Magnetoplasma resonances and nonlinear mode coupling in pools of ions trapped below the surface of superfluid helium

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The results are reported of experiments on magnetoplasma modes of oscillation in two-dimensional circular pools of ions trapped below the free surface of superfluid helium. The theory of such modes is summarized, and experiments are described in which the modes are excited by an external oscillating electric field and detected by the current induced in a neighboring electrode. This technique for direct detection is satisfactory for modes that are axisymmetric, but proves to be rather insensitive for those that are not. A method of detection based on a nonlinear coupling between the two types of modes is described, and the application of this method to the study of the full range of magnetoplasma modes (including edge modes) is shown to lead to experimental results that are in good agreement with theory.

I. INTRODUCTION

This paper is concerned with the behavior of quasi-two-dimensional pools of ions trapped below the free surface of superfluid ^{4}He. Such pools have already been studied in some detail in our laboratory, and an account of this earlier work was given in Ref. 1.

Two types of stable charged particle, or "ion," can be produced easily in liquid helium, by applying a suitable potential to a field emission or field ionization tip immersed in the liquid. The negative ion is an electron in a bubble; at low pressures the bubble has a radius of about 1.9 nm and a temperature-independent effective mass (M) of about $237m_4$, where m_4 is the mass of a helium atom. The positive ion is probably a He_2^+ ion embedded in a small volume of solid helium, formed by electrostriction; the radius of the solid is about 0.55 nm; the effective mass of the whole object depends on temperature and is equal to about $30m_4$ at the lowest temperatures.

Both types of ion can be trapped just below the free surface of superfluid helium by the combination of an external electric field E_0 that forces the ion towards the surface and an image potential that forces the ion away from the surface. In practice the liquid surface is situated approximately midway between two parallel electrodes (spacing h) that produce the field E_0. With a suitable fringing field a circular pool can then be trapped, the density of ions (n_0) being almost constant over the area of the pool except within a distance of order $h/2$ of its edge, where it falls rapidly to zero. The density n_0 cannot exceed $2\varepsilon_0 E_0/e$, which is typically 10^{12} m^{-2}. At temperatures above about 3 mK, there is some thermally excited vertical motion of the ions, but its amplitudes is very small compared with a typical ionic separation. The system is therefore two dimensional in much of its behavior, and it can support two-dimensional plasma waves in which the ionic motion is essentially horizontal. In a circular pool standing plasma waves can form, giving rise to plasma resonances.

In earlier papers we reported detailed studies of axisymmetric plasma resonant modes, both in the absence[1] and the presence[2] of a steady vertical magnetic field. In this paper we report an extension of our work to modes that are not axisymmetric, and we concentrate particularly on plasma modes in the presence of a vertical magnetic field (magnetoplasma modes). We also report the observation of nonlinear coupling between different magnetoplasma modes, and the way in which this coupling can be used to facilitate the observation of nonaxisymmetric modes.

Plasma resonances of the type that we describe here were first observed in two-dimensional pools of electrons trapped *above* the surface of superfluid helium.[3] Magnetoplasma resonances in these electron pools were first observed and explained by Glattli *et al.*[4] and by Mast, Dahm, and Fetter,[5] and our paper is in part a confirmation of the existence of these modes, and of the theory underlying them, in the two-dimensional ion system. However, the experiments that rely on nonlinear coupling between the modes are new and have not, to our knowledge, been carried out on the electron system.

The paper is arranged as follows. In Sec. II we summarize the theory of the magnetoplasma modes, and in Sec. III we present a few rather unsatisfactory data obtained by direct detection of these modes. In Sec. IV we present experimental data on the nonlinear response of a single plasma mode and, more importantly, on the nonlinear coupling that we find between one mode and another. Section V is devoted to the results of "double-drive" experiments in which we exploit the existence of this coupling to detect the nonaxisymmetric magnetoplasma modes in a way that is more satisfactory. In order to account for the exact frequencies of these modes it is necessary to take full account of the actual density profile in the pool, and this is discussed in Sec. VI. Experimental data on the dependence of the damping of magnetoplasma modes on the magnitude of the steady magnetic field are presented and explained in Sec. VII. A preliminary report of this work has already been published.[6]

