\delta| \tag{1.13}$$

for small negative δ implies that

$$M_0 \geq C|\delta|, \tag{1.14}$$

with a distribution-dependent coefficient C.

In the disordered phase, the mean end-point susceptibility

$$\overline{\chi_1} = \left.\frac{\partial \overline{M}_1}{\partial H_1}\right|_{H_1=0} \tag{1.15}$$

likewise provides a lower bound for the bulk susceptibility. McCoy found the *a priori* very surprising result that $\overline{\chi}_1$ is infinite for a whole *range* of $\Delta > \Delta_c$, implying that the bulk χ diverges in at least as wide a range.

If *all* the h_i's are larger than all the J_i's, then it can be shown straightforwardly that χ is finite, and so at least for narrow distributions $\rho(h)$ and in fact more generally, there is a value of Δ_h, $\Delta_\chi > \Delta_c$, which separates the regimes with finite and infinite susceptibility. The striking difference between this behavior and that for conventional systems has given rise to some confusion as to whether the transition in the MW model is "sharp"; nevertheless, it *is* sharp, and the true transition occurs at Δ_c. As we shall see, Δ_χ is not really a particularly special point; rather there is a weakly disordered Griffiths "phase"[6] which occurs when $\max\{J_i\} > \min\{h_j\}$ but $\Delta_h > \Delta_c$, in which $M(H)$ is singular at $H = 0$. This is discussed in detail in Sec. IV.

In the ordered phase, there is also a Griffiths region. If $\min\{J_i\} > \max\{h_j\}$, then there exists at zero temperature a nonzero interfacial energy S_∞ defined as the limit of the difference

$$S_L \equiv F_{+-} - F_{++} \tag{1.16}$$

between the ground state energies (or free energies at $T > 0$) of chains of length L with boundary conditions on the ends that are opposite $(+-)$ versus equal $(++)$. If $\min\{J_i\} < \max\{h_j\}$, then $S_\infty = 0$ with probability *one*, even though Δ_h may be less than Δ_c so that there is a spontaneous magnetization.[7] The properties of this weakly ordered Griffiths "phase" are discussed in Sec. IV.

B. Renormalization-group transformation

At this point, it is useful to ask why more information has not been obtainable from the methods of MW and SM. Partially, this is due to the usual difficulty of obtaining spin correlations in Ising systems that arises from the nonlocal relationship between the spins and the free fermion operators in terms of which the zero-field thermodynamics becomes trivial. In addition, there are of course the difficulties associated with dealing with random rather than uniform systems which substantially complicate the analysis. But there is a third reason why asymptotic, low-energy or long-distance properties near to criticality are difficult to obtain, and this involves interesting new physics.

By multiplying transfer matrices in the obvious (or any other predetermined) order, it is hard to keep track of the possible development of rare anomalous regions of the system which may dominate the low-energy properties.

Instead, one is forced to use central-limit-theorem-like results which, by their nature, only deal with almost all of the cases, perhaps leaving out a negligible fraction which could affect the physics. The purpose of this paper is to develop and utilize an approximate method which focuses on the important degrees of freedom which dominate the physics at low energies. Many new results will thereby be obtained. By combining the present techniques with the exact formulation of SM, it may in the future be possible to put our results on a much firmer footing by multiplying transfer matrices in a clever order that depends on the specific realization of the randomness; this possibility is discussed briefly in Appendix E.

The main goal of this paper is to analyze the consequences of a simple approximate renormalization-group (RG) treatment of the random transverse-field Ising chain that was introduced earlier.[8] Surprisingly, many of the results will turn out to be exact in the scaling regime near Δ_c, as well as yielding qualitative understanding of the weakly ordered and weakly disordered Griffiths phases that occur near Δ_c. The reasons for this remarkable property of the RG treatment, as well as direct comparison of some of the results with those of McCoy, will be discussed in Secs. V and VI.

We are interested in the low-energy properties of the system and would thus like to systematically get rid of high-energy degrees of freedom. A nice simple way of doing this was introduced by Ma, Dasgupta, and Hu[9] for random Heisenberg antiferromagnetic spin chains and developed extensively by this author.[8,10] The main idea is to take the strongest coupling in the system, find the ground states of the associated part of the Hamiltonian, treat the coupling to the rest of the system perturbatively, and then throw out the excited states involving the strong coupling, yielding a new effective Hamiltonian $\tilde{\mathcal{H}}$. The procedure is then iterated *ad physicum.*

For the transverse-field chain in a small or zero applied field H, we thus choose first the largest of the set of couplings,

$$\Omega_I \equiv \max\{J_i, h_j\} \tag{1.17}$$

(which without real loss of generality of the procedure we take to be finite), and set the energy scale Ω, which will gradually be reduced, to its initial value Ω_I. If the largest coupling is an h_j, say, h_2, then the associated part of $\mathcal{H}$ is simply $-h_2\sigma_2^x$ which has a ground state $|\rightarrow_2\rangle$ and excited state $|\leftarrow_2\rangle$ separated by a gap $2h_2$. The coupling to the rest of the system, $-J_1\sigma_1^z\sigma_2^z - J_2\sigma_2^z\sigma_3^z$, can then be treated by second-order degenerate perturbation theory in the four-dimensional space of states $|\sigma_1 \rightarrow_2 \sigma_3\rangle$ with $\sigma_{1(3)} = \uparrow$ or $\downarrow$. This yields an effective Hamiltonian

$$\tilde{\mathcal{H}}_{13} = -\tilde{J}_1\tilde{\sigma}_1^z\tilde{\sigma}_3^z , \tag{1.18}$$

with the effective exchange

$$\tilde{J}_{13} \approx \frac{J_1J_2}{h_2} + O\left(\frac{J_1^3J_2}{h_2^3}, \frac{J_1^2J_2^2}{h_2^3}, \frac{J_1J_2^3}{h_2^3}\right) . \tag{1.19}$$

Note that the effective spin operators $\tilde{\sigma}_1^z$ and $\tilde{\sigma}_3^z$ in Eq. (1.18) are not quite the same as σ_1^z and σ_3^z, but the differences will be small if $h_2 \gg J_1, J_2$.

We now make the approximation of throwing out the excited state of σ_2 entirely, replacing $\tilde{J}_1$ by the lowest-order expression in Eq. (1.19) and replacing $\tilde{\sigma}_i^z$ by σ_i^z. We now have a new chain with one less spin and one modified bond with $\tilde{J} < \Omega$ that has length

$$\tilde{\ell}_B = \ell_{B1} + \ell_{B2} + \ell_{S2} . \tag{1.20}$$

Here we have divided, for later convenience, the length up so that initially 1/2 is associated with each bond ℓ_{Bi} and 1/2 with each spin ℓ_{Si}. Thus after the first step above, $\tilde{\ell}_{B1} = 3/2$.

If the largest coupling in the original Hamiltonian were a bond, J_i, say, J_2, then the associated part of $\mathcal{H}$ is $-J_2\sigma_2^z\sigma_3^z$ which has two degenerate ground states $|\uparrow\uparrow\rangle$ and $|\downarrow\downarrow\rangle$. The neighboring transverse fields $h_{2,3}$ can now be treated perturbatively, yielding an effective transverse field

$$\tilde{h}_2 \approx \frac{h_2h_3}{J_2} , \tag{1.21}$$

which flips *coherently* the *spin cluster* of $\sigma_2 + \sigma_3$. This yields an effective Hamiltonian

$$\tilde{\mathcal{H}}_2 \approx -\tilde{h}_2\tilde{\sigma}_2^x - H\tilde{m}_2\tilde{\sigma}_2^z , \tag{1.22}$$

where we have explicitly included a small z field H to show the new feature: The spin *cluster* has a *magnetic moment*

$$\tilde{m}_2 = m_1 + m_2 ; \tag{1.23}$$

i.e., in the present case, $\tilde{m}_2 = 2$. (Note that the effects on the RG procedure of a nonzero H will be analyzed later.)

The spin cluster has length

$$\tilde{\ell}_{S2} = \ell_{S2} + \ell_{B2} + \ell_{S3} ; \tag{1.24}$$

i.e., here, $\tilde{\ell}_{S2} = 3/2$. We now throw out the excited states of the cluster, $|\uparrow\downarrow\rangle$ and $|\downarrow\uparrow\rangle$, which are separated by energy $2J_2$ from the ground states, and note that, for $J_2 \gg h_2, h_3$, the effective couplings $-J_1\sigma_1^z\sigma_2^z$, and $-J_3\sigma_3^z\sigma_4^z$ become simply $-J_1\sigma_1^z\tilde{\sigma}_2^z$ and $-\tilde{J}_2\tilde{\sigma}_2^z\sigma_4^z$ with $\tilde{J}_2 = J_3$. The latter follows as we chose to label effective fields and bonds by the leftmost spin. Thus again, as after decimating a strong h_i, we have a new effective Hamiltonian with one less spin degree of freedom and all couplings $< \Omega_I$. Note the duality between h and J explicit in the recursion relations Eqs. (1.19), (1.21) and those for the lengths Eqs. (1.20) and (1.24).

There are several crucial features of the RG transformation. On the one hand is the *independence* from site to bond, site to site, and bond to bond, of the effective couplings: The only modified couplings are associated with the eliminated couplings, an eliminated h also eliminating the two neighboring J's to give one new J, and vice versa. On a given bond or site, on the other hand, the sets of variables $(\tilde{h}_i, \tilde{m}_i, \tilde{\ell}_{Si})$ or $(\tilde{J}_j, \tilde{\ell}_{Bj})$ are correlated.

As the procedure described above is iterated, Ω is gradually lowered and more and more bonds and spins are replaced by effective new bonds or spin clusters. But because of the above-mentioned independence, we need only keep track of two joint distributions

$$\tilde{\pi}(\tilde{J}, \ell_B; \Omega) \tag{1.25}$$

and

$$\tilde{\rho}(\tilde{h}, \ell_S, m; \Omega) , \tag{1.26}$$

where we have displayed the implicit dependence on the maximum energy scale Ω of the effective couplings (and any remaining original couplings).

The distributions $\tilde{\pi}$ and $\tilde{\rho}$ must be rescaled to keep them properly normalized. Thus information about the number density $n(\Omega)$ of spin clusters or bonds at energy scale Ω is not directly available from them. However, since each decimation reduces the number of clusters (and bonds) by 1, we have

$$\frac{-\partial n}{\partial \Omega} = -\left[\int d\ell_B \tilde{\pi}(J = \Omega, \ell_B; \Omega) + \int d\ell_S \int dm \; \tilde{\rho}(h = \Omega, \ell_S, m; \Omega)\right] n(\Omega) , \tag{1.27}$$

the two quantities in the square brackets being, respectively, $1/d\Omega$ times the fraction of bonds or spin clusters that are eliminated because of the strong couplings when Ω is decreased to $\Omega - d\Omega$. The number density of spin clusters, $n(\Omega)$, contains information about the connection between a characteristic length scale $\ell(\Omega) \sim 1/n(\Omega)$ and the energy scale.

The recursion relations Eqs. (1.19), (1.20), (1.21), (1.23), and (1.24) are analyzed in detail starting in the next section. We first give a qualitative picture of the RG flows.

C. Qualitative flows

Much can be learned about the behavior of random transverse-field chains by simple qualitative consideration of the RG flows under the transformation introduced above.[18] Since we are interested in the critical behavior, we first consider the self-dual case where the J and h distributions are identical; this property will obviously be preserved by the flow.

1. At criticality

After many steps of renormalizing, effective $\tilde{J}$'s, for example, will be ratios of a product of many original J_i's to a product of many original h_i's:

$$\tilde{J}_i \approx \frac{J_i J_{i+1} \cdots J_{i+\ell-\frac{1}{2}}}{h_{i+1} h_{i+2} \cdots h_{i+\ell-\frac{1}{2}}} , \tag{1.28}$$

with the bond length ℓ an integer plus $\frac{1}{2}$. If the J_i's and h_j's involved were chosen independently, then we would expect $\ln\tilde{J}_i$ to be a Gaussian random variable with mean Δ_J and variance

$$\text{var}(\ln\tilde{J}_i) \equiv \overline{(\ln\tilde{J}_i)^2} - (\overline{\ln\tilde{J}_i})^2 \sim \ell V_I , \tag{1.29}$$

where

$$V_I \equiv \text{var}(\ln J)_I + \text{var}(\ln h)_I \tag{1.30}$$

is defined from the variances of the original (initial) variables with distributions π and ρ.

The quantity V_I is a dimensionless measure of the strength of the randomness. From Eq. (1.28), we can guess that the $\tilde{J}$ and $\tilde{h}$ become broadly distributed on length scales larger than

$$\ell_V \equiv \frac{2}{V_I} \tag{1.31}$$

(the "2" chosen for later convenience) and the system will cross over to strongly random behavior. However, the simple independence assumption above is clearly incorrect. We know, from the definition of the RG transformation, that $\tilde{J}_i \leq \Omega < \Omega_I$; therefore, $\ln\tilde{J}_i$ is clearly *not* Gaussian and its mean is *not* Δ_J, in spite of the independence of the original J_i's. This is because of the subtle way in which the J_i and h_j that enter a renormalized $\tilde{J}$ are chosen, or concomitantly, which effective bonds $\tilde{J}_i$ exist and what their lengths ℓ_{B_i} are.

Nevertheless, the distribution of $\ln\tilde{J}$ does indeed broaden with decreasing Ω, corresponding to increasing ℓ, and its variance will turn out to scale as Eq. (1.29). (This could be guessed from knowledge of the statistics of extrema of random walks.[11]) In fact, at criticality, the *mean* of $\ln\tilde{J}$ will turn out to grow in the same way as the width of its distribution, i.e., $\sqrt{\text{var}(\ln\tilde{J})}$, so that

$$-\overline{\ln\tilde{J}} \sim \sqrt{\ell V_I} , \tag{1.32}$$

suggesting, since $\ln\tilde{J} \leq \ln\Omega$, that effective bonds of length ℓ will be typical in the critical regime when

$$\ln(\Omega_V/\Omega) \sim \sqrt{\ell V_I} \sim \sqrt{\ell/\ell_V} \tag{1.33}$$

with Ω_V an energy scale of order Ω_I.[12] These heuristic arguments thus yield the relation between energy and length scales at the critical point, with the basic microscopic length scale being ℓ_V from Eq. (1.31).

The dramatic broadening of the distributions of $\ln\tilde{J}$ and $\ln\tilde{h}$ under renormalization—with widths that diverge at low energies at the critical point—is the key feature that makes our RG procedure work. Initially, errors will be made by the second-order perturbation approximation of Eqs. (1.19) and (1.21). But once the typical bond and cluster lengths become much larger than ℓ_V, the breadth of the distributions of $\ln\tilde{J}$ and $\ln\tilde{h}$ will make the RG decimations less and less likely to be problematic, as the neighbors of the strongest remaining effective coupling will typically be much weaker and hence can be treated perturbatively. Thus, up to nonuniversal coefficients of order 1 (or more generally, powers of V_I) which depend on the small-scale high-energy physics, the RG should produce asymptotically exact results for universal quantities in the scaling limit of small δ, low energies, low temperature, and long length scales.

The effects of a small applied magnetic field H can be analyzed by keeping track of the distribution of the magnetic moments m of spin clusters. In our simple approximation, the magnetic moment of a cluster is simply the number of *active* spins it contains, i.e., the spins which were not decimated at earlier stages of the RG. At some scale Ω_H, the strongest couplings in the system will, in the presence of an applied field H, no longer be effective transverse fields or bonds; instead the magnetic couplings to H will dominate. But this will result in the remaining spin clusters being aligned by the applied field, resulting in a magnetization that is, roughly, the number of active spins at scale Ω_H. This will be used in Sec. III to obtain the magnetization as a function of the applied field.

The scaling behavior for small H will be determined by the scaling of the cluster moments in zero field. Like the cluster lengths, at criticality the typical moments m will, from Eq. (1.23), grow as a power of $\ln(1/\Omega)$, and hence, also as a power of the cluster lengths ℓ_S. We will see that, in fact, at criticality

$$m \sim \left[\ln(1/\Omega)\right]^{\phi} , \tag{1.34}$$

with the exponent

$$\phi = \frac{1+\sqrt{5}}{2} \tag{1.35}$$

equal to the golden mean; ϕ will determine various measurable critical exponents.

2. Off criticality

Before summarizing some of the main results of our RG treatment, we try to develop a heuristic picture of the behavior of the RG flows away from, but near to, criticality.

For later convenience, we define the distance from criticality $\Delta_c = \Delta_J$ by

$$\delta \equiv \frac{\Delta_h - \Delta_c}{V_I} = \frac{(\overline{\ln h})_I - (\overline{\ln J})_I}{\mathrm{var}(\ln h)_I + \mathrm{var}(\ln J)_I} . \tag{1.36}$$

Since δ is the ratio of the mean per unit length to the variance per unit length of $\sum_i(\ln h_i - \ln J_i)$, it should be preserved exactly by the flow; this will turn out to be the case provided the effects of the distributions of bond and cluster lengths (and their correlations with the J_i and h_i) are taken into account; see Sec. II D.

3. Disordered phase

We first consider the weakly disordered phase, i.e., $\delta > 0$. At a length scale

$$\tilde{\xi} \sim \frac{\ell_V}{\delta} , \tag{1.37}$$

the mean $\overline{\ln\tilde{h}}$ will be bigger than $\overline{\ln\tilde{J}}$ by an order of unity. One might expect that $\tilde{\xi}$ would be the true correlation length, and indeed, it is the length that sets the typical correlations for asymptotically large $|i-j|$ as in Eq. (1.6). However, at length scale $\tilde{\xi}$ the width of the $\ln\tilde{J}$ distribution is of order $(\tilde{\xi}V_I)^{1/2} \sim \frac{1}{\delta^{1/2}} \gg 1$. Thus the mean of $\ln\tilde{J}$ characterizes its distribution very poorly. Indeed, almost half of the $\tilde{J}_i$'s will still be larger than their neighboring $\tilde{h}_i$'s and regions of the system of size $\tilde{\xi}$ will not "know" that they are noncritical. This is the root of the inequality of Harris and Chayes *et al.*[13,14] that the "true" correlation length ξ must diverge with an exponent ν, which is large enough that regions of size ξ "know" that they are in the disordered phase with probability substantially greater than 1/2. This implies, in the simple picture outlined here, that ξ should be determined by

$$(\xi V_I)^{\frac{1}{2}} \sim \xi\delta V_I , \tag{1.38}$$

i.e.,

$$\xi \sim \frac{\ell_V}{\delta^{\nu}} , \tag{1.39}$$

with $\nu = 2$, saturating the Harris inequality $\nu \geq 2/d = 2$.[15]

On scales larger than ξ, most of the $\tilde{h}$ are bigger than most of the $\tilde{J}$, and the physics is noncritical and can be described more simply. Beyond this scale, most of the eliminated couplings will be transverse fields so that clusters will be eliminated, yielding progressively longer and weaker bonds between the remaining clusters. But the strengths of the remaining bonds will *not* appreciably affect the physics, which will be dominated by the properties of the remaining *clusters*.

A simple approximation at low energies in the disordered phase is thus to simply set all the remaining $\tilde{J}$'s to zero at length scale ξ and be left with a problem of decoupled clusters whose properties will be given by the number density $n(\Omega_\delta)\tilde{\rho}(\tilde{h}, \ell_S, m; \Omega_\delta)$ of the clusters with $\tilde{h}$, ℓ, and m at the scale Ω_δ at which

$$n(\Omega_\delta) \sim V_I\delta^2 \sim \xi^{-1} , \tag{1.40}$$

so that the effective $\tilde{\mathcal{H}}(\Omega_\delta)$ is substantially away from criticality. As we shall see in Sec. IV, an exponential tail of the distribution of $\ln(1/\tilde{h})$—i.e., a power law tail for $\tilde{\rho}(\tilde{h})$—leads naturally to singular magnetization at small fields with

$$M_{\mathrm{sing}}(H) \sim H^{\kappa} \tag{1.41}$$

and a divergent susceptibility at low T,

$$\chi(T) \sim \frac{1}{T^{\gamma}} , \tag{1.42}$$

with $\gamma < 1$. The exponents κ and γ depend *continuously* on δ via the parameters of the $\ln(1/\tilde{h})$ distribution.

The exponential tail in $\ln(1/\tilde{h})$ arises from an exponentially small probability of large ℓ_S's and the tendency, as we have seen, for $\ln(1/\tilde{h}) \sim \ell_S$. The long rare clusters with anomalously small $\tilde{h}$ in fact *dominate* the low-energy physics in the weakly disordered Griffiths phase that we have been discussing. What is needed from the RG transformation is information on how the tail of the $\tilde{h}$ distribution develops, and what its parameters are at scale Ω_δ; this we analyze in Sec. IV and the end of Appendix A.

4. Ordered phase

In the ordered phase for δ small and negative, similar considerations lead to a characteristic scale $\xi \sim \frac{1}{\delta^2}$ beyond which the system is well ordered. Since most $\tilde{J}$'s will be bigger than most $\tilde{h}$'s at scale ξ, beyond this scale only a few clusters will be eliminated while most of them will be joined together into bigger clusters; eventually at $\Omega = 0$ an infinite cluster will have formed. This infinite cluster will contain as "active" spins a substantial fraction of those that were active at scale ξ. Since only the active spins can contribute to the spontaneous magnetization M_0, we can obtain an estimate of this by the fraction of spins active at the energy scale Ω_δ that corresponds to length scale ξ. The order parameter exponent β will thus be determined by ν and ϕ, from Eqs. (1.34) and (1.39).

By analogy with the weakly disordered phase, the weakly ordered phase for δ small and negative should have at scale ξ an exponential tail of the distributions of bond lengths ℓ_B and effective couplings $\ln(1/\tilde{J})$. This translates into a *power law* distribution of $\tilde{J}$. If, as we may do to a crude but good approximation—see Sec. IV A—we ignore all the remaining $\tilde{h}$'s at scale ξ, then the effective $\tilde{\mathcal{H}}$ is just a *classical* random bond Ising model with a power law distribution of $\tilde{J}$'s. The interfacial tension S_L of a system of length L will then just be the weakest $\tilde{J}$ in length L. We thus see that S_L should vanish as $L^{-\alpha}$ for large L with $\alpha(\delta)$ depending on the parameters of the distribution at length scale ξ. Similarly, at *positive* temperatures there will be a thermal correlation length $\xi_T \sim T^{-\alpha}$ and a susceptibility

$$\chi(T) \sim \frac{1}{T^{1+\alpha}} , \tag{1.43}$$

which is now larger than the Curie susceptibility due to the developing correlations. Again, the task of the RG analysis near the critical point is to obtain the form of the distributions and hence $\alpha(\delta)$.

In Sec. IV, we will see that the simple toy models that we have sketched above capture most of the low-energy properties of the weakly ordered and disordered Griffiths' phases. The small δ dependence of α and κ will be obtained *exactly* from the RG.

D. Summary of main results

Before proceeding with the concrete calculations, we summarize some of the main results of this paper. These primarily concern the critical region with δ, the dimensionless measure of the distance to criticality, Eq. (1.36), the temperature T, the applied ordering field H, and wave vectors or inverse distances all small.

At zero temperature, the *magnetization* in a small (positive) applied ordering magnetic field H has the scaling form

$$M(\delta, H) \approx \bar{\mu}[\ln(D_H/H)]^{\phi-2}\mathcal{M}\left[\delta \ln\left(\frac{D_H}{H}\right)\right] \tag{1.44}$$

for $|\delta|$ and $1/|\ln H|$, both small, but their ratio γ tending to any fixed constant.

Here the exponent $\phi = \frac{1}{2}(1+\sqrt{5})$ while $\bar{\mu}$ and D_H are nonuniversal dimensional constants. The *universal scaling function*

$$\mathcal{M}(\gamma) = \frac{\gamma^2 \propto (\gamma)}{\sinh^2\gamma} + \frac{e^{-\gamma}}{\sinh\gamma}\left[\phi\gamma\alpha(\gamma) + \gamma^2 \frac{d\alpha}{d\gamma}\right] , \tag{1.45}$$

with $\alpha(\gamma)$ obeying the second-order differential equation (3.49) with $\alpha(0) = 1$, which can be solved, yielding

$$\alpha(\gamma) \propto \frac{1}{|\gamma|^\phi} Q_{\phi-1}(\coth\gamma) , \tag{1.46}$$

with $Q_{\phi-1}$ a Legendre function.[16]

Note that the analogous magnetization scaling function for the *pure* Ising model is not known.

At the critical point $\delta = 0$,

$$M(H) \sim \frac{1}{|\ln H|^{2-\phi}} \tag{1.47}$$

for small H, while in the ordered phase $\delta < 0$, the spontaneous magnetization

$$M_0(\delta) = \lim_{H\to 0^+} M(\delta, H) \sim (-\delta)^\beta , \tag{1.48}$$

with

$$\beta = 2 - \phi = \frac{3-\sqrt{5}}{2} . \tag{1.49}$$

In both phases $M(H)$ is very singular for small H near the critical point. In particular, in the disordered phase $\delta > 0$, the scaling function yields a continuously variable power law singularity for small H,

$$M(H) \sim \delta^{3-\phi} H^{2\delta} |\ln H| , \tag{1.50}$$

so that the linear susceptibility χ is *infinite* for a range of δ. At asymptotically low fields at *fixed* δ in the disordered phase there are nonscaling corrections to Eq. (1.44) which change slightly the exponents of H [probably by $O(\delta^2)$] and $\ln H$ in Eq. (1.50); see Sec. IV B. For δ sufficiently large, χ becomes finite but there will still be a weaker power law singularity in $M(H)$ for a wider range of δ.

We also compute the scaling function for the magnetization of the first spin in a semiinfinite chain as a function of a z field H_1 applied to this end-point spin only. This was previously known both for the pure Ising system and, thanks to McCoy,[3] for the random chain. Our results agree exactly with McCoy's in the scaling limit, both for the scaling function of the sample averaged *end-point magnetization*, which is dominated by rare samples, and the very broad distribution of the end-point spontaneous magnetization just into the ordered phase.

By our methods, the scaling behavior of the bulk linear low-temperature susceptibility $\chi(\delta, T)$ is also obtained, and its low-temperature limit analyzed in the two phases and at the critical point, although the exact scaling function has not yet been computed.

The behavior of the two-point spin-spin correlation function

$$C(x) = \langle \sigma_j^z \sigma_{j+x}^z \rangle \tag{1.51}$$

is studied in more detail. At the critical point, $C(x)$ is very broadly distributed, with the typical behavior

$$-\ln C(x) \sim \sqrt{x} \ , \tag{1.52}$$

with the coefficient in Eq. (1.52) having a *fixed* (but so far unknown) distribution in the limit of large x. Conversely, in the disordered phases the typical correlations decay as

$$-\ln C(x) \approx x/\tilde{\xi} \ , \tag{1.53}$$

with probability one for large x with the typical correlation length

$$\tilde{\xi} \sim \frac{1}{|\delta|^{\tilde{\nu}}} \ , \tag{1.54}$$

with

$$\tilde{\nu} = 1 \ . \tag{1.55}$$

Nevertheless, the *mean* correlations, measurable by neutron scattering, are dominated by the pairs of spins with atypically large correlations that are of order unity. The scaling function of the *mean correlations*, $\overline{C}(x)$, at zero temperature is reduced to analysis of the solutions of a second-order linear ordinary differential equation (ODE). Although this has not been solved in closed form, all the interesting limits can be analyzed. At the critical point, power law decay of the mean correlations

$$\overline{C}(x) \sim \frac{1}{x^{2-\phi}} \tag{1.56}$$

is found, while in the disordered phase for distances x much larger than the *true* correlation length

$$\xi \sim \frac{1}{\delta^{\nu}} \ , \tag{1.57}$$

with

$$\nu = 2 \ , \tag{1.58}$$

the mean correlations for small δ and $x \gg \xi$ decay as

$$\overline{C}(x) \sim \delta^{4-2\phi} \left(\frac{\xi}{x}\right)^{5/6} e^{-C_\xi (x/\xi)^{1/3}} e^{-x/\xi} \ , \tag{1.59}$$

with C_ξ a computable coefficient, Eq. (3.112).

Thus we see that the mean correlations decay exponentially with a much longer correlation length ξ than do the correlations between a typical pair of widely separated points. Both the long-distance mean correlations and the low-field magnetization are dominated by spins which remain active down to very low energies.

The decay of the mean correlation function in the ordered phase (at $H = T = 0$) to its long-distance form of M_0^2 are also computed, as is the behavior of the correlation length in nonzero magnetic field and nonzero temperature. Again, these collectively represent more information than is known exactly for the pure Ising model. Many other properties—essentially all the behavior of mean multipoint correlations when all of T, H, δ, and inverse distances are small—are computable, in principle, by the methods presented here. These are left for future researchers.

II. RENORMALIZATION-GROUP FLOWS

In this section, the renormalization-group flows outlined in the Introduction are analyzed and fixed points, eigenvalues, and scaling functions computed. We postpone a detailed discussion of the justification of the approximations made in the decimation procedure to Sec. VI.

It is convenient to work in logarithmic variables defining

$$\Gamma \equiv \ln(\Omega_I/\Omega) \ , \tag{2.1}$$

$$\zeta \equiv \ln(\Omega/J) \geq 0 \ , \tag{2.2}$$

and

$$\beta \equiv \ell n(\Omega/h) \geq 0 \ , \tag{2.3}$$

where the maximum coupling is initially Ω_I but after renormalization becomes $\Omega < \Omega_I$. The variable Γ is the flow parameter that is analogous to ℓ in the rescaling of the momentum cutoff Λ to $\Lambda e^{-\ell}$ in conventional momentum-space renormalization-group transformations. The logarithmic variables ζ and β are defined to be positive; large ζ or β correspond to small J or h, respectively.

As Ω is reduced to $\Omega - d\Omega$, Γ is increased to $\Gamma + d\Gamma = \Gamma + \frac{d\Omega}{\Omega}$ and bonds with ζ within $d\Gamma$ of zero are decimated away and similarly spin clusters with β within $d\Gamma$ of zero. If, for example, cluster 2 with $\beta_2 = 0$ is decimated, the new effective bond coupling cluster 1 to cluster 3 becomes

$$\tilde{\zeta} = \zeta_1 + \zeta_2 \ ; \tag{2.4}$$

likewise if a bond 1 is decimated, the new cluster has

$$\tilde{\beta} = \beta_1 + \beta_2 \ . \tag{2.5}$$

After the decimations, all β's and ζ's are reduced by $d\Gamma$ corresponding to the effects of the change in Ω included in Eqs. (2.2) and (2.3). We will generally drop tildes on the effective J's and h's, using them primarily to define the RG transformation as above. For convenience, one can renumber the new bonds and clusters sequentially; which spin cluster contains which original spins is thus not kept track of directly.

In order to obtain information on correlations, in particular on the relationship between length and energy scales, it is therefore necessary to keep track of the lengths of the bonds and clusters. For convenience, we assign half of the original unit distance between two sites to the spins and half to the bonds to preserve the duality. Initially, all bond and cluster lengths are thus

$$\ell_{Bi} = \frac{1}{2} , \tag{2.6}$$

$$\ell_{Si} = \frac{1}{2} , \tag{2.7}$$

for all i. Under decimation of, e.g., cluster 2, the length of the new bond is

$$\tilde{\ell}_B = \ell_{B1} + \ell_{S2} + \ell_{B2} \tag{2.8}$$

and similarly for the length of the new cluster when a bond is decimated.

After renormalization, the bond strengths and their lengths are correlated; as we shall see, long bonds are likely to be weaker. Nevertheless, because of the simple structure of the RG, the properties of the remaining effective bonds and clusters at any scale are *independent*, provided they are independent initially. (In Appendix D it is shown that, in fact, weak short-range correlations are irrelevant in the RG sense at the critical fixed point.)

We thus must keep track of the renormalization of two joint distributions: those of the effective bonds and clusters that exist at scale Γ. These we denote $P(\zeta,\ell;\Gamma)d\zeta d\ell$ and $R(\beta,\ell;\Gamma)d\beta d\ell$, respectively, treating the lengths ℓ as continuous variables for notational convenience. We will usually *not* keep track of the magnetic moments of the clusters directly. The distributions P and R are both normalized to unity. We will often drop the explicit Γ dependence and use several shorthand notations

$$P(\zeta) \equiv P\left(\zeta, \textstyle\int\right) \equiv \int d\ell P(\zeta,\ell) \text{ , etc. ,} \tag{2.9}$$

the former used only when unambiguous, and

$$P_0 \equiv P(0) = P\left(0, \textstyle\int\right) , \tag{2.10}$$

$$R_0 \equiv R\left(0, \textstyle\int\right) , \tag{2.11}$$

so that the probability that a bond is decimated on going from Γ to $\Gamma + d\Gamma$ is $P_0 d\Gamma$. Because of the additive nature of the recursion relations, Eqs. (2.4), (2.5), and (2.8), convolutions of P's and R's will naturally occur. These we denote, for example,

$$R \underset{\beta\ell}{\otimes} R \equiv \int_0^\infty d\ell' \int_0^\infty d\beta' R(\beta-\beta',\ell-\ell')R(\beta',\ell') \tag{2.12}$$

or if, e.g., only one variable is being convoluted,

$$P(0,\cdot) \underset{\ell}{\otimes} R \equiv \int_0^\infty d\ell' P(0,\ell')R(\beta,\ell-\ell') , \tag{2.13}$$

the centerdot denoting the variable (here ℓ) to be convoluted.

