

Nearly localized states in weakly disordered conductors

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The time dispersion of the averaged conductance $G(t)$ of a mesoscopic sample is calculated in the long-time limit when t is much larger than the diffusion traveling time t_D . In this case the functional integral in the effective supersymmetric field theory is determined by the saddle-point contribution. If t is shorter than the inverse level spacing Δ ($\Delta t/\hbar \ll 1$), then $G(t)$ decays as $\exp[-t/t_D]$. In the ultra-long-time limit ($\Delta t/\hbar \gg 1$) the conductance $G(t)$ is determined by the electron states that are poorly connected with the outside leads. The probability to find such a state decreases more slowly than any exponential function as t tends to infinity. It is worth mentioning that the saddle-point equation looks very similar to the well known Eilenberger equation in the theory of dirty superconductors.

We consider long-time relaxation phenomena in a disordered conductor that is attached to ideal leads. For simplicity we assume that the electrons in this conductor do not interact with each other, the temperature is zero ($T = 0$) and there are no inelastic processes. The total current $I(t)$ at time t depends upon the voltage according to the Ohm law,

$$I(t) = \int_{-\infty}^t dt' G(t-t') V(t').$$

We are interested in the asymptotic form of the conductance $G(t)$ as $t \rightarrow \infty$.

The same problem has been considered earlier by Altshuler, Kravtsov, and Lerner¹ (AKL). Our initial goal was to obtain their results by means of a more direct calculation. At present, we can neither confirm nor disprove the AKL results. We have found an intermediate range of times, where the conductance $G(t)$ decays more slowly than it was predicted in Ref. 1. The AKL asymptote could be valid at longer times (see discussion below).

There are three time scales in the problem.

(1) The mean-free time $\tau = l/v_F$, where v_F is the Fermi velocity and l is the mean-free path. This time scale determines the dispersion of the Drude conductivity $\sigma_0 \sim e^{-t/\tau}$.

(2) The time of diffusion through the sample $t_D = L^2/D$, where $D = l^2/3\tau$ is the diffusion coefficient, and L is the sample size.

(3) The inverse mean level spacing $\hbar/\Delta = \hbar\nu V$, where ν is the density of states and V is the volume of the sample.

In a macroscopic sample the inequality $\tau \ll t_D \ll \hbar/\Delta$ is valid, provided that $L \gg l$ and the disorder is weak. Indeed, the product Δt_D is connected with the dimensionless conductance of the sample $g = 2\pi\hbar/(t_D\Delta)$, which is large for a weak disorder. The times t_D and \hbar/Δ enter into time dispersion only due to quantum corrections to conductivity.

At times $t \ll \hbar/\Delta$ an electron can be considered a wave packet of many superimposed states propagating

semiclassically. Therefore, it is natural to assume that the conductance $G(t)$ is proportional to the probability of finding a Brownian trajectory that remains in the sample for the time t . For $t \gg t_D$ such a probability decays as $\exp[-t/t_D]$. Our calculations confirm this result.

In the opposite limit, for $t \gg \hbar/\Delta$, the conductance $G(t)$ is proportional to the probability of finding an electron state with the lifetime t . In order to trap an electron for a long time the state must be poorly connected with the leads (nearly localized). We show that the probability of finding such a state decays nonexponentially with time. Namely, $G(t) \sim \exp[-g \ln^2(t\Delta)]$ for $d = 1$ and $G(t) \sim (t\Delta)^{-g}$ for $d = 2$. These results are not valid in the very-long-time limit. We discuss this later together with the question of dimensional crossover.

Instead of calculating the conductance as a function of time, we could have worked in the frequency representation. In that way we would have found a singularity in $G(\omega)$ as $\omega \rightarrow 0$. This singularity, however, does not affect the value of the dc conductance and, therefore, has an obscure physical meaning, while the time domain results have the direct interpretation.