II. ELEMENTARY THEORY OF THE MAGNETOPLASMA MODES

The theory of these modes was first given in outline in Refs. 4 and 5, and the basis of the theory as applied to the ion pools was set out in a convenient form in Ref. 1. We have a circular ion pool of radius R with a uniform magnetic-flux density B applied in a direction normal to the plane ($z=d$) of the ions [we use cylindrical polar coordinates (r,θ,z)]. The pool is situated between two plane highly conducting electrodes, at $z=0$ and h. The pool will have plasma resonant modes, the frequencies of which are determined by the boundary conditions at the edge of the pool. In this section we shall assume that the edge of the pool is sharp, the pool density dropping abruptly from the constant value n_0 to zero, and that, as discussed in Ref. 1, the radial component of the ionic velocity v_r is zero at the edge. The perturbed charge density associated with any mode then has the form

$$J_m(r_{mn}r)\exp i(m\theta-\omega t)\ , \tag{2.1}$$

where m is a positive or negative integer and the discrete set of wave numbers k_{mn} are determined by the boundary condition at $r=R$. The frequency ω and the wave number k are related by the dispersion relation

$$\omega^2=\omega_P^2(k)+\omega_c^2\ , \tag{2.2}$$

where ω_c is the cyclotron frequency eB/M, and

$$\omega_P^2(k)=\left[\frac{n_0e^2k}{2\varepsilon_0M}\right]F(k)\ . \tag{2.3}$$

The factor $F(k)$, given by

$$F(k)=\frac{2\sinh kd\,\sinh(h-d)}{\sinh kh+(\varepsilon-1)\cosh kd\,\sinh k(h-d)}\ , \tag{2.4}$$

where ε is the dielectric constant of the liquid helium, allows for the screening effects of the electrodes on the Coulomb interaction between the ions. The boundary condition, $v_r=0$ at $r=R$, leads to the condition

$$(\omega-\omega_c)mJ_m(kR)-\omega kRJ_{m+1}(kR)=0\ , \tag{2.5}$$

the solutions of this equation yielding the values of k_{mn}, which depend in general on both ω and ω_c. A particular mode is described by the pair of integers (m,n). In zero magnetic field the modes are essentially longitudinal; in a finite field the Lorentz force leads to some transverse motion.

As explained in Ref. 7, the ion pools can probably exist in three phases, depending on the temperature: a crystal phase, a hexatic phase, and a fluid phase. The crystal phase has a shear modulus, which gives rise to the possibility of shear modes, which generally have a much lower frequency than do the plasma modes; the hexatic and fluid phases have conventional or effective viscosities. The shear modulus present in the crystal phase has only a very small effect on the plasma mode frequencies; the viscosity gives rise to a very small contribution to the damping of the plasma modes, most of the observed damping arising from the finite mobility of the ions, as explained in Ref. 1.

The axisymmetric $(0,n)$ modes are nondegenerate for all values of B. The nonaxisymmetric modes are degenerate with respect to the sign of m for $B=0$, but this degeneracy is lifted by a finite magnetic field. We see from Eq. (2.5) that the values of k_{mn} for an axisymmetric mode are independent of B. The frequency of each axisymmetric mode therefore increases with B in accord with Eq. (2.2), where ω_P is now independent of B, the frequency tending asymptotically to ω_c. In the case of the nonaxisymmetric modes, however, the values of k_{mn} do depend on B, and the resonant frequencies vary with B in a more complicated way. The case of modes with no nodes in the radial direction ($n=1$) is particularly interesting. It can be shown that the frequency of a mode with negative m increases with increasing B, and again tends to ω_c in the limit of large B. In contrast, the frequency of a mode with positive m decreases with increasing B. As we see from Eq. (2.2), when the frequency falls below ω_c, $\omega_P(k)$ must become imaginary, and then k must also be imaginary. This means that the wave becomes evanescent in its dependence on $(R-r)$. The wave propagates round the edge of the pool, but is strongly localized near this edge. It is then called an *edge magnetoplasma wave.* Such waves were observed in two-dimensional *electron* systems trapped *above* the surface of superfluid helium.[4,5]

We emphasize that the quantitative results of this section apply only if the edge of the pool is abrupt and the boundary condition is the vanishing of v_r at this edge. We discuss the effects of relaxing these unrealistic assumptions in Sec. VI. We emphasize that we have also assumed that the pool has exact circular symmetry.