From the recursion relations, Eqs. (2.4), (2.5), and (2.8), we thus have

$$\frac{\partial P}{\partial \Gamma} = \frac{\partial P}{\partial \zeta} + R(0,\cdot) \underset{\ell}{\otimes} P \underset{\zeta\ell}{\otimes} P - 2R_0 P + (P_0 + R_0)P , \tag{2.14}$$

$$\frac{\partial R}{\partial \Gamma} = \frac{\partial R}{\partial \beta} + P(0,\cdot) \underset{\ell}{\otimes} R \underset{\beta\ell}{\otimes} R + (R_0 - P_0)R . \tag{2.15}$$

The first term in Eq. (2.14) arises from the change in the definition of ζ as Γ increases, the second term from new bonds created when a cluster with $\beta = 0$ is decimated, the third from the elimination of the two neighboring bonds of the decimated cluster (which combine to form the new bond), and the last term from an overall rescaling of the probability to keep it normalized, i.e., to compensate for the net loss of a fraction $(P_0 + R_0)d\Gamma$ of the bonds to decimation. The evolution of the number density n_Γ of the remaining clusters at scale Γ (or equivalently bonds) is thus

$$\frac{dn_\Gamma}{d\Gamma} = -[P_0(\Gamma) + R_0(\Gamma)]n_\Gamma , \tag{2.16}$$

where we have displayed the Γ dependence explicitly and used the notation n_Γ rather than the $n(\Omega)$ of the Introduction. The initial condition in Eq. (2.16) is just $n_{\Gamma=0} = 1$. The duality is manifest in Eqs. (2.14) and (2.15).

The probability that a given spin is the first (i.e., leftmost) spin of a cluster that exists at scale Γ, and has field strength β and length ℓ, is

$$n_\Gamma R(\beta,\ell;\Gamma) . \tag{2.17}$$

Note, however, that this is *not* the probability that the given spin is an active member of *some* such cluster; the latter requires information on the number of active spins in a cluster which we are not yet keeping track of.

A. Critical fixed point

The first task is to try to find a critical fixed point of the system of equations (2.14) and (2.15). In order to do this, we must allow for a rescaling of the variables by, one would guess, a power of Γ. We postpone until later the consideration of the lengths, and first concentrate on the distributions of ζ and β alone, $P(\zeta)$ and

$R(\beta)$. Anticipating the effects of duality, we rescale β and ζ similarly, defining $\eta \equiv \zeta/\Gamma^{\psi}$ and $\theta \equiv \beta/\Gamma^{\psi}$ with ψ to be chosen to find a well-behaved fixed point. The distributions of the rescaled variables are denoted $Q(\eta)$ and $B(\theta)$, respectively. From the expected self-duality at the critical point, we are led to look for a self-dual fixed point, i.e., with $B = Q$. The resulting recursion relation for Q becomes

$$\Gamma \frac{\partial Q}{\partial \Gamma} = \psi \left[Q + \eta \frac{\partial Q}{\partial \eta} \right] + \Gamma^{1-\psi} \left[\frac{\partial Q}{\partial \eta} + Q_0 Q \overset{\otimes}{\eta} Q \right] , \tag{2.18}$$

with $Q_0 \equiv Q(0)$. Fixed points Q^* are solutions of Eq. (2.18) with no explicit Γ dependence for $\Gamma \to \infty$.

For $\psi > 1$, the terms in the first set of brackets in Eq. (2.18) dominate for large Γ, yielding the unphysical fixed point solution $Q^* \propto 1/\eta$. Conversely, for $\psi < 1$, the terms in the second set of brackets dominate, yielding (via Laplace transforms) a Q^* which oscillates in sign for large η and is thus also unphysical. Therefore, we must have $\psi = 1$ so that

$$\eta \equiv \zeta/\Gamma \tag{2.19}$$

and

$$\theta \equiv \beta/\Gamma \tag{2.20}$$

give the correct rescaling at the critical point. In the Appendix of Ref. 10, it was shown that for $\psi = 1$, Eq. (2.18) has a family of fixed-point solutions with different Q_0. However, almost all of these correspond to functions Q with power law tails in η which can only arise from very singular initial distributions [Prob $(J) \sim \frac{1}{J(\ln J)^y}$ for small J]. The only well-behaved fixed point has $Q_0 = 1$, corresponding to

$$Q^*(\eta) = e^{-\eta}\Theta(\eta) , \tag{2.21}$$

with Θ the Heaviside step function.

The behavior of the distribution $P(\zeta)$ for large Γ at criticality is implied by the fixed point

$$P(\zeta;\Gamma) \approx \frac{1}{\Gamma} e^{-\zeta/\Gamma} \tag{2.22}$$

[and $R(\beta;\Gamma)$ likewise]. This yields preliminary justification for the approximate decimation procedure: For large Γ, the distributions of the effective J_i and h_i are so broad that the probability that the neighbors of a to-be-decimated $J_i \approx \Omega$ are a significant fraction of Ω is very small, of order $1/\Gamma$.

The form of the fixed-point distribution also yields information on the distribution of cluster and bond lengths. From the RG equation for the cluster density n_Γ, Eq. (2.16), we see that, at the critical fixed point,

$$P_0 = R_0 \approx \frac{1}{\Gamma} \tag{2.23}$$

and hence

$$n_\Gamma \sim \frac{1}{\Gamma^2} \tag{2.24}$$

for large Γ, implying that the typical bond and cluster lengths are

$$\bar{\ell} = \frac{1}{2n_\Gamma} \sim \Gamma^2 \sim \ln^2\left(\frac{\Omega_I}{\Omega}\right) , \tag{2.25}$$

so that lengths scale logarithmically with energy. It is the associated extremely broad distribution of energy scales that makes the whole analysis of this paper both tractable and valid.

The significance of the fixed point found above depends on its stability. We must thus consider the effects of small perturbations away from the fixed point. With the rescaling of Eqs. (2.19) and (2.20), we have RG flows for $Q(\eta)$ and $B(\theta)$:

$$\Gamma \frac{\partial Q}{\partial \Gamma} = Q + (1+\eta)\frac{\partial Q}{\partial \eta} + B_0 Q \otimes Q + (Q_0 - B_0) Q \tag{2.26}$$

and

$$\Gamma \frac{\partial B}{\partial \Gamma} = B + (1+\theta)\frac{\partial B}{\partial \theta} + Q_0 B \otimes B + (B_0 - Q_0) B , \tag{2.27}$$

where the $\eta \frac{\partial Q}{\partial \eta}$ and Q terms arise from the rescaling Eq. (2.19).

To analyze the stability of the fixed point, we define

$$Q = Q^* + q = e^{-\eta} + q , \tag{2.28}$$

$$B = B^* + b = e^{-\theta} + b . \tag{2.29}$$

The detailed analysis of the flow for small q and b is carried out in Appendix A. It is found that there are exactly two physical eigenperturbations which behave as

$$q = q_\Lambda(\eta)\Gamma^\Lambda ,$$

$$b = b_\Lambda(\theta)\Gamma^\Lambda . \tag{2.30}$$

The only *relevant* eigenvalue is

$$\Lambda = 1 , \tag{2.31}$$

with eigenvector

$$q_1 = (\eta - 1)e^{-\eta} , \tag{2.32}$$

$$b_1 = -(\theta - 1)e^{-\theta} , \tag{2.33}$$

corresponding to going *off* criticality, with growth of the perturbation as $\Gamma^\Lambda \sim \Gamma$. The *irrelevant* eigenvalue

$$\Lambda = -1 \tag{2.34}$$

is associated with flow on the critical manifold to the critical fixed point. It corresponds to a symmetric, i.e., self-dual, eigenvector

$$q_{-1}(\eta) = (\eta - 1)e^{-\eta} , \tag{2.35}$$

$$b_{-1}(\theta) = (\theta - 1)e^{-\theta} . \tag{2.36}$$

By differentiating the fixed point $P(\zeta;\Gamma) = \frac{1}{\Gamma}e^{-\zeta/\Gamma}$ with respect to Γ, and rescaling to η variables, it can be seen that the irrelevant eigenvector just corresponds to a constant shift of the origin of Γ: $\Gamma \to \Gamma + \delta\Gamma$. Surprisingly, there are no other physical eigenperturbations. A generic perturbation, as shown in Appendix A, will yield a projection onto the relevant and irrelevant eigenvectors (above) and a remainder which decays faster than any inverse power of Γ.

We thus see that the fixed point Eq. (2.21) indeed has the correct stability properties to be the critical fixed point that controls the transition between the paramagnetic and ferromagnetic phases at zero temperature. The magnitude of the relevant eigenperturbation should thus be proportional to the distance from criticality δ for small $|\delta|$.

We now go back to the full distributions of lengths and couplings. At the fixed point, the lengths must be rescaled as

$$\lambda \equiv \frac{\ell}{\Gamma^2} \tag{2.37}$$

to be consistent with the result for n_Γ, Eq. (2.24). From the structure of Eqs. (2.14) and (2.15), it is clear that the Laplace transforms

$$\hat{P}(\zeta, y) \equiv \int_0^\infty d\ell e^{-y\ell} P(\zeta, \ell) \tag{2.38}$$

and similarly $\hat{R}(B, y)$ have *decoupled* flow equations for different y, except that the $y = 0$ parts that correspond to $P(\zeta, \int)$ enter the other equations. This property will be used extensively later.

For finding the fixed point, we Laplace transform (LT) in $\lambda \to \tilde{y}$ to the scaled distributions

$$\hat{Q}(\eta, \tilde{y}) \quad \text{and} \quad \hat{B}(\theta, \tilde{y}) , \tag{2.39}$$

which should be equal at the fixed point by duality. From either Ref. 10 or the detailed analysis of the flows in the next subsections, one obtains a fixed-point distribution unique up to the overall length scale (i.e., $\tilde{y} \to \text{const} \times \tilde{y}$)

$$Q^*(\eta, \lambda) = LT_{\tilde{y}}^{-1} \frac{\sqrt{\tilde{y}}}{\sinh\sqrt{\tilde{y}}} e^{-\eta\sqrt{\tilde{y}}\coth\sqrt{\tilde{y}}} , \tag{2.40}$$

which can be seen to decay exponentially for $\lambda \gg 1$, i.e., $\ell \gg \Gamma^2$, as the nearest singularity to the origin of $\tilde{y}$ occurs at $\tilde{y} = -\pi$. On integrating over η, we find that

$$\text{Prob}(\lambda) = \sum_{n=-\infty}^{\infty} \left(n + \frac{1}{2}\right)\pi(-1)^n e^{-\pi^2\lambda(n+\frac{1}{2})^2} , \tag{2.41}$$

which behaves as $e^{-\frac{\pi^2\lambda}{4}}$ for large λ and as $\lambda^{\frac{-3}{2}}e^{-1/4\lambda}$ for small λ. [Note that by considering Eq. (2.41) and its Poisson-resummed form, it can be readily shown by the method of Appendix B that Prob(λ) is strictly positive as it must be.]

From Eq. (2.40), one can show that, as expected, λ and η are positively correlated. In particular, bonds with anomalously large η (corresponding to anomalously small J) will also be anomalously long with $\lambda \approx C_\lambda \eta + O(\sqrt{\eta})$. With the choice of normalization of lengths implicit in Eq. (2.40), the coefficient $C_\lambda = 2/3$.

B. Off-critical flows and special solution

So far, we have concentrated on the critical fixed point and linearized perturbations away from it. But in order to get information about the behavior slightly off critical, in particular to obtain the scaling functions, we need to analyze where the flows in the relevant eigendirection go. Motivated by the properties of the fixed-point solution, in particular the decoupling of different Laplace transform components in ℓ, and the special properties of exponential functions of η, we look for solutions to the RG equations in a particular simple form. This form, as we shall see, includes both the relevant and irrelevant eigenperturbations discussed above and has the needed asymptotic properties to yield off-critical scaling functions. From the analysis of Appendix A, we anticipate that the system will converge rapidly to this form for all but pathological initial conditions that correspond to singular distributions of initial couplings and/or lengths.

We work with the Laplace transform of the unrescaled distributions of $P(\zeta, \ell)$ and $R(\beta, \ell)$: $\hat{P}(\zeta, y)$ and $\hat{R}(\beta, y)$. The RG flow equations are simply, from Eqs. (2.14) and (2.15),

$$\frac{\partial \hat{P}(\zeta, y)}{\partial \Gamma} = \frac{\partial \hat{P}(\zeta, y)}{\partial \zeta} + \hat{R}(0, y)\hat{P}(\cdot, y) \overset{\otimes}{\zeta} \hat{P}(\cdot, y) + [\hat{P}(0,0) - \hat{R}(0,0)]\hat{P}(\zeta, y) \tag{2.42}$$

and similarly for $\frac{\partial \hat{R}}{\partial \Gamma}$ by duality. Note that $\hat{R}(0, y = 0) = R_0$. The coupling of different y's only involves the ζ-independent $y = 0$ parts, R_0 and P_0, which are just functions of Γ whose determination nevertheless involves analyzing the ζ dependence of $\hat{P}(\zeta, 0)$ and $\hat{R}(\zeta, 0)$.

We look for solutions to Eq. (2.42) in the form

$$\hat{P}(\zeta, y; \Gamma) = \Upsilon(y; \Gamma)e^{-\zeta u(y;\Gamma)} , \tag{2.43}$$

$$\hat{R}(\beta, y; \Gamma) = T(y; \Gamma)e^{-\beta\tau(y;\Gamma)} , \tag{2.44}$$

where we have displayed the Γ dependence explicitly. For $y = 0, \hat{P}(\zeta, 0) = P(\zeta)$ [as from Eq. (2.10)] which must be normalized so that

$$\Upsilon(y = 0; \Gamma) = u(y = 0; \Gamma) \equiv u_0(\Gamma) = P_0(\Gamma)$$

and (2.45)

$$T(y = 0; \Gamma) = \tau(y = 0; \Gamma) \equiv \tau_0(\Gamma) = R_0(\Gamma) ,$$

yielding

$$P(\zeta) = u_0 e^{-\zeta u_0} \quad \text{and} \quad R(\beta) = \tau_0 e^{-\beta\tau_0} \; . \tag{2.46}$$

Substituting Eqs. (2.45) and (2.47) into the RG flow equations, we see that they are indeed a solution if the following nonlinear ordinary differential equations are satisfied for each y:

$$\frac{\partial u}{\partial \Gamma} = -\Upsilon T \; , \tag{2.47}$$

$$\frac{\partial \tau}{\partial \Gamma} = -\Upsilon T \; , \tag{2.48}$$

$$\frac{\partial \Upsilon}{\partial \Gamma} = (u_0 - \tau_0 - u)\Upsilon \; , \tag{2.49}$$

$$\frac{\partial T}{\partial \Gamma} = (\tau_0 - u_0 - \tau) T \; . \tag{2.50}$$

These can be integrated explicitly to yield

$$\tau(y) = \delta(y) + \Delta(y) \coth\{[\Gamma + C(y)]\Delta(y)\} \; , \tag{2.51}$$

$$u(y) = \tau(y) - 2\delta(y) \; , \tag{2.52}$$

$$T(y) = \frac{\Delta(y)}{\sinh\{[\Gamma + C(y)]\Delta(y)\}} e^{D(y) + [2\delta_0 - \delta(y)]\Gamma} \; , \tag{2.53}$$

$$\Upsilon(y) = \frac{\Delta(y)}{\sinh\{[\Gamma + C(y)]\Delta(y)\}} e^{-D(y) - [2\delta_0 - \delta(y)]\Gamma} \; , \tag{2.54}$$

where

$$\delta(y), C(y), D(y), \text{ and } \Delta(y) = \sqrt{\gamma(y) + \delta^2(y)} \tag{2.55}$$

are y-dependent integration constants with

$$\delta_0 \equiv \delta(y=0) \quad \text{and} \quad C_0 \equiv C(y=0), \text{ etc.} \tag{2.56}$$

To satisfy the normalization conditions, Eq. (2.45), we require that

$$D(0) = C(0)\delta(0) \tag{2.57}$$

and

$$\Delta(0) = |\delta_0|, \tag{2.58}$$

i.e.,

$$\gamma(0) = 0 \; . \tag{2.59}$$

Thus for $y = 0$, the expressions Eqs. (2.53) and (2.54) simplify to

$$\tau_0 = \delta_0\{1 + \coth[(\Gamma + C_0)\delta_0]\} = \frac{2\delta_0}{1 - e^{-2\delta_0(\Gamma + C_0)}} \; ,$$

$$u_0 = \delta_0\{-1 + \coth[(\Gamma + C_0)\delta_0]\} = \frac{2\delta_0}{e^{2\delta_0(\Gamma + C_0)} - 1} = \tau_0 - 2\delta_0 \; . \tag{2.60}$$

C. Scaling solution

We now construct a scaling solution, which will contain the whole critical regime, from the above special solution. If $D(y) = 0$ and $\delta(y) = 0$, the solution is self-dual and thus should correspond to the critical point on which we focus first. For $y = 0$, this yields simply

$$\tau_0 = u_0 = 1/(\Gamma + C_0) \; , \tag{2.61}$$

which is exactly the scaling solution at the *critical point* found earlier, with $C_0 \neq 0$ corresponding to the irrelevant eigenperturbation Eqs. (2.35) and (2.36), as can be seen by expanding in C_0.

At low-energy scales, we are interested in long-length scales corresponding to small y. In the critical region, with the scaled variable $\tilde{y} = y\Gamma^2$ corresponding to Eq. (2.37), the only properties of the integration constants $C(y)$ and $\gamma(y)$ that should matter is their leading small-y behavior. By integrating over ζ to obtain the Laplace transform of the distribution of bond lengths, it can be seen that any well-behaved distribution of lengths [e.g., $\delta(\ell - \frac{1}{2})$] implies smoothness for small y. Thus we must have

$$\gamma(y) \approx \gamma_1 y \tag{2.62}$$

for small y and $C(y) \approx C_0$. The coefficient C_0 disappears in the scaled solution, and so we set it equal to zero to eliminate the irrelevant perturbation. The coefficient γ_1 in Eq. (2.62) sets the overall length scale which can be defined away. Thus, to obtain the critical scaling solution, we may choose

$$\gamma(y) = y \tag{2.63}$$

and

$$C(y) = 0 \; , \tag{2.64}$$

thereby yielding the scaled critical fixed point distribution Eq. (2.40).

From the above discussion, it should be apparent that to obtain scaling distributions just *off critical*, we should again focus on the small-y behavior, and again set $C(y) = 0$ and $\gamma(y) = y$. Since δ_0 can now be nonzero—indeed, as we shall see, δ_0 is just a measure of the deviation from criticality and will thus be small—we can take, for the scaling solution, $\delta(y)$ to be y independent:

$$\delta(y) = \delta_0 \equiv \delta \; . \tag{2.65}$$

Furthermore, since $D_0 = C_0\delta_0 = 0$, we should also set

$$D(y) = 0 \ . \tag{2.66}$$

We thus obtain a *scaling solution* which should be valid for Γ large, y small, and δ small:

$$\Delta(y) = \sqrt{y + \delta^2} \ , \tag{2.67}$$

$$\tau(y) = \delta + \Delta(y) \coth[\Gamma\Delta(y)] \ , \tag{2.68}$$

$$u(y) = \tau(y) - 2\delta \ , \tag{2.69}$$

$$T(y) = \frac{\Delta(y)}{\sinh[\Gamma\Delta(y)]} e^{\delta\Gamma} \ , \tag{2.70}$$

$$\Upsilon(y) = \frac{\Delta(y)}{\sinh[\Gamma\Delta(y)]} e^{-\delta\Gamma} \ . \tag{2.71}$$

This scaling solution is unique up to the overall length scale which is discussed in the next subsection. For $y = 0$, we have

$$\tau_0 \equiv \tau(0) = T(0) = \delta + \delta\coth\Gamma\delta = \frac{2\delta}{1 - e^{-2\Gamma\delta}} \tag{2.72}$$

and

$$u_0 \equiv u(0) = \Upsilon(0) = -\delta + \delta\coth\Gamma\delta = \frac{2\delta}{e^{2\Gamma\delta} - 1} \ . \tag{2.73}$$

Expanding the scaling solution for $y = 0$ in δ and rescaling ζ and β to η and θ, we see that the small-δ perturbation corresponds precisely to the relevant eigenperturbation away from criticality, Eqs. (2.32) and (2.33).

The analysis of Appendix A strongly suggests that general near-critical distributions will converge to this scaling solution in the limit that $\delta, 1/\Gamma$, and y are all small with $\delta\Gamma$ and $\tilde{y} = y\Gamma^2$ fixed. As we shall see, scaling functions of observable physical quantities can be obtained by calculating, in an analogous manner, the scaling limit of other distributions.

At this point, however, we can already anticipate that there will exist a characteristic correlation length ξ near the critical point, which diverges as

$$\xi \sim |\delta|^{-\nu} \ , \tag{2.74}$$

with

$$\nu = 2 \tag{2.75}$$

arising from the scaling of δ^2 with the inverse length y explicit in $\Delta(y)$, Eq. (2.67). This is a consequence of the scaling of lengths with Γ^2 at the critical point, and the relevant eigenvalue $\Lambda = 1$ for perturbations away from the critical point.

D. Normalization of δ and lengths

We have seen that $\delta = \delta_0$ is the parameter that controls the distance from criticality. But at this point the normalization of δ appears arbitrary; indeed we have chosen to define it with a factor of 2 in Eq. (2.69). This normalization, in fact, turns out to correspond *exactly* for small δ to that chosen in the Introduction:

$$\delta \equiv \frac{\Delta_h - \Delta_J}{V_I} \ , \tag{2.76}$$

with V_I the sum of the variances of the original $\ln h$ and $\ln J$ variables, Eq. (1.30).

To see this, we consider the mean and variance of the sum

$$\Sigma = \sum_i (\ln\tilde{h}_i - \ell n\tilde{J}_i) \tag{2.77}$$

over the set of effective spin clusters and bonds at scale Γ which start exactly at (original) site 0 and end exactly at (original) site L so that

$$L = \sum_i (\ell_{Si} + \ell_{Bi}) \ . \tag{2.78}$$

Because of the additive nature of the recursion relations, any decimation which does not destroy the first or last coupling in the segment $[0, L]$ will leave Σ invariant. Thus, naively, we would expect Σ to have a distribution independent of the stage of renormalization but depending, of course, on L. But this is not quite right: The condition denoted $\{S0\text{–}LB\}$ that an effective cluster starts at 0 and an effective bond ends at L means that the segment $[0, L]$ is *not* quite typical in a way that depends on L. Nevertheless, this should be primarily an end effect so that for

$$L \gg \bar{\ell} \equiv \bar{\ell}_S + \bar{\ell}_B \ , \tag{2.79}$$

with $\bar{\ell}$ the mean length of a single cluster-bond pair, we should expect the distribution of Σ to be essentially independent of Γ; indeed, it should, by the central limit theorem, be Gaussian with mean $\overline{\Sigma}$ and variance $\mathrm{var}(\Sigma)$ both proportional to L. The conditional distribution $\Pr(\Sigma|L)$ of Σ given the event $\{S0\text{–}LB\}$ that a spin cluster starts at 0 and a bond ends at L is computable from P and R.

For the special solution, the double transform of this conditional distribution, Laplace in $L \to y$ and Fourier in $\Sigma \to s$, has the simple form

$$\Pr(\Sigma|L) = \frac{n_\Gamma \mathrm{LT}^{-1}\mathrm{FT}^{-1}\left\{1 - \frac{T(y)\Upsilon(y)}{[is+\tau(y)][-is+u(y)]}\right\}^{-1}}{\Pr\{S0 - LB\}} \ , \tag{2.80}$$

with

$$\Pr\{S0 - LB\} = n_\Gamma \mathrm{LT}^{-1}\left[1 - \frac{T(y)\Upsilon(y)}{\tau(y)u(y)}\right]^{-1} \ . \tag{2.81}$$

Direct computation shows that in the limit $L \to \infty$ the distribution of

$$\frac{\Sigma - \overline{\Sigma}}{\sqrt{L}} \tag{2.82}$$

becomes Gaussian with variance

$$\frac{\text{var}(\Sigma)}{L}=\frac{2}{\gamma_1}-\frac{8\delta_0^2\gamma_2}{\gamma_1^3}+\frac{8\delta_1\delta_0}{\gamma_1^2}+O\left(\frac{1}{L}\right) \tag{2.83}$$

and

$$\bar{\sigma}\equiv\frac{\overline{\Sigma}}{L}=\frac{2\delta_0}{\gamma_1}+O\left(\frac{1}{L}\right), \tag{2.84}$$

where we have expanded for small y,

$$\delta(y)=\delta_0+\delta_1 y+O(y^2) \tag{2.85}$$

and

$$\gamma(y)=\gamma_1 y+\gamma_2 y^2+O(y^3). \tag{2.86}$$

The mean and variance of Σ per unit length are thus both explicitly independent of the renormalization scale for large L. [Note, however, that, not surprisingly, there are $O(1)$ corrections to $\overline{\Sigma}-\bar{\sigma}L$ (which are also readily calculable) that *do* depend on Γ as should be expected from the conditioning of the end points of the segment.] In the regime in which we are most interested—near the critical point—δ_0 is small but γ_1, γ_2, and δ_1 are all of order 1 (or smaller) so that, for small $\delta\equiv\delta_0$,

$$\frac{\overline{\Sigma}}{\text{var}(\Sigma)}\approx\delta+O(\delta^2). \tag{2.87}$$

Thus to leading order in δ, the normalization of δ in the Introduction, Eq. (1.36), corresponds exactly to that in Eqs. (2.65)–(2.69).

Since any well-behaved initial distribution will converge to the special solution Eqs. (2.43)–(2.56) for large Γ, if it is initially near critical (see Appendix A), the "conservation" of $\overline{\Sigma}$ and var(Σ) for large L guarantees that δ defined in terms of the initial distributions will indeed be the *same* δ in the special solution to which the initial distributions converge. One might worry that the errors made in the RG approximation at early stages would invalidate this equivalence; however, the discussion of the transfer matrix solution in Appendix E shows that, due to the exact solvability of the random transverse-field Ising chain with nearest neighbor couplings, these errors exactly cancel at later stages. We thus believe that asymptotically near criticality, δ will be given *exactly* in terms of the original variables by Eq. (1.36).[17] The agreement discussed in Sec. V with McCoy's exact results[3] confirms this claim.

Before turning to the direct computation of physical properties, it is useful to note that the special solution Eqs. (2.43) and (2.44) is, in fact, quite general, at least as far as distributions of lengths. In particular, a chain with all spins and bonds having initial (i.e., at $\Gamma=\Gamma_I=0$) lengths $\ell_{Bi}=\ell_{Si}=1/2$, and power law distributions of initial J's and h's,

$$\pi(J)=\frac{u_I}{J}\left(\frac{\Omega_I}{J}\right)^{u_I}\Theta(\Omega_I-J) \tag{2.88}$$

and

$$\rho(h)=\frac{\tau_I}{h}\left(\frac{\Omega_I}{h}\right)^{\tau_I}\Theta(\Omega_I-h), \tag{2.89}$$

corresponds to y-independent δ and D,

$$\delta(y)=\delta_0=\frac{1}{2}(\tau_I-u_I), \tag{2.90}$$

$$D(y)=D_0=C_0\delta_0, \tag{2.91}$$

with the lengths fixed by

$$\gamma(y)=u_I\tau_I(1-e^{-y}) \tag{2.92}$$

and

$$C(y)=\frac{1}{2\Delta(y)}\ln\left[\frac{u_I+\tau_I+2\Delta(y)}{u_I+\tau_I-2\Delta(y)}\right], \tag{2.93}$$

with

$$\Delta(y)=\sqrt{\delta_0^2+\gamma(y)}. \tag{2.94}$$

This yields

$$\frac{\overline{\ln h}-\overline{\ln J}}{V_I}=\delta_0\frac{u_I\tau_I}{\frac{1}{2}(u_I^2+\tau_I^2)}, \tag{2.95}$$

which is $\approx\delta_0$ for small δ_0 and can be checked with the more general expression Eq. (1.36).

The choice $\gamma_1=1$ (from $\gamma=y$) in the scaling solution Eqs. (2.67)–(2.71) generally corresponds to measuring all lengths in units of

$$\ell_V=\frac{2}{V_I}, \tag{2.96}$$

the basic length scale beyond which the randomness dominates. From now on, lengths in the scaling limit will thus be expressed in units of ℓ_V; from the above analysis, the *coefficients* of lengths will then become exact. We now turn to computation of physically measurable properties.

E. Scaling of cluster moments

To obtain information about the scaling of spin correlation functions, we need to have information about the distribution of the moments of spin clusters as well as their lengths. The approach we will use in the next section enables mean correlation functions to be obtained directly; however, a simpler approach yields the correct scaling exponents.

The RG flow equations for the joint distribution of the effective field and the moment of a spin cluster, i.e., β and m, can be obtained simply from the recursion relation for the moment of a cluster,

$$\tilde{m}=m_1+m_2, \tag{2.97}$$

when a strong bond 1 is decimated. Unfortunately, in this case the joint distributions cannot be found explicitly. But using the methods of Ref. 10, one can show that the only well-behaved fixed point has typical moments scaling as

$$m \sim \Gamma^{\phi} , \tag{2.98}$$

with

$$\phi = \frac{1+\sqrt{5}}{2} \tag{2.99}$$

the golden mean. We shall later see how this arises from other quantities that can be calculated more explicitly; however, it is instructive, first, to see how the scaling *forms* of average correlations can be obtained from Eq. (2.98).

In the ordered phase $\delta < 0$, some fraction of the spins will never be decimated. These will eventually form an infinite cluster whose moment yields the spontaneous magnetization of the system. The scaling of the density of spins in this infinite cluster, which is simply proportional to the spontaneous magnetization density M_0, can be guessed by stopping the renormalization at a scale of order the correlation length, i.e., when $\Gamma \sim \Gamma_\delta \sim 1/|\delta|$, at which point the scaling solution is well away from criticality. At this scale, the clusters will still have lengths $\ell \sim \Gamma_\delta^2$ and moments $m \sim \Gamma_\delta^{\phi}$, and the natural guess (see Sec. IV) is that only a finite fraction of spins remaining at this scale will be decimated at lower energies. Hence, we guess that

$$M_0 \sim \Gamma_\delta^{\phi-2} \sim (-\delta)^{\beta} , \tag{2.100}$$

with

$$\beta = 2-\phi = \frac{3-\sqrt{5}}{2} . \tag{2.101}$$

The explicit computation of Sec. III A verifies this conjecture.

Similarly, the mean untruncated spin-spin correlation function $\overline{C}(x)$ should scale so that for $\delta < 0$ and $x \gg \xi$, it becomes M_0^2. This suggests that

$$\overline{C}(x) \sim \frac{1}{x^{\beta}} \mathcal{C}_{\pm}\left(\frac{x}{\xi}\right) , \tag{2.102}$$

with $\xi \sim |\delta|^{-2}$ and the scaling functions $\mathcal{C}_{\pm}$ obtaining for $\delta \gtrless 0$. In Sec. III B we will obtain the scaling function $\mathcal{C}_{\pm}$ in terms of the solution to a second-order linear differential equation which can be exhaustively analyzed, in particular to obtain the asymptotic limits $x \gg \xi$ in both phases.

III. MAGNETIZATION AND CORRELATION SCALING FUNCTIONS

To obtain information about the mean correlation functions is rather more complicated than what we have done so far. As discussed at the end of the last section and in Ref. 8, the *scaling forms* of the spontaneous magnetization and the spin-spin correlations can be guessed from analysis of the scale dependence of the typical number of spins that are active in a cluster, i.e., its moment m at the critical fixed point. Yet this does not provide information on the actual correlation functions. Even to answer seemingly simple questions such as whether these decay exponentially in the disordered phase requires a much more detailed analysis. In addition, we would like to obtain information about the magnetization in small fields; we will see that this will appear as a by-product of the calculations below.

The simplest route that has been found so far for obtaining mean correlation functions is via the function

$$G(\beta,x;\Gamma)d\beta dx \equiv d\ \text{Prob}\{\text{spin } 0\ \in\ (\text{cluster with right end at } x \text{ and } \ln(\Omega/\tilde{h}) = \beta) \text{ at logarithmic energy scale } \Gamma\} , \tag{3.1}$$

where "spin $\in$ cluster" means that it is *active* in the cluster, i.e., not yet decimated. The event in Eq. (3.1) is shown schematically in Fig. 1. For notational convenience, we have assumed a continuous position variable x and will sometimes be sloppy with dx's.

The function G contains a lot of information. If $x = 0$, then G is just the probability that the right end of a cluster with logarithmic field β is at the origin, i.e.,

$$G(\beta, x=0;\Gamma) = n_\Gamma R(\beta;\Gamma) . \tag{3.2}$$

Since the events {spin 0 $\in$ (cluster ending at x)} and {spin 0 $\in$ (cluster ending at x')} are disjoint for $x \neq x'$, integrating G over x yields the probability that spin 0 is in *some* cluster with β at scale Γ. Equivalently, by noting that $G(\beta,x)$ is also the probability that spin $(-x)$ $\in$ (cluster ending at 0 with β), we see that

$$\int_0^\infty dx G(\beta,x,\Gamma) = \overline{m}(\beta;\Gamma) n_\Gamma R(\beta;\Gamma) , \tag{3.3}$$

where $\overline{m}(\beta;\Gamma)$ is the mean moment (i.e., number of active spins) in clusters with β at scale Γ. Integrating over β,

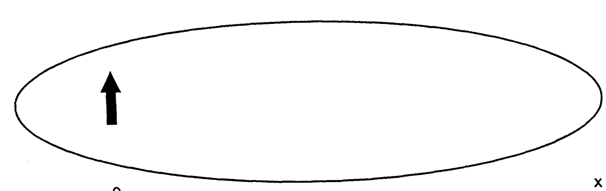

FIG. 1. The "event" whose probability defines $G(\beta,x;\Gamma)$ in Eq. (3.1). The spin at site zero is active in a spin cluster that ends at site x and has effective transverse field $h = \Omega e^{-\beta}$ at scale $\Gamma \equiv \ln(\Omega_I/\Omega)$.

we see that

$$N_A(\Gamma) = \int_0^\infty d\beta \int_0^\infty dx G(\beta, x; \Gamma) \quad (3.4)$$

is the density of spins that are *active* in any cluster at Γ.