Since the long-time asymptote corresponds to the rare events when the electron is nearly trapped in the sample, it is natural to use the saddle-point approximation. We carry out the following program:

(1) We express the averaged conductance² as a functional integral over supermatrices³ (see Refs. 4 and 5 for a review):

$$G(t) = G_0 e^{-t/\tau} + \int \frac{d\omega}{2\pi} e^{-i\omega t} \int \mathcal{D}Q(r) P\{Q\} \exp[-A],$$

$$A = \frac{\pi\nu}{8} \int dr \text{Str}\{D(\nabla Q)^2 + 2i\omega\Lambda Q\}. \quad (1)$$

(2) We vary the action A with respect to Q , taking into account the constraint $Q^2 = 1$, and obtain the saddle-point condition which recalls the diffusion limit of the Eilenberger equation⁶

$$2D\nabla(Q\nabla Q) + i\omega[\Lambda, Q] = 0. \quad (2)$$

(3) We derive the condition at the boundary with the lead

$$Q|_{\text{lead}} = \Lambda. \quad (3)$$

(4) We perform the integration over ω in Eq. (1) and obtain the self-consistency condition

$$\int \frac{dr}{V} \text{Str}(\Lambda Q) = -\frac{4t\Delta}{\pi\hbar} \quad (4)$$

which allows us to exclude ω from Eq. (2).

(5) We substitute the solution of Eq. (2) with boundary conditions (3) in Eq. (1) and obtain the results with exponential accuracy.

The 8×8 supermatrix Q has commutative and anti-commutative matrix elements. Since $Q^2 = 1$ it can be chosen in the form⁷:

$$Q = V^{-1}HV, \quad V = \begin{pmatrix} u & 0 \\ 0 & v \end{pmatrix},$$

$$\Lambda = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad H = \begin{pmatrix} \cos \hat{\theta} & i \sin \hat{\theta} \\ -i \sin \hat{\theta} & -\cos \hat{\theta} \end{pmatrix}, \quad (5)$$

$$\hat{\theta} = \begin{pmatrix} \theta_1 & 0 & 0 & 0 \\ 0 & \theta_1 & 0 & 0 \\ 0 & 0 & i\theta & 0 \\ 0 & 0 & 0 & i\theta \end{pmatrix}.$$

This decomposition allows us to present the action A in the form

$$A = \frac{\pi\nu}{8} \int dr \text{Str}\{D(\nabla H)^2 + DM^2 + 2i\omega\Lambda H\}, \quad (6)$$

where $M = [V^{-1}\nabla V, H]$. The minimum action is reached for $V = \text{const}$, and Eq. (6) may be expressed in terms of θ variables only,

$$A = \frac{\pi\nu}{2} \int dr \{[D(\nabla\theta)^2 - 2i\omega \cosh \theta] + [D(\nabla\theta_1)^2 + 2i\omega \cos \theta_1]\}. \quad (7)$$

Consequently, Eq. (2) has the form

$$D\nabla^2\theta + i\omega \sinh \theta = 0, \quad (8a)$$

$$D\nabla^2\theta_1 + i\omega \sin \theta_1 = 0. \quad (8b)$$

The boundary condition (3) follows from the fact that Q does not fluctuate in the bulk electrodes, $Q = \Lambda$. Hence, at the boundary with the ideal lead $\theta = \theta_1 = 0$.⁸ The time decay of the conductance $G(t) \sim \exp(-i\omega t)$ corresponds to real and positive values of $i\omega$. The permitted values of frequency ω in Eq. (8b) are bounded from below by the value $\omega_1 \sim 1/t_D$, which corresponds to the linearized form of Eq. (8b). For smaller frequencies $\omega < \omega_1$, which will turn out to be the only relevant ones, Eq. (8b) has only trivial solutions $\theta_1 = 0$. Thus, the self-consistency equation (4) has the form

$$\int \frac{dr}{V} \{\cosh \theta - 1\} = \frac{t\Delta}{\pi\hbar}. \quad (9)$$

The solutions of Eq. (8a) depend on the sample geometry. We start by considering a one-dimensional wire of length L , attached to ideal leads at $x = \pm L/2$. If $t\Delta \ll 1$, then, to satisfy the self-consistency condition (9) we must choose $\theta \ll 1$. Therefore, Eq. (8a) can be linearized. The solutions that satisfy the boundary conditions are

$$\theta = C \cos(\pi n x/L), \quad \omega_n = \frac{-i\pi^2 n^2}{t_D}, \quad (10)$$

where n is an arbitrary integer. The above formula for the frequency implies that in the discussed regime

$$G(t) \sim e^{-i\omega_1 t} = \exp\left(-\frac{\pi^2 t}{t_D}\right). \quad (11)$$

To obtain this result we determine the amplitude C from the linearized self-consistency equation, and then substitute (10) into the action A .