III. DIRECT DETECTION OF THE MAGNETOPLASMA RESONANCES

Our apparatus and experimental techniques are similar to those described in Ref. 1. The cell is in the form of a pill box. The upper and lower plane faces are two plane electrodes that provide the dc holding field E_0. The cylindrical "wall" of the box is a third electrode that provides the fringing field required to confine the pool in a circular shape. The plasma modes are excited by applying a small ac potential of the required frequency to the wall electrode. They are detected by amplifying the current induced on a central part of one of the circular plane electrodes, with lock-in detection to achieve acceptable signal-to-noise ratios. The frequency of the $(0,1)$ mode is typically 40 kHz for negative ion pools; and typically 120 kHz for positive ions.

The cell used in our present work differs from that described in Ref. 1 in that the wall electrode is split into four equal segments to which different potentials can be applied. If the four segments are at the same potential the cell has cylindrical symmetry, within the accuracy with which it was constructed, and the extent to which the nonaxisymmetric modes can be either excited or detected ought to be small. If different potentials are applied to the segments, the cylindrical symmetry is breached, and excitation and detection of the nonaxisymmetric modes ought to be easier. We have checked that this is indeed the case. However, departures from cylindrical symmetry would complicate the theory of Sec. II in a way that we did not attempt to pursue, and would therefore invalidate a comparison of our experimental results with this theory. We have therefore used the cell in a mode in which the cylindrical symmetry is breached only as a result of imperfections in its construction. As a result the direct detection of the nonaxisymmetric modes proves to be insensitive, and we have not devoted much time to it.

A few experimental results for a positive-ion pool are shown in Fig. 1. The experimentally observed frequencies are qualitatively in agreement with theory, but agreement is less good than is the case with the results we report in Sec. V. This failure may be associated with the fact that the pool had to be driven very hard in order to detect the nonaxisymmetric modes, so that the results may have been affected significantly by nonlinear effects. It is also the case that for these results we were using a pool with what was probably too large a radius, so that the density profile in the pool was affected by the liquid meniscus close to the wall electrode.

We note that the frequencies of the different unperturbed modes often cross as the applied magnetic field is changed. However, the modes never become degenerate at these level crossings, as is shown clearly in the inset to Fig. 1. The residual lack of cylindrical symmetry of the cell perturbs the modes, and the resulting hybridization near a level crossing lifts the degeneracy. The degeneracy between the modes $(\pm m, n)$ in zero magnetic field is also removed. When hybridization with an axisymmetric mode occurs, a nonaxisymmetric mode is more easily generated and detected.

It must of course by emphasized that a different design of cell would have facilitated the direct observation of the nonaxisymmetric modes, so that the indirect method, which we describe in Sec. V, may seem unnecessary. However, as we shall explain, the indirect method has enabled us to confirm the existence of the magnetoplasma modes without our having to build a new cell. More importantly, it provides a technique that is useful when efficient detection cannot be achieved by simply redesigning the cell; this is the case with shear modes, which we mentioned briefly in Ref. 7 and which shall be discussing in more detail in a later paper.

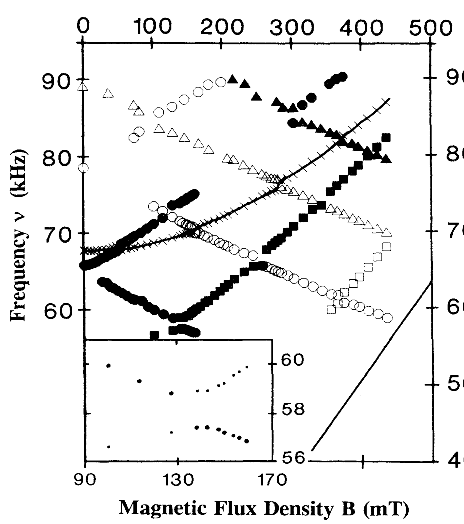

FIG. 1. The frequencies of magnetoplasma modes excited in a pool of positive ions, plotted against magnetic-flux density. $T=15$ mK; $n_0=2.29\times10^{11}$ m^{-2}. □ $(+1,1)$; ■ $(+2,1)$; ● $(\pm3,1)$; ○ $(\pm4,1)$; △ $(-5,1)$; ▲ $(-6,1)$; × $(0,1)$. Inset. Mode crossing: the frequencies of the $(-3,1)$ and $(+2,1)$ modes plotted against magnetic-flux density, for the same positive-ion pool.