In the ordered phase, we expect, as discussed earlier, a single infinite cluster to form as $\Gamma \to \infty$. Thus

$$M_0 = \overline{\mu} N_A(\Gamma = \infty) \quad (3.5)$$

is the spontaneous magnetization density, with the coefficient $\overline{\mu} < 1$ the finite mean reduction of the moment of the active spins due to the nonuniversal high-energy small-scale fluctuations. This will be a relatively smooth function of δ whose δ dependence will only lead to subdominant corrections to M_0, so that we can, to the accuracy at which we are working, set $\overline{\mu} = \overline{\mu}(\delta = 0)$.

Most importantly, however, the mean spin-spin correlation function can also be obtained from G. From the discussion in the Introduction, the mean correlation function

$$\overline{C}(x) \equiv \overline{\langle \sigma_j^z \sigma_{j+x}^z \rangle} , \quad (3.6)$$

with, for convenience, x positive, is proportional to the probability that spins j and $j + x$ are active in the *same* cluster at *some* Γ; if so they will fluctuate together, and hence, for large x, contribute on average $\overline{\mu}^2$ to the mean correlation function. Thus

$$\overline{C}(x) \approx \overline{\mu}^2 K_\infty(x) \equiv \overline{\mu}^2 \int_0^\infty d\Gamma \; k(x, \Gamma) , \quad (3.7)$$

where

$$k(x,\Gamma) d\Gamma \equiv \text{ Prob } \{\text{spin 0 } \textit{and} \text{ spin } x \textit{ become} \text{ active in the } \textit{same} \text{ cluster when } \Gamma \to \Gamma + d\Gamma\}. \quad (3.8)$$

The process by which spin 0 and x *become* active in the same cluster at Γ can be represented schematically as shown in Fig. 2. For this to happen, σ_0 and σ_x must both be active in *separate* clusters at scale Γ, which are coupled together by a bond that is about to be decimated at Γ. If the right end of the cluster containing 0 is at x_1, which occurs with probability $G(\int, x_1)$, there must thus be a bond beginning at x_1 and ending at some x_2 which has $\beta < d\Gamma$; this occurs with probability $P(0, \ell = x_2 - x_1) d\Gamma$. Furthermore, σ_x must be in the cluster beginning at x_2. The probability that this occurs *given* that a bond ends (and hence a cluster begins) at x_2 is $n_\Gamma^{-1} G(\int, x - x_2)$ since n_Γ is the probability that a cluster begins at a particular point, and we have used the reflection symmetry. [Note that from Eq. (3.2), $G(\int, 0)/n_\Gamma = 1$ by the normalization of $R(\beta)$.] We thus see that

$$k(x,\Gamma) = n_\Gamma^{-1} \int_0^x dx_1 \int_{x_1}^x dx_2 G\left(\int, x_1, \Gamma\right) \times P(0, x_2 - x_1; \Gamma) G\left(\int, x - x_2; \Gamma\right) . \quad (3.9)$$

Defining the Laplace transforms

$$\hat{G}(\beta, y) \equiv \int_0^\infty dx \; e^{-yx} G(\beta, x) \quad (3.10)$$

and

$$\hat{g}(y) \equiv \int_0^\infty d\beta \; \hat{G}(\beta, y) , \quad (3.11)$$

and similarly $\hat{k}(y)$ and

$$\hat{K}_\infty(y) = \int_0^\infty d\Gamma \hat{k}(y, \Gamma), \quad (3.12)$$

we see that

$$\hat{k}(y) = n_\Gamma^{-1} \hat{P}(0, y) [\hat{g}(y)]^2 . \quad (3.13)$$

The structure factor

$$S(q) \equiv \int_{-\infty}^\infty \overline{C}(x) e^{-iqx} \quad (3.14)$$

is then simply

$$S(q) \approx 2\overline{\mu}^2 Re \hat{K}_\infty(y = iq) . \quad (3.15)$$

We shall later see that the magnetization and correlation functions in a small field and/or low positive temperature can also be obtained from G. Thus, we see that the function G contains a great deal of information.

But how do we obtain G? From the previous section, it should be clear that we can derive an integro-differential RG equation for $\frac{\partial G}{\partial \Gamma}$; the simplifying feature is that this will be a *linear* equation which can be analyzed rather fully. The renormalization of G under decimation has several contributions: (a) The value of β of each cluster (with $\beta > d\Gamma$) is decreased by $d\Gamma$ because of the redefinition of β. (b) If $\beta < d\Gamma$, the cluster is decimated; (a) and (b) are both included in the effects of a $\frac{\partial G}{\partial \beta}$ term

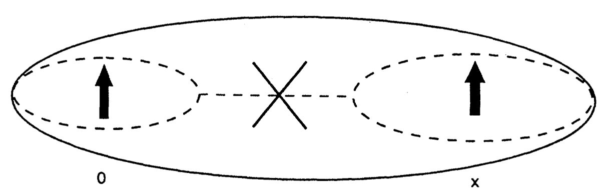

FIG. 2. The process by which spins 0 and x *become* active in the same cluster. The bond connecting the two clusters (dotted lines) is decimated (indicated by X) and a larger cluster (shown by solid line) formed. The probability of this occurring as the scale is changed from Γ to $\Gamma + d\Gamma$ defines $k(x, \Gamma)$, Eq. (3.8).

in $\frac{\partial G}{\partial \Gamma}$. In addition, a cluster can be eliminated if either of the bonds adjoining it is decimated. However, if this occurs, then a new larger cluster will be formed. For a cluster ending at x and containing an active σ_0, there is an asymmetry between (c) its left-hand bond being decimated, which results in a longer cluster with a different β but still ending at x, and (d) its right-hand bond of length ℓ being eliminated which results in a cluster ending at $\ell + x$ and containing σ_0, thereby contributing to $G(\beta, x+\ell)$. These processes are illustrated schematically in Fig. 3. The resulting RG flow for $\hat{G}$ is given by

$$\frac{\partial \hat{G}}{\partial \Gamma} = \frac{\partial \hat{G}}{\partial \beta} - 2P_0\hat{G} + P_0\hat{G}(\cdot, y) \underset{\beta}{\otimes} R(\cdot) + \hat{P}(0,y)\hat{G}(\cdot,y) \underset{\beta}{\otimes} \hat{R}(\cdot,y) \ . \tag{3.16}$$

From the previous section, we know $\hat{P}(\beta,y)$, $\hat{R}(\beta,y)$, $P_0 = \hat{P}(0,0)$, and $R(\beta) = \hat{R}(\beta,0)$ in the scaling limit, and therefore have a linear flow equation with known coefficients for $\hat{G}(y)$ with each y *independent.* This needs to be supplemented by some information on the "initial" condition, i.e., the behavior for small Γ.

We postpone this for now, and try to solve Eq. (3.16). Following the strategy of the previous section, we look for a special solution in the form

$$\hat{G}(\beta,y;\Gamma) = a(y;\Gamma)\left[\frac{e^{-\beta\tau(y;\Gamma)} + e^{-\beta\tau_0(\Gamma)}}{2}\right] + b(y;\Gamma)\left[\frac{e^{-\beta\tau_0(\Gamma)} - e^{-\beta\tau(y;\Gamma)}}{\tau(y;\Gamma) - \tau_0(\Gamma)}\right] , \tag{3.17}$$

with $\tau(y;\Gamma)$ and $\tau_0(\Gamma) \equiv \tau(0;\Gamma)$ given by the special solution Eqs. (2.44) and (2.51) for $\hat{R}$. By substituting Eq. (3.17) into the flow equation (3.16), we see that it is indeed a solution, provided that

$$\frac{\partial a}{\partial \Gamma} = -a\frac{\partial \Phi}{\partial \Gamma} + b \tag{3.18}$$

and

$$\frac{\partial b}{\partial \Gamma} = -b\frac{\partial \Phi}{\partial \Gamma} + Wa \ , \tag{3.19}$$

with

$$\frac{\partial \Phi}{\partial \Gamma} = 2u_0 + \frac{1}{2}\tau + \frac{1}{2}\tau_0 \tag{3.20}$$

and

$$W = \frac{1}{4}(\tau - \tau_0)^2 + \frac{1}{2}u_0\tau_0 + \frac{1}{2}\Upsilon T = \frac{1}{4}(\tau - \tau_0)^2 - \frac{1}{2}\frac{\partial \tau_0}{\partial \Gamma} - \frac{1}{2}\frac{\partial \tau}{\partial \Gamma}, \tag{3.21}$$

the latter equality for W from using Eq. (2.48). For notational simplicity, we have not displayed the y and Γ dependence. Equations (3.18) and (3.19) are written in a form such that they can immediately be solved in terms of $\Phi(y;\Gamma)$ and a function $A(y;\Gamma)$ by

$$a = e^{-\Phi}A \ , \tag{3.22}$$

$$b = e^{-\Phi}\frac{\partial A}{\partial \Gamma} \ , \tag{3.23}$$

with

$$\frac{\partial^2 A}{\partial \Gamma^2} = WA \ . \tag{3.24}$$

Note that Φ is defined up to an additive y-dependent constant (or equivalently a y-dependent coefficient multiplying A) which we will later choose for convenience.

From the analysis in Appendix A, it can be shown that the Laplace transform of any $G(\beta,x)$ which decays at least exponentially for large x will converge, at large Γ, to the special form Eq. (3.17); thus we expect that the scaling functions will be given by $\hat{G}$ of the form Eq. (3.17), with the appropriate "initial" conditions which we now consider.

On the critical manifold, we expect all quantities in the scaling limit of large Γ and small y to be powers of

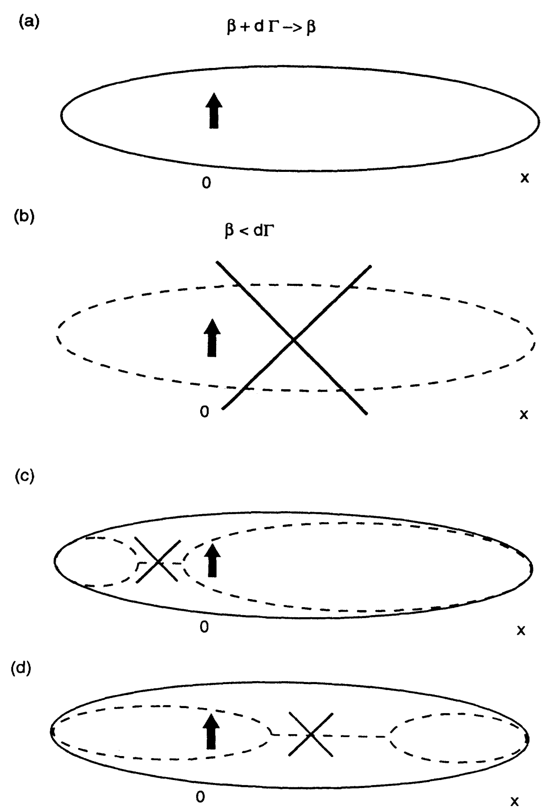

FIG. 3. Processes that contribute to the renormalization of $G(\beta,x;\Gamma)$ as in Eq. (3.16): (a) β is decreased because of the redefinition of $\Gamma \to \Gamma + d\Gamma$; (b) a cluster with $\beta < d\Gamma$ is eliminated; (c) the bond to the left of the cluster containing active spin 0 is decimated; and (d) the bond to the right of the cluster containing active spin 0 is decimated resulting in a longer cluster ending at x.

Γ times functions of $\tilde{y} = y\Gamma^2$ as in Sec. II A. Initially, $G(\beta, x)$ will vanish for all but small x and thus $\hat{G}(\beta, y)$ will be roughly independent of y for small y. This will persist until $n_\Gamma \sim y$, i.e., $\Gamma \sim \frac{1}{\sqrt{y}}$. Thus for $\tilde{y} \to 0$, $\hat{G}$ should be independent of y. It is therefore convenient to choose the integration constants in Φ so that Φ is y independent as $\tilde{y} \to 0$. Using the scaling forms of u and τ, Eqs. (2.68) and (2.69), we can integrate $\frac{\partial \Phi}{\partial \Gamma}$ Eq. (3.20) to obtain, at criticality,

$$e^{-\Phi_c} = \frac{1}{\Gamma^{5/2}} \left\{ \frac{\sqrt{\tilde{y}}}{\sinh[\Gamma\sqrt{y}]} \right\}^{\frac{1}{2}} . \tag{3.25}$$

Since we are using scaling forms, the small-$\tilde{y}$ limit is equivalent to $\Gamma \to 0$ in which limit $e^{-\Phi_c} \approx \Gamma^{-3}$.

In the same $\Gamma \to 0$ limit at criticality, $W = W_c \approx \frac{1}{\Gamma^2}$ independent of y. Thus the equation for A becomes

$$\frac{\partial^2 A}{\partial \Gamma^2} = \frac{1}{\Gamma^2} A , \tag{3.26}$$

which will manifestly have a y-independent solution. In this limit, there are thus two linearly independent solutions $A \sim \Gamma^\phi$ and $A \sim \Gamma^{1-\phi}$ with $\phi = \frac{1}{2}(1+\sqrt{5})$. These yield $a \sim \Gamma^{\phi-3}, b \sim \Gamma^{\phi-4}$ and $a \sim \Gamma^{-2-\phi}, b \sim \Gamma^{-3-\phi}$, respectively. The mean number of spins in a cluster at Γ is given in terms of $\hat{G}$ by Eq. (3.3). The two small-Γ solutions for A, above, yield, respectively, $\overline{m}_\Gamma \sim \Gamma^\phi$ and $\Gamma^{1-\phi}$. The latter is clearly unphysical as the number of spins must grow as the clusters combine. Hence, the correct solution must have $A \sim \Gamma^\phi$ for $\Gamma \to 0$ at criticality. Since this is the smaller solution as $\Gamma \to 0$, we see that in this limit the correct solution must be *purely* of this form *for all* y, with no mixing in of the $\Gamma^{1-\phi}$ solution.

Thus the appropriate boundary condition for the scaling functions at criticality is that, for all y, the small-Γ limit of the solution to Eq. (3.27) is $A \sim \Gamma^\phi$ with a y-independent coefficient that is nonuniversal and related to the small-scale physics, such as the lattice constant, etc.

We are now in a position to understand the appropriate conditions in the full critical region with δ nonzero but small. For fixed small $|\delta|$ and y, the behavior at "high" energies with $\Gamma \ll \frac{1}{|\delta|}$ will be characteristic of the critical point. Thus the scaling functions can be determined again, up to an overall constant coefficient, by the condition that for all y

$$A \approx \Gamma^\phi \quad \text{for} \quad \Gamma \to 0 , \tag{3.27}$$

with Φ chosen to be y and δ independent in this limit:

$$e^{-\Phi} = e^{\delta\Gamma} \left(\frac{\delta}{\sinh\Gamma\delta} \right)^{5/2} \left[\frac{\Delta(y)}{\sinh\Gamma\Delta(y)} \right]^{1/2} . \tag{3.28}$$

We have thus reduced the determination of the scaling form of $\hat{G}$ to the solution of Eq. (3.24) with the boundary condition Eq. (3.27) and

$$W = -\frac{1}{2}(\Delta^2 + \delta^2) + \frac{3}{4}\Delta^2 \coth^2(\Gamma\Delta) + \frac{3}{4}\delta^2 \coth^2(\Gamma\delta) - \frac{1}{2}\Delta\delta \coth(\Gamma\Delta) \coth(\Gamma\delta) , \tag{3.29}$$

with the y dependence arising from $\Delta(y) = \sqrt{y + \delta^2}$.

Note that W is *even* in δ; the asymmetry between the phases thus comes entirely from the $e^{\delta\Gamma}$ factor in Eq. (3.28) and from the dependence of $\tau(y)$ on δ in Eq. (2.68). For obtaining correlation functions, we only need

$$\hat{g}(y;\Gamma) \equiv \hat{G}\left(\textstyle\int, y; \Gamma\right) = \frac{e^{-\Phi}}{\tau\tau_0} \left[\frac{1}{2}(\tau + \tau_0)A + \frac{\partial A}{\partial \Gamma} \right] . \tag{3.30}$$

A. Magnetization scaling

The simplest property of the spin correlations that can be obtained from the methods of the previous section is the probability that a given spin is active in some cluster at scale Γ,

$$N_A(\Gamma) \equiv \hat{g}(y = 0; \Gamma) , \tag{3.31}$$

which is given by Eq. (3.30) with $\tau = \tau_0$ and

$$A = A(y = 0) \equiv A_0 . \tag{3.32}$$

In the limit $\Gamma \to \infty$, N_A is simply proportional to the spontaneous magnetization. Furthermore, the dependence of $N_A(\Gamma)$ on Γ yields, as explained qualitatively in the Introduction, the magnetization as a function of an applied field H, which we take to be positive.

The crucial observation is that at a scale $\Gamma \gg 1$, most of the remaining couplings are *much* smaller than Ω in the vicinity of the critical point, since the distribution of the logarithmic couplings is broad. The effect of an applied field on a cluster is to split the energy of the up and down configurations by $E_H = 2\overline{\mu} m H$ with m the number of active spins in the cluster. As discussed in Sec. II E, the cluster moments are typically of order $m \sim \Gamma^\phi$ in the critical region with a distribution with width of the same order that decays exponentially for m/Γ^ϕ large.[10] On a logarithmic scale, the magnetic energies are thus

$$\ln(\overline{\mu} m H) \approx \ln H + \phi \ln\Gamma + O(1) , \tag{3.33}$$

with the $O(1)$ term random with a distribution derivable from that for the moments m. But, since the distributions of the logarithmic couplings for large Γ have width of order Γ, we see that on this scale, essentially all the E_H are either much larger than the couplings if $\ln H$ is substantially larger than $\ln\Omega$, or much smaller than the strongest couplings Ω if $\ln H$ is substantially smaller than $\ln\Omega$. Thus to a good approximation that is asymptotically exact as far as scaling functions which involve $\delta \ln\Omega$, we can stop the renormalization when

$$\Gamma \approx \Gamma_H = \ln(\Omega_I / H) \tag{3.34}$$

and be assured that the field is strong enough to fully align virtually all of the remaining clusters. Furthermore, we can be assured that on scales $\Gamma < \Gamma_H$ the field energies will have almost always been much less than the strengths of the eliminated couplings and thus not have played an appreciable role. The $O(\ln\Gamma)$ uncertainty in the appropriate Γ_H implied by Eq. (3.33) translates into negligible $O\left(\frac{\ln\Gamma_H}{\Gamma_H}\right)$ corrections in the scaling limit.

Thus we can obtain the scaling function for $M(\delta, H)$ directly from $N_A(\Gamma_H)$:

$$M(\delta, H) \approx \bar{\mu} N_A(\delta, \Gamma_H) \ , \tag{3.35}$$

which will be valid in the limit $\delta \to 0, \Gamma_H \to \infty$ with $\delta\Gamma_H \to$ any fixed constant. We have displayed the implicit δ dependence of N_A in Eq. (3.35). As we shall see, in the scaling limit, the above argument for the form of Γ_H, Eq. (3.34), will be valid even off critical provided $\delta\Gamma_H$ is fixed. In the potentially problematic limit of $\delta\Gamma_H$ large and negative, M will tend to the spontaneous magnetization and the $\delta\Gamma_H$ dependence will only give corrections; we will discuss the justification in this limit in more detail in Sec. IV, as well as the form of the nonscaling corrections.

The calculation of the magnetization scaling function thus reduces, from Eqs. (3.30) and (3.31), to evaluation of the function $A_0(\Gamma)$ which satisfies from Eqs. (3.21) and (3.24) the differential equation

$$\frac{d^2 A_0}{d\Gamma^2} = \frac{\delta^2}{\sinh^2(\Gamma\delta)} A_0 \ , \tag{3.36}$$

with the boundary condition

$$A_0(\Gamma) \approx \Gamma^{\phi} \tag{3.37}$$

for small Γ. Here we have used $W_0 = \delta^2 / \sinh^2(\Gamma\delta)$ from the scaling forms of u_0 and τ_0.

We can put Eq. (3.36) in an explicitly scaling form by defining

$$A_0(\delta, \Gamma) = |\delta|^{-\phi} \tilde{A}_0(\gamma) \ , \tag{3.38}$$

with

$$\gamma \equiv \delta\Gamma \tag{3.39}$$

and

$$\frac{d^2 \tilde{A}_0}{d\gamma^2} = \frac{\tilde{A}_0}{\sinh^2\gamma} \ , \tag{3.40}$$

with

$$\tilde{A}_0 \approx |\gamma|^{\phi} \tag{3.41}$$

for small γ. The function $\tilde{A}_0$ is explicitly an even function of γ which we hence study for positive γ.

Since $\tilde{A}_0$ and $\frac{d\tilde{A}_0}{d\gamma}$ are positive for small γ, from Eq. (3.40) $\tilde{A}_0$ and $\frac{d\tilde{A}_0}{d\gamma}$ are increasing functions for all γ. Hence for large γ, for which $(\sinh\gamma)^{-2} \to 0$ implying $\tilde{A}_0 \sim \gamma$ or $\tilde{A}_0 \sim$ constant, the correct asymptotic solution must be

$$\tilde{A}_0 \approx C_A \gamma \tag{3.42}$$

for large γ. By making the substitution

$$z = \coth\gamma \ , \tag{3.43}$$

we see that

$$\left[(z^2 - 1)\left(\frac{d}{dz}\right)^2 + 2z\frac{d}{dz} - 1\right]\tilde{A}_0 = 0 \ . \tag{3.44}$$

But this is the differential equation for the Legendre function $Q_\nu(z)$ with $\nu = \phi - 1 = (\sqrt{5} - 1)/2$. Thus we see that[18]

$$\tilde{A}_0(\gamma) = C_A Q_{\phi-1}(\coth\gamma) \ , \tag{3.45}$$

with C_A a constant coefficient,

$$C_A = \frac{2^{\phi}}{\sqrt{\pi}} \frac{\Gamma(\phi + \frac{1}{2})}{\Gamma(\phi)} \ , \tag{3.46}$$

with $\Gamma(x)$ the usual gamma function.

The scaling function is most usefully expressed in terms of

$$\alpha(\gamma) \equiv \frac{\tilde{A}_0(\gamma)}{|\gamma|^{\phi}} \ , \tag{3.47}$$

which is an even function of γ that is smooth for small γ with

$$\alpha(0) = 1 \tag{3.48}$$

and

$$\gamma^2 \frac{d^2\alpha}{d\gamma^2} + 2\phi\gamma\frac{d\alpha}{d\gamma} + \alpha\left(1 - \frac{\gamma^2}{\sinh^2\gamma}\right) = 0 \ . \tag{3.49}$$

This has the advantage that the scaling functions in terms of α are manifestly smooth as a function of δ for fixed Γ_H. In terms of α, we obtain the *exact critical scaling function*

$$M(\delta, \Gamma_H) \approx \bar{\mu}\Gamma_H^{\phi-2}\left[\frac{\gamma^2\alpha}{\sinh^2\gamma} + \frac{e^{-\gamma}}{\sinh\gamma}\left(\phi\gamma\alpha + \gamma^2\frac{d\alpha}{d\gamma}\right)\right] , \tag{3.50}$$

valid in the limit $\delta \to 0$ and $\Gamma_H \to \infty$ with

$$\gamma \equiv \delta\Gamma_H \approx \delta \ln(D_H/H) \tag{3.51}$$

fixed, where we have allowed for a general scale factor D_H for the magnetic field to make the arguments of the ln in Eq. (3.51) dimensionless. But note that changes of D_H by multiplicative factors represent really only *corrections* to scaling; indeed these are smaller than the corrections to scaling that arise from the neglect of all but the $\ln H$ term in Eq. (3.33).

We now consider various limits of the scaling function.

1. Critical point

At the critical point, $\gamma = 0$ so that $\alpha = 1$ and $\frac{d\alpha}{d\gamma} = 0$. The expression in brackets in Eq. (3.50) thus simply becomes $1+\phi$ so that

$$M(\delta = 0, H) \sim \frac{1}{[\ln(D_H/H)]^{2-\phi}} . \tag{3.52}$$

2. Ordered phase

The ordered phase in the limit of small fields corresponds to $\gamma \to -\infty$. For large $|\gamma|$,

$$\alpha \approx C_A |\gamma|^{1-\phi} , \tag{3.53}$$

and hence the bracketed expression in Eq. (3.50) becomes $2C_A|\gamma|^{2-\phi}$, yielding

$$M(\delta < 0, H = 0) \sim (-\delta)^{2-\phi} , \tag{3.54}$$

as guessed earlier from scaling.

The leading correction to this result for small H can be obtained from the behavior of $\tilde{A}_0$ or α for large $|\gamma|$: The general large $|\gamma|$ form of $\tilde{A}_0$ is, from Eq. (3.40),

$$\tilde{A}_0 = C_A|\gamma| + C'_A + O(|\gamma| e^{-2|\gamma|}) , \tag{3.55}$$

so that $\frac{d\tilde{A}_0}{d\gamma}$ has only exponentially small corrections to C_A. From the relation Eq. (3.45) and the connection between Legendre functions and hypergeometric functions, we see that

$$\alpha(\gamma) = \left(\frac{\tanh\gamma}{\gamma}\right)^{\phi} F\left(\frac{\phi}{2} + \frac{1}{2}, \frac{\phi}{2}, \phi + \frac{1}{2};\ \tanh^2\gamma\right) , \tag{3.56}$$

from which C_A and C'_A can be obtained from the behavior of $F(a, b, a+b; x)$ for $x \nearrow 1$,[16] yielding C_A in Eq. (3.46) and

$$\frac{C'_A}{C_A} = \psi(1) - \psi(\phi) \approx -0.719 , \tag{3.57}$$

with ψ the digamma function.

Thus we see that for small H in the ordered phase, the scaling function yields the form

$$M(\delta < 0, H) \approx M_0(\delta)\left[1 + O\left(H^{2|\delta|}\delta \ln H\right)\right] . \tag{3.58}$$

Close to the transition, the susceptibility thus remains infinite with a continuously variable exponent parametrizing the small H singularity in $M(H)$. As we shall see in Sec. IV A, this behavior is associated with a classical ordered fixed line. It obtains with only subtle corrections even if we take H to 0 at fixed negative δ so that $\delta \ln H$ is infinite and hence not strictly in the scaling limit.

3. Disordered phase

In the limit of small fields in the disordered phase, it can easily be seen that $N_A(\delta, \Gamma = \infty) = 0$. The leading large-$\gamma$ behavior of N_A is

$$N_A(\delta, \Gamma) \approx \Gamma^{\phi-2}\left[4C_A\gamma^{3-\phi}e^{-2\gamma} + (2C_A + 4C'_A)\gamma^{2-\phi}e^{-2\gamma} + O(\gamma^{3-\phi}e^{-4\gamma})\right] , \tag{3.59}$$

yielding for small H

$$M(\delta > 0, H) \sim \delta^{2-\phi}\left[H^{2\delta}\left(\delta \ln\frac{1}{H} + \frac{1}{2} + \frac{C'_A}{C_A}\right) + O\left(H^{4\delta}\delta \ln\frac{1}{H}\right)\right] . \tag{3.60}$$

Thus we see that $M(H)$ has a *power law singularity* with a *continuously variable exponent* 2δ. As we shall see in Sec. IV, this behavior is associated with a disordered fixed line of almost disconnected clusters. The *form* of Eq. (3.60) with corrections to the exponent which are higher order in δ and hence not in the scaling function will be seen to obtain along this fixed line even outside the scaling limit, i.e., $H \ll e^{-c/\delta}$.

The fact that the scaling function for the magnetization can be obtained is quite remarkable; the analogous function, with scaling variable $\delta/H^{\frac{1}{\beta+\gamma}} = \delta/H^{8/15}$ instead of $\delta \ln H$, is *not* known for the pure Ising model. Thus, in a sense, the random Ising chain in the scaling limit is more solvable than the pure system due to the wide separation of energy scales.

Although it may well be possible to make the approximations used here rigorous asymptotically, this has not been done and one might thus question whether the scaling function so obtained is really correct. The best evidence for its validity, at this point, is the agreement between an analogous scaling function for the end-point magnetization of a semi-infinite system as a function of a field applied only to the spin at the end. This is calculated in Sec. V and found to agree *exactly* in the scaling limit with the exact results of McCoy[3] for a particular class of distributions of couplings.

B. Mean correlation functions

Earlier in this section, we saw how the mean correlation functions in the scaling limit could be obtained from the function $\hat{g}(y, \Gamma) \equiv \hat{G}(\int, y, \Gamma)$. This is given by Eq. (3.30) with $A(y, \Gamma)$ satisfying the second-order linear differential equation (3.24) and $\hat{P}(0, y) = \Upsilon(y)$ given by Eq. (2.71).

1. Critical point

We first analyze the behavior of the mean correlations at the critical point $\delta = 0$. For $\delta = 0$, the expressions for $\hat{g}$, $\hat{k}$, A, etc., are powers of y times functions of the scaled variable

$$\tilde{\Gamma} \equiv \Gamma\sqrt{y} . \tag{3.61}$$

In particular, we can write

$$A(y, \Gamma) = y^{-\phi/2}\mathcal{A}(\tilde{\Gamma}) , \tag{3.62}$$

with

$$\frac{d^2\mathcal{A}}{d\tilde{\Gamma}^2} = \mathcal{W}(\tilde{\Gamma})\mathcal{A} \tag{3.63}$$

and

$$\mathcal{A} \approx \tilde{\Gamma}^{\phi} \tag{3.64}$$

for small $\tilde{\Gamma}$. The function in Eq. (3.63) is given by

$$\mathcal{W} = \frac{W(\delta = 0, y, \Gamma)}{y} = -\frac{1}{2} + \frac{3}{4}\coth^2\tilde{\Gamma} - \frac{1}{2\tilde{\Gamma}}\coth\tilde{\Gamma} + \frac{3}{4\tilde{\Gamma}^2} \ . \tag{3.65}$$

For large $\tilde{\Gamma}$, $\mathcal{W} \approx \frac{1}{4}$, implying $\mathcal{A} \sim e^{\frac{1}{2}\tilde{\Gamma}}$. Using

$$n_\Gamma = \frac{1}{\Gamma^2} = \frac{y}{\tilde{\Gamma}^2} \ , \tag{3.66}$$

$$\tau_0 = \frac{1}{\Gamma} = \frac{\sqrt{y}}{\tilde{\Gamma}} \ , \tag{3.67}$$

$$\tau = \sqrt{y}\coth\Gamma\sqrt{y} = \sqrt{y}\coth\tilde{\Gamma} \ , \tag{3.68}$$

$$\Upsilon = \frac{\sqrt{y}}{\sinh\Gamma\sqrt{y}} = \frac{\sqrt{y}}{\sinh\tilde{\Gamma}} \ , \tag{3.69}$$

and

$$e^{-\Phi} = \frac{1}{\Gamma^{\frac{5}{2}}}\left(\frac{\sqrt{y}}{\sinh\Gamma\sqrt{y}}\right)^{\frac{1}{2}} = \frac{y^{\frac{3}{2}}}{\tilde{\Gamma}^{\frac{5}{2}}}(\sinh\tilde{\Gamma})^{-\frac{1}{2}} \tag{3.70}$$

at the critical point, we obtain

$$\hat{k} = y^{\frac{3}{2}-\phi}\frac{1}{\tilde{\Gamma}\cosh^2\tilde{\Gamma}}\left[\frac{1}{2}\left(\frac{1}{\tilde{\Gamma}} + \coth\tilde{\Gamma}\right)\mathcal{A} + \frac{d\mathcal{A}}{d\tilde{\Gamma}}\right]^2 . \tag{3.71}$$

As $\tilde{\Gamma} \to 0$, $\hat{k} \approx \Gamma^{2\phi-3}$ independent of y as it should since the correlations will be short range in x in this "high"-energy limit. Conversely, as $\tilde{\Gamma} \to \infty$, $\hat{k} \sim y^{\frac{3}{2}-\phi}e^{-\tilde{\Gamma}}/\tilde{\Gamma}$. Thus, we see that the scaled integral to obtain the correlations,

$$\hat{K}_\infty(y) = \int_0^\infty d\Gamma\hat{k}(y,\Gamma) = y^{1-\phi}\int_0^\infty d\tilde{\Gamma}\left\{\frac{\hat{k}(\tilde{\Gamma})}{y^{\frac{3}{2}-\phi}}\right\} , \tag{3.72}$$

is well behaved both for small and large $\tilde{\Gamma}$ and hence yields a finite constant, implying

$$\hat{K}_\infty(y) \sim y^{1-\phi} \ . \tag{3.73}$$

Therefore, for large x at the critical point, the mean correlation function

$$\overline{C}(x) \sim K_\infty(x) \sim \frac{1}{x^{2-\phi}} \ . \tag{3.74}$$

This has the form that was guessed earlier, Eq. (2.102): It is of order the probability that σ_0 and σ_x are both active at the length scale $\ell_\Gamma \sim x$, in which case one would expect that it is reasonably likely that they are both active in the *same* cluster. Indeed, the integral in Eq. (3.72) is dominated by $\tilde{\Gamma} \sim 1$, i.e., $\Gamma^{-2} \sim n_\Gamma \sim y$, corresponding roughly to $\ell_\Gamma \sim 1/n_\Gamma \sim x \sim 1/y$.

The behavior for $x \gg \ell_\Gamma$ is controlled by the singularities, in y, of $\hat{k}(y,\Gamma)$. For fixed Γ, $\hat{k}$ is not singular as $y \to 0$. Its singularities in the complex y plane occur when the prefactor of Eq. (3.71) is singular, i.e., $\cosh\Gamma\sqrt{y} = 0$, or when $\mathcal{A}$ is singular, which, since $\mathcal{A}$ satisfies a linear differential equation, occurs only when W is singular, i.e., at nonzero solutions of $\sinh\Gamma\sqrt{y} = 0$. The nearest singularity to the origin of $\hat{k}(y,\Gamma)$ in the complex y plane is thus controlled by the prefactor in Eq. (3.71) and is a double pole at

$$y = y_s = \frac{-\pi^2}{4\Gamma^2} \ . \tag{3.75}$$

Thus for fixed Γ at the critical point, as $x \to \infty$,

$$k(x,\Gamma) \sim x\Gamma^{2\phi-7}e^{-\frac{x\pi^2}{4\Gamma^2}} \ , \tag{3.76}$$

so that the probability of spins σ_0 and σ_x *becoming* active in the same cluster is exponentially small for $x \gg \ell_\Gamma$. As we shall see in Sec. III C, the finite-temperature correlations can be obtained from $\hat{k}(y,\Gamma)$; their long-distance behavior is controlled by singularities in the complex y plane at the scale $\Gamma = \Gamma_T = \ln(\Omega_I/T)$.