For arbitrary times Eqs. (8a) and (9) in dimensionless coordinates have the form

$$\frac{d^2\theta}{dz^2} + \frac{\gamma^2}{2} \sinh \theta = 0, \quad z = \frac{x}{L}, \quad (12)$$

$$\int_{-1/2}^{1/2} dz [\cosh \theta - 1] = \frac{\Delta t}{\pi\hbar}, \quad \gamma^2 = 2i\omega t_D. \quad (13)$$

The solution of (12) is symmetric $\theta(z) = \theta(-z)$, and in the region $z > 0$ is given by the quadrature:

$$z = \frac{1}{\gamma} \int_{\theta(z)}^{\theta_0} \frac{d\theta'}{\sqrt{\cosh \theta_0 - \cosh \theta'}}, \quad (14)$$

$$\theta_0 = \theta(0) = 2 \ln \frac{1}{\gamma} + 2 \ln \ln \frac{1}{\gamma} \quad \text{for } \gamma \ll 1. \quad (15)$$

The function $\theta(z)$ is almost linear $\theta = \theta_0(1 - 2|z|)$ everywhere except in the region $|z| < 1/\ln(1/\gamma) \ll 1$. Substituting Eq. (14) into Eqs. (9) and (1) we get

$$i\omega = \frac{2g}{t} \ln \frac{t\Delta}{\hbar}, \quad G(t) \sim \exp\left[-g \ln^2 \frac{t\Delta}{\hbar}\right]. \quad (16)$$

As mentioned earlier, the contribution from the individual nearly localized states dominates in $G(t)$ whenever $t\Delta/\hbar \gg 1$. The square modulus of the wave function for such a state $|\Psi|^2$ equals $\cosh \theta$. As we can see, this value decays exponentially towards the leads, where $|\Psi(x = \pm L/2)|^2 = \cosh \theta(\pm 1/2) = 1$. Because of the latter condition, the current through the wire is equal to unity. Therefore, the escape time t is proportional to the normalization integral. This is exactly what is stated in the self-consistency condition (13) for $\theta \gg 1$. To summarize, the wave function is localized in the region $|x| \ll \xi \ll L$ with the localization length $\xi = L/\ln(t\Delta/\hbar)$ and the probability to find such a state is given by Eq. (16). For very long times, when ξ becomes less than the transverse size of the sample, the one-dimensional regime crosses over to a two- or three-dimensional one.

In the two-dimensional case we consider a mesoscopic

disk of radius R surrounded by a well conducting electrode. The Laplacian operator in Eq. (8) is now two dimensional and the boundary condition is $\theta(R) = 0$ at the circumference of the disk. It is natural to assume that the minimal action corresponds to θ that depends on the radius only and, therefore, obeys the equation

$$\theta'' + \theta'/z + i\omega t_D \sinh \theta = 0, \quad \theta(1) = 0, \quad (17)$$

where $z = r/R$ and $t_D = R^2/D$.

For $t \ll \hbar/\Delta$, Eq. (17) can be linearized. Its solution is the Bessel function

$$\theta = C J_0(\gamma z), \quad \gamma = \sqrt{i\omega t_D} = \mu_n,$$

where μ_n denotes the n th zero of the Bessel function. The conductance is

$$G(t) \sim \exp\left(-\frac{\mu_1^2 t}{t_D}\right), \quad t_D \ll t \ll \hbar/\Delta. \quad (18)$$

For a long-time tail $t \gg \hbar/\Delta$, the nonlinear term in Eq. (17) is large near the origin and can be neglected elsewhere. As a result,

$$\theta(z) = C \ln \frac{1}{z} \quad (19)$$

for all but very small z . On the other hand, for $z \ll 1$, the parameter θ is large and $\sinh \theta = e^\theta/2$. With this approximation the solution of Eq. (17) can be found⁹ having the asymptote

$$\theta(z) = -\theta(0) + 6 \ln 2 + \ln \frac{4}{\gamma^2} + 4 \ln \frac{1}{z}, \quad (20)$$

for $\gamma \ll z \leq 1$. Comparing with Eq. (19), we have $\theta(0) = 6 \ln 2 + \ln(4/\gamma^2)$ and $C = 4$. To calculate the integral in the self-consistency equation

$$\frac{\Delta t}{2\pi\hbar} = \int_0^1 \{\cosh \theta(z) - 1\} z dz \quad (21)$$

we multiply Eq. (17) by z , integrate in the limits 0 and 1, and obtain

$$z \frac{d\theta}{dz} \Big|_0^1 + i\omega t_D \int_0^1 \sinh \theta(z) z dz = 0. \quad (22)$$