IV. NONLINEAR EFFECTS AND MODE COUPLING

The response of any plasma mode becomes markedly nonlinear as the drive is increased, as has been previously reported.[8,9] Examples are shown in Fig. 2, where we see that, in general, there is a broadening and a shift in the resonance.

The theory of magnetoplasma wave propagation given in Sec. I was based on the equation of motion for the ions

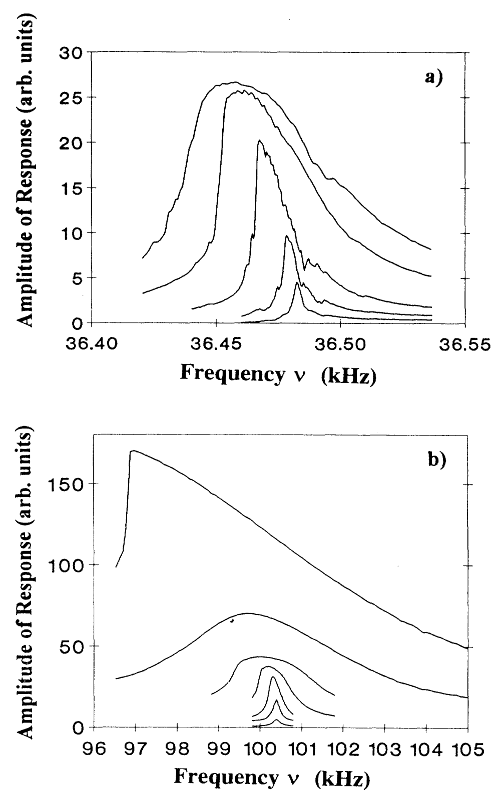

FIG. 2. Nonlinear effects: plasma mode line shapes for different drive levels. (a) Negative ions: drive levels: 0, -5, -10, -15, -25, -35 dB relative to 10 mV p to p on wall electrode; $T=14$ mK; trapping depth: 24.2 nm. (b) Positive ions: drive levels: $+20$, $+10$, 0, -10, -20, -30 dB relative to 10 mV p to p on wall electrode; $T=30$ mK; trapping depth: 27.3 nm.

$$\frac{\partial \mathbf{v}}{\partial t}+(\mathbf{v}\cdot\nabla)\mathbf{v}=-\left[\frac{e}{M}\right](\nabla_{\perp}\phi-\mathbf{v}\times\mathbf{B}) , \quad (4.1)$$

where $\nabla_{\perp}$ is the two-dimensional gradient operator and ϕ is the electrostatic potential, and the equation of continuity

$$\frac{\partial \sigma}{\partial t}+\nabla_{\perp}\cdot(\sigma\mathbf{v})=0 , \quad (4.2)$$

where σ is the charge density in the pool. Both these equations contain nonlinear terms that were neglected in Ref. 1. Part of the nonlinear response is due to these terms.[10] In particular, the second term on the left-hand side of (4.1) gives rise to a nonvanishing change in the average ion density, through what is essentially the Bernoulli effect, and this shifts the resonant frequency of a pool to a lower value. It also leads to a coupling of modes, as we shall see more clearly later, and coupling to the thermally excited plasma modes of high frequency can increase the attenuation of the type of low-frequency mode that we observe.

At least in the case of positive ions there is probably an additional source of nonlinear response. It is known from experiments on the ion pools that the effective mass of the positive ion is quite strongly temperature dependent. It is probable that this temperature dependence is due fundamentally to a velocity dependence of the effective mass (Ref. 1 and references therein), and such a velocity dependence would contribute to the nonlinear plasma response when the ionic flow velocity becomes comparable with the ionic thermal velocities. The nonlinear response does indeed set in when these two velocities are comparable in magnitude. (See Fig. 3.)