2. Spontaneous magnetization

We next consider the long-distance behavior of the mean correlations in the ordered phase. At long distances, the mean $\overline{C}(x)$ should tend to M_0^2, which is proportional to $[N_A(\Gamma = \infty)]^2$. Thus, we must have

$$K_\infty(y) \approx [N_A(\infty)]^2/y \tag{3.77}$$

for $y \to 0$ in the ordered phase. If we set $y = 0$, then $\hat{g}(y = 0, \Gamma) \approx N_A(\Gamma) \to N_A(\infty)$ for large Γ and we see that with $\delta < 0$ and $\Gamma \gg 1/|\delta|$

$$\hat{k}(y = 0, \Gamma) \approx \frac{1}{2|\delta|}e^{2\Gamma|\delta|}[N_A(\infty)]^2 \tag{3.78}$$

and hence the integral to obtain $\hat{K}(y = 0)$ from Eq. (3.7) indeed diverges. It is not *a priori* clear how this divergence will be cut off for small y.

A natural guess is that for $y \ll \delta^2$ the dominant Γ will be $\Gamma \sim \Gamma_y$ defined by

$$n_{\Gamma_y} \sim y \ . \tag{3.79}$$

Since

$$n_\Gamma \approx 4\delta^2 e^{-2\Gamma|\delta|} \tag{3.80}$$

for $\Gamma \gg 1/|\delta|$,

$$\Gamma_y \approx \frac{1}{2|\delta|}\ln\left(\frac{\delta^2}{y}\right) \ . \tag{3.81}$$

We can now attempt to expand $\hat{g}(y;\Gamma)$ in y and keep all terms that are non-negligible for $\Gamma \sim \Gamma_y$. We have

$$\Gamma\Delta(y) \approx \Gamma|\delta| + \frac{1}{2}\frac{y\Gamma}{|\delta|} + O(y^2) \ , \tag{3.82}$$

but even the second term in Eq. (3.82) is negligible for $\Gamma \sim \Gamma_y$. Indeed, one can see that

$$A(y,\Gamma_y) \approx A(y=0,\Gamma_y) \ . \tag{3.83}$$

The only problem with setting y to zero in Eq. (3.30) is thus in the function that appears in the denominator,

$$\begin{aligned}\tau(y) &= \delta + \Delta \ \coth\Gamma\Delta \\ &\approx 2|\delta|e^{-2\Gamma|\delta|} + \frac{1}{2}\frac{y}{|\delta|} \\ &\approx \tau_0\left[1 + y/n_\Gamma + O\left(\frac{y^2}{\delta^2 n_\Gamma}, \frac{y\Gamma}{\delta}\right)\right] \ ,\end{aligned} \tag{3.84}$$

with

$$\tau_0 \approx 2|\delta|e^{-2\Gamma|\delta|} \ , \tag{3.85}$$

which is very small at low-energy scales. The y/n_Γ term in Eq. (3.84) is clearly needed for large Γ. In the needed limit,

$$\frac{\partial A}{\partial \Gamma} \gg (\tau + \tau_0)A \ . \tag{3.86}$$

Thus, to the required accuracy for $\Gamma \sim \Gamma_y$, we have, from Eq. (3.30),

$$\hat{g}(y,\Gamma) \approx \hat{g}(0,\infty)\frac{1}{1+y/n_\Gamma} \tag{3.87}$$

and hence, from Eq. (3.13),

$$\hat{k}(y,\Gamma) \approx \frac{2|\delta|n_\Gamma}{(y+n_\Gamma)^2}N_A^2(\infty) \ . \tag{3.88}$$

By changing variables from $d\Gamma$ to dn_Γ and noting that $dn_\Gamma \approx -2|\delta|n_\Gamma d\Gamma$, we see that the integral over Γ of Eq. (3.88) yields Eq. (3.77). The neglect of the other regions of the Γ integration, $\Gamma \leq 1/|\delta|$ and $\Gamma \geq |\delta|/y$ for which the approximation Eq. (3.86) is not valid, can readily be shown to be justified for $y \ll \delta^2$. Thus we see that, as it must, the long-distance form of the mean correlations in the ordered phase reduces to the square of the mean spontaneous magnetization. Note that

$$\overline{\langle\sigma_0\sigma_x\rangle} \to \left(\overline{\langle\sigma\rangle}\right)^2 \tag{3.89}$$

rather than $\overline{\langle\sigma\rangle^2}$, because the $\{\langle\sigma_j\rangle\}$ are approximately independent for widely separated spins.

In order to obtain the corrections to the asymptotic form Eq. (3.89), i.e., the form of the *decay* of the correlations, the effects of the singularity in $\hat{K}_\infty$ near $y=0$ need to be removed and the singularities of the function $\hat{K}_\infty(y) - \frac{N_A^2(\infty)}{y}$ studied. A straightforward study of the form of the neglected terms in the above analysis would lead one to expect $\ln y$ corrections to $\hat{K}_\infty(y)$ for small y. But, as we shall see, these cancel exactly along with all other singularities near the origin $y = 0$. In fact, the mean correlations decay *exponentially* to their asymptotic limit associated with a singularity *away* from the origin of the complex y plane. Before showing this, we turn to the somewhat simpler behavior in the disordered phase. This will set the stage for a fuller analysis of the analytic structure of $\hat{K}_\infty(y)$ in the ordered phase, as well as enabling correlations to be computed at finite temperatures.

3. *Disordered phase*

In the disordered phase, we would expect the mean correlations to decay exponentially. As we shall see, this is true but the actual detailed form of the decay is quite interesting; we focus on the long-distance behavior here.

We first note that the singularities in $\hat{g}(y;\Gamma)$ near $y=0$ that arose in the ordered phase from the vanishing of $\tau(y;\Gamma)$ do *not* occur in the disordered phase. Instead, the nearest singularities to the origin occur near $y=-\delta^2$, i.e., $\Delta = 0$. These arise from two sources: First, $\tau(y,\Gamma)$ can still vanish in the disordered phase; it will do so when $\Delta(y)$ is purely imaginary with the closest singularity to $y=0$ in the range $\frac{\pi}{2\Gamma} < |\mathrm{Im}\Delta| < \frac{\pi}{\Gamma}$ corresponding to a singularity at

$$y = y_{s\tau} = -\delta^2 - \frac{a_s(\delta\Gamma)}{\Gamma^2} \ , \tag{3.90}$$

with $a_s(\delta\Gamma)$ in the range $\frac{\pi^2}{4} \leq a_s \leq \pi^2$. But, in addition, $\hat{g}$ will be singular if A is singular. As mentioned earlier, since A satisfies a linear differential equation, it will be singular only when the function $W(y,\Gamma)$ is singular. Hence, we expect singularities in $\hat{g}$ at

$$y_{sn}(\Gamma) \approx -\delta^2 + \frac{n^2\pi^2}{\Gamma^2} \tag{3.91}$$

for $n = 1,2,3\ldots$. As $\Gamma \to \infty$, these singularities, as well as $y_{s\tau}$, approach $-\delta^2$ and hence upon integrating $\hat{k}$ over Γ we anticipate a *cut* in $\hat{K}_\infty(y)$ on the negative real axis ending at $y_\infty = -\delta^2$.

From the analysis below, it can be seen that $\hat{K}_\infty(y)$ will be analytic except on this cut. Thus the contour in the inverse Laplace transform to obtain $K_\infty(x)$ can be deformed to an integral around this cut, yielding

$$K_\infty(x) = e^{-\delta^2 x}\int_0^\infty d\eta \ e^{-\eta x}\rho_K(\eta) \ , \tag{3.92}$$

where the "spectral density"

$$\begin{aligned}\rho_K(\eta) \equiv -\frac{1}{2\pi i}\big[&\hat{K}_\infty(-\delta^2-\eta+i\theta) \\ &-\hat{K}_\infty(-\delta^2-\eta-i\theta)\big] \ ,\end{aligned} \tag{3.93}$$

with θ a positive infinitesimal, is given by the discontinuity across the cut. From Eq. (3.92) we immediately obtain the correlation length

$$\xi \approx \frac{1}{\delta^2} \tag{3.94}$$

in the scaling limit.

In order to analyze the needed behavior of $\hat{K}_\infty$ across the cut, it is useful to separate out the effects of the singularities in $1/\tau$ from those in A. This can be done by an appropriate integration by parts of $K_\Gamma(y)$. Using the fact that $\frac{\partial^2 A}{\partial\Gamma} = WA$, we can write

$$\begin{aligned}\hat{K}_\Gamma &\equiv \int_0^\Gamma d\Gamma' \hat{k}(y;\Gamma') \\ &= \frac{1}{\delta + \Delta\ \coth\Gamma\Delta}\ \frac{2\delta}{e^{2\Gamma\delta}-1}[I_1(y,\Gamma)]^2 \\ &\quad + \int_0^\Gamma d\Gamma' \hat{k}'(y,\Gamma')\ ,\end{aligned} \tag{3.95}$$

with

$$\hat{k}'(y,\Gamma) = \frac{2\delta}{e^{2\Gamma\delta}-1} I_1(y,\Gamma) I_2(y,\Gamma)\ , \tag{3.96}$$

with

$$I_1(y,\Gamma) = \frac{\partial A}{\partial\Gamma} + \left(\delta + \frac{1}{2}\Delta\ \coth\Gamma\Delta + \frac{1}{2}\delta\ \coth\Gamma\delta\right) A \tag{3.97}$$

and

$$I_2(y,\Gamma) = -\frac{\partial A}{\partial\Gamma} + \left(\delta - \frac{1}{2}\Delta\ \coth\Gamma\Delta + \frac{3}{2}\delta\ \coth\Gamma\delta\right) A\ . \tag{3.98}$$

In the desired limit $\Gamma \to \infty$, the first term in Eq. (3.95) can be shown to vanish for any Δ in the right half plane, i.e., any y off the cut of $\hat{K}_\infty(y)$. Thus $\hat{K}_\infty(y)$ is not affected by the singularities of τ and we can determine its properties by analyzing only the singularities in $A(y,\Gamma)$.

The mean correlations will decay exponentially for large $x \gg \xi = 1/\delta^2$ as follows immediately from the form of Eq. (3.91). But for $x \gg \xi$, the form of the decay will be dominated by the behavior of $\hat{K}_\infty$ near the end of the cut at $y = -\delta^2$, i.e., $\rho_K(\eta)$ for $\eta \ll \delta^2$. We are thus led to analyze the behavior of the solution to the differential equation for A for $|\Delta| = |\sqrt{y+\delta^2}| \ll \delta$, where $\Delta = \pm i\sqrt{\eta} + \theta$ for just above or just below the cut. We will see that the dominant values of Γ are of order $1/|\Delta|$ because of the singularities in the complex Γ plane that correspond to those of Eq. (3.96) in the complex y plane. Thus we need the behavior of A for $\Gamma\delta \gg 1$ but with general $\Gamma\Delta$.

In this "outer" limit, $\delta\Gamma \gg 1$, but $\Gamma\Delta$ of order unity,

$$\begin{aligned}W \approx W_+ = &-\frac{1}{2}\Delta^2 + \frac{1}{4}\delta^2 + \frac{3}{4}\Delta^2\ \coth^2\Gamma\Delta \\ &-\frac{1}{2}\delta\Delta\ \coth\Gamma\Delta + O\left(\delta^2 e^{-2\Gamma\delta}\right)\ .\end{aligned} \tag{3.99}$$

Ignoring the exponentially small corrections to W_+, a general solution to

$$\frac{\partial^2 A_+}{\partial\Gamma^2} = W_+ A_+ \tag{3.100}$$

can be found. This must be matched onto the solution for A_- which obtains in the "inner" regime $\delta\Gamma \sim 1$, but $|\Delta|\Gamma \ll 1$, to obtain the correct coefficients of the two linearly independent solutions to Eq. (3.100). From Appendix C, it is found that

$$A_+ \approx C_{1-}\delta^{-\phi-\frac{1}{2}} e^{\frac{1}{2}\delta\Gamma} \left(\frac{\Delta}{\sinh\Gamma\Delta}\right)^{\frac{1}{2}} \left[1 + O\left(\delta\Gamma e^{-\delta\Gamma}\right)\right]\ , \tag{3.101}$$

where C_{1-} is an $O(1)$ coefficient obtained from the solution to the scaled inner equation

$$\frac{d^2\tilde{A}_-}{d\gamma^2} = \tilde{W}_-\tilde{A}_-\ , \tag{3.102}$$

with

$$\tilde{A}_- \approx \gamma^\phi \qquad \text{as } \gamma \to 0 \tag{3.103}$$

and

$$\tilde{W}_- = -\frac{1}{2} + \frac{3}{4}\ \coth^2\gamma - \frac{1}{2\gamma}\ \coth\gamma + \frac{3}{4\gamma^2}\ . \tag{3.104}$$

The function

$$\tilde{A}_-(\gamma) = \delta^\phi A_- \approx \delta^\phi A\left(\Gamma = \gamma/\delta;\ |\Delta| \ll \frac{1}{\Gamma}\right) \tag{3.105}$$

represents the solution for $\Gamma\delta \sim 1$ and $\Gamma|\Delta| \ll 1$. It has the form

$$\tilde{A}_- \approx C_{1-}\gamma^{-\frac{1}{2}} e^{\frac{1}{2}\gamma}(1 + C_{2-}e^{-\gamma} + \cdots) \tag{3.106}$$

for $\gamma \to \infty$, corresponding to the intermediate regime $1/\delta \ll \Gamma \ll 1/|\Delta|$, which therefore matches onto Eq. (3.101) for $\Gamma|\Delta| \ll 1$, thereby determining the coefficient in Eq. (3.101).

The required regime of the integration of Eq. (3.95) can now be obtained simply from Eq. (3.101). Since in this limit

$$\frac{\partial A}{\partial\Gamma} \approx \left(\frac{1}{2}\delta - \frac{1}{2}\Delta\ \coth\Gamma\Delta\right) A\ , \tag{3.107}$$

we see that with $\Gamma\delta \gg 1$

$$I_1 \approx I_2 \approx 2\delta A\ . \tag{3.108}$$

The integrand in Eq. (3.95) thus becomes

$$\hat{k}'(\Gamma) \approx 8\delta^{2-2\phi} C_{1-}^2 e^{-\Gamma\delta} \frac{\Delta}{\sinh\Gamma\Delta} \tag{3.109}$$

for $\Gamma\delta \gg 1$. Since the singularities in y of the integral $\hat{K}_\infty(y)$ are far from the end of the y plane cut for small Γ, we see that the singular behavior near the end of the cut is dominated by large Γ, and for the purposes of determining the spectral density $\rho_K(\eta)$ for small η, we can thus replace $\hat{K}_\infty(y)$ by an integral over the asymptotic expression in Eq. (3.109).

We are hence interested in

$$\int^{\infty} d\Gamma \hat{k}'(y,\Gamma) \qquad (3.110)$$

for $y = -\delta^2 - \eta \pm i\theta$, i.e., for $\Delta = \pm i\sqrt{\eta} + \theta$ with θ a positive infinitesimal. For $\Delta = i\sqrt{\eta} + \theta$, $\hat{k}'$ has singularities in Γ at and near $\Gamma\Delta = in\pi$, i.e., for $\Gamma = \frac{n\pi}{\sqrt{\eta}} + i\theta$, while for $\Delta = -i\sqrt{\eta} + \theta$, which, due to the symmetry of $\hat{k}'$ under $\Delta \to -\Delta$, is equivalent to $\Delta = i\sqrt{\eta} - \theta$, the singularities are at and near $\Gamma = \frac{n\pi}{\sqrt{\eta}} - i\theta$. In the first case, the singularities can be put on the real Γ axis and the Γ integration contour can be deformed to just *below* the real axis, while in the second case it can be similarly moved to just *above* the real axis. The needed *difference* between these two, for Eq. (3.93), is hence a closed contour integral in the Γ plane running out just above and back just below the positive real axis and hence enclosing the poles in Γ of $\hat{k}'$, Eq. (3.109) (but not those which occur at $\Gamma = i\frac{n\pi}{\delta}$ in the $\delta\Gamma \sim 1$ part of A). The function $\rho_K(\eta)$ is thus simply given by minus the sum over the residues of the poles in $\hat{k}'$ *as a function of* Γ for fixed η with $\Delta = i\sqrt{\eta}$. From Eq. (3.109) for $\hat{k}'$ we hence obtain

$$\rho_K(\eta) \approx 8\delta^{2-2\phi} C_{1-}^2 e^{-\frac{\pi\delta}{\sqrt{\eta}}} \qquad (3.111)$$

for $\eta \ll \delta^2$.

The long-distance behavior of the mean correlation function can now be obtained by evaluating the integral over η in Eq. (3.92) as a saddle-point integration. We find that, for $x \gg \xi = 1/\delta^2$,

$$\overline{C}(x) \approx \bar{\mu}^2 C_{1-}^2 \frac{16\pi^{5/6}}{2^{1/3}\sqrt{3}} \delta^{4-2\phi} \left(\frac{\xi}{x}\right)^{5/6} e^{-\frac{3}{2}(2\pi^2 x/\xi)^{\frac{1}{3}}} e^{-x/\xi} . \qquad (3.112)$$

4. Ordered phase

We now return to the ordered phase and consider the *corrections* to the asymptotic constant long-distance behavior of the mean correlations. The two terms in Eq. (3.95) for $\hat{K}_\Gamma(y)$ are both badly behaved for large Γ in the ordered phase and are hence not directly useful. Nevertheless, if an appropriate expression such as

$$\delta A(\Gamma)[I_1(\Gamma) - I_2(\Gamma)] \qquad (3.113)$$

is added to the first term in Eq. (3.95) and its derivative subtracted from $\hat{k}'$, both terms become well behaved for large Γ and hence more useful. We thus have, with implicit y dependence of all quantities, but displaying the δ dependence,

$$\hat{K}_\Gamma = \frac{1}{\delta + \Delta\,\coth\Gamma\Delta}\,\frac{2\delta}{e^{2\Gamma\delta} - 1}[I_1(\Gamma)]^2 + \delta A(\Gamma)[I_1(\Gamma) - I_2(\Gamma)] + \int_0^\Gamma d\Gamma' \hat{k}''(\Gamma') , \qquad (3.114)$$

with

$$\begin{aligned}\hat{k}''(\delta,\Gamma) &= \frac{-2\delta}{e^{-2\Gamma\delta} - 1}\,[I_1(\delta,\Gamma) - 2\delta A(\delta,\Gamma)] \\ &\quad\times [I_2(\delta,\Gamma) - 2\delta A(\delta,\Gamma)] \\ &= \frac{2|\delta|}{e^{2\Gamma|\delta|} - 1}\, I_1(|\delta|,\Gamma)\, I_2(|\delta|,\Gamma) = \hat{k}'(|\delta|,\Gamma) ,\end{aligned} \qquad (3.115)$$

using the fact that $A(-\delta,\Gamma) = A(\delta,\Gamma)$ and the definitions Eqs. (3.97) and (3.98) of $I_{1,2}$. Thus we see that $\hat{K}_\infty(y)$ in the ordered phase differs from that in the disordered phase with $\delta \to |\delta|$ only by the first two terms in Eq. (3.114) which are obtainable from the $\Gamma \to \infty$ behavior of A.

Since $W \approx \frac{1}{4}(|\delta| - \Gamma)^2$ for real Δ as $\Gamma \to \infty$, the solution for A has the form

$$A \approx a_- e^{\frac{1}{2}(\Delta - |\delta|)\Gamma} + a_+ e^{-\frac{1}{2}(\Delta - |\delta|)\Gamma} \qquad (3.116)$$

as $\Gamma \to \infty$ with $a_+(\Delta, |\delta|)$ and $a_-(\Delta, |\delta|)$ smooth except at $\Delta = |\delta|$ and on the imaginary Δ axis, corresponding, respectively, to $y = 0$ and y real and less than $-\delta^2$, and therefore on the cut; A is analytic in the rest of the y plane. From Eqs. (3.97), (3.98), and (3.116), we see that the first two terms in Eq. (3.114) yield

$$2\delta(\Delta - |\delta|) a_+ a_- \qquad (3.117)$$

as $\Gamma \to \infty$. Thus singularities in $\hat{K}_\infty(y)$ also occur only at $y = 0$ and on the $y \in (-\infty, -\delta^2)$ cut, the latter involving contributions from the integral over $\hat{k}''$ in Eq. (3.114). The $y = 0$ singularity can readily be recovered by noting that, for $\Delta = |\delta|$,

$$A \approx C_A \Gamma |\delta|^{1-\phi} \qquad (3.118)$$

for $\Gamma \to \infty$ from Eqs. (3.38) and (3.42) so that

$$-a_+ \approx +a_- \approx \frac{C_A |\delta|^{1-\phi}}{\Delta - |\delta|} \qquad (3.119)$$

for $\Delta \to |\delta|$, yielding

$$\hat{K}_\infty(y) \approx \frac{4 C_A^2 |\delta|^{4-2\phi}}{y} \qquad (3.120)$$

for small y in agreement with Eq. (3.77).

To obtain the long-distance behavior of the decay of $\overline{C}(x)$ to $\overline{\langle\sigma\rangle}^2$, as for the disordered phase, the discontinuities across the cut in $\hat{K}_\infty(y)$ are needed for y just less than $-\delta^2$, i.e., for Δ small and imaginary. To obtain these, it is actually more convenient to work with Eq. (3.95) rather than Eq. (3.114) which is not problematic in this regime since the integral in Eq. (3.95) converges for $|\Delta| \ll |\delta|$. The asymptotic solution for A in the limit $|\Delta| \ll |\delta|$ that is derived in Appendix C can be used. The contribution from the first term in Eq. (3.95) vanishes exponentially for large Γ and hence does not enter $\hat{K}_\infty$. If the asymptotic expression from Eq. (3.109) is used in determining $\hat{k}'$, the singularities from the $(\sinh\Gamma\Delta)^{-1/2}$ are found to cancel almost exactly, with only $O(e^{-3\Gamma|\delta|})$ residual terms. Thus it is necessary to go *beyond* the dominant terms calculated in Appendix C. The next leading correction discussed there is sufficient to obtain the small-η behavior of the spectral density $\rho_K(\eta)$ of the singularities in a manner analogous to that for the disordered phase.

The result from Appendix C involves *both* the coefficients C_{1-} and C_{2-} from the inner solution A_- of Appendix C and Eq. (3.106). From the signs of the coefficients in the differential equation for

$$E = \left(\frac{\delta\sinh\Gamma\Delta}{\Delta\sinh\Gamma\delta}\right)^{\frac{1}{2}} A \ , \tag{3.121}$$

it can be seen that C_{2-} is *negative*. We obtain

$$\hat{\rho}_K(\eta) \approx -16C_{1-}^2 C_{2-}|\delta|^{2-2\phi} e^{\frac{-2\pi|\delta|}{\sqrt{\eta}}} \ , \tag{3.122}$$

which is of the same form as in the disordered phase, Eq. (3.111), but with an extra factor of 2 in the exponent. The mean correlations will thus decay more *rapidly* than in the disordered phase but have the same form as Eq. (3.10) with modified coefficients:

$$\overline{C}(x) - M_0^2 \approx \bar{\mu}^2\left(\frac{-32C_{1-}^2C_{2-}}{\sqrt{3}}\,\pi^{\frac{5}{6}}\right)|\delta|^{4-2\phi}\left(\frac{\xi}{x}\right)^{\frac{5}{6}} \times e^{-3(\pi x/\xi)^{\frac{1}{3}}} e^{-x/\xi} \ , \tag{3.123}$$

with, again, $\xi \approx 1/\delta^2$.

Note that the correlation function Eq. (3.123) is *not* the mean of the conventionally defined truncated correlation function

$$C_{0x}^t = \langle\sigma_0^z\sigma_x^z\rangle - \langle\sigma_0^z\rangle\langle\sigma_x^z\rangle \ , \tag{3.124}$$

but differs from it by the "disconnected" correlation function of the local magnetizations,

$$\overline{C}_{0x}^d = \overline{\langle\sigma_0^z\rangle\langle\sigma_x^z\rangle} - \overline{\langle\sigma^z\rangle}^2 \ , \tag{3.125}$$

which is of course zero in the disordered phase, as all $\langle\sigma_i^z\rangle$ are zero.

In our simple picture, the disconnected correlation function is dominated by (at long distances) the correlations in the *positions* of the spins that *remain* active down to zero energy and hence have non-negligible $\langle\sigma_i^z\rangle$. Conversely, the truncated correlations are dominated by strongly correlated pairs of spins that do *not* remain active all the way to zero energy. In principle, the correlations in the ensemble of random couplings of the $\langle\sigma_i^z\rangle$ can be computed. In particular, $\overline{\langle\sigma_0^z\rangle\langle\sigma_x^z\rangle}$ can be obtained from

$$D(\beta,x;\Gamma) \equiv \text{Prob}\{\sigma_0 \text{ and } \sigma_x \text{ active in same cluster with } \beta \text{ at scale } \Gamma\}. \tag{3.126}$$

This obeys an *inhomogenous* linear differential RG equation

$$\frac{\partial D}{\partial\Gamma} = \frac{\partial D}{\partial\beta} + k(\beta,x;\Gamma) + 2P_0 D(\cdot,x) \underset{\beta}{\otimes} R(\cdot) - 2P_0 D \ , \tag{3.127}$$

with the inhomogeneous term arising from the joining of two clusters at scale Γ, one cluster containing σ_0 active and the other with σ_x active, while the last two terms arise from the combining of the cluster containing 0 and x with another cluster. Here, $k(\beta,x;\Gamma)$ is defined in the same way as $k(x;\Gamma) = \int_0^\infty d\beta\, k(\beta,x;\Gamma)$ but with the extra condition that the cluster formed has logarithmic field of β; it is thus given in terms of G by Eq. (3.9) with the integrals over β in the two G factors replaced by β' and $\beta-\beta'$, respectively, with an integral only over β'.

From the solution to Eq. (3.127), with the appropriate initial conditions on the Laplace transform $\hat{D}(\beta,y;\Gamma)$ for small Γ, one could obtain

$$\overline{\langle\sigma_0^z\rangle\langle\sigma_x^z\rangle} \approx \bar{\mu}^2\int_0^\infty d\beta\, D(\beta,x;\infty) \ . \tag{3.128}$$

The truncated correlations are then obtainable by noting that

$$\frac{\partial K_\Gamma(x)}{\partial\Gamma} - \frac{\partial D(\int,x;\Gamma)}{\partial\Gamma} = D(0,x;\Gamma) \ , \tag{3.129}$$

yielding

$$\overline{C^t}(x) \approx \bar{\mu}^2\int_0^\infty d\Gamma\, D(0,x;\Gamma) \ . \tag{3.130}$$

A preliminary examination of the structure of the Laplace transform of the flow equation (3.127) for $\hat{D}$ suggests that even the special solution will reduce to a fourth-order linear ODE which we do not attempt to analyze here. Although we have not computed $\hat{D}(y)$, it is clear that it will have a cut on the negative y axis ending at $y=-\delta^2$ and hence yield a $D(x)$ decaying exponentially with the same correlation length as $K_\infty(x)$ but possibly different subexponential factors than Eq. (3.123). Thus, we expect that $\overline{C^t}(x)$ and $\overline{C}^d(x)$ will both decay exponentially as $e^{-x/\xi}$ but probably with different subexponential prefactors. Note that one could put useful bounds on both mean correlation functions in terms of simpler quantities that we have calculated, without explicitly computing D.

C. Correlation functions at $T > 0$

The finite-temperature correlations can be obtained in the scaling limit by stopping the renormalization at the scale $\Omega = T$, i.e.,

$$\Gamma_T = \ln(\Omega_I/T) \ . \tag{3.131}$$

At this scale, almost all the remaining clusters will have couplings and effective transverse fields much less than T and hence the clusters—which are internally made rigid by effective couplings $\gg T$—will be essentially independent. Conversely, entropic effects will have been negligible for almost all decimations at scales $\Omega > T$. The mean long-distance correlation function for small T close to the critical point is thus

$$\overline{C}(x,T) \approx \bar{\mu}^2 K_{\Gamma_T}(x) \ . \qquad (3.132)$$

The long-distance behavior of $K_{\Gamma_T}(x)$ is dominated by the nearest singularity to the origin of the complex y plane of $\hat{K}_{\Gamma_T}(y)$. Since $\hat{k}'(y,\Gamma < \Gamma_T)$ [defined in Eq. (3.96)] is only singular for $y \leq -\delta^2 - \frac{\pi^2}{\Gamma_T^2}$, the nearest singularity to the origin in $\hat{K}_{\Gamma_T}(y)$ arises from the first term in Eq. (3.95) for $\delta > 0$ and the first term in Eq. (3.114) for $\delta < 0$, which are the *same*. Thus, at low temperatures, the correlation length ξ_T will be minus the inverse of the closest zero to the origin, y_T, of

$$\tau(y;\Gamma_T) = \delta + \Delta(y)\ \coth[\Gamma_T \Delta(y)] \ . \qquad (3.133)$$

Since the singularity in $\hat{K}_{\Gamma_T}$ at y_T is a simple pole, the decay will be a simple exponential

$$\overline{C}(x,T) \approx \bar{\mu}^2 \Xi_T(\delta) e^{-x/\xi_T} \ , \qquad (3.134)$$

with the coefficient Ξ_T a function of δ and T given by the residue of the pole in y at y_T from the first term in Eq. (3.95). We can evaluate $\xi_T(\delta)$ and $\Xi_T(\delta)$ in various limits.

1. Critical point

At the critical point, the dominant pole is at $\Delta = \frac{i\pi}{2\Gamma_T}$, yielding

$$\xi_T = -y_T^{-1} = \frac{4\Gamma_T^2}{\pi^2} \ . \qquad (3.135)$$

The coefficient

$$\Xi_T = C_c \Gamma_T^{2\phi-4} \ , \qquad (3.136)$$

with the numerical coefficient C_c obtainable from the behavior of the scaled function $\mathcal{A}(\tilde{\Gamma})$, Eq. (3.62), and its first derivative both evaluated at $\tilde{\Gamma} = \frac{i\pi}{2}$.

2. Disordered phase

In the disordered phase, the pole of y_T will move away from the origin as δ increases. For $\Gamma_T\delta \ll 1$, it will be near the critical y_T, Eq. (3.135), while for $\Gamma_T\delta \gg 1$, it approaches

$$\xi_T^{-1} \approx \delta^2 + \frac{\pi^2}{\Gamma_T^2} - \frac{2\pi^2}{\delta\Gamma_T^3}\left[1 + O\left(\frac{1}{\delta\Gamma_T}\right)\right] \ . \qquad (3.137)$$

Note that in addition to the pole in $\hat{K}_{\Gamma_T}(y)$ at y_T there is also a cut ending at $y = -\delta^2 - \frac{\pi^2}{\Gamma_T^2}$ which means that x/ξ_T must be much larger than $(\delta\Gamma_T)^3$ for the correlations to be dominated by the simple exponential decay. By analyzing the residue of the pole in the $\delta\Gamma_T \gg 1$ limit, we obtain that the coefficient in Eq. (3.134) is

$$\Xi_T \approx 2\pi^2 C_{1-}^2 \frac{\delta^{1-2\phi}}{\Gamma_T^3} e^{-\delta\Gamma_T} \qquad (3.138)$$

at low temperatures in the disordered phase.

3. Ordered phase

In the ordered phase the correlation length will diverge strongly at low temperatures due to the development of long-range order at zero temperature. The correlation length in the limit $|\delta|\Gamma_T \to \infty$ is found to be

$$\xi_T \approx \frac{e^{2\Gamma_T|\delta|}}{4\delta^2} \approx \frac{1}{4\delta^2}\left(\frac{\Omega_I}{T}\right)^{2|\delta|} \ , \qquad (3.139)$$

which diverges as a continuously variable power of T associated with the fixed line that controls the ordered phase, as discussed in Sec. IV A. The coefficient Ξ_T in this limit is

$$\Xi_T \approx 4|\delta|^{4-2\phi} C_A^2 \approx [N_A(\infty)]^2 \approx \frac{M_0^2}{\bar{\mu}^2} \ , \qquad (3.140)$$

as should be expected since the very long clusters which dominate at low T have magnetization which is $\pm$ their length times the mean spontaneous magnetization density. Thus, for $1/\delta^2 \ll x \ll \xi_T$ in the ordered phase, the mean correlations should be just $\approx M_0^2$.

D. Correlations in a magnetic field

We now briefly discuss the correlations in a small applied magnetic field H at zero temperature. We focus only on the mean correlations; again as in Sec. III B the mean *truncated* correlations, in which $\overline{\langle\sigma_i^z\rangle\langle\sigma_j^z\rangle}$ rather than $\overline{\langle\sigma_i^z\rangle}^2$ is subtracted from $\langle\sigma_i^z\sigma_j^z\rangle$, require more work.

Following the analysis of Sec. III B, we must compute the probability that spins i and $j = i + x > i$ are both active at the scale $\Gamma_H \approx \ln(1/H)$ beyond which the applied field dominates and aligns all the remaining active spins. The Laplace transform of this probability is

$$\text{LT Prob}\{\sigma_i \text{ and } \sigma_{i+x} \text{ active at } \Gamma_H\}$$

$$= \hat{K}_{\Gamma_H}(y) + \hat{K}^D_{\Gamma_H}(y) \ , \quad (3.141)$$

the first from the probability that the spins are active in the same cluster, the second $\hat{K}^D$ being the transform of the probability that they are active in *distinct* clusters. We have already studied $\hat{K}_\Gamma(y)$ in the previous subsection.

By summing up the probabilities that there are 0, 1, 2,..., ∞ intervening clusters between those in which i and j are active, one can readily derive an expression for $\hat{K}^D$:

$$\hat{K}^D_\Gamma(y) = \frac{\frac{1}{n_\Gamma}\left[\hat{g}(y;\Gamma)\right]^2 \hat{P}(\int, y;\Gamma)}{1-\hat{P}(\int,y;\Gamma)\hat{R}(\int,y;\Gamma)} \ . \tag{3.142}$$

For y small at fixed H, the second term in Eq. (3.141) dominates and one can show that

$$\hat{K}^D_{\Gamma_H}(y\to\infty) \approx \frac{N_A^2(\Gamma_H)}{y} \approx \frac{M^2(H)}{\bar{\mu}^2 y} \ , \tag{3.143}$$

so that, as expected, the mean long-distance correlations approach $\left(\overline{\langle\sigma_i^z\rangle}\right)^2$ at long distances. From an analogous analysis to that in Sec. III B, one can see that this $1/y$ pole is the only singularity at $y=0$; the mean correlations then decay exponentially towards $\overline{\langle\sigma^z\rangle}^2$ with a correlation length ξ_H given in terms of the nearest nonzero singularity to the origin, y_s, in the complex y plane of Eq. (3.95):

$$\xi_H = \frac{-1}{y_s(\Gamma_H)} \ . \tag{3.144}$$

As in the case of thermal correlations, in the scaling limit, the nearest singularity in both terms in Eq. (3.141) is again a simple pole at $y_{s\tau}$ that arises from the closest zero to the origin of

$$\tau(y;\Gamma_H) = \delta + \Delta(y)\coth\left[\Gamma_H\Delta(y)\right] \ . \tag{3.145}$$

Thus we see that the correlation length $\xi_H(\Gamma_H)$ has *identical functional form* to $\xi_T(\Gamma_T)$.

At the critical point, we therefore have

$$\xi_H \approx \frac{4\ln^2(1/H)}{\pi^2} \ , \tag{3.146}$$

while well into the disordered phase, for $\delta\Gamma_H \gg 1$,

$$\xi_H^{-1} \approx \delta^2 + \frac{\pi^2}{\Gamma_H^2} \ . \tag{3.147}$$

In the ordered phase, the behavior of the correlation length as $H\to 0$ reflects the development of long-range order, so that

$$\xi_H \approx \frac{e^{2\Gamma_H|\delta|}}{4\delta^2} \sim H^{-2|\delta|}, \tag{3.148}$$

i.e., a nonuniversal power law dependence on H reflecting that in $M(H)-M_0$, Eq. (3.58).

In Sec. IV D, the behavior of the magnetization when *both* H and T are nonzero is discussed briefly. One could also analyze the crossovers in the correlations in this case, but we have not carried this out.

IV. ORDERED AND DISORDERED PHASES AND PHASE DIAGRAMS

The RG flows of Sec. II can be used, for small $|\delta|$, all the way into the ordered or disordered phases, i.e., for $|\delta\Gamma| \gg 1$. Although this enables the scaling functions to be calculated exactly, one must be careful with the exchange of limits in using these to infer properties of the noncritical phases. In this section, we will focus on the behavior for *fixed* small $|\delta|$ in the limit of low energies, *not* strictly in the scaling regime, and one would therefore like to have a low-energy description of the weakly off-critical phases which can be "matched" onto the large $|\delta\Gamma|$ RG flows. This will also give us substantial extra insight into the behavior near the critical point, in particular the cause of the singularities in $M(H)$ and the low-temperature linear susceptibility $\chi(T)$.

A. Weakly ordered phase

We first study the weakly ordered phase as this is somewhat simpler conceptually. We thus focus on fixed small negative δ. In the ordered phase, one expects quantum fluctuations to be irrelevant at low energies so that the system should be approximately describable by a *classical* effective Hamiltonian. To see this, consider the behavior of the J and h distributions for $|\delta|\Gamma \gg 1$, corresponding to $\Omega \ll$ the crossover scale Ω_δ. In this limit, almost all the J's will be much larger than almost all the h's as $1/u_0 \approx 1/(2|\delta|)$ is the typical $\ln(\Omega/J)$ while $1/\tau_0 \approx \frac{1}{2|\delta|}e^{2|\delta|\Gamma}$ is the typical $\ln(\Omega/h)$. For $\Gamma \gg \Gamma_\delta \sim 1/|\delta|$, almost all the decimations are thus of *bonds* causing larger and larger clusters to form, with smaller and smaller h/Ω. Only very occasionally will a cluster be eliminated. Thus the natural classical limit which will obtain at low energies is that with *all remaining h's zero* and with nearest neighbor effective exchanges distributed as

$$\pi_-(J;\Omega) \approx \frac{\Theta(\Omega-J)}{z\Omega}\left(\frac{\Omega}{J}\right)^{1-1/z} \ , \tag{4.1}$$

with, from the large Γ limit of u_0, the exponent z given by

$$z \approx \frac{1}{2|\delta|} \tag{4.2}$$

for small $|\delta|$.

We expect corrections to Eq. (4.2) as $|\delta|$ grows due to corrections to scaling, effects of errors in the RG, etc., although these will not affect the critical scaling functions calculated in Sec. III. Nevertheless, the *form* of the distribution Eq. (4.1) should obtain in the limit of low energies. This can be readily seen by considering the decimation RG transformation which becomes trivial in the classical limit, as it just corresponds to cutting off the $J\in[\Omega-d\Omega,\Omega]$ part of the distribution and rescaling the remaining J's.

The family of distributions, Eq. (4.1), is trivially a *fixed line*, parametrized by z of this RG. The quantum fluctuations—roughly parametrized by τ_0—are *irrelevant* on this fixed line. Thus to a first approximation, we may ignore them. At zero temperature, the magnetization is then just M_0 sgn$|H|$ with $M_0 \sim |\delta|^{2-\phi}$ given by the moment per unit length of the clusters as set (up to corrections that are small for small $|\delta|$) by the critical scaling function Eq. (3.50).

At energy scale $\Omega \ll \Omega_\delta$, corresponding to length scales much longer than ξ, the density of clusters will be

$$n(\Omega) \sim \Omega^{\frac{1}{z}} \sim e^{\frac{-\Gamma}{z}} \quad (4.3)$$

as obtainable trivially from the distribution Eq. (4.1). Thus z is the exponent that relates energy scales and length scales $\ell(\Omega) = 1/n(\Omega)$ on the weakly ordered fixed line:[19]

$$\Omega \sim \ell(\Omega)^z \; . \quad (4.4)$$

At nonzero temperature, the behavior of the weakly ordered phase will be nontrivial due to competition between the thermal fluctuations and the exchange. Crudely, we can still stop the RG at $\Omega = \Omega_T = T$, as in Sec. III C, and consider the remaining spin clusters to be rigid but free. The linear susceptibility will then be

$$\chi \sim \frac{1}{T} n(\Omega = T)\overline{m^2(\Omega = T)} \quad (4.5)$$

in terms of the mean square moment of the remaining clusters. Since in the ordered phase $m(\Omega) \sim M_0 \ell(\Omega)$ [indeed one can show that the variations from this are only $O(\ell(\Omega)^{1/2})$ for $\Omega \to 0$], we see that

$$T\chi \sim M_0^2 \ell(\Omega = T) \; , \quad (4.6)$$

yielding, for small T,

$$\chi \sim \frac{|\delta|^{2-2\phi}}{T^{1+1/z}} \; , \quad (4.7)$$

i.e., a continuously variable power law with stronger than Curie divergence due to the buildup of correlations. In obtaining the prefactor of Eq. (4.7), we have used the coefficient relating the length to frequency scales as obtained from the critical scaling functions:

$$\ell_\Gamma \sim \frac{1}{\delta^2} e^{2|\delta|\Gamma} \; , \quad (4.8)$$

where the prefactor here is just the critical correlation length.

We thus see that the behavior of $\chi(T)$ as the weakly ordered fixed line is approached depends only on the properties of the fixed line. On the other hand, in order to obtain the singular *corrections* to $M(H)$ at zero temperature, we must analyze the irrelevant quantum fluctuations. The excess magnetization $M(H) - M_0$ in a very small field H can be estimated by noting how many clusters are eliminated between the scale Γ_H and $\Gamma = \infty$. Most of those eliminated in this range will be eliminated in the first factor of 2 or so in length scale, as beyond that the remaining transverse fields will have become even smaller due to the combining of clusters. Thus a rough estimate of the fraction of clusters eliminated between Γ_H and ∞ is the fraction eliminated as half of the bonds are eliminated. This is roughly

$$R_0(\Gamma_H)/P_0(\Gamma_H) \approx \frac{r_0(\Gamma_H)}{u_0(\Gamma_H)} \approx e^{-2|\delta|\Gamma_H} \; . \quad (4.9)$$

The excess magnetization for small H is thus of order $e^{-\Gamma_H/z} M_0$. But we must now be more careful in obtaining Γ_H or, equivalently, Ω_H. As in the critical region, we stop when $Hm(\Omega_H) \sim \Omega_H$. But now $m(\Omega_H) \sim M_0 \ell(\Omega_H) \sim M_0 \Omega_H^{-1/z}$ so that

$$\Omega_H \sim H^{\frac{1}{1+1/z}} \; , \quad (4.10)$$

yielding

$$M(H) - M_0 \sim H^{\frac{1}{1+z}} (\ln H)^x \; , \quad (4.11)$$

where we have allowed for unknown $\ln H$ multiplicative factors, which appear in the scaling function Eq. (3.58) and may also appear in the asymptotic (nonscaling) low-H limit of interest here; we have not computed these.

In the above discussion, we have ignored the difference between z and $1/2|\delta|$. Since the definition of δ is unambiguous only to leading order for δ small, we must generally allow for nonuniversal corrections so that

$$z = \frac{1}{2|\delta|} + O(1) \; . \quad (4.12)$$

The advantage of using z in physical expressions becomes clear when Eq. (4.11) is examined: the exponent in the excess magnetization is $2|\delta|$ to lowest order, but in general *differs* from $1/z$.

The relations between the exponents in $\chi(T)$, Eq. (4.7), and $M(H) - M_0$, Eq. (4.11), are properties of the weakly ordered fixed line and therefore should be exact. The discrepancy from the critical scaling function that occurs in Eq. (4.11) formally represents only corrections to scaling since the extra factor of $H^{\frac{-4\delta^2}{1+2|\delta|}}$ in Eq. (4.11) [with $z = 1/(2|\delta|)$] only becomes substantially different from unity when $\delta^2 \ln H \sim 1$, i.e., $\delta \ln H \sim 1/\gamma$ which is *not* in the scaling function Eqs. (3.50) and (3.58). Similarly, corrections from Eq. (4.12) will also not appear in scaling functions.

We end discussion of the weakly ordered phase by noting that the length-dependent stiffness Eq. (1.16) typically is

$$S_L \sim \frac{1}{L^z} \; , \quad (4.13)$$

being dominated by the weakest effective bond in a system of length L. As $-\delta$ is increased out of the critical region, z decreases so that the susceptibility $\chi(T)$ diverges more and more rapidly and S_L decays less rapidly. Beyond the lower Griffiths' point, where the smallest original J_i becomes larger than the largest original h_j, S_L approaches a nonzero constant, there is a gap in the spectrum, and $\chi(T)$ diverges exponentially rapidly as $T \to 0$. This is the more conventional *strongly ordered phase.* Its character can be qualitatively seen even in our simple approximate RG: In the strongly ordered phase, no spins are ever decimated and no very weak bonds are generated; the fixed point is then a classical *uniform* ordered Ising chain with no fluctuations. In this regime, $M(H)$ will have only essentially singular "droplet" singularities for small H, like those of the pure Ising system.[20]

B. Weakly disordered phase

At low energies in the disordered phase, almost all the effective bonds will be much weaker than almost all the effective transverse fields. Thus we guess that, asymptotically, the flows will approach a fixed line consisting of *uncoupled* clusters with a distribution of lengths ℓ_S, effective transverse fields h, and moments m. The distribution of these will be roughly, up to small modifications of δ, that given from the limit of the critical scaling functions.

Under the trivial RG transformation that in this limit just eliminates clusters with $h \in [\Omega - d\Omega, \Omega]$, a power law distribution of transverse fields is seen to be a fixed line:

$$\rho_+(h;\Omega) \approx \frac{\Theta(\Omega - h)}{z\Omega}\left(\frac{\Omega}{h}\right)^{1-1/z} , \tag{4.14}$$

parametrized by an exponent z with the number density of clusters $n(\Omega) \sim \Omega^{1/z}$, so that again z relates length and energy scales. This is exactly the form found from the $\delta\Gamma \gg 1$ limit of the critical scaling function with

$$z \approx \frac{1}{2\delta} , \tag{4.15}$$

a relation which, like that in the ordered phase discussed above, will have corrections of higher order in δ. The similarity between Eqs. (4.1) and (4.14) reflects the duality.

Unlike in the ordered phase, the clusters in the disordered phase will only occupy a small fraction of the length at low energies, while the bonds connecting them will grow in length, as $\ell_B \sim 1/n(\Omega)$. Nevertheless, the distribution of cluster lengths can be computed from the special solution: $T(y)/\tau_0$ is the Laplace transform of the probability distribution of lengths of clusters that are eliminated at scale Γ. By a saddle-point evaluation of the inverse transform, it can be seen that, for $\delta\Gamma \gg 1$,

$$\ell_S = \frac{\Gamma}{2\delta} + O\left(\frac{\Gamma}{\delta^3}\right)^{\frac{1}{2}} ; \tag{4.16}$$

$\ell_S(\Gamma)$ obeys a central limit theorem, and the long clusters hence have transverse fields of order $e^{-2\delta\ell_S}$.

The mean moment of clusters being eliminated can also be computed to be

$$\overline{m} \sim \Gamma\delta^{1-\phi} \sim \ell_S\delta^{2-\phi} . \tag{4.17}$$

We conjecture that, in fact, there will typically only be $\sqrt{\Gamma}$ variations in m around its mean, as for ℓ_S in Eq. (4.16). We have not computed the distribution of m directly in this limit, but at the critical point, the distribution of clusters with anomalously small h have $m \sim \ell_S \sim \beta = \ln(\Omega/h)$ with only $O(\sqrt{\beta})$ variations.[10] Since for δ small and positive these clusters are more likely to survive well into the disordered regime than typical ones, we expect similar behavior for the distribution of m in the weakly disordered phase. Physically, this arises from the composition of the anomalously long clusters which consist of many roughly independent more typical clusters that are joined together when many atypically strong bonds connecting them are decimated. A long tail of the cluster distribution of this type will occur, as discussed at the end of Appendix A, as soon as any fraction of the J's is decimated. Thus the power law tail for small effective h in Eq. (4.14) will already be well formed when $\delta\Gamma \sim 1$.

The form of Eq. (4.17) has a simple physical interpretation: The magnetization per unit length of the anomalous clusters that remains in the disordered phase at low energies is of order $\delta^{2-\phi}$ which means that they are like *typical* clusters that would occur for δ the same distance from criticality on the *ordered* side of the transition where the spontaneous magnetization density is $M_0 \sim |\delta|^{2-\phi}$. This is a consequence of the properties of the distribution of segments of length of order $\xi \sim \frac{1}{\delta^2}$.

The width of the distribution of the sum over *original* couplings $\Sigma = \sum_i(\ln h_i - \ln J_i)$ of segments of length ξ is the same order as its mean; thus some finite fraction of them for small δ "think" that they are in the *ordered* phase, and hence "want" to have magnetization density $M_0 \sim |\delta|^{2-\varphi}$ even though the system as a whole is in the disordered phases. If a large number of such strongly coupled segments occur sequentially, then an anomalously long, anomalously low h cluster is likely to form in that region. The probability of such a cluster with length ℓ will thus be roughly $e^{-c_s\ell/\xi}$, since Σ for each segment of length ξ will be roughly independent. From the probability $\sim e^{-2\delta\Gamma}$ of a cluster surviving from scale $\Gamma_\delta \sim 1/\delta$ (where the clusters and bonds have lengths $\sim \xi$) down to a much lower-energy scale Γ and the dependence of $\ell_S(\Gamma)$ or Γ, we see that indeed the distribution of the lengths of such long-surviving clusters is exactly of this form (with the coefficient $c_s = 4$ with length units of ℓ_V in which $\xi = 1/\delta^2$). This simple picture has consequences in higher dimensions as we shall argue in Sec. VII.

The thermodynamic properties of the weakly disordered phase can be found straightforwardly in terms of the above picture of the distribution of independent clusters.

At zero temperature, the magnetization will be singular for small H. Stopping the renormalization at Γ_H given by $Hm_{\Gamma_H} \sim e^{-\Gamma_H}$ and noting that the magnetization of the remaining clusters is dominated by those with $h \leq \Omega_H$, we get

$$M(H) \sim \delta^{3-\phi}H^{\frac{1}{z}}|\ln H|^{1+1/z} , \tag{4.18}$$

where the extra $[\ln(1/H)]^{1/z}$, which is not in the critical scaling function Eq. (3.60), is a consequence of the dependence of Γ_H on $m_{\Gamma_H} \sim \ln\Gamma_H$.

At nonzero temperature, the thermal fluctuations compete with the quantum fluctuations. The linear susceptibility can be estimated by stopping at scale $\Omega_T = T$ and treating the very weakly coupled remaining clusters as free spins with moments

$$m_{\Gamma_T} \sim \delta^{1-\phi}\ln\left(\frac{1}{T}\right) . \tag{4.19}$$

We thus find

$$\chi(T) \sim \frac{[\ln(1/T)]^2}{T^{1-1/z}}\delta^{4-2\phi}\ , \quad (4.20)$$

i.e., a slower than Curie divergence due to the "freezing out" of most of the clusters at low energies.

As for the ordered phase, it is instructive to consider what happens as δ is increased and the system moves away from criticality. At some point, the exponent z will decrease through 1. Beyond this point, the susceptibility at zero temperature will be finite and will no longer be dominated by the rare anomalously low-energy clusters but rather by the more typical ones. Nevertheless, there will still be a singular part of the magnetization at zero temperature,

$$M_{\text{sing}}(H) \sim H^{1/z}\ , \quad (4.21)$$

but perhaps with different logarithmic prefactors than Eq. (4.18). If beyond some (Griffiths') point all the original h_j's are larger than all the original J_i's, then in this regime there will be a gap above the ground state and $M(H)$ will be analytic; this is the *strongly disordered phase.*

1. Toy model

It is instructive pedagogically to consider a simple toy model of the disordered phase: The J_i's are taken to be 1 with probability p and 0 with probability $1-p$, while the h_i's are all equal to $\epsilon < 1$. This model can obviously not be ordered except at $p = 1$, but it nevertheless has all the principle thermodynamic properties of the weakly disordered phase discussed above—including the logarithmic factors. This can be seen by observing that the number density of connected segments—i.e., clusters—of length ℓ, is $\sim p^\ell$ while the splitting between the two lowest-energy states of the cluster—"mostly up" and "mostly down"—is of order ϵ^ℓ; i.e., the effective transverse field is of order ϵ^ℓ, and the moment m is of order ℓ. This corresponds to the weakly ordered phase discussed above with the exponent

$$z = \frac{\ln\epsilon}{\ln p}\ . \quad (4.22)$$

Since the computations involve only finite-size sections of the pure Ising model, the details are left to the reader [who will, unfortunately, not be able to produce a closed form expression for $M(H)$ due to difficulties with the pure Ising model].

C. Susceptibility and specific heat scaling

The scaling function of the linear susceptibility at low temperatures—a potentially measurable quantity—could be obtained in the critical region of small $|\delta|$ and low T by keeping track of the mean square magnetic moment of clusters, in addition to the mean moment that was studied in Sec. III A. We have not carried this out (except at $\delta = 0$; see Ref. 10), but there appear to be no difficulties of principle. The *form* of the scaling function can, however, immediately be deduced from the scaling of cluster moments and their number density:

$$\chi(T) \sim \frac{[\ln(1/T)]^{2\phi-2}}{T}\, X[\delta\ln(1/T)]\ , \quad (4.23)$$

with $X(\gamma)$ a scaling function. At *criticality* $\delta = 0$,

$$T\chi(T) \sim [\ln(1/T)]^{2\phi-2}\ , \quad (4.24)$$

while the *form* of the low-temperature limits for $|\delta|$ fixed but small can be obtained from the results of Secs. IV A and IV B; these correspond, up to unknown numerical prefactors, to the behavior of the scaling function $X(\gamma)$ as $\gamma \to \pm\infty$.

In the *disordered phase* as $T \to 0$, we have

$$T\chi \sim \delta^{4-2\phi}T^{2\delta}[\ln(1/T)]^2\ , \quad (4.25)$$

which vanishes at $T \to 0$. Conversely, in the *ordered phase* $\delta < 0$,

$$T\chi \sim T^{2\delta}(-\delta)^{2-2\phi}\ , \quad (4.26)$$

which *diverges* as $T \to 0$.

The zero-field specific heat $C_v(T)$ is obtained much more easily. At low temperatures, the entropy is just ln2 per free cluster, yielding an entropy density of simply

$$S(T) \approx \ln 2\ n(\Omega = T) \quad (4.27)$$

and hence

$$C_v(T) \approx \ln 2 \frac{\partial}{\partial \ln T} n(\Omega = T)\ . \quad (4.28)$$

Using

$$n \sim \frac{\delta^2}{\sinh^2(\Gamma\delta)}\ , \quad (4.29)$$

we have

$$C_v \sim \frac{1}{\Gamma_T^3}\left(\frac{\delta\Gamma_T}{\sinh(\delta\Gamma_T)}\right)^3 \cosh(\delta\Gamma_T)\ , \quad (4.30)$$

with

$$\Gamma_T = \ln(D_T/T)\ , \quad (4.31)$$

D_T being a nonuniversal energy scale.[21] At the *critical* point $\delta = 0$,

$$C_v \sim \frac{1}{\ln^3(D_T/T)}\ , \quad (4.32)$$

while for $\delta \neq 0$, the duality under $\delta \leftrightarrow -\delta$ is explicit and, for low T off critical,

$$C_v \sim |\delta|^3\left(\frac{T}{D_T}\right)^{2|\delta|}\left[1 + O\left(\frac{T}{D_T}\right)^{2|\delta|}\right]\ . \quad (4.33)$$

In general, in the weakly ordered or disordered phases, the $T^{2|\delta|}$ in Eq. (4.33) will be replaced by $T^{1/z}$. This can be seen to be a direct consequence of the scaling of lengths ℓ_T as $T^{1/z}$ and a hyperscaling form for the singular part of the free energy density,

$$f_{\text{sing}}(T) \sim \ell_T^{-(d-\theta)} , \qquad (4.34)$$

with $\theta = -\frac{1}{z}$ and $d = 1$.[22]

D. Scaling with nonzero H and T

If the magnetic field and temperature are *both* nonzero but small, then some information on the scaling behavior can still be obtained. Basically, one stops the renormalization at the smaller of Γ_T and Γ_H, i.e., the *larger* of the energy scales $\Omega_T = e^{-\Gamma_T} = T$ and

$$\Omega_H = e^{-\Gamma_H} \sim H m_{\Gamma_H} , \qquad (4.35)$$

with m_Γ a typical magnetic moment of a cluster at scale Γ, so that $m_\Gamma \sim (\ln\Gamma)^\phi$ at the critical point. It is important to note that Ω_H is only defined up to a multiplicative coefficient which depends on small-scale physics and is also ambiguous because of the distribution of cluster moments m. Conversely, Γ_H is well defined up to an *additive* constant of order unity. Even if the deviations of Γ_H from $\ln(\Omega_I/H)$ are ignored, Γ_H still only has corrections that are

$$O(\delta \ln H) \text{ and } O(\ln|\ln H|) , \qquad (4.36)$$

so that in the scaling function for $M(H, T = 0)$, Eq. (3.50), which is a function of the combination $\delta\Gamma_H$, the corrections to $\delta\Gamma_H$ are negligible. (They do, however, affect the off-critical behavior at asymptotically low fields as discussed above in Secs. IV A and IV B.)

If $\Gamma_H \gg \Gamma_T$, then the thermal effects will be negligible and

$$M(H,T) \approx M(H, T = 0) . \qquad (4.37)$$

In fact, as long as

$$\Omega_H \gg T , \qquad (4.38)$$

this will be true. This corresponds just to $\Gamma_H - \Gamma_T \gg 1$, a much weaker condition than $\Gamma_H \gg \Gamma_T$. Note that for Eq. (4.38) the ambiguous coefficient in Eq. (4.35) is clearly unimportant.

The other limit

$$\Omega_H \ll T \qquad (4.39)$$

is also simple. In this case, the magnetic energy mH of the remaining clusters at energy scale $\Omega = T$ will be much less than T, and so they will exhibit a Curie susceptibility and

$$M \approx H\chi(T) , \qquad (4.40)$$

with the susceptibility $\chi(T)$ having the forms discussed in Sec. IV C.

Strictly speaking, Eqs. (4.37) and (4.40) imply that scaling functions with the arguments $\delta\Gamma_T$ and $\delta\Gamma_H$ will have nonanalyticities at $\delta\Gamma_T = \delta\Gamma_H$. This is because the scale of the crossover from field dominated to temperature dominated is of $O(1)$ in Γ, and hence $O(\delta)$ and therefore formally negligible in the scaling variable $\delta\Gamma$. This sharp crossover will be rounded in a way that is technically not in the critical scaling function but can be analyzed.

First note that the forms of $M(H, T = 0)$ from the scaling function, and $M(H,T)$ from the linear susceptibility, Eq. (4.40), valid for $\Omega_H \ll T$, are the same, up to a constant multiple, at $\Omega_H = \Omega_T$ if Eq. (4.35) is used for Ω_H. Therefore, the magnetization only changes by a constant factor in the crossover regime. But now we observe that as long as H, T, and $|\delta|$ are all small, almost all the remaining couplings at $\Omega = T$ will be much smaller than T so that the picture of *independent spin clusters* interacting only with the applied field H is valid. If the distribution of magnetic moments of these clusters,

$$\hat{R}_M(m; \Gamma) \equiv \int d\beta \int d\ell \; R(\beta, \ell, m; \Gamma) , \qquad (4.41)$$

were known, this would immediately yield the crossover as Ω_H goes from $\ll T$ to $\gg T$ via

$$M(H,T) \approx n_{\Gamma_T} \int dm \; \hat{R}(m;\Gamma)\bar{\mu} m \; \tanh\frac{\bar{\mu}Hm}{T} , \qquad (4.42)$$

with $\bar{\mu}$ the nonuniversal coefficient from suppression of magnetic moments on small scales that enters, e.g., Eq. (3.50). At this point, $\hat{R}_M(m)$ is only known *at* the critical point in terms of the solution of a nonlinear ODE as discussed in the Appendix of Ref. 10. But off criticality, information on it should also be obtainable, at least numerically. It is readily seen that Eq. (4.42) has the right limits for $\Omega_H \ll T$ and $\Omega_H \gg T$.

We note that any experiments are likely to involve this crossover regime and indeed will probably probe only slightly into the regimes where Γ_H and Γ_T differ substantially.

E. Phase diagram

A qualitative phase diagram as a function of Δ_h, T and H is shown in Fig. 4, showing the behavior as $T, H \to 0$ in the various regimes. Note that for $|\delta|$ small and T and H small, but not so small that $|\delta \ln T|$ or $|\delta \ln H|$ is large, the behavior will be dominated by the quantum critical point, as shown in the figure.

The schematic RG flows in the critical region are shown in Fig. 5, indicating the weakly ordered and weakly disordered fixed lines. In order to show even schematically the Griffiths' points at which these lines end, it would be necessary to include, on another axis, some information about the lower and upper bounds (if any) on the J and h distributions. We have not indicated this in Fig. 5. Note that whether or not the Griffiths' points that separate the weakly from the strongly ordered or disordered phases are considered true phase transitions is largely a

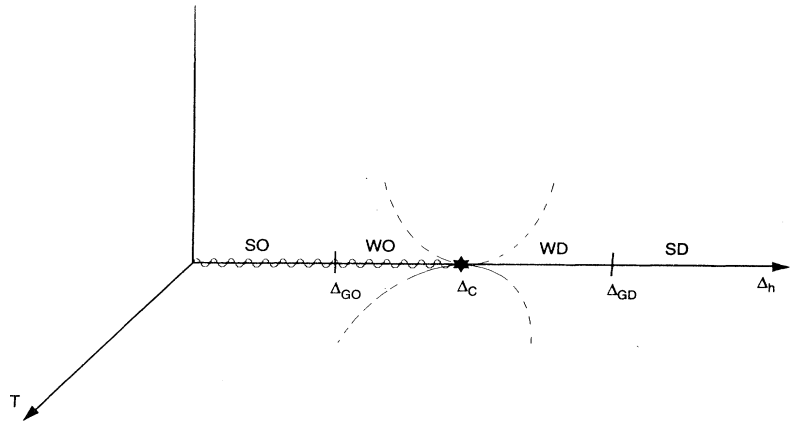

FIG. 4. A schematic phase diagram as a function of T, H, and $\Delta_h \equiv \overline{\ln h}$ for a system with a nonzero lower bound and a finite upper bound for the distribution of couplings $\{h_i\}$ and $\{J_i\}$. On the line $H = T = 0$, there is a phase transition at a critical $\Delta_h = \Delta_c$, from a paramagnetic disordered phase to a ferromagnetic ordered phase that exists only on the wavy line shown. Near the transition are weakly disordered, "WD," and weakly ordered, "WO," phases whose properties are dominated by anomalous rare regions of the sample. Further away from the transition, these become, respectively, more conventional strongly disordered, "SD," and strongly ordered, "SO," phases that behave like their pure system counterparts. The points Δ_{GD} and Δ_{GO} separating the WD from the SD and the WO from the SO phases are not real transitions, but rather "Griffiths'" points, were some rare regions start to behave as if they are in the *opposite* (ordered vs disordered) phase. As an applied field H tends to zero at zero temperature, the magnetization is analytic for SD, vanishes in a singular way for WD, tends to a spontaneous magnetization for WO with, and for SO without, a power law singularity. As $T \to 0$, the linear susceptibility $\chi(T)$ is smooth for SD, has a sub-Curie power law T dependence for WD, but diverges in a range $\Delta_c < \Delta_h < \Delta_\chi$ of the paramagnetic phase. For WO, $\chi(T)$ diverges more rapidly than $1/T$ and it diverges exponentially for SO. In the region "QC," roughly bounded by the dotted lines, the behavior will be dominated by the quantum critical point.

matter of taste. There is certainly no broken symmetry associated with them, but the properties change enough that perhaps, like the Kosterlitz-Thouless transition of the classical 2D XY model, they might be considered actual transitions.

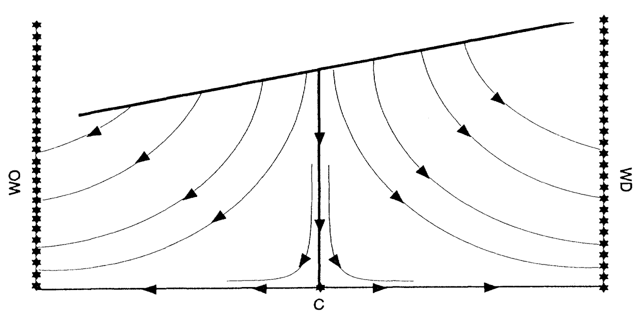

FIG. 5. Schematic renormalization-group flows indicating the critical fixed point C and the weakly disordered, "WD," and weakly ordered, "WO," fixed lines along which the exponents (e.g., z that relates time and length scales) vary continuously, increasing to infinity at the bottom of the lines. The Griffiths' points and the strongly ordered and strongly disordered regimes are not shown. The initial physical line is shown bold with the transverse fields increasing to the right.

V. END-POINT MAGNETIZATION

So far, we have concentrated entirely on bulk properties of random Ising chains, i.e., on the behavior of spins far from free ends. In contrast, McCoy was able to calculate the properties of the first spin σ_1 in a semi-infinite $i \in (1,\infty)$ chain as a function of a (positive) field H_1, applied only to the boundary spin:

$$\mathcal{H} = -\sum_{i=1}^{\infty}(J_i\sigma_i^z\sigma_{i+1}^z + h_i\sigma_i^x) - H_1\sigma_1^z \ . \qquad (5.1)$$

We can also use our methods to compute the end-point magnetization $M_1 \equiv \langle\sigma_1^z\rangle$ and thereby compare with McCoy's exact results. Since the approximations made will be the same as those in the bulk, the resulting agreement in the scaling limit provides strong support for the claim that the bulk properties we have obtained are also exact in the scaling limit. We will first focus on the mean $\overline{M}_1(H_1)$ and then discuss its distribution in the limit of $H_1 \to 0$ in the ordered phase, i.e., the distribution of the spontaneous end-point magnetization.

A. Mean end-point magnetization

Up to a multiplicative factor of order unity from the small-scale high-energy physics, the mean end-point magnetization is simply given by the probability that σ_1 survives to scale

$$\Gamma_E \equiv \ln(\Omega_I/H_1) \ . \tag{5.2}$$

This can be found by analyzing $E(\beta;\Gamma)d\beta$, the probability that σ_1 survives to scale Γ and is in a cluster with $\ln(\Omega/h) = \beta$. Since the end cluster can only join with clusters on its right, this obeys a somewhat different, but related, RG flow equation to $\int_0^\infty G(\beta,x;\Gamma)dx$:

$$\frac{\partial E(\beta)}{\partial \Gamma} = \frac{\partial E}{\partial \beta} - P_0 E + P_0 E \overset{\otimes}{\beta} R \ , \tag{5.3}$$

the first term arising as usual from the redefinition of β and elimination of the end cluster with $\beta = 0$, and the second and third terms from the combining of the end cluster with the one to its right.