These nonlinear effects can be expected to lead not

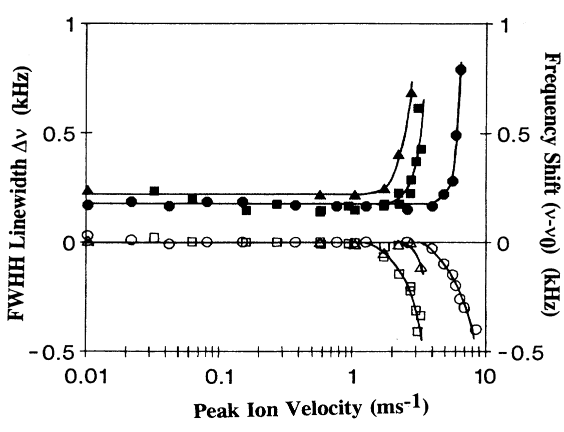

FIG. 3. Nonlinear effects: plasma mode linewidths and frequency shifts plotted against drive level expressed as a peak ion velocity. Positive ions; $T=30$ mK. ○, ●: response of the (0,1) mode; □, ■: response of the (0,2) mode; △, ▲: response of the (0,1) mode driven at low amplitude in the presence of the (0,2) mode driven at the level indicated by the plotted peak ion velocity.

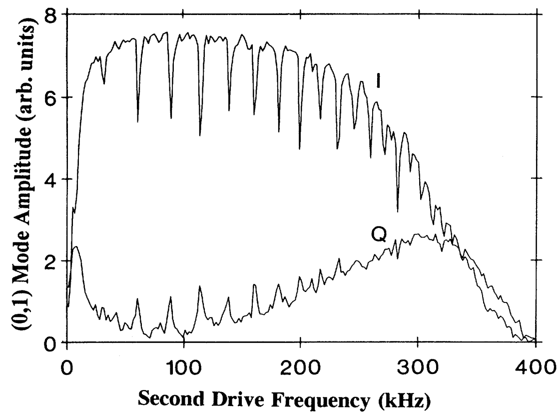

FIG. 4. Response of the (0,1) mode, driven at small amplitude, when a second drive at relatively large amplitude is swept through a range of frequencies: the in-phase and quadrature responses of the (0.1) mode plotted against the frequency of the second drive. Positive-ion pool; $T=30$ mK; $B=3$ T; the frequency of the (0,1) mode is 416.02 kHz.

only to a nonlinear response of a particular plasma mode, but also to a coupling between two modes, the excitation of one mode at a relatively large amplitude having an effect on the response of another mode excited at small amplitude. In general the frequency of maximum response of the second mode will presumably be shifted, and its linewidth will be increased. We have confirmed that these coupling effects do exist, and that they set in when the strongly driven mode is itself starting to exhibit a nonlinear response. The effects are illustrated in Fig. 3 for the case of a positive-ion pool, where we show the response of both the (0,1) and the (0,2) modes separately, together with the effect on the low-amplitude response of the (0,2) mode when the (0,1) mode is driven hard. In the double-drive experiment both modes are driven from the wall electrode, but the lock-in detection records only the response the weakly driven mode, in this case the (0,2) mode.

V. THE APPLICATION OF MODE COUPLING TO THE DETECTION OF NONAXISYMMETRIC MODES

We have already noted that our experimental cell is ill-suited to the direct detection of the nonaxisymmetric modes. The poor coupling of these modes to the electrodes means that *both excitation and detection* are inefficient. However, the nonlinear coupling of one of these modes to one of the axisymmetric modes is not diminished as a result of the cell design. It follows that if this coupling could be used to *detect* the response of a nonaxisymmetric mode, then we would be limited only by the inefficient *excitation.* Observation of the nonaxisymmetric modes ought then to be easier. Furthermore, although the nonaxisymmetric mode has still to be driven fairly hard, it does not have to be driven as hard as it would in a direct detection experiment, so that the mode will be affected less by nonlinearities.

We now present experimental results that confirm the validity of these suggestions. The (0,1) mode is driven continuously with small amplitude from the wall electrode at its normal resonant frequency, $\omega(0,1)$, and the in-phase and quadrature components of the response are observed with the lock-in detector. At the same time a second driving signal of larger amplitude, and frequency ω, is applied to the same wall electrode, and its frequency is swept through a range that includes the frequencies $\omega(m,n)$ of other plasma modes of the system. When ω is close to $\omega(m,n)$, so that the (m,n) mode is strongly excited, the response of the (0,1) mode is modified. As we see from Fig. 4, the shift in resonant frequency and the broadening in the resonant peak give rise to changes in both the in-phase and the quadrature responses. The sequence of magnetoplasma modes is clearly visible.

We emphasize that this technique for the detection of the magnetoplasma modes can be very sensitive. This is especially so at low temperatures, when the damping of the modes is small, so that a very small increase in damping or a very small shift in resonant frequency can produce a large change in the response of the (0,1) mode. There is no need to drive the (m,n) mode at a very large amplitude; it need be driven only at a level that produces a small nonlinear coupling. As we anticipated, this indirect detection of nonaxisymmetric modes proves to be much more sensitive than the direct technique described in Sec. III.

The results of a study by this technique of the magnetoplasma modes in a pool of positive ions are summarized in Fig. 5. They are in good agreement with those obtained by Glattli *et al.*[4] for an electron pool above the helium surface.

This type of double-drive technique can be used to detect other modes of oscillation of the ion pools when direct detection is difficult or impossible. Its use in the probable detection of shear modes in the crystalline phase of the pools has already been reported briefly.[7]

VI. DETAILED THEORY OF THE MAGNETOPLASMA MODE FREQUENCIES

The theory set out in Sec. II was based on the assumption that at the edge of the pool the ion density drops abruptly from the constant value n_0 to zero. In reality the pool density falls to zero more gradually, in the manner described in Ref. 1. The calculation of the magnetoplasma mode frequencies then becomes significantly more complicated. A method of carrying out this more complicated calculation is described in an appendix. This problem had previously been addressed by Glattli *et al.*,[4] and we shall compare the results of the two calculations.

When we compare our computational results with the experimental results described in the last section, we find that there is generally good agreement, as shown in Fig. 6, although the measured frequencies are consistently higher than the predicted ones by about 2%. Such a discrepancy may well be associated with an error in the measured total charge in the pool, which is used, as explained in Ref. 1, to determine the charge density in the pool.

We have compared our computational results with the theoretical predictions of Glattli *et al.*[4] The agreement is effectively perfect for $B=0$. For modes having a frequency that increases with increasing B the differences are small: less than 1% at $B=1$ T. For the edge modes $(\omega<\omega_c)$ the discrepancy is larger, the predictions of Ref. 4 exceeding our own by some 4–5%. However, the analysis by Glattli *et al.* of the edge modes has not been

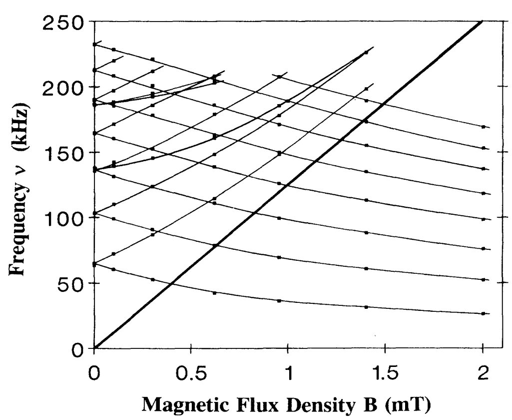

FIG. 5. The frequencies of magnetoplasma modes obtained by the double-drive technique, plotted against magnetic-flux density. Positive ions; $T=30$ mK; $n_0=8.51\times10^{11}$ m^{-2}; pool radius: 12.79 mm. The modes at zero magnetic field with increasing frequency are $(\pm1,1)$, $(\pm2,1)$, $(\pm3,1)$, $(0,1)$, $(\pm4,1)$, $(\pm1,2)$, $(\pm5,1)$, $(\pm6,1)$, $(\pm7,1)$. The thick solid line shows the cyclotron frequency.