Note that the flow equation for the *conditional* distribution of β of the end cluster if it survives, $E(\beta)/\left[\int d\beta' E(\beta')\right]$, is *identical* to that for $R(\beta)$ from Eq. (2.15); this is a reflection of the *independence* of the renormalized bonds and the equivalence of the end spin to a bulk spin which happens to be next to a bond with $J = 0$, i.e., $\zeta = \infty$. Of course, the survival probability of such a spin depends on its anomalous neighbor, as will be reflected in the difference between $\overline{M}_1$ and $\overline{M}$, but *if* it survives, it will be in a typical cluster.

A special solution to Eq. (5.3) can be guessed, analogous to those for the bulk problem:

$$E(\beta,\Gamma) = \bar{E}(\Gamma)\tau_0 e^{-\beta\tau_0} \ , \tag{5.4}$$

with

$$\frac{\partial \bar{E}}{\partial \Gamma} = -\bar{E}\tau_0 \ , \tag{5.5}$$

yielding

$$\bar{E}(\Gamma) = \int_0^\infty d\beta E(\beta;\Gamma) = \frac{C_E}{e^{2\delta\Gamma}-1} \propto u_0(\Gamma) \ . \tag{5.6}$$

To give the correct δ-independent behavior for $\delta\Gamma \ll 1$, we must have the coefficient $C_E \propto \delta$, therefore, up to an unknown nonuniversal coefficient

$$\bar{E}(\Gamma) = \frac{2\delta}{e^{2\delta\Gamma}-1} = u_0(\Gamma) \ . \tag{5.7}$$

As for bulk quantities (see Appendix A), convergence towards this special solution for large Γ can be shown.

Equations (5.2), (5.4), and (5.7) yield a scaling function for the *mean end-point magnetization* as a function of the field applied to the end spin, for δ and $1/\ln(1/H_1)$ small,

$$\overline{M}_1(H_1) \approx \frac{\bar{\mu}_1}{[\ln(D_1/H_1)]}\left[\frac{2\delta\ln(D_1/H_1)}{\left(\frac{D_1}{H_1}\right)^{2\delta}-1}\right] \ , \tag{5.8}$$

where $\bar{\mu}_1$ is a nonuniversal suppression of the mean magnetic moment of the first spin due to small-scale physics, and $D_1 \sim \Omega_I$ is a nonuniversal magnetic energy scale. Note that Eq. (5.8) is explicitly in scaling form with the quantity in square brackets just a function of $\delta\ln|D_1/H_1|$.

In the ordered phase $\delta < 0$, the spontaneous end-point magnetization is

$$\overline{M}_1(H_1 \to 0^+) \approx 2\bar{\mu}_1|\delta|^{\beta_1} \ , \tag{5.9}$$

with the exponent

$$\beta_1 = 1 \tag{5.10}$$

and $H_1^{2|\delta|}$ corrections to Eq. (5.7) for small H_1. At the critical point,

$$\overline{M}_1(H_1) \approx \frac{\bar{\mu}_1}{\ln(D_1/H_1)} \ , \tag{5.11}$$

while in the disordered phase, for small H_1,

$$\overline{M}_1(H_1) \approx \bar{\mu}_1 2\delta\left(\frac{H_1}{D_1}\right)^{2\delta} \ , \tag{5.12}$$

i.e., a power law dependence of the same form as the *bulk* $M(H)$, although with a different prefactor.

The scaling form for the end-point magnetization can also be extracted from McCoy's exact calculations.[3] McCoy restricts consideration to a specific form of the distributions of $\{J_i\}$ and $\{h_i\}$, the former of the form Eq. (4.1), but with z very small, and the latter a δ function. He defines a dimensionless measure of the distance to the critical point δ, which corresponds *exactly* to our general definition Eq. (1.36) for these distributions. The scaling form for $\overline{M}_1(H_1)$ extracted from Eqs. (2.15) and (2.19) in Ref. 3 is *identical* to our Eq. (5.8) but with a specific form for the nonuniversal scale factors D_1 and $\bar{\mu}_1$. As predicted, however, with the correct definition of δ there is *no* nonuniversal coefficient in the scaling combination $\delta\ln(D_1/H_1)$.[23]

The agreement between McCoy's exact results and those obtained by our seemingly questionable methods provides strong support for the validity of the approximations made in this paper.

B. Distribution of end-point magnetization

Further support for our methods is provided by the *distribution* of the spontaneous end-point magnetization

$$M_{1,0} = \lim_{H_1\to 0^+} M_1(H_1) \tag{5.13}$$

in the ordered phase. More generally, we consider the distribution of $M_1(H_1)$ for small H_1 and small $|\delta|$. Since we expect M_1 to be very broadly distributed in this regime, we study

$$\Lambda_1 \equiv \ln(1/M_1) \ ; \tag{5.14}$$

we will later consider how much information the scaling limit of the distribution of Λ_1 gives about the distribution of M_1 itself.

If the end-point spin remains active until scale Γ_E, then M_1 will be of order unity and such cases will dominate the mean $\overline{M}_1$ computed above. Since $\overline{M}_1$ is small in the critical region, we know that most boundary spins do not remain active down to low-energy scales. But for nonzero H_1 (or $H_1 = 0$ but $\delta < 0$) *some* spins in the bulk of the sample will remain active all the way to scale Γ_E and hence have expectations of order unity. Since σ_1 is correlated to the other spins through the intervening effective bonds, M_1 will be of roughly the same magnitude as the correlation between σ_1 and the leftmost spin σ_A that remains active at Γ_E, i.e.,

$$M_1 \sim \langle \sigma_1 \sigma_A \rangle \ , \tag{5.15}$$

where for convenience we drop the z superscripts of σ_i's. Typically, σ_A will be *far* from the boundary and M_1 will be very small.

Let us first consider a simple case in which, after the scale Γ_1 at which the cluster $S_1(\Gamma_1)$ containing σ_1 is eliminated, the next leftmost cluster $S_2(\Gamma_1)$ remains active until Γ_E; in this case, σ_A will be the first spin in $S_2(\Gamma_1)$. The correlation $\langle \sigma_1 \sigma_A \rangle$ can then be estimated perturbatively from the effective bond $\tilde{J}_1(\Gamma_1)$ which couples $S_1(\Gamma_1)$ and $S_2(\Gamma_1)$ at the scale Γ_1. Since until a cluster is eliminated each active spin in the cluster is roughly equal to the effective cluster spin variable, e.g., $\tilde{\sigma}_\alpha(\Gamma_1)$, of the cluster $S_\alpha(\Gamma_1)$, we have for the simple case above

$$\langle \sigma_1 \sigma_A \rangle \ \sim \ \langle \tilde{\sigma}_1(\Gamma_1) \tilde{\sigma}_2(\Gamma_2) \rangle \ \sim \frac{\tilde{J}_1(\Gamma_1)}{\tilde{h}_1(\Gamma_1)} \tag{5.16}$$

from a simple perturbative calculation. But since the elimination of S_1 at Γ_1 implies $\tilde{h}_1 = \Omega_1 = \Omega_I e^{-\Gamma_1}$, we see that

$$\langle \sigma_1 \sigma_A \rangle \ \sim e^{-\zeta_1}, \tag{5.17}$$

where

$$\zeta_1 = \ln[\Omega_1 / \tilde{J}_1(\Gamma_1)] \tag{5.18}$$

is distributed according to the distribution $P(\zeta_1; \Gamma_1)$.

One can now guess how to generalize to the cases where not only is $S_1(\Gamma_1)$ eliminated at $\Gamma_1 < \Gamma_E$, but the cluster $S_2(\Gamma_2)$ is also eliminated at a later scale $\Gamma_2 < \Gamma_E$. The cluster $S_2(\Gamma_2)$ consists of $S_2(\Gamma_1)$ and possibly extra spin clusters to the right of this that have been joined to it between scales Γ_1 and Γ_2. The correlation of active spins in $S_2(\Gamma_2)$ to active spins in the next leftmost cluster $S_3(\Gamma_2)$ that is active beyond Γ_2 is, by an argument analogous to that above, of order $\tilde{J}_2(\Gamma_2)/\Omega_2 = e^{-\zeta_2}$ with ζ_2 distributed as $P(\zeta_2; \Gamma_2)$.

Hence, the correlation of σ_1 to active spins in $S_3(\Gamma_2)$ is of order

$$\begin{aligned} \langle \sigma_1 \tilde{\sigma}_3(\Gamma_2) \rangle &\sim \langle \tilde{\sigma}_1(\Gamma_1) \tilde{\sigma}_2(\Gamma_1) \rangle \langle \tilde{\sigma}_2(\Gamma_2) \tilde{\sigma}_3(\Gamma_2) \rangle \\ &\sim e^{-\zeta_1} e^{-\zeta_2} \ . \end{aligned} \tag{5.19}$$

In general, by iterating this procedure, we see that the correlation between σ_1 and the leftmost spin σ_A (or indeed any other active spin) of the leftmost cluster $S_{n_A}(\Gamma_E)$ remaining at Γ_E will be

$$\langle \sigma_1 \sigma_A \rangle \sim \ \exp\left[- \sum_{\alpha=1}^{n_A - 1} \zeta_\alpha \right] \ , \tag{5.20}$$

where the leftmost cluster has been eliminated $n_A - 1$ times between the initial scale $\Gamma_I = 0$ and the final scale Γ_E. This yields

$$\Lambda_1 = \sum_{\alpha=1}^{n_A - 1} \zeta_\alpha(\Gamma_\alpha) \ , \tag{5.21}$$

where we have explicitly noted the elimination scales Γ_α at which the logarithmic bond strengths ζ_α occur.

The probability that the leftmost remaining cluster $S_\alpha(\Gamma_\alpha)$ is eliminated at Γ_α is, from the independence of remaining transverse fields, simply $R_0(\Gamma_\alpha) d\Gamma_\alpha$; in the scaling limit $R_0(\Gamma_\alpha) = \tau_0(\Gamma_\alpha)$ given by Eq. (2.72). In order to obtain the distribution of the sum Eq. (5.21), with n_A itself being a random variable that is *not* independent of the others, it is useful to define a set of infinitesimal *independent* random variables $\{d\hat{\zeta}_\Gamma\}$ with distributions

$$d\hat{\zeta}_\Gamma = \zeta \ \text{ with probability } \ P(\zeta; \Gamma) R_0(\Gamma) d\zeta d\Gamma$$

and

$$d\hat{\zeta}_\Gamma = 0 \ \text{ with probability } \ 1 - R_0(\Gamma) d\Gamma \ . \tag{5.22}$$

Then

$$\Lambda_1 = \int_{\Gamma_I}^{\Gamma_E} d\hat{\zeta}_\Gamma \ , \tag{5.23}$$

with $\Gamma_I (= 0)$ the initial scale, and we can compute the Laplace transform $\hat{\mathcal{L}}(z)$ of the distribution $\mathcal{L}(\Lambda_1) d\Lambda_1$ since Λ_1 is the sum of the independent random variables. In terms of the Laplace transform of the distribution of $d\hat{\zeta}_\Gamma$ which is

$$\hat{d}(z; \Gamma) = 1 + R_0(\Gamma) d\Gamma [\hat{\hat{P}}(z; \Gamma) - 1] \ , \tag{5.24}$$

with

$$\hat{\hat{P}}(z; \Gamma) \equiv \int_0^\infty d\zeta e^{-z\zeta} P(\zeta; \Gamma) \ , \tag{5.25}$$

we have

$$\begin{aligned} \hat{\mathcal{L}}(z) &= \prod_{\Gamma = \Gamma_I}^{\Gamma_E} \hat{d}(z; \Gamma) \\ &= \exp\left(\int_{\Gamma_I}^{\Gamma_E} d\Gamma R_0(\Gamma) \left[\hat{\hat{P}}(z; \Gamma) - 1 \right] \right) \ . \end{aligned} \tag{5.26}$$

From the special solution for $P(\zeta; \Gamma) = u_0(\Gamma) e^{-\zeta u_0(\Gamma)}$, the Laplace transform in ζ is

$$\hat{\hat{P}}(z; \Gamma) = \frac{u_0(\Gamma)}{z + u_0(\Gamma)} \ , \tag{5.27}$$

and using $\frac{\partial u_0}{\partial \Gamma} = -u_0 \tau_0$, we obtain

$$\hat{\mathcal{L}}(z) = \frac{1 + z/u_0(\Gamma_I)}{1 + z/u_0(\Gamma_E)} \ . \tag{5.28}$$

In the scaling limit, we expect Λ_1 to be large which is controlled by z small so that the initial $u_0(\Gamma_I) \gg z$, while $u_0(\Gamma_E)$ will be small. We then have

$$\hat{\mathcal{L}}(z) \approx \frac{u_0(\Gamma_E)}{z + u_0(\Gamma_E)} , \qquad (5.29)$$

corresponding to

$$\mathcal{L}(\Lambda_1) \approx u_0(\Gamma_E) e^{-\Lambda_1 u_0(\Gamma_E)} \Theta(\Lambda_1) . \qquad (5.30)$$

Before discussing the result Eq. (5.30), we must examine the approximations made. The correlations between an individual active spin and the effective spin of the cluster in which it is active are reduced at low-energy scales by a (random) factor of order unity due to nonuniversal high-energy physics. This will *add* a factor of order unity to Λ_1. Concomitantly, the scaling form will *not* be valid for Λ_1 of order unity, as in this limit, the high-energy physics embodied in the $u_0(\Gamma_I)$ factor in Eq. (5.28) and the approximations made will affect the results. Thus, the scaling distribution Eq. (5.30) is valid in the limit $\Lambda_1 \to \infty$, $u_0(\Gamma_E) \to 0$, and $\Lambda_1 u_0(\Gamma_E) \to$ any constant. It corresponds, in a looser sense, roughly to the power law distribution of M_1:

$$\text{Prob}[M_1(H_1)] \approx \frac{dM_1}{M_1} \Theta(\mu_1' - M_1) \frac{u_0[\ln(D_1/H_1)]}{\left(\frac{M_1}{\mu_1'}\right)^{u_0[\ln(D_1/H_1)]}} , \qquad (5.31)$$

valid essentially for $M_1 \ll \mu_1'$ with μ_1' a nonuniversal scale factor.

In spite of these limitations, the *form* of the mean $\overline{M}_1$ can be extracted, up to a nonuniversal prefactor, from Eq. (5.31), yielding

$$\overline{M}_1(H_1) \sim u_0(\Gamma_E) \qquad (5.32)$$

for small u_0 which is of identical form to that derived above Eq. (5.8) by considering only the active spins. The domination of the mean magnetization by the active spins that we have assumed throughout this paper is thus verified explicitly by the knowledge of the form of distribution Eq. (5.31). Higher moments of M_1 can also be derived, roughly, from Eq. (5.31); however, they will all be dominated by values of Λ_1 of order unity and hence all be of order $u_0(\Gamma_E)$. The ratios of the moments will *not* be given correctly as they are all dominated by the large M_1 limit in which Eq. (5.31) is not valid.

McCoy[3] calculated all the integer moments of the spontaneous end-point magnetization $M_{1,0}$ in the ordered phase. Since these calculations were performed with a subtle exchange of limits, he did not obtain the exact distribution. But from his results he extracted a form of the distribution for small negative δ which is, up to an $O(1)$ coefficient, *identical* in the limit of small M_1 to that of Eq. (5.31) with $u_0(\Gamma_E = \infty) = 2|\delta|$ (as it should be). Because of information about the large M_1 behavior contained in the moments, McCoy also obtained a smooth multiplicative cutoff function for large M_1, which goes beyond the simple sharp cutoff used in Eq. (5.31).

Nevertheless, the striking agreement of our universal results with McCoy's exact results via the computation of different quantities (the moments) by completely different methods again supports the claim that the results of the present paper are exact in the scaling limit.

VI. JUSTIFICATION OF RESULTS AND PROSPECTS

So far, in this paper, we have presented various results from the simple RG transformation and claimed that many of them are exact in the critical region in which one might expect universal behavior. In Appendix A, issues of convergence towards the guessed special solutions Eqs. (2.43) and (2.44) of the RG flow equations are dealt with and found *not* to give rise to problems for well-behaved initial distributions. In this section, we discuss the thornier problems of potential difficulties with the RG approximation itself. We must note, however, that at this point, the best evidence for the *validity* of the approximations is the exact agreement of the scaling function of the end-point magnetization and the distribution of the spontaneous end-point magnetization of a semi-infinite chain, with McCoy's exact results.[3] Nevertheless, one might worry that even if the approximations made for the end-point magnetization seem to be the same as those for the bulk, there could be extra problems with the latter. Thus it is useful to consider the approximations made more critically.

The presence or development of correlations in nearby couplings in the early stages of renormalization is shown to be unimportant in Appendix D. Potentially more problematic are the effects of spin renormalization factors that we have not taken into account and the effects of "bad" decimations in which the perturbative approximation is not good. We will later show that the effects of the bad decimations, in particular their *lack* of effects on the typical low-energy scale effective couplings, can be derived from a transfer matrix representation of the spin chain. But first we consider spin renormalizations.

A. Spin renormalizations

In pure systems, the spin renormalizations that occur at each stage of, say, a momentum shell RG transformation are what give rise collectively to nontrivial power law decay at critical points. But we have argued that in our system these factors only give rise to a finite suppression of the moments of the spins that are still active at low energies. This can be seen to be correct from the structure of the perturbative RG.

When two spin clusters are combined together to form a larger cluster, the effective spin operators of the new cluster [$\{\tilde{\sigma}\}$ in Eqs. (1.18) and (1.22)] are not, in general, simply related to the spin operators of the two clusters from which it forms. In particular, there will be perturbative suppression of the overlap of these operators arising from matrix elements between the ground states of the new cluster of the old spin operators $\{\sigma_1\}$. Perturbatively, these give rise to suppression factors of order $1 - c\frac{h_{1,2}}{J_1}$ when the strong bond $J_1 = \Omega$ is decimated. In the bad cases when h_1 or h_2 is close to J_1, the reduction factor will be of order unity. However, since at low energies $h_{1,2} \ll \Omega$ almost always, this will only occur rarely.

Indeed, the logarithm of the geometric mean moment suppression factor from decimations of a given scale can easily be seen to be of order the probability that either h_1 or h_2 is a substantial fraction of J_1; this is of order $R_0(\Gamma) \approx \tau_0(\Gamma)$, which is just the probability of a bad decimation. But recall that at the critical point $\tau_0 \sim 1/\Gamma$ for large Γ and therefore bad decimations become increasingly rare at low energies. Since the probability that a bond is decimated as $\Gamma \to \Gamma + d\Gamma$ is $u_0(\Gamma)d\Gamma$, the total geometric mean suppression factor is of order

$$\mu \sim \exp\left(-c\int^{\infty} u_0\tau_0 d\Gamma\right) . \tag{6.1}$$

Both at the critical point and off critical, we see that the integral over Γ in Eq. (6.1) converges, yielding a total suppression factor μ of order unity, arising primarily from small-scale high-energy physics. This is equivalent to the observation that a given spin is likely to be involved in only a *finite* total number of bad decimations. Thus the effects of spin renormalization do not appear to cause problems.

Note, however, that for weak randomness the crossover length scale ℓ_V [Eq. (1.31)] is long and there will be substantial suppression and smearing of the moments of spin clusters out to scales $\sim \ell_V$ due to the critical fluctuations of the pure system. The crossover from the weak to strong randomness regimes is thus clearly *not* accessible by the present methods, except qualitatively.

B. Effects of bad decimations

Physically the main reason that rare bad decimations do not affect the qualitative aspects of the simple RG is because even badly formed spin clusters will have two closely spaced lowest-energy eigenstates that correspond roughly to the even and odd combinations of the up- and down-spin clusters. Nevertheless, one might expect that the deviations of the splitting of these levels from the naive value of twice the effective field $\tilde{h}$ on the cluster, would invalidate some of the more quantitative predictions.

In particular, one might doubt the claim that the definition of δ in terms of the initial distributions Eq. (1.36) yields the exponents that control the weakly ordered and disordered phases *without* any nonuniversal prefactor and the concomitant claimed absence of nonuniversal prefactors of $\delta\Gamma$ in scaling functions.

In Sec. II D, we saw that these features arose from the behavior of products of many original h_i/J_i that appear in effective fields at low energies. Therefore, the exact identification of δ can only be correct if the errors made in bad decimations are somehow canceled almost exactly at lower-energy scales. But, in general, one would expect that a better treatment of the bad decimations would alter the distributions of the effective couplings at intermediate scales—and introduce short-range correlations in them—both of which would then, like the general perturbations discussed in Appendix A, lead to some change in the low-energy parameter δ. For many of the random antiferromagnetic spin chains discussed in Ref. 10, this will indeed be the case. But the exact solvability of the transverse-field Ising system, in particular its equivalence to a free fermion system, yields severe extra constraints that we now briefly discuss; these are enough to yield δ that is exactly given by Eq. (1.36).

C. Transfer matrices

Shankar and Murthy[4] have analyzed the McCoy-Wu model by transfer matrix techniques, transferring in the random direction which corresponds to the space direction for the quantum transverse-field chain. This corresponds to writing the Hamiltonian as a sum over local terms quadratic in fermion operators which are related, via nonlocal ordering operators, to σ_j^z and σ_j^x. The full transfer matrix is then factorizable into an outer product of 2×2 transfer matrices at different frequencies that correspond to the one-particle energies of the free fermion system. Although spin correlations are hard to obtain due to the nonlocal ordering operators, the zero-field thermodynamics and correlations of, e.g., transverse spin components σ_j^x can be obtained at least in principle.[4]

In Appendix E, we show that the transfer matrices can be cast in a form suitable for our decimation RG. The decimation procedure then corresponds to multiplying the frequency-dependent transfer matrices in a nontrivial order: When, e.g., a bond is decimated, the corresponding transfer matrix is multiplied by its neighboring matrices that correspond to the transverse fields on either ends of the bond. This order of multiplication keeps the low-frequency behavior of interest remarkably simple; indeed for good decimations the form of the product matrix is essentially exactly that corresponding to an effective field on the resulting spin cluster.

Rather remarkably, as shown in Appendix E for a particular case, it appears that errors that occur due to bad decimations *cancel* out of both the effective couplings and the form of the low-frequency product transfer matrices at much lower-energy scales. We conjecture that this is true generally.

It is this remarkable property that causes the form of the low-energy spectrum to be very well described in terms of effective couplings derived from the naive approximate RG. The agreement of the *exponent* 2δ of the mean end-point magnetization in the weakly ordered and disordered phases between McCoy's exact results and that obtained from our approximations Eq. (5.8) provides strong support for this conjecture. We leave its possible more general verification for future work.

Recently, Mikheev[18] has developed an exact RG treatment from the "frequency" dependence of the transfer matrices by a quite different approach. The qualitative behavior that he finds is similar to that found here—in particular the exponent ν and the structure near criticality in terms of locally ordered and disordered regions. Although spin correlations and much of the more detailed structure found here have not been obtainable, at least so far, from Mikheev's method, the combination of this with our approach holds promise for further progress.

The behavior of spin correlation functions is much more difficult to extract from fermions due to the ordering operators that relate spins to fermions. Nevertheless, it appears that the approximate ground state wave functions that correspond to our picture of spin clusters may be approximate eigenstates of certain combinations of the ordering operators. If this is the case, then it may even be possible to obtain—or at least justify—some of our results for spin correlations by use of fermion techniques. But this, also, we must leave for the probably rather more distant future.

Overall, it appears that the combination of the above arguments and the agreement with McCoy's results[3] does rather strongly suggest that our results for the asymptotic critical region are indeed exact.

VII. DISCUSSION AND EXTENSIONS

In this paper, we have focused primarily on the computation of physical properties of random transverse-field Ising chains in the critical region, with digressions to interpret physically some of the results. In this last section, we briefly discuss how the RG analysis of this paper can be viewed in a more general framework—in particular by considering the origins of the peculiar scaling properties. We then consider extensions to other one-dimensional random quantum systems, as well as possible applications to the understanding of random quantum transitions in higher dimensions, and finally draw some general lessons about random systems.

A. Nature of critical fixed-point and renormalization-group flows

The critical RG fixed point that we have studied is evidently very strange. This is suggested by the extreme difference between typical and average quantities, the scaling relating logarithms of frequency to lengths and the extremely broad distribution of energy scales, the existence of two different correlation length exponents (ν and $\tilde{\nu}$), and the asymptotically sharp division at low energies of the spins into "active" and "frozen."

Some of these features are vaguely reminiscent of things that occur near to first-order phase transitions in classical systems with randomness, which we now briefly summarize. Consider cutting large finite-size samples of volume L^d from an infinite d-dimensional system at a first-order transition at temperature T_c, between an ordered phase with, say, a broken Ising symmetry that does not couple to the randomness, and a disordered phase. In a collection of such samples, roughly half will be more strongly coupled—by amounts $\varepsilon \sim L^{-d/2}$ per unit volume from the central limit theorem—than the average sample, while the other half will be more weakly coupled. Since the randomness will couple asymmetrically to the two phases, the free energy cost of a sample that "thinks" that it is ordered to fluctuate into the disordered phase is $\varepsilon L^d \sim L^{d/2}$. Thus only a small fraction—those with anomalously small ε of order $T/L^{d/2}$—of the samples will be "uncertain" as to which phase they are in. Almost all samples for large L will therefore either be almost completely disordered or be almost completely ordered with very slow fluctuations—with a rate τ_L—between the two symmetry related "up" and "down" states. Since a flip from up to down in the ordered phase requires passage of a domain wall, with interfacial free energy density σ, through the sample, the rate will have an Arrhenius form:

$$\tau_L \sim e^{c\sigma L^{d-1}/T} . \tag{7.1}$$

If the temperature is changed on a single sample, then the width ΔT_1 of the temperature range over which the sample will change from mostly ordered to mostly disordered is

$$\Delta T_1 \sim L^{-d} , \tag{7.2}$$

since the free energy difference between the phases is linear in $T - T_c$. But if the *distribution* of samples is considered and, say, the mean square magnetization measured, it will, from the above discussion, cross over from ordered to disordered only over a much wider temperature interval

$$\Delta T_D \sim L^{-d/2} . \tag{7.3}$$

One can invert Eqs. (7.2) and (7.3) to define two distinct finite-size characteristic lengths, one of them

$$\tilde{\xi} \sim |T - T_c|^{-\tilde{\nu}} \text{ with } \tilde{\nu} = 1/d \tag{7.4}$$

describing the rounding of the transition in a *typical* sample, the other

$$\xi \sim (T - T_c)^{-\nu} \text{ with } \nu = 2/d \tag{7.5}$$

describing the rounding of the *distribution* of samples. This latter length ξ must satisfy the general bound of Chayes *et al.*[14] for probabilistically defined finite-size correlation lengths in random systems,

$$\nu \geq 2/d , \tag{7.6}$$

which is thus saturated for a first-order transition at which the randomness couples asymmetrically.

In a RG framework, the reason for the anomalous behavior described above is that first-order transitions are described by fixed points at which the fluctuations that take the system from one phase to another are *dangerously irrelevant.* Formally, there are *no* fluctuations at the fixed point, since as $L \to \infty$ *all* samples will act either fully ordered or fully disordered. However, physical quantities such as certain susceptibilities, specific heats, and truncated correlations vanish at $T = 0$ and thus one cannot set T to zero in computing these properties. The dangerously irrelevant fluctuations will give rise to anomalous scaling of various quantities and to several different characteristic lengths.[22] "Activated" dynamic scaling of times with lengths as in Eq. (7.1) is also characteristic of the behavior near such fluctuationless fixed points.[22]

In classical random-field Ising magnets—in which the random fields couple to the order parameter—the *critical* fixed point is at zero temperature with the random variations dominating the thermal fluctuations at long length scales. This causes, for example, violation of hyperscaling laws, the existence of two different exponents for correlations at criticality, and broadly distributed time scales at long-length scales with[24]

$$\ln\tau_L \sim L^{\psi} , \tag{7.7}$$

which is a more precise statement of the scaling Eq. (7.1) when the coefficient c is random.

Several features discussed above are reminiscent of some of the properties of our random transverse-field Ising chain critical point. In particular, the observation that at low energies near the critical point almost all clusters are active with only very small effective fields, or frozen with negligible fluctuations in their magnetization, is reminiscent of the behavior near the random classical first-order transitions discussed above. Furthermore, the quantum dynamics, the rare flipping of spin clusters, is exponentially slow at long-length scales with broadly distributed time scales

$$\ln\tau_L \sim L^{1/2} \tag{7.8}$$

at the critical point. This suggests that the quantum fluctuations from one phase to the other are asymptotically *absent* at the fixed point; i.e., in a sense, *Planck's constant* $\hbar$, which controls quantum fluctuations, *is dangerously irrelevant* with the different parts of the low-energy effective Hamiltonian almost commuting with each other. The dynamics of the spin clusters is, of course, still controlled by the remaining quantum fluctuations.

The flipping of a spin cluster proceeds by tunneling of a domain wall through the cluster with a rate proportional to a typical effective field, $\tilde{h}$ with $\ln\tilde{h} \sim \Gamma \sim L^{1/2}$, at length scale L. This behavior and the resulting scaling of logarithmic frequency and logarithmic energy scales with powers of length scales we dub *"tunneling dynamic scaling"* by analogy with the "activated dynamic scaling" in classical systems controlled by fluctuationless fixed points.[22] Note that the exponent ψ_D controlling the *dynamics* via $\ln\tau_L \sim L^{\psi_D}$ is the *same* as the exponent controlling the static energy scaling of the couplings that maintain the order:

$$-\ln\tilde{J}_L \sim L^{\psi_S} \quad , \tag{7.9}$$

with

$$\psi_D = \psi_S = 1/2 \; . \tag{7.10}$$

This should probably have been expected for a quantum system as the dynamics and energetics are inextricably linked in contrast to classical systems, for which the static and dynamic exponents are often independent.

The behavior of our quantum system is both far richer and now far better understood than the somewhat related random classical systems with activated dynamic scaling.[22] Off criticality, rather than just flowing to simple ordered or disordered fixed points, the RG flows of the random Ising chain go to low-energy behavior characterized by ordered and disordered *fixed lines* with continuously variable exponents. Asymptotically close to criticality, the flows eventually go to the bottom of these fixed lines as shown schematically in Fig. 5; these, like the critical fixed point, also correspond to points with very singular scaling.

B. Extensions

1. *Transverse-field Ising chains*

In addition to the behavior discussed in this paper—and extensions of it to, for example, higher spin correlations and higher moments of the two-spin correlations, both of whose scaling forms can readily be guessed—one could also study the behavior of correlations in real or imaginary *time*. The decay of correlations in imaginary time is unphysical for the quantum system but corresponds to the decay in the uniform direction of the McCoy-Wu model and is thus natural to measure numerically; it could also be explored by the present methods. The distributions of the local autocorrelation in imaginary time τ will exhibit so-called "multifractal" behavior[25] which is not much more here (and in many cases) than the observation that the basic variable is $\ln\tau$ rather than τ.

Real time or real frequency properties are more interesting physically. Again, response and correlation functions at low frequencies ω in the critical regime can probably be extracted. The potentially problematic "hydrodynamic" regime $\omega \ll T$ is the most interesting. Whether the free Fermi nature of the Ising system will substantially simplify the physics and the computations, or whether the complicated connection between the spin and Fermi operators will make dynamic spin correlations hard to analyze, we leave as an outstanding open question.

After this paper was completed, an unpublished paper was received[26] that analyzed the McCoy-Wu model *numerically*, specifically studying the finite-size scaling properties of systems of length $L \leq 16$ in the spatial direction as a function of their "length" L_τ in the imaginary time direction. Crisanti and Rieger[26] compute properties at the (exactly known) critical point, using relatively strong randomness so that the crossover length $\ell_V \approx 2$. From fits of various quantities they quote dynamic scaling exponents z in the range 1.5–1.75 rather than the infinite value of z expected from the RG flows and the exact solution. Nevertheless, other data they present seem to indicate an apparent z which increases with increasing strength of randomness. Thus it appears likely that in the range of sizes studied, their systems are in a crossover region in which the apparent z should increase with system size. Nevertheless, from the scaling of the absolute value of the magnetization, an estimate of $\beta/\nu \approx 0.18 \pm 0.01$ is obtained, seemingly in good agreement with the predicted value of $(3 - \sqrt{5})/4 \approx 0.19$.

Resolution of the apparent discrepancy between the behavior found numerically for small systems[26] and the results of the present paper we must leave for future work. In order to analyze these and other numerical simulations of random quantum Ising systems it would also be useful to compute distributions of properties of systems with finite spatial and temporal extent. This should be straightforward by the present techniques. At this point, it should be noted that the singular dynamical scaling with $\ln L_\tau \sim L^{1/2}$ is a direct consequence of the exact solution[2,4] and does *not* depend on the approximation in the present paper. Thus the failure of Ref. 26 to observe singular dynamic scaling in small-size numerical simulations makes one worry about the analysis of higher-dimensional systems, in particular the transverse-field Ising spin glass discussed below.[27,28]

As mentioned earlier, one might also study the weak to strong randomness crossover which may be partially analyzable—along with other properties—via transfer matrices.

More generally, one should ask how universal are the critical properties found here for random transverse-field Ising chains. Further neighbor or more complicated interactions, correlations among the couplings, and certain types of extra degrees of freedom should result, at least in some parameter ranges, in a para-to-ferromagnetic transition that is in the same universality class as the simple nearest neighbor model studied here. The critical scaling functions and the other qualitative features should then still be valid. But the extra information that we obtained from the exact solvability—in particular the exact normalization of δ and the concomitant absence of a nonuniversal prefactor on the scaling variable $\gamma = \delta\Gamma$—will no longer be valid.

Note that a common mechanism for changing the nature of phase transitions by them becoming first order *cannot* occur here due to the general impossibility of first-order phase transitions in random one- (and two-) dimensional systems in the presence of randomness that couples asymmetrically to the two phases (here para and ferro).[29]

2. Other one-dimensional systems

Renormalization-group transformations like those used in this paper can be used to treat other random quantum systems in one dimension. The crucial ingredient for the validity of the technique is the development of an asymptotically infinitely broad distribution of couplings at low-energy scales. Then, at any given low-energy scale Ω, almost all effective couplings will be either much larger than Ω and hence "satisfied" or much smaller than Ω and hence treatable perturbatively.

A variety of random antiferromagnetic spin chains have been analyzed in Ref. 10, including ones which are not even partially exactly solvable. Of particular interest, for our present purposes, is a transition in a spin chain system from an XY "random singlet" phase to an Ising antiferromagnetic phase in the presence of a conserved quantity, the total magnetization in the uniaxial direction. Not surprisingly, it is found to be in a different universality class than the Ising transition with no auxiliary conserved quantities analyzed here.[10]

Some of the antiferromagnetic chains are related to interacting spinless fermions with random hopping which can also be studied. The most interesting possibility would be to analyze the low-frequency transport properties at low but finite temperatures, with $\omega \ll T$. Whether the separation of scales and the formal irrelevance of interactions will enable progress to be made in this regime—where one might expect variable range hopping[30]—is an intriguing question. In any case, some new tricks will probably be necessary to handle the interesting very long-time behavior.

3. Higher dimensions

Some of the most intriguing open questions are whether or not various physical effects found here in one dimension can persist in higher dimensions. One example is already known: The random singlet phase of random antiferromagnetic Heisenberg spin-$\frac{1}{2}$ chains was first investigated by Dasgupta and Ma[9] and more extensively in Ref. 10. Formally, it is quite similar to the critical point of the random transverse-field Ising chain. But a similar random-singlet phase is also known to exist in higher dimensions, in particular, as shown by Bhatt and Lee,[31] in the insulating phase of Si:P.

Here we are more interested in the behavior near random quantum phase transitions, and so we focus on the behavior of random transverse-field Ising ferromagnets in dimensions $d \geq 2$, potentially the simplest random quantum system. We first ask whether the apparently "split" nature of the zero-temperature transition which occurs in 1D can occur in $d > 1$. The answer to this is certainly yes: For transverse-field distributions which are strictly in the disordered phase at zero temperature, there will be a range of parameters in which at least some derivative of the magnetization M with respect to an ordering field H will diverge. This is the hallmark of Griffiths' singularities in a random quantum Ising system and can be seen simply as follows.

As long as the smallest h's, say, h_-, are sufficiently less than some (lattice-dependent) multiple of the largest J's, say, J_+, such that a system consisting solely of J_+ bonds and h_- fields would be in the ordered phase, then $M(H)$ will exhibit a power law singularity at $H = 0$. This can be seen by a simple generalization of the argument in Sec. IV B. The probability that a compact region of volume L^d is strongly enough coupled to act as if it were in the ordered phase is at least of order p^{L^d} for some $p < 1$, since roughly each coupling in the region must lie in a particular favorable range. Such a quasiordered region will behave like a spin cluster with an effective field that collectively flips it of magnitude $\tilde{h}_L \sim \epsilon^{L^d}$ for some ϵ. In anisotropic classical language, this is just a statement that the interfacial free energy in the "time" direction between an "up-" and a "down"-spin cluster of area L^d is of order L^d. Quantum mechanically, the ϵ^{L^d} can be thought of as arising from L^dth-order perturbation theory which is the lowest order that couples the symmetric and antisymmetric states of the spin cluster. At zero temperature, a magnetic field will align the cluster if $H \geq \tilde{h}_L$. There will thus be a singular part of $M(H)$ which can be crudely estimated to be

$$M_{\rm sing}(H) \sim \int_{L_H}^{\infty} p^{L^d} dL \ , \qquad (7.11)$$

with the length scale L_H determined by

$$\epsilon^{L_H^d} \sim H \ . \qquad (7.12)$$

Equation (7.11) is just the contribution from the rare large ($L > L_H$) almost fully aligned spin clusters. Ignoring logarithms, this yields

$$M_{\rm sing}(H) \sim H^{\kappa} \ , \qquad (7.13)$$

with

$$\kappa = \frac{\ln p}{\ln \epsilon} \ , \qquad (7.14)$$

as in Sec. IV B. Physically, the important open question is whether or not, for $d > 1$, there are locally ordered regions that are sufficiently common (not too small p) and sufficiently well ordered (small ϵ) that some low, and hence measurable, derivative of the magnetization diverges, i.e., if there exists a class of rare, quasiordered regions with $\kappa(p, \epsilon)$ small.

Very recent numerical studies of random transverse-field Ising *spin glasses* have found that the leading nonlinear susceptibility $\chi_3 \equiv \frac{\partial^3 M}{\partial H^3}$ —which is related to the conventional spin-glass susceptibility in classical systems— does indeed diverge at zero temperature in the disordered phase, near but *away from* the transition in both two[27] and three[28] dimensions. However, the linear susceptibility remains finite in three dimensions[28] although it diverges in 2D.[27] One might thus be guardedly optimistic about experimental verification of a power law $M_{\rm sing}$ at low temperatures.

In one dimension, we have shown that κ becomes smaller and smaller as the transition is approached, vanishing as $\kappa \sim \delta$. But this is caused by properties of the rare—but not *so* rare—regions *at* the critical point. It is also associated with the anomalous tunneling dynamic scaling in 1D. Physically, as discussed in Sec. IV B, the *rare regions* which dominate for δ small and positive are *in the scaling distributions*—albeit in their tails. These anomalously ordered regions of size $L \gg \xi \sim 1/\delta^2$ are spin clusters that behave as if they are a distance $\delta_r \approx -c\delta$ into the ordered region [with $c = O(1)$] and have magnetic moments of order

$$m \sim LM_0(\delta = \delta_r) \sim (-\delta_r)^{\beta} \ , \qquad (7.15)$$

effective fields of order

$$-\ln \tilde{h}_L \sim L\delta \ , \qquad (7.16)$$

and probability

$$-\ln p_L \sim L/\xi \sim L\delta^2 \ . \qquad (7.17)$$

Because p_L vanishes much more slowly than $\tilde{h}_L$ for large L near the critical point, the dominant of these rare regions give rises to a strongly singular magnetization

$$M \sim H^{2\delta} \ . \qquad (7.18)$$

The difference between the δ in Eq. (7.16) and δ^2 in Eq. (7.17) that gives rise to Eq. (7.18) is a reflection of the anomalous scaling at the critical fixed point, in particular of the dangerous irrelevance of quantum fluctuations.

In higher dimensions, we are naturally led to consider rare regions which are similarly in the scaling distribution, i.e., which occur with probabilities

$$\ln p_L \sim \left(\frac{L}{\xi}\right)^d \ , \qquad (7.19)$$

so that for $L \sim \xi$ they are *not* rare. Such regions can act as if they are in the ordered phase by an amount

$$\delta_r \sim -\delta \qquad (7.20)$$

and have effective fields, i.e., flipping rates

$$-\ln \tilde{h}_L \sim \delta^{\mu} L^d \ . \qquad (7.21)$$

Here μ is most likely to be the exponent for the interfacial tension of the ordered phase for an interface normal to the "time" direction in the equivalent classical system. If the scaling at the critical point is conventional, then the analog of the Widom scaling law would imply

$$\mu = d\nu \ . \qquad (7.22)$$

This would then result in κ *saturating* to a fixed and (in the absence of domination by much rarer regions) universal value as $\delta \to 0^+$.

Thus we see that the vanishing of κ at the critical point is, not surprisingly, closely tied to anomalous scaling. If this occurs because

$$\mu < d\nu \qquad (7.23)$$

as in 1D, then κ will vanish as

$$\kappa \sim \delta^{d\nu - \mu} \qquad (7.24)$$

near the transition and any derivative of $M(H)$ will diverge. This would probably also give rise to tunneling dynamic scaling associated with the slow flipping rates of the clusters of size of order ξ,

$$-\ln \tilde{h}_{\xi} \sim \xi^{d-\mu/\nu}, \qquad (7.25)$$

i.e., a scaling of times with length scales in the critical regime of

$$\ln \tau_L \sim L^{\psi} \ , \qquad (7.26)$$

with

$$\psi = d - \mu/\nu \; . \tag{7.27}$$

Note, however, that scaling laws such as Eq. (7.27), which work in 1D with $\mu = 1$, $\nu = 2$, and $\psi = \frac{1}{2}$, may be too naive in higher dimensions even if tunneling dynamic scaling does occur.

The natural alternative, probably associated with $\kappa \to$ const, is conventional dynamical scaling in which rare regions would dominate less of the physics and there would be a dynamic critical exponent z, with

$$\tau_L \sim L^z \tag{7.28}$$

and scaling functions of, for example, $\omega\xi^z$ and $H/|\delta|^\Delta$ rather than the $\delta\ln\omega$ and $\delta\ln H$ that appear in 1D. Whether χ diverges before the transition would then depend on the sign of $\kappa - 1$.

At this point, the numerical computations on transverse-field Ising spin glasses in two and three dimensions seem to be more consistent with conventional scaling with a reasonably small value of $z \sim 1.5$.[27,28] But the methods of analyzing these have not yet been tested in the 1D system for which the behavior is now known, and more work is clearly warranted. Indeed, the apparent problems in the analysis of Ref. 26 in 1D discussed above suggest that a substantial degree of caution is in order.

A priori, it does not seem inconsistent that there might be random quantum transitions in higher dimensions that have some of the more exotic features of the 1D transverse-field Ising system. If so, then the development of order could be viewed as a kind of quantum percolation of larger and larger ordered clusters. This might suggest possible approximate RG treatments of such a system in a way which would, hopefully, indicate its own failings if this scenario were incorrect.

C. Conclusions

In this paper, we have seen that a simple approximation which contains some of correct physics leads to a surprising amount of information and understanding—much of it exact—about one of the simplest realistic random models. Unfortunately, physically transparent, controllable methods for treating other random systems have been sorely lacking; rather the field has been dominated by formal calculations on sometimes pathological models, particularly via "replicas," and phenomenological scaling arguments, with occasional rigorous results on which to attempt to build foundations.

Huse and the present author, along with others, have developed quite a general picture for understanding, phenomenologically, the properties of phases and phase transitions in random systems that are governed by zero-temperature fixed points.[22,24,32] Unfortunately, many of the features that emerge—domination of much of the physics by rare regions, activated dynamic scaling, hypersensitivity of states to changes in parameters, and nonequilibrium dynamic effects controlled by extremely broad separation of time scales—have not been derivable by controlled methods on any but toy models. Thus the analogs of some of these effects that occur in the quantum dynamics of the random transverse-field Ising chain should lend substantial additional credence to their existence in models that are not analytically tractable. Perhaps one might hope that, as has often been the case with exactly solvable models, insights from the solution of the random transverse-field Ising chain will lead to better approximate—or even systematically controllable—methods for treating other random systems.

For the present, we close with a challenge to those who seem to believe that replicas are the route to all the interesting statistical mechanics of random systems: to derive *any* of the exotic properties of the random transverse-field Ising chain by such methods.

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APPENDIX A: CONVERGENCE TO THE SPECIAL SOLUTION

In this appendix, the convergence of general well-behaved initial distributions to the special solutions, and hence the scaling solutions analyzed in the main text, is discussed. Linear stability about the special solutions is analyzed first, using the behavior at the critical point as a detailed example, with the more general behavior only summarized. This naturally leads to the emergence of certain singular distributions which fall *outside* the universality class discussed in the text. Finally, the nonlinear development of the exponential tails of distributions, crucial for the asymptotic low-energy properties away from criticality, is analyzed.

1. Eigenperturbations at the fixed point

In Sec. II A, the eigenperturbations for the rescaled distributions of the logarithmic bonds and logarithmic fields $Q(\eta)$ and $B(\theta)$ about the fixed point,

$$Q^*(\eta) = e^{-\eta},$$

$$B^*(\theta) = e^{-\theta} \; , \tag{A1}$$

were given. Defining the linear perturbations

$$q = Q - Q^* \tag{A2}$$

and

$$b = B - B^* \ , \tag{A3}$$

the linearized RG flows are given by

$$\Gamma \frac{\partial q}{\partial \Gamma} = q + (\eta + 1)\frac{\partial q}{\partial \eta} + 2q \otimes Q^* + b_0 \eta e^{-\eta} + e^{-\eta}(q_0 - b_0),$$

$$\Gamma \frac{\partial b}{\partial \Gamma} = b + (\theta + 1)\frac{\partial b}{\partial \theta} + 2b \otimes B^* + q_0 \theta e^{-\theta} + e^{-\theta}(b_0 - q_0) \ , \tag{A4}$$

with

$$b_0 \equiv b(0) \ ,$$

$$q_0 \equiv q(0) \ . \tag{A5}$$

The duality is reflected in the symmetry of Eq. (A4) under interchange of q and b. We can thus restrict consideration to perturbations which are symmetric, $q = b$, or antisymmetric, $q = -b$, under this interchange. We analyze here the *antisymmetric* perturbations. The symmetric perturbations can be handled similarly.

We thus look for eigensolutions to Eq. (A4) with $b = -q$ with eigenvalue Λ, i.e., with

$$q(\eta;\Gamma) = q(\eta)\Gamma^{\Lambda} \ . \tag{A6}$$

It is convenient to work with the function

$$a(\eta) \equiv q(\eta)e^{\eta} = q(\eta)/Q^*(\eta) \ . \tag{A7}$$

Differentiating the resulting eigenequation for a, we obtain

$$(1+\eta)\frac{d^2 a}{d\eta^2} + \frac{da}{d\eta}(-\Lambda + 1 - \eta) + a - a_0 = 0 \tag{A8}$$

and the subsidiary condition for the undifferentiated equation at $\eta = 0$,

$$-\Lambda a(0) + \left.\frac{da}{d\eta}\right|_{\eta=0} + 2a_0 = 0 \ . \tag{A9}$$

The boundary condition at $\eta = 0$ is

$$a(0) = a_0 \ , \tag{A10}$$

but for the time being we ignore this and let $a(0)$ take any value. By expanding in η, a simple solution for $\Lambda = 1$ can be found,

$$a = a^{(1)} = a_0(1 - \eta) \ , \tag{A11}$$

which satisfies $a(0) = a_0$ and corresponds to the one relevant perturbation Eq. (A12), the flow away from criticality parametrized in the special solution by δ.

For $\Lambda \neq 1$, there is similarly a simple solution to the differential equation

$$a = a_0 + [a(0) - a_0]\left(1 - \frac{\eta}{1-\Lambda}\right) \ , \tag{A12}$$

with

$$a(0)\left(\Lambda + \frac{1}{1-\Lambda}\right) = a_0\left(2 - \frac{1}{1-\Lambda}\right) \tag{A13}$$

from the subsidiary condition Eq. (A9). But this solution can only satisfy $a(0) = a_0$ if $\Lambda = 2$. This corresponds to an unphysical perturbation of a non-normalizable probability distribution $q^{(2)} = e^{-\eta}a^{(2)} = a_0 e^{-\eta}$ which does *not* have norm 0 as it must for Q to have norm 1.

For $\Lambda \neq 1$ or 2, we must add in an admixture of the other independent solution to the differential equation. This is

$$a_B(\eta) = (1 - \Lambda - \eta)\int_0^{\eta} d\xi \ \frac{e^{\xi}(1+\xi)^{\Lambda-2}}{(1-\Lambda-\xi)^2} \ , \tag{A14}$$

the integral in the principle-part sense. For large η,

$$a_B \sim e^{\eta}\eta^{\Lambda-3} \ . \tag{A15}$$

Thus, for $\Lambda \neq 1, 2$, there are *no* eigenperturbations with $q(\eta)$ decaying exponentially. Indeed, by use of the boundary conditions Eq. (A9) and $a(0) = a_0$, and the probability normalization condition that $\int_0^{\infty} e^{-\eta}a(\eta) = 0$, one can show that there are in fact no other eigenperturbations at all.

For the solutions *symmetric* under $q \leftrightarrow b$, a similar analysis yields exactly one eigenfunction with eigenvalue $\Lambda = -1$, corresponding to irrelevant perturbations on the critical manifold; this has eigenfunction

$$q^{(-1)} = b^{(-1)} = (\eta - 1)e^{-\eta} \ . \tag{A16}$$

The nature of the RG operator linearized about the fixed point, Eqs. (A4), will become clearer from the analysis below: A general perturbation will be projected onto the unique stable and unstable eigenperturbations with the remainder an exponentially rapidly decaying perturbation. Thus the special solution Eqs. (2.51)–(2.54) which contains both the irrelevant and relevant perturbations will turn out to be strongly stable.

2. General initial conditions at the critical point

To give a better understanding of the convergence to the special solution, we specialize to the critical manifold and consider only symmetric perturbations. It is more instructive to work with the unrescaled distributions, writing

$$P(\zeta;\Gamma) = u(\Gamma)e^{-\zeta u(\Gamma)} + p(\zeta;\Gamma) \ ,$$

$$R(\beta;\Gamma) = \tau(\Gamma)e^{-\beta\tau(\Gamma)} + r(\beta;\Gamma) \ , \tag{A17}$$

where we have dropped the subscript 0 on u and τ that occurs in Sec. II B, as we are not yet considering the distribution of lengths. The special solution about which p and r are perturbations is, at the *critical point*,

$$u = \tau \ , \tag{A18}$$

with

$$\frac{du}{d\Gamma} = -u\tau \ , \tag{A19}$$

so that

$$u = \tau = \frac{1}{C+\Gamma} \ . \tag{A20}$$

Symmetric perturbations have

$$r = p \ , \tag{A21}$$

and we hence work just with p with

$$p_0 \equiv p(0) \tag{A22}$$

and

$$\frac{\partial p}{\partial \Gamma} = \frac{\partial p}{\partial \zeta} + 2u^2 p \ \otimes \ e^{-\zeta u} + p_0 u^2 \zeta e^{-\zeta u} \ . \tag{A23}$$

In principle, one can solve Eq. (A23) by Laplace transforming in ζ, but the presence of p_0 complicates the matter and makes the analysis rather complicated. It is, nevertheless, instructive to look for solutions which are sums of exponentials. Because of the special role of $e^{-\zeta u}$, we look for a solution of the form

$$p = a(\Gamma)e^{-\kappa(\Gamma)\zeta} + b(\Gamma)e^{-\zeta u(\Gamma)} + c(\Gamma)\zeta e^{-\zeta u(\Gamma)} \ , \tag{A24}$$

which from Eq. (A23) yields differential equations for the coefficients $a(\Gamma)$, etc., which can be solved.

In particular, we see that

$$\frac{d\kappa}{d\Gamma} = 0 \ , \tag{A25}$$

so that a general exponential perturbation $e^{-\kappa\zeta}$ produces under renormalization only itself and terms like those of the special solution which decay more and more slowly in ζ as Γ increases. The coefficient $a(\Gamma)$ and the magnitude of the feeding of this into $b(\Gamma)$ and $c(\Gamma)$ all decay as $e^{-\kappa/u(\Gamma)} \sim e^{-\kappa\Gamma}$ for large Γ. With the normalization condition that $\int_0^\infty p d\zeta = 0$, the b and c terms appear in the combination

$$p_- = -u^2 e^{-\zeta u} + u^3 \zeta e^{-\zeta u} \tag{A26}$$

for large Γ plus terms decaying as $e^{-\kappa\Gamma}$. By inspection, we see that, from Eq. (A19),

$$p_- = \frac{\partial}{\partial \Gamma}(u e^{-\zeta u}) \tag{A27}$$

and therefore a perturbation proportional to p_- is just a shift in C in Eq. (A20), i.e., a shift in the origin of Γ. In the rescaled variables $\eta = \zeta/\Gamma$, etc., p_- corresponds exactly to the irrelevant eigenperturbation Eq. (A16) with eigenvalue $\Lambda = -1$; this eigenvalue arises from the fact that, for a typical $\zeta \sim 1/u$, p_- is of order $u \sim 1/\Gamma$ times the scaling solution $u e^{-\zeta u} \approx e^{-\zeta/\Gamma}/\Gamma$.

Thus an exponentially decaying perturbation results for large Γ in a finite amount of the irrelevant eigenperturbation plus other parts that decay exponentially in Γ. In the rescaled variables, these residual parts are very strongly irrelevant with, crudely, "eigenvalue" $\Lambda = -\infty$.

From solutions of the form Eq. (A24), a solution to Eq. (A23) can be constructed for general initial conditions $p_I(\zeta)$. It is rather complicated, and we will not display it here. It consists of two parts: first, the irrelevant perturbation p_- with a coefficient that depends only on the first three moments of p_I, and second, parts which depend only on the tail of $p_I(\zeta)$ for $\zeta > \Gamma$; note that these must exist as there will still be original couplings remaining at scale Γ that originally had $\zeta > \Gamma$. The second kind of parts thus decays with Γ in the same way as the decay of p_I with ζ, i.e., exponentially for well-behaved distributions p_I that decay exponentially in ζ. As long as the third moment of p_I is finite, the p_- part will thus be the most slowly decaying part. On the other hand, if the third moment of p_I is infinite but the second moment finite, e.g., $p(\zeta) \sim \frac{1}{\zeta^{3+\sigma}}$ for large ζ with $0 < \sigma < 1$, then the perturbation will decay (relative to the special solution) as $1/\Gamma^{1-\sigma}$ rather than the $1/\Gamma$ decay associated with generic irrelevant eigenperturbations p_-.

If the *second moment* of p_- is *infinite*, then the critical point is in a *different universality class.* This should have been anticipated from the qualitative discussion of the Introduction and the use there of the central limit theorem for sums of ζ_i.

The conditions for the validity of our results are thus that

$$\overline{(\ln J)^2} < \infty$$

and

$$\overline{(\ln h)^2} < \infty \ . \tag{A28}$$

It is amusing to note that in the case of distributions of J and h that are bounded away from zero and infinity, the details of the original distribution are completely forgotten below some finite-energy scale, for the self-dual case at the critical point that we have analyzed here. In this case, the only features that affect low energies are the first three moments of $\ln J$.

3. General convergence

The behavior of the convergence to the special solution that we have discussed above for the coupling distributions on the critical manifold holds much more generally as can be shown directly by analyzing linear deviations $[p, r]$ from the special solutions for the joint distributions of couplings and lengths $P(\zeta, \ell), R(\beta, \ell)$. We are also interested in the convergence of the function $\hat{G}(\beta, y)$ to the special solution Eq. (3.17). Since this obeys a linear RG equation (3.16), we are thus interested in its general solution. This can be analyzed by methods similar to those used above at the critical point and leads again to rapid convergence to the special solution; we will not discuss it in detail.

Here we focus on the behavior of linear perturbations from the special solution for the Laplace transforms $\hat{P}(\zeta, y)$ and $\hat{R}(\beta, y)$, noting that each y is independent. The structure is similar to that discussed above for convergence of $P(\zeta) = \hat{P}(\zeta, y = 0)$ at the critical point. But here, for each y, we have a four-parameter special solution with $C(y)$, $D(y)$, $\delta(y)$, and $\gamma(y)$ [or, equivalently, $\Delta(y) = \sqrt{\gamma(y) + \delta^2(y)}$] rather than the one parameter $C = C(y = 0)$ of Eq. (A20) for the $y = 0$ distribution at the critical point discussed above. We thus try to express a general linear perturbation as the sum of four perturbations which correspond to derivatives of the special solution $[\hat{P}_s, \hat{R}_s]$ with respect to C, D, δ, and γ, respectively.

We thereby arrive at a general linear perturbation expressed as linear combinations with different κ and ν of solutions of the form

$$\hat{p} = a(\kappa, \nu, \Gamma)e^{-\kappa\zeta} + b(\kappa, \nu, \Gamma)e^{-u(\Gamma)\zeta} + c(\kappa, \nu, \Gamma)\zeta e^{-u(\Gamma)\zeta} ,$$

$$\hat{r} = h(\kappa, \nu, \Gamma)e^{-\nu\beta} + j(\kappa, \nu, \Gamma)e^{-\tau(\Gamma)\beta} + k(\kappa, \nu, \Gamma)\beta e^{-\tau(\Gamma)\beta} , \quad \text{(A29)}$$

where we have suppressed the y dependence and $u(\Gamma)$ and $\tau(\Gamma)$ are given by the special solution Eqs. (2.51) and (2.52) in terms of C, D, δ, γ, and Γ. For each (Γ independent) κ and ν, an independent solution of the form Eq. (A29) can be found with a, b, c, h, j, and k obeying ordinary differential RG flow equations in Γ. [Note that a and b here should not be confused with the coefficients used in Eq. (3.17).]

The perturbation associated with C corresponds to the irrelevant perturbation at the critical point and it vanishes generally as $1/\Gamma$ (relative to the special solution). Conversely, the perturbation associated with δ corresponds to the relevant perturbation at the critical point and thus grows as Γ away from criticality, as it should. The perturbations associated with γ and D, which vanish at $y = 0$, are, in a sense explained below, also irrelevant except for a portion of $\gamma(y)$ which is "redundant."

The residual perturbations, i.e., those that are not associated with the special solution, *decay exponentially* as the slower of $e^{-(\kappa+2\delta)\Gamma}$ or $e^{-(\nu-2\delta)\Gamma}$. At $y = 0$, corresponding to the distributions of just ζ and β, this implies that perturbations will only decay if

$$\kappa + 2\delta_0 > 0$$

and

$$\nu - 2\delta_0 > 0 . \quad \text{(A30)}$$

These conditions are very physical: In the disordered phase, a perturbation with a more slowly decaying tail in β than the $\Gamma \to \infty$ special distribution

$$\hat{R}(\beta, y = 0; \Gamma \to \infty) = \tau_0 e^{-\beta\tau_0} , \quad \text{(A31)}$$

with

$$\tau_0(\Gamma \to \infty) = 2\delta_0 , \quad \text{(A32)}$$

will clearly change the behavior at low energies as the remaining fields will be dominated by the residual ones with large β originally that have not yet been decimated. Similarly in the ordered phase the condition is on the bond distribution Eq. (2.44) with $u_0(\Gamma \to \infty) = 2|\delta_0|$ for $\delta_0 < 0$. Note, however, that as long as κ and ν are positive, even if the conditions Eq. (A30) are violated, the *critical* behavior will not change; only off critical with $|\delta_0|$ sufficiently large will the behavior of the weakly disordered and/or weakly ordered phases change. Thus any power law tail for small couplings in the initial distributions of bonds or fields will yield the same critical behavior.

We now consider the distributions of bond and cluster *lengths*. In the low-energy regime under study, we are interested in long-length scales which will be dominated by small y. Indeed, in Sec. II C we argued that the scaling solution Eqs. (2.65)–(2.73) only depended on *one* parameter of the y-dependent quantities C, D, γ, and δ beyond their values at $y = 0$: This is simply the coefficient γ_1 of $\gamma(y)$ for small y, $\gamma(y) \approx \gamma_1 y + O(y^2)$, as in Eq. (2.86). The perturbations which correspond to all the y dependence of C, D, γ, and δ except γ_1 are thus *irrelevant* at long-length scales by inverse powers of the typical length scale $\bar{\ell}_\Gamma = 1/n_\Gamma$.

The parameter γ_1 sets the nonuniversal coefficient of the overall length scale; it has no other effect and is thus termed "redundant." Perturbations in γ_1 can thus be scaled away by rescaling y and thereby all lengths.

A general solution for $[\hat{p}, \hat{r}]$ can in principle be constructed from Eq. (A29) by contour integrals over imaginary κ and ν which can be deformed to yield only real parts of κ and ν that are greater than or equal to those parametrizing the decay for large ζ and β, respectively, of the initial perturbations. With the exceptions of the special cases discussed above, all perturbations are irrelevant except for the shift in the distance δ from criticality and a change in the overall length scale. In general, there is thus rapid convergence of initial distributions towards the special solution.

4. Development of exponential tails

The formal convergence of linear perturbations about the special solution analyzed above does not directly tell us the degree of uniformity of this convergence, especially not if the initial conditions differ strongly in some regimes from the special solution. Of particular importance is the development of the exponential tails in the distributions of β, ζ, ℓ, and m which dominate the low-energy physics off criticality. A simple physical argument shows that these tails develop immediately, as soon as any arbitrarily small but finite fraction of the couplings have been decimated.

Consider increasing Γ by a small amount (from its initial value of zero), in particular by less than the widths of the initial distributions R_I and P_I of β and ζ, respectively. With the first decimation of a bond, there is some probability that the new effective bond generated is larger than, say, twice the median $\hat{\beta}$ of the β. If the bond next to this is also decimated as the scale is changed from 0 to Γ, then the resulting bond can be bigger than 3 times $\hat{\beta}$ as it is the sum of the old effective β and an original β.

Generally, if a string of n consecutive original bonds is decimated by scale Γ without any of the intervening fields being decimated, the resulting effective cluster will have length $\ell_c = n + \frac{1}{2}$, moment $m = n$, and logarithmic effective field β which is greater than $n\hat{\beta}$ with probability greater than 2^{-n}, i.e., the probability that all of the original β's decimated were greater than $\hat{\beta}$. The probability of this occurring is roughly

$$\Pr(\text{cluster } > n) \sim \left(\frac{\Gamma P_{0I}}{2}\right)^n , \tag{A33}$$

with ΓP_{0I} being roughly the probability that a given bond is decimated before scale Γ. Thus we have explicitly generated exponential tails for the distributions of β, ℓ_c, and m. Similarly, one generates exponential tails in the distributions $P(\zeta, \ell_B)$ of the bonds. Furthermore, one sees the root of observations like "anomalously large β also tend to have m and ℓ of order β." Rare large clusters with this behavior are of the type that give rise to the power law Griffiths' singularities of the weakly disordered phase (Sec. IV B). They exist as long as some fraction of the original J_i's are bigger than some fraction of the original h_j's.

The argument above only gives a lower bound on the probability of large clusters. But, in fact, the exponential form of Eq. (A33), albeit with a different coefficient that is greater than $\Gamma P_{0I}/2$, is the correct form for the tail provided that the initial distributions decay at least exponentially. The case of particular interest in which the original distributions are *bounded* (or decay more rapidly than exponentially) can be analyzed directly.

For simplicity, we focus only on the behavior of self-dual distributions of β and ζ at the critical point, thus studying $P(\zeta;\Gamma)$ which obeys Eq. (2.14) with $R = P$. Laplace transforming in ζ, we have

$$\frac{\partial \tilde{P}(z)}{\partial \Gamma} = z\tilde{P} - P_0 + P_0\tilde{P}^2 . \tag{A34}$$

Exponential tails are dominated by the singularity in $\tilde{P}(z)$ with largest (i.e., least negative) Rez which for non-negative $P(\zeta)$ must be on the negative real z axis. With an initial distribution $P_I(\zeta)$ that decays more rapidly than exponentially, $\tilde{P}_I(z)$ is analytic except for Rez $= -\infty$ where it diverges exponentially (or more rapidly) for Re$z \to -\infty$. The inverse $1/\tilde{P}(z)$ is thus small in the regime of interest and we have

$$\begin{aligned} -\frac{\partial \tilde{P}^{-1}}{\partial \Gamma} &= z\tilde{P}^{-1} + P_0 - P_0\tilde{P}^{-2} \\ &\approx z\tilde{P}^{-1} + P_0 , \end{aligned} \tag{A35}$$

ignoring the $\tilde{P}^{-2}$ term. With, for simplicity, $P_I(\zeta)$ smooth for small ζ, we have for small Γ

$$\tilde{P}(z;\Gamma) \approx \frac{\tilde{P}_I(z)}{1 - \Gamma P_{0I}\tilde{P}_I(z)} , \tag{A36}$$

which thus develops a simple pole at $z_s(\Gamma)$ for arbitrarily small Γ. Near this pole $\tilde{P}^{-2} \ll \tilde{P}^{-1}$ so that its neglect in Eq. (A35) is justified. Indeed, the $\tilde{P}^{-2}$ in Eq. (A35) will only slightly shift z_s and the residue of the pole for small Γ but cannot change the form of the singularity.

For a distribution $P_I(\zeta)$ with a maximum value μ at which P_I drops discontinuously from a value $P_I(\mu^-)$ to zero, we have, for small Γ,

$$z_s(\Gamma) \approx -\frac{1}{\mu}\left[\ln\left[1/\Gamma P_{0I}\mu P_I(\mu^-)\right] + \ln\ln\left(\frac{1}{\Gamma}\right) + o(1)\right] , \tag{A37}$$

so that for large $\zeta \gg -\frac{1}{z_s(\Gamma)}$

$$P(\zeta;\Gamma) \approx A(\Gamma)e^{\zeta z_s(\Gamma)} , \tag{A38}$$

the behavior of the tail being dominated by the very largest initial ζ's. From the form of Eq. (A35), RG equations for the position and amplitude $A(\Gamma)$ of the pole once it is formed can be derived: Writing, near the pole,

$$\tilde{P}(z;\Gamma) \approx \frac{A(\Gamma)}{z + v(\Gamma)} + B(z;\Gamma) , \tag{A39}$$

with $v(\Gamma) \equiv -z_s(\Gamma)$ positive, we obtain

$$\frac{dv}{d\Gamma} = -P_0(\Gamma)A(\Gamma) \tag{A40}$$

and

$$\frac{dA}{d\Gamma} = -v(\Gamma)A(\Gamma) + 2P_0B(z = -v;\Gamma) . \tag{A41}$$

The special solution corresponds to

$$v = A = P_0 = u_0 \tag{A42}$$

and

$$B = 0 , \tag{A43}$$

with

$$\frac{du_0}{d\Gamma} = -u_0^2 . \tag{A44}$$

APPENDIX B: DISTRIBUTION OF LENGTHS

In this appendix, we consider the distribution of bond or cluster lengths, prove that the special solution indeed corresponds to a non-negative probability distribution, and analyze the form of its small-ℓ limit. We will focus on the distribution of the lengths of bonds which are about to be eliminated; it is somewhat simpler than the full distribution of bonds (discussed at the critical point in Sec. II A). The cluster length distribution in the scaling limit is related to that of bond lengths by duality.