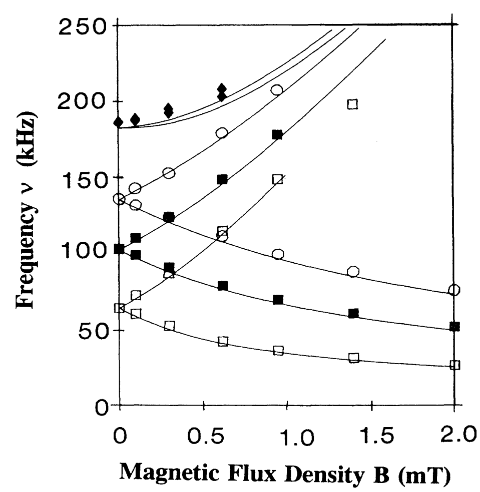

FIG. 6. Comparison of the theory of Sec. VI and the Appendix with some of the experimental results exhibited in Fig. 6. The smooth curves are from the theory; the data points from experiment. □ $(\pm1,1)$; ■ $(\pm2,1)$; ○ $(\pm3,1)$; ♦ $(\pm1,2)$.

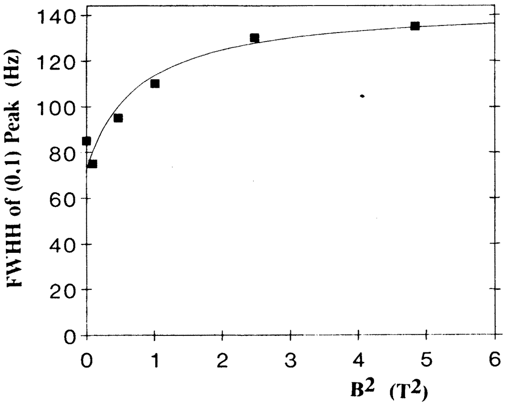

FIG. 7. Dependence of plasma mode linewidth of magnetic field: linewidth plotted against the square of the magnetic-flux density for the (0, 1) mode. Positive ions; $T=15$ mK; trapping depth 43.5 nm. The solid line shows the predicted dependence (7.1).

published in detail, and we have therefore been unable to trace the origin of this discrepancy.

VII. DAMPING OF THE MAGNETOPLASMA WAVES

The damping of the magnetoplasma modes is due to the finite mobility, μ, of the ions, as explained in Ref. 1. This finite mobility can be taken into account by adding a term $(e\mathbf{v}/M\mu)$ to the left-hand side of the equation of motion (4.1). It is then easy to show that in a linear approximation the plasma resonant linewidth (full width at half height in angular frequency) ought to be given by

$$\Delta\omega=\left[\frac{\omega_P^2+2\omega_c^2}{\omega_P^2+\omega_c^2}\right]\frac{e}{M\mu} . \tag{7.1}$$

The linewidth ought therefore to increase by a factor of 2 as the applied magnetic field is raised from zero to a very large value. Experimental confirmation of this predicted dependence of linewidth on applied magnetic field is shown in Fig. 7.

VIII. CONCLUSIONS

We have carried out an experimental investigation of the plasma resonant modes that can be excited in a circular two-dimensional pool of ions trapped below the free surface of superfluid ^{4}He in the presence of a vertical magnetic field. Modes with and without axisymmetry have been studied, and those without have included "edge magnetoplasma modes" in which motion is confined to the edge of the pool. Excitation of the modes has been detected both directly and through a nonlinear coupling to one particular axisymmetric mode, the use of this nonlinear coupling proving to be a more sensitive method for the detection of modes that lack axial symmetry. The observed frequencies of the modes and the dependence of linewidth on magnetic field agree with the results of theoretical analysis.

ACKNOWLEDGMENTS

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APPENDIX

In this appendix we describe a computational study of the magnetoplasma mode frequencies that takes account of the real density profile in the pool.