The distribution of lengths of about-to-be eliminated bonds is

$$F_0(\ell;\Gamma) = \frac{P(0,\ell;\Gamma)}{P_0(\Gamma)} = \mathrm{LT}^{-1}\frac{\Upsilon(y;\Gamma)}{u_0(\Gamma)} , \tag{B1}$$

the latter equality obtaining in the scaling solution of Sec. II C. We must thus study the inverse Laplace transform

$$F_0(\ell) = \frac{\sinh(\Gamma\delta)}{\delta}\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} dy\, e^{y\ell}\frac{\Delta(y)}{\sinh[\Gamma\Delta(y)]} , \tag{B2}$$

with

$$\Delta(y) = \sqrt{\delta^2 + y} . \tag{B3}$$

By changing variables from y to $\Gamma\Delta$, the integral can be done by deforming the contour, yielding, for $\ell > 0$,

$$F_0(\ell) = \frac{\sinh\Gamma\delta}{\delta}e^{-\delta\ell}\frac{\pi^2}{\Gamma^3}\sum_{n=-\infty}^{\infty}(-1)^{n-1}n^2 e^{-\ell\pi^2 n^2/\Gamma^2} . \tag{B4}$$

Note that the δ dependence of this is very simple. For large ℓ, the behavior is dominated by the nearest pole in y (here duplicated as $n = \pm 1$), yielding

$$F_0(\ell) \sim e^{-(\delta + \pi^2 n^2/\Gamma^2)\ell} . \tag{B5}$$

Note, however, that Eq. (B4) is not obviously positive for all ℓ, although by combining terms, it can easily be seen to be positive for

$$\ell > \frac{\ln 4}{3}\Gamma^2 . \tag{B6}$$

For small ℓ, the Poisson resummed version of Eq. (B4) is more useful:

$$F_0(\ell) = \frac{\sinh\Gamma\delta}{\delta}e^{-\delta\ell}\frac{1}{2\sqrt{\pi}\ell^{3/2}}\sum_{m=-\infty}^{\infty}\left[\frac{\Gamma^2}{2\ell}(2m+1)^2 - 1\right] \times e^{\frac{-\Gamma^2}{4\ell}(2m+1)^2} . \tag{B7}$$

For small $\ell \ll \min\left(\frac{1}{\delta}, \Gamma^2\right)$ we see that

$$F_0(\ell) \sim \frac{1}{\ell^{5/2}}e^{-\Gamma^2/4\ell} ; \tag{B8}$$

thus the probability of original bonds surviving (with $\ell \sim 1$) is exponentially small in Γ^2. Equation (B7) is also useful for showing positivity for small ℓ: This follows straightforwardly for

$$\ell < \frac{1}{2}\Gamma^2 . \tag{B9}$$

Since $\frac{1}{2} > \frac{\ln 4}{3}$, from Eqs. (B6) and (B9), $F_0(\ell)$ is positive for all positive ℓ as it must be.

One can similarly show the positivity of the distribution of lengths of all bonds in the critical region. The positivity of the full joint distribution of ζ and ℓ is more complicated due to the essential singularities in the complex y plane as in Eq. (2.40); we have not attempted to analyze this here. Note that it would suffice to prove positivity at the critical point (i.e., $\Gamma \ll 1/|\delta|$); preservation of positivity of the flows would then guarantee positivity for all Γ.

Note that we have here treated ℓ as a continuous variable; if ℓ is restricted to half-integers, there will be periodicity in the complex y plane and similar results will follow with the discreteness not playing a significant role in the scaling limit.

APPENDIX C: ASYMPTOTICS FOR CORRELATION FUNCTIONS

In this appendix, some of the details of the asymptotic analysis of the solutions to the differential equation (3.24) for the function A are given, focusing on the parts needed to obtain the asymptotic long-distance mean correlation functions.

As explained in Sec. II B, the long-distance correlations are dominated by the singularities of $\hat{K}_\infty(y)$ for $y \approx -\delta^2 - \epsilon^2 \pm i0$, i.e., just above and just below the cut in the complex y plane. This corresponds to $\Delta = i\epsilon \pm 0$ small and just to the left or right of the imaginary Δ axis. The singularities in $\hat{K}_\infty(y)$ come from those in A, but the latter is singular only when W diverges, i.e., when $\sinh\Gamma\Delta = 0$. Since we are interested in fixed δ, we need to analyze these singularities in the limit $|\Delta| \ll |\delta|$.

The most useful expression for $\hat{K}_\infty(y)$ in this limit is Eq. (3.95). The first term vanishes as $\Gamma \to \infty$ for either sign of δ with $|\Delta| \ll |\delta|$. The remaining term can be cast in a simpler form for analyzing the singularities by defining

$$E = \left(\frac{\delta\sinh\Gamma\Delta}{\Delta\sinh\Gamma\delta}\right)^{\frac{1}{2}} A , \tag{C1}$$

whereupon

$$I_{1,2} = \left(\frac{\Delta\sinh\Gamma\delta}{\delta\sinh\Gamma\Delta}\right)^{\frac{1}{2}}\left[\frac{\delta e^{-\Gamma\delta}}{\sinh\Gamma\delta}E \pm \frac{\partial E}{\partial\Gamma}\right] , \tag{C2}$$

the + and − signs obtaining for I_1 and I_2, respectively, so that

$$\hat{K}_\infty(y) = \int_0^\infty \hat{k}'(y,\Gamma) = \int_0^\infty d\Gamma\frac{\Delta}{\sinh\Gamma\Delta}e^{-\Gamma\delta}\left[\frac{\delta^2 e^{-2\Gamma\delta}}{\sinh^2\Gamma\delta}E^2 - \left(\frac{\partial E}{\partial\Gamma}\right)^2\right]. \tag{C3}$$

From Eqs. (C1), (3.24), and (3.29), we see that E satisfies the differential equation

$$\frac{\partial^2 E}{\partial\Gamma^2} = \frac{\delta^2}{\sinh^2\Gamma\delta}E + (\Delta\coth\Gamma\Delta - \delta\coth\Gamma\delta)\frac{\partial E}{\partial\Gamma} , \tag{C4}$$

with the boundary condition for small Γ,

$$E(\Gamma) \approx \Gamma^\phi . \tag{C5}$$

Because it will go to a constant for large Γ in the desired regime $|\Delta| \ll |\delta|$, and $\frac{\partial E}{\partial \Gamma}$ satisfies a first-order differential equation in this regime, E is more convenient to study for the present purposes than A. Note that $E(\delta,\Gamma)$ like $A(\delta,\Gamma)$ is a function of $|\delta|$ but independent of the sign of δ. The difference between the ordered and disordered phases is entirely due to the explicitly δ-dependent factors in Eq. (C3).

From the discussion in the main text, we only need the behavior of the integrand $\hat{k}'(y,\Gamma)$ in Eq. (C3) near to its singularities at

$$\Gamma_n = \frac{i\pi n}{\Delta} \tag{C6}$$

for Δ purely imaginary and n a nonzero integer. Since A and E are even in $\Delta \to -\Delta$, we can restrict consideration to Δ near the positive imaginary axis and hence $n > 0$. For obtaining the needed singular parts of $\hat{K}_\infty(y)$ near the cut, we can replace Eq. (C3) by

$$\hat{K}_\infty^{\rm sing}(y) = \int_{\tilde{\Gamma}}^{\infty} \hat{k}'(y,\Gamma)d\Gamma \ , \tag{C7}$$

with $\tilde{\Gamma}$ chosen to be in the range

$$\frac{1}{|\delta|} \ll \tilde{\Gamma} \ll \frac{1}{|\Delta|} \ . \tag{C8}$$

We are thus interested in the behavior of $E(\delta,\Delta\Gamma)$ for $\Gamma \geq \tilde{\Gamma}$, but to get this, we need to integrate the differential equation for E, Eq. (C4), from its boundary condition at small Γ out to $\tilde{\Gamma}$.

For $\Gamma \ll \frac{1}{|\Delta|}$, we can use the inner solution A_- in scaled form, Eq. (3.105), which is not known explicitly but its *form* for $|\delta|\Gamma \gg 1$ and analytic properties are known, Eq. (3.106). In particular, at $\Gamma = \tilde{\Gamma}$, A, and hence E are smooth functions of Δ with, for $\Delta/|\delta| \to 0$,

$$E(\tilde{\Gamma}) \approx E_-(\tilde{\Gamma}) = |\delta|^{\phi-\frac{1}{2}} C_{1-} \tilde{\Gamma}^{-\frac{1}{2}} e^{\frac{1}{2}|\delta|\tilde{\Gamma}} [1 + C_{2-}(1+|\delta|\tilde{\Gamma})e^{-|\delta|\tilde{\Gamma}} + O(e^{-2|\delta|\tilde{\Gamma}})]. \tag{C9}$$

Since for $\Gamma > \tilde{\Gamma}$, $\delta \coth\Gamma\delta \approx |\delta|[1 + O(e^{-2\Gamma|\delta|})]$ and $(\sinh\Gamma\delta)^{-2} \sim e^{-2\Gamma|\delta|}$ to first approximation, we can replace the equation for E in this regime by

$$\frac{\partial^2 E_+}{\partial \Gamma^2} = (\Delta\coth\Gamma\Delta - |\delta|)\frac{\partial E_+}{\partial \Gamma} \ , \tag{C10}$$

which can immediately be integrated to yield

$$E_+(\Gamma) = E_{0+} + E_{1+}(\Gamma) \ , \tag{C11}$$

with

$$E_{1+} = \epsilon_{1+} e^{-|\delta|\Gamma}[\Delta\cosh(\Gamma\Delta) + |\delta|\sinh(\Gamma\Delta)] \tag{C12}$$

and E_{0+} and ϵ_{1+} constants.

More generally, we can expand $E(\Gamma)$ for $\Gamma > \tilde{\Gamma}$ in powers of $e^{-|\delta|\tilde{\Gamma}}$ with E_{0+} and E_{1+} being the first two terms. Because

$$\frac{\partial E_+}{\partial \Gamma} = \epsilon_{1+}(\Delta^2 - \delta^2)e^{-|\delta|\Gamma} \sinh\Gamma\Delta \ , \tag{C13}$$

it would appear that the $\left(\frac{\partial E}{\partial \Gamma}\right)^2$ term in $\hat{k}'$ in Eq. (C3) is nonsingular; as we shall see, this is *not* correct in general. Indeed, examination of the behavior of Eq. (C4) near its singularities at Γ_n, Eq. (C6), indicates that

$$E^{\rm sing}(\Delta,\Gamma) \sim \left(\Gamma - \frac{\pi i n}{\Delta}\right)^2 \ln\left(\Gamma - \frac{\pi i n}{\Delta}\right) \ , \tag{C14}$$

with, for Δ just off the imaginary axis, the singularities occurring just off the real Γ axis.

In the disordered phase, however, to obtain the leading behavior of $\hat{K}_\infty^{\rm sing}$ for small Δ, we need go no further than writing $E \approx E_+$ and matching the coefficients of Eq. (C12) at $\tilde{\Gamma}$ to those from Eq. (C9), obtaining

$$E_{0+} = |\delta|^{-\phi}\sqrt{2}C_{1-}\left[1 + O\left(\tilde{\Gamma}|\delta|, \frac{\Delta}{|\delta|}\right)\right] \ , \tag{C15}$$

$$\epsilon_{1+} = |\delta|^{-\phi}\sqrt{2}\frac{C_{1-}C_{2-}}{\Delta}\left[1 + O\left(\tilde{\Gamma}|\delta|, \frac{\Delta}{|\delta|}\right)\right] \ . \tag{C16}$$

The dominant singularities in $\hat{k}'$ are then those from the E^2 term in Eq. (C3) which yields the ρ_K of Eq. (3.111), just arising from the E_{0+}^2 contribution to E^2 in Eq. (C3). For the ordered phase, in contrast, the prefactor of the E^2 term in Eq. (C3) vanishes as $e^{-3\Gamma|\delta|}$, and so we must examine the possibility of larger terms arising from *corrections* to the $\left(\frac{\partial E}{\partial \Gamma}\right)^2$ term.

To go beyond Eq. (C12), it is simplest to define the coefficients E_{0+} and ϵ_{1+} so that

$$E(\tilde{\Gamma}) = E_+(\tilde{\Gamma}) \tag{C17}$$

and

$$\frac{\partial E}{\partial \Gamma}(\tilde{\Gamma}) = \frac{\partial E_+}{\partial \Gamma}(\tilde{\Gamma}) \ , \tag{C18}$$

yielding Eqs. (C11), (C12) with, *crucially*, E_{0+} and ϵ_{1+} smooth functions of Δ for $|\Delta|$ small. To $O(e^{-2|\delta|\tilde{\Gamma}})$, we can then write for $\Gamma > \tilde{\Gamma}$

$$E(\Gamma) \approx E_+(\Gamma) + E_{2+}(\Gamma) \ , \tag{C19}$$

with

$$\frac{\partial E_{2+}}{\partial \Gamma} = 4\delta^2 E_{0+} e^{-|\delta|\Gamma} \sinh\Gamma\Delta \int_{\tilde{\Gamma}}^{\Gamma} \frac{e^{-|\delta|\gamma}}{\sinh\gamma\Delta} d\gamma \tag{C20}$$

and $E_{2+}(\Gamma)$ obtained from integrating Eq. (C20) up from the matching point $\tilde{\Gamma}$ to Γ. Note that since the integral in Eq. (C20) has logarithmic singularities at $\Gamma = \frac{\pi n}{\Delta}$, E_{2+} will have singularities like that of the full E, i.e., of the form Eq. (C14); the expression Eq. (C20) thus yields the lowest-order expression for the amplitudes of the singularities in E. From Eqs. (C3) and (C20), we then see that the dominant amplitude of the corresponding singularities of the form $(\Gamma-\pi in/\Delta)\ln(\Gamma-\pi in/\Delta)$ in $\hat{k}'_\infty$ arise from the $-2\frac{\partial E_{1+}}{\partial\Gamma}\frac{\partial E_{2+}}{\partial\Gamma}$ cross term in Eq. (C3). From Eqs. (C3) and (C20) we have

$$\hat{K}^{\text{sing}}_\infty(y) \approx (-2)\Delta(\Delta^2-\delta^2)\epsilon_{1+}\int_{\tilde{\Gamma}}^{\infty}\frac{\partial E_{2+}}{\partial\Gamma}d\Gamma \ , \quad \text{(C21)}$$

since the Γ dependence of the prefactors in Eqs. (C3) and (C20) cancels that of $\frac{\partial E_{1+}}{\partial\Gamma}$ from Eq. (C12).

Doing the double integral to obtain $E_{2+}(\Gamma)$ by parts, we obtain, after substituting for the coefficients from Eqs. (C15) and (C16),

$$\hat{K}^{\text{sing}}_\infty \approx 16|\delta|^{2-2\phi}C_{1-}^2C_{2-} \times\int_{\tilde{\Gamma}}^{\infty}e^{-2\Gamma|\delta|}\ \frac{\Delta\cosh\Gamma\Delta+|\delta|\sinh\Gamma\Delta}{\sinh\Gamma\Delta}d\Gamma \ , \quad \text{(C22)}$$

with the $|\delta|\sinh\Gamma\Delta$ part not contributing to the singularities and the $n=1$ singularity with amplitude $\sim e^{-2\pi|\delta|/\epsilon}$ dominating for small $\epsilon=\Delta/i$ and yielding Eq. (3.122) for the spectral density ρ_K from Eq. (3.93). Since $\cosh\Gamma\Delta$ is *negative* at this first singularity, the coefficient C_{2-} must be *negative.* This can readily be seen by examining the differential equation for E in the inner region $|\Delta\Gamma|\ll 1$. The boundary condition forces E and $\frac{\partial E}{\partial\Gamma}$ to be positive for small Γ. If one of these changes sign for larger Γ, the first to do so must be $\frac{\partial E}{\partial\Gamma}$. But if this is zero at some Γ, then, since E will still be positive, $\frac{\partial^2 E}{\partial\Gamma^2}>0$ from Eq. (3.4) which is contradictory. Thus, $\frac{\partial E}{\partial\Gamma}$ is positive out to $\tilde{\Gamma}$, implying $C_{2-}<0$.

We close with a brief note on the reason for defining E_{2+} by integrating up from $\tilde{\Gamma}$, as in Eq. (C20), rather than the more natural integrating down from infinity which would make $E_{2+}(\Gamma)=O(e^{-2|\delta|\Gamma})$ for large Γ rather than $O(e^{-2|\delta|\tilde{\Gamma}})$. If E_{2+} had been defined the latter way, then E_{0+} and ϵ_{1+} would have weak singularities as functions of Δ which would cause singular parts of the integral over $\hat{k}'$ for all $\Gamma>\tilde{\Gamma}$. Instead, by integrating E_{2+} up from $\tilde{\Gamma}$, all the singularities are put into E_{2+}. Of course, the calculations could have been done the other way, but considerable care would have been needed to cancel parts of the singularities and end up with the result Eq. (C22) which does not depend on $\tilde{\Gamma}$.

APPENDIX D: CORRELATIONS AMONG THE COUPLINGS

So far, we have completely ignored possible correlations between nearby couplings. Indeed, the independence of couplings presumed in the approximate RG has been crucial to the analysis. If the results are really universal, as we have claimed, then one should be able to show the irrelevance at the critical fixed point of, at the least, weak short-range correlations in the couplings. Note that even if such correlations were not present initially, they would likely be generated by a more careful treatment at the early stages of the renormalization of higher-order perturbative effects, in particular from the effects of bad cases when the neighboring couplings are comparable to the decimated coupling.

In this appendix, we consider the simplest case of weak correlations between nearest neighbor couplings, i.e., between a bond ζ_i and two transverse fields on either side of it, β_i and β_{i+1}. We define the joint distribution of a neighboring ζ,β (i.e., bond-field) pair to be

$$J(\zeta_i,\beta_i)\equiv\Pr(\zeta_i,\beta_i)=P(\zeta_i)R(\zeta_i)+K(\zeta_i,\beta_i) \quad \text{(D1)}$$

and similarly for $\Pr(\zeta_i,\beta_{i+1})$ so that K is a measure of the correlations and is hence zero if ζ and β are independent. In general, from the definition of K,

$$\int K(\zeta,\beta)d\zeta=0 \qquad \text{for all } \beta$$

and

$$\int K(\zeta,\beta)d\beta=0 \qquad \text{for all } \zeta \ . \quad \text{(D2)}$$

If K is small, there is a precise sense in which other correlations can be even smaller and hence ignorable. Specifically, we assume that the conditional probability of, e.g., ζ_0 given everything to the right of it depends only on its immediate neighbor β_1:

$$\Pr(\zeta_0|\beta_1,\zeta_1,\beta_2,\zeta_2,\ldots)=\Pr(\zeta_0|\beta_1)=J(\zeta_0,\beta_1)/R(\beta_1) \ . \quad \text{(D3)}$$

Then the joint probability of neighboring *bonds*,

$$\Pr(\zeta_0,\zeta_1)=P(\zeta_0)P(\zeta_1)+\int\frac{K(\zeta_0,\beta_1)K(\zeta_1,\beta_1)}{R(\beta_1)}d\beta_1 \ , \quad \text{(D4)}$$

so that the correlations of neighboring bonds are of order K^2 and similarly correlations between further neighbors are higher powers of K. This also implies that K "small" should be defined more precisely as

$$\frac{K(\zeta,\beta)}{P(\zeta)R(\beta)} \ \text{uniformly small in } \zeta \text{ and } \beta \ . \quad \text{(D5)}$$

This is equivalent to

$$\left[\frac{\Pr(\zeta_i|\beta_i)}{\Pr(\zeta_i)}-1\right] \ \text{small for all } \zeta_0 \text{ and } \beta_i \quad \text{(D6)}$$

and likewise for $\Pr(\beta_i|\zeta_i)/\Pr(\beta_i)$. The results below are probably true for weaker senses of K being small, but some extra work would be needed to show this.

In general, if all terms of order K^2 are ignored, then it can be shown that further neighbor correlations are *not* generated and a linear RG flow equation for K can be derived:

$$\frac{\partial K(\zeta,\beta)}{\partial\Gamma} = \frac{\partial K}{\partial\zeta} + \frac{\partial K}{\partial\beta} + K(\zeta,0)R(\beta) + K(0,\beta)P(\zeta) + P_0K(\zeta,\cdot) \overset{\otimes}{\beta} R(\cdot) + R_0K(\cdot,\beta) \overset{\otimes}{\zeta} P(\cdot) , \quad \text{(D7)}$$

where the dots are dummy variables of the convolution. With the critical scaling solution $R(\beta) = \frac{1}{\Gamma}e^{-\beta/\Gamma}$ and $P(\zeta) = \frac{1}{\Gamma}e^{-\zeta/\Gamma}$, a general solution of Eq. (D8) with the conditions Eq. (D2) can be expressed as linear combinations of

$$k_{\kappa\nu}(\Gamma)\frac{1}{\Gamma^2}\left[e^{-\kappa\zeta-\nu\beta} - \frac{1}{\kappa\Gamma}e^{-\zeta/\Gamma-\nu\beta} - \frac{1}{\nu\Gamma}e^{-\kappa\zeta-\beta/\Gamma} + \frac{1}{\kappa\nu\Gamma^2}e^{-\zeta/\Gamma-\beta/\Gamma}\right] , \quad \text{(D8)}$$

with

$$\frac{\partial k_{\kappa\nu}}{\partial\Gamma} = \frac{k_{\kappa\nu}}{\Gamma}\left(2 - \kappa\Gamma - \nu\Gamma - \frac{1}{\kappa\Gamma-1} - \frac{1}{\nu\Gamma-1}\right) . \quad \text{(D9)}$$

For large Γ, we thus see that

$$k_{\kappa\nu} \sim \Gamma^2 e^{-(\kappa+\nu)\Gamma} , \quad \text{(D10)}$$

so that initial $K(\zeta,\beta)$ with well-behaved (i.e., exponential) tails will decay exponentially for large Γ. Thus the *joint distribution* $J(\zeta,\beta)$ *converges rapidly* to the *independent product distribution* $J(\zeta,\beta) \to P(\zeta)R(\beta)$. Note that this analysis also shows explicitly that no correlations of different effective couplings develop if they are not present initially; this is indicated by the absence of inhomogeneous terms in Eq. (D7).

APPENDIX E: TRANSFER MATRICES

In this appendix, some of the special properties of the transfer matrices that make many of the results of this paper more exact than might be expected are discussed briefly. Since we will not deal here with spin correlations—indeed at this point is not clear how to do so—we will focus on the properties of the low-energy *spectrum* of the Hamiltonian.

The simplest way to obtain results for the random transverse-field Ising chain, as shown by Shankar and Murthy,[4] is to transfer in the *space* direction and write the full transfer matrix as an outer product of transfer matrices at all possible frequencies. Because of the free fermion nature of the 2D Ising system, each frequency component can be treated separately, most conveniently in terms of a pair of Majorana (real) fermions at each site. The transfer matrix from site j to site $j+1$ along the chain can then be written in the form

$$T_j(\omega) = \begin{pmatrix} \frac{-h_j}{J_j} & \frac{-2\omega}{J_j} \\ \frac{2\omega h_j}{J_j h_{j+1}} & \frac{-J_j}{h_{j+1}}\left(1-\frac{\omega^2}{4J_j^2}\right) \end{pmatrix} , \quad \text{(E1)}$$

where we have rescaled the frequency so that for a single site with a field h the excitation energy (i.e., gap) is $\omega = h/2$. The transfer matrix Eq. (E1) is not in a useful form for decimation. But products of T_j's, $T_nT_{n-1}\cdots T_m$, can be written by factoring T_j's and recombining factors in terms of products of matrices $\hat{J}_n(\omega)\hat{h}_n(\omega)\hat{J}_{n-1}(\omega)\hat{h}_{n-1}(\omega)\cdots\hat{J}_m(\omega)\hat{h}_m(\omega)$ where we have introduced the matrices

$$\hat{J}_j(\omega) \equiv \begin{pmatrix} \frac{1}{J_j} & \frac{\omega}{J_j} \\ \frac{-\omega}{J_j} & J_j - \frac{\omega^2}{J_j} \end{pmatrix} \quad \text{(E2)}$$

and

$$\hat{h}_j(\omega) \equiv \begin{pmatrix} h_j - \frac{\omega^2}{h_j} & \frac{\omega}{h_j} \\ \frac{-\omega}{h_j} & \frac{1}{h_j} \end{pmatrix} . \quad \text{(E3)}$$

The eigenvalue condition for free fermion excitations ω (>0) of a chain $m\cdots n$ can then be simply written as

$$(1\ \omega)\hat{h}_n(\omega)\hat{J}_{n-1}(\omega)\hat{h}_{n-1}(\omega)\cdots\hat{J}_m(\omega)\hat{h}_m(\omega)\begin{pmatrix} 1 \\ -\omega \end{pmatrix} = 0. \quad \text{(E4)}$$

Equation (E4) is in a useful form to perform a decimation transformation by simply selecting the largest coupling, say, J_1, (equal to the energy scale Ω) and multiplying the corresponding matrix in Eq. (E4) by its nearest neighbor matrices. If we are interested in a low frequency $\omega \ll \Omega$, then the resulting matrix can be expanded in $\frac{\omega}{\Omega}$, and for a "good" decimation which must have h_1/J_1 and h_2/J_1 small, the resulting product matrix $\hat{\tilde{h}}(\omega)$ has, up to subdominant corrections, *exactly* the form of Eq. (E3) with h_j replaced by

$$\tilde{h} = \frac{h_1h_2}{J_1} . \quad \text{(E5)}$$

At zero frequency $\omega = 0$, this result is of course trivial as all the matrices are diagonal; however, at nonzero frequency, it is a nontrivial result. From the $\omega = 0$ behavior, one can guess that in some sense the basic decimation result Eq. (E5) is exact [and indeed it is this that determines the form of *typical* energy (or σ^x) correlations as computed by Shankar and Murthy[4]]. But to really show that the decimation procedure is valid, one needs to know that the small-ω frequency dependence of the renormalized matrix $\hat{\tilde{h}}(\omega)$ is of the correct form and that "bad" decimations (with h_1/J_1 and/or h_2/J_1 not small) do not ruin the decimation procedure. Surprisingly, this can in fact be shown explicitly. Specifically, we consider a bad decimation with at least one of, say, h_1 and h_2 of order $\Omega = J_1$.

If one focuses on frequencies of order ω so as to obtain the excitation energy $\tilde{h}/2$ associated with the difference between the even and odd combinations of the "up-" and "down"-spin cluster, then the approximation of replacing the spin cluster matrix product with a matrix like Eq. (E3) with $\tilde{h}$ given by Eq. (E5) is quite bad. Indeed, even if one is interested in low frequencies $\omega \ll \Omega$, as we are, the product matrix of the spin cluster will have substantial corrections. In particular the off-diagonal terms will have multiplicative corrections of $1+2\left(\frac{h_{1,2}}{\tilde{J}_1}\right)^2$ which is 3 in the worst case. Nevertheless, the decimation procedure is saved because of what happens at a later stage. At some lower-energy scale $\Omega \sim \tilde{h}$ (but perhaps not exactly this due to the earlier errors), the "bad" spin cluster will be decimated either by combining with another cluster or by being eliminated; we consider the behavior in the latter case.

If, as typically will be the case, the decimation of the bad $\tilde{h}$ cluster is "good," then it will involve neighboring (effective) bonds $\tilde{J}_0$ and $\tilde{J}_1$ of the cluster which are both much smaller than $\tilde{h}$. In the usual case that these neighboring effective bonds are not themselves results of bad decimations, the resulting effective bond matrix can be directly computed. Amazingly, it has again the form Eq. (E2) with

$$\tilde{J} = \frac{\tilde{J}_0 \tilde{J}_1}{\tilde{h}} \tag{E6}$$

exactly, and negligible corrections to the matrix elements which are reduced by factors of $(\tilde{J}_{1,2}/\tilde{h})^2$ and $(\omega/\tilde{h})^2$ (albeit with coefficients which are larger than if $\tilde{h}$ had been a good cluster). Thus for $\omega \ll$ the original scale $\Omega = J_1$ of the earlier bad decimation, we recover *exactly* at a later stage from the errors made earlier.

We conjecture on the basis of this analysis of one type of bad case that this behavior is true generally. More precisely, if in the critical region the decimation is carried out by the naive rules—i.e., by replacing the cluster matrices by those of the form Eq. (E3) with $\tilde{h}$ given by Eq. (E5) and likewise for bonds—then at low-energy scales Ω and $\omega \ll \Omega$ the resulting transfer matrices will, with high probability, be very close to those obtained by multiplying exactly the same combinations of the original matrices. Furthermore, we expect the main (but rare) source of errors to be bad decimations which have occurred at scales not much larger than Ω; these occur with probability $\sim 1/\ln(\Omega_I/\Omega)$ in the critical region, and $\sim |\delta|$ at low energies just off critical.

To actually get the eigenvalues near some ω, the decimation should be stopped when Ω/ω is some large factor. But then with Ω small, the distribution of $\tilde{h}$'s and $\tilde{J}$'s will be so broad that the eigenvalue condition Eq. (E4) is, to a good approximation, a local condition in a region where some $\tilde{h}$ or $\tilde{J} \approx 2\omega$. Although we have not attempted to prove the basic conjecture stated above, it may well be possible to do so which would, of course, begin to place the results of this paper on a very firm footing. Note also the recent work of Mikheev[18] from a somewhat different direction.

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[1] See, e.g., K. Binder and A.P. Young, Rev. Mod. Phys. **58**, 801 (1986) for a review of spin glasses.

[2] B.M. McCoy and T.T. Wu, Phys. Rev. **176**, 631 (1968); **188**, 982 (1969).

[3] B.M. McCoy, Phys. Rev. **188**, 1014 (1969).

[4] R. Shankar and G. Murthy, Phys. Rev. B **36**, 536 (1987).

[5] N.F. Mott and W.D. Twose, Adv. Phys. **10**, 107 (1961).

[6] R.B. Griffiths, Phys. Rev. Lett. **23**, 17 (1969).

[7] M. Campanino, A. Klein, and J.F. Perez, Commun. Math. Phys. **135**, 499 (1991).

[8] D.S. Fisher, Phys. Rev. Lett. **69**, 534 (1992).

[9] S.K. Ma, C. Dasgupta, and C.-K. Hu, Phys. Rev. Lett. **43**, 1434 (1979); C. Dasgupta and S.K. Ma, Phys. Rev. B **22**, 1305 (1980).

[10] D.S. Fisher, Phys. Rev. B **50**, 3799 (1994); Note that J.E. Hirsh and J.V. José [J. Phys. C **13**, L53 (1980); Phys. Rev. **22**, 5339 (1980); **22**, 5355 (1980)] earlier obtained some of the results of the above paper on antiferromagnetic chains by a different approximate RG transformation. Reference to these was inadvertently omitted from the above paper.

[11] The scaling of the statistics of *extreme* of random walks can easily be seen by "absorbing wall" methods, to be the same as that of typical parts of the walks.

[12] The cutoff in Eq. (1.33), Ω_V, will be reduced from Ω_I by a power of V_I^{-1} for small V_I, i.e., for weak randomness.

[13] A.B. Harris, J. Phys. C **7**, 1671 (1974).

[14] J.T. Chayes, L. Chayes, D.S. Fisher, and T. Spencer, Phys. Rev. Lett. **57**, 299 (1986); Commun. Math. Phys. **120**, 501 (1989).

[15] The appropriate dimension d here is the number of *random* dimensions, i.e., $d=1$ for random quantum spin chains.

[16] See, e.g., *Handbook of Mathematical Functions*, edited by M. Abramowitz and I.A. Stegun, Natl. Bur. Stand. Appl. Math., Ser. No. 55 (U.S. GPO, Washington, D.C., 1965).

[17] For random transverse-field Ising chains that have, e.g., further neighbor interactions but still have a phase transition in the same universality class, one can presumably no longer obtain the exact normalization of δ.

[18] L.V. Mikheev (unpublished and private communication) based on the work of L.V. Mikheev and M.E. Fisher [Phys. Rev. B **49**, 378 (1994)] has recently developed an RG approach which yields qualitative behavior similar to some of that discussed in Sec. I C.

[19] Note that strictly speaking there is no dynamics here, but since the system is quantum mechanical, we have used the notation z rather than the classical notation $\theta = -1/z$ where $-\theta$ is the RG eigenvalue of temperature at a zero-temperature fixed point.

[20] See, e.g., M.E. Fisher, Physics **3**, 255 (1967).

[21] As for the magnetic field scale factor D_H in Eq. (3.51), changes in D_T represent only corrections to scaling. But if lengths are measured in units of ℓ_V, then the proportionality *coefficients* of the specific heat in Eq. (4.30) can be found exactly using the results of Sec. II D, since these imply that the proportionality coefficient in Eq. (4.29) is, in fact, $1/\ell_V$.

[22] D.S. Fisher, J. Appl. Phys. **61**, 3672 (1987), and Refs. 24 and 32.

[23] Note that, equivalently, we could have used the variable $\exp[-\delta \ln(D_1/H_1)] = c_{H_1} H_1^{\delta}$ with $C_{H_1} \to 1$ in the scaling limit so that D_1 represents only *corrections* to scaling, as did D_H in bulk scaling functions.

[24] D.S. Fisher, Phys. Rev. Lett. **56**, 416 (1986).

[25] See, e.g., A.W.W. Ludwig, Nucl. Phys. B **330**, 639 (1990) for a good discussion of multifractal behavior in random magnets.

[26] A. Crisanti and H. Rieger (unpublished).

[27] H. Rieger and A.P. Young, Phys. Rev. Lett. **72**, 4141 (1994).

[28] M. Guo, R.N. Bhatt, and D.A. Huse, Phys. Rev. Lett. **72**, 4137 (1994).

[29] See, e.g., K. Hui and A.N. Berker, Phys. Rev. Lett. **62**, 2507 (1989) and references therein.

[30] See, e.g., B.I. Shklovskii and A.L. Efros, *Electronic Properties of Doped Semiconductors* (Springer-Verlag, New York, 1984).

[31] R.N. Bhatt and P.A. Lee, Phys. Rev. Lett. **48**, 344 (1982) and references therein.

[32] For a brief review of phenomenological scaling governed by zero-temperature fixed points, see D.S. Fisher, G. Grinstein, and A. Khurana, Phys. Today **41** (12), 56 (1988).