Let $\sigma_0(r)$ describe the equilibrium charge-density profile of the ion pool $[\sigma_0(r)=en_0(r)]$. In the presence of a magnetoplasma mode the total charge density can then be written

$$\sigma(r,\theta,t)=\sigma_0(r)+\sigma_1(r)\exp i(m\theta-\omega t) , \tag{A1}$$

where the suffix 1 denotes a perturbation associated with the magnetoplasma mode. We assume that all time varying quantities have the dependence on time and azimuthal angle indicated in (A1). The linearized version of the equations of motion (4.1) may then be solved for the radial and azimuthal components of the ionic drift velocity in terms of ϕ_1 and $E_1=-\partial\phi_1/\partial r$, where ϕ_1 is the perturbation in the electrostatic potential due to σ_1. When the results are substituted in the continuity equation (4.2) we obtain

$$\frac{M}{e}(\omega^2-\omega_c^2)r\sigma_1=\frac{d}{dr}(r\sigma_0E_1)+\frac{m^2\sigma_0}{r}\phi_1+\frac{m\omega_c}{\omega}\sigma_0'\phi_1 . \tag{A2}$$

An alternative form of this equation, avoiding explicit mention of the derivative σ_0' (which diverges at the edge of the pool) is obtained by integrating both sides between 0 and r:

$$\frac{M(\omega^2-\omega_c^2)}{2\pi e}\int_0^r 2\pi r\sigma_1 dr =\sigma_0(r)\left\{rE_1+\frac{m\omega_c}{\omega}\phi_1\right\} +m\int_0^r dr\,\sigma_0(r)\left\{\frac{m\phi_1}{r}+\frac{\omega_c}{\omega}E_1\right\} \tag{A3}$$

Computational techniques have been used to find eigenfunctions σ_1 of this equation and their corresponding eigenfrequencies ω.

As already explained in Sec. II, the perturbation σ_1 is expected to have the form $J_m(kr)$ near the center of the pool. However, by analogy with the exact solution available for the same problem in semi-infinite Cartesian geometry, σ_1 is expected to diverge like $(R-r)^{-1/2}$ near the edge of the pool.[11] A flexible form of σ_1 capable of modeling this behavior is provided by expression σ_1 as a linear spline multiplied by r^m. The region $0\le r\le R$ is divided into N annuli, and on each annulus $\sigma_1(r)/r^m$ is assumed to have a linear variation. This is equivalent to expressing $\sigma_1(r)/r^m$ as a sum of B splines B_l each of which is nonzero only on two annuli:[12]

$$\sigma_1(r)=\sum_{l=1}^{N+1} u_l B_l(r)\times r^m , \tag{A4}$$

where the u_l are parameters to be determined. Better convergence as a function of N is obtained by including an explicit divergence at the edge of the pool in place of the final B spline. The form $r^m/(R^2-r^2)^{1/2}$ enables integrals involved in deriving the corresponding potential to be evaluated explicitly. It should be noted that the field corresponding to this term is finite at the edge of the pool. Using the known Green function, the potential and field can be written as sums corresponding to the above form of σ_1.

$$\phi_1(r)=\sum_{l=1}^{N+1} u_l\phi_l(r), \quad E_1(r)=\sum_{l=1}^{N+1} u_l E_l(r) . \tag{A5}$$

Substituting (A4) and (A5) into (A3) we find

$$\lambda \sum_{l=1}^{N+1} u_l Q_l(r)=\sigma_0(r)\sum_{l=1}^{N+1} u_l \left[rE_l+m\frac{\omega_c}{\omega}\phi_l \right] +m\sum_{l=1}^{N+1} u_l\eta_l(r) , \tag{A6}$$

where

$$Q_l(r)=\int_0^r 2\pi r B_l(r)dr ,$$

$$\eta_l(r)=\int_0^r \sigma_0(r)\left[\frac{m\phi_l}{r}+\frac{\omega_c}{\omega}E_l\right]dr ,$$

and

$$\lambda=\frac{M}{2\pi e}(\omega^2-\omega_c^2) .$$

The integrals η_l must be evaluated numerically, since we have no analytic form for $\sigma_0(r)$;$\sigma_0(r)$ is obtained by numerical modeling as described in Ref. 1. For best numerical stability (A6) is applied at the centers of the annuli and at the edge of the pool. We then obtain $(N+1)$ equations in the form of a generalized eigenvalue problem, which can be solved by standard routines. Note that ω_c/ω must be specified as a parameter; we then obtain corresponding values of $(\omega^2-\omega_c^2)$ and hence extract ω as a function of ω_c.

Stable eigenfrequencies are obtained with N of the order of 50. The method does not depend on (h/R) being small and can be used for all values of the magnetic field. The procedure could, in principle, handle any form of the equilibrium profile.

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