Since $\theta(0) \gg 1$, we neglect the difference between the integrals in Eqs. (21) and (22), and with asymptote (19) finally get $i\omega = 4g/t$. The action A is dominated by the contribution of the tail (19)

$$A = 4g \ln \frac{t\Delta}{2\pi\hbar}, \quad G \sim \left(\frac{\hbar}{\Delta t}\right)^{4g}. \quad (23)$$

The characteristic size of the averaged 2D wave function is $\xi = \gamma R = R(\hbar/t\Delta)^{1/2}$. The crossover to a 3D case occurs when ξ becomes comparable with the film thickness.

The consideration of the 3D case makes relevant the question of the validity of the diffusion approximation. As before, we consider a disordered drop of radius R sur-

rounded by a well conducting lead. Analogously to what has been done in the 2D case, the function θ depends on the radius r only and obeys Eq. (8a), where the Laplace operator is substituted by its 3D radial component. The boundary condition is $\theta(r = R) = 0$. The analysis of the linear regime is similar to that for 1D and 2D cases and gives for $t_D = R^2/D \ll t \ll \hbar/\Delta$,

$$A = \pi^2 t/t_D, \quad G(t) \sim \exp(-\pi^2 t/t_D). \quad (24)$$

The nonlinear in θ regime leads to the equation

$$\frac{d^2\theta}{dz^2} + \frac{2}{z} \frac{d\theta}{dz} + i\omega t_D \sinh \theta = 0, \quad z = \frac{r}{R}. \quad (25)$$

The analysis of this equation shows that the permitted values of ω are larger than a certain value $\omega_0 > 0$, and that the integral in Eq. (9) remains finite even for the solutions of Eq. (25) with $\theta(r = 0) \rightarrow \infty$. As a result, the self-consistency equation cannot be satisfied for sufficiently long time $t \geq \hbar/\Delta$. Thus, for $|\omega t_D| \ll 1$, all nontrivial solutions of Eq. (25) satisfying the condition $\theta(1) = 0$ are singular at $z \rightarrow 0$. Therefore, the derivative $d\theta/dr$ becomes comparable with the inverse mean-free path $1/l$ for a certain radius r_* . The diffusion approximation inevitably breaks down for smaller distances, where nonlocal corrections become important.

We do not try to solve the kinetic problem now but assume that the mentioned nonlocality smoothes out the singularity at the origin. We also assume that, similarly to what has happened in the 1D and 2D cases, the nonlinear term in Eq. (25) can be neglected at $r > r_*$ and is important for $r \sim r_*$. Thus

$$\theta(r) \sim C \left(\frac{R}{r} - 1\right), \quad r > r_*,$$

$$1 = l \frac{d\theta}{dr} \Big|_{r=r_*} = \frac{ClR}{r_*^2},$$

and $r_* = (ClR)^{1/2}$. Then $\theta_* = \theta(r_*) = (CR/l)^{1/2}$ and, finally,

$$\omega t_D \exp(\theta_*) \sim \frac{\theta_* R^2}{r_*^2} \sim \frac{1}{\theta_*} \left(\frac{R}{l}\right)^2,$$

which gives

$$\theta_* = \ln \left[\frac{1}{\omega t_D} \left(\frac{R}{l}\right)^2 \right], \quad C = \frac{l}{R} \ln^2 \left[\frac{1}{\omega t_D} \left(\frac{R}{l}\right)^2 \right].$$

Using the self-consistency condition we express the frequency ω through the time t and obtain a rough estimate for the action

$$A \sim \left(\frac{l}{\lambda}\right)^2 \ln^\alpha \left(\frac{t}{\tau}\right), \quad G(t) \sim \exp[-A(t)], \quad (26)$$

where $\lambda = \hbar/p_F$ is the Fermi wavelength and the exponent $\alpha \sim 1$ can only be determined from the solution of the kinetic problem. Equation (26) looks similar to the AKL result if $\alpha = 2$. The action (26) becomes comparable with that from Eq. (24) at

$$t = t_* = t_D \left(\frac{l}{\lambda} \right)^2 = \frac{\hbar l}{\Delta R}, \quad t_D \ll t_* \ll \frac{\hbar}{\Delta}. \quad (27)$$

Therefore, we have two independent contributions to the conductance $G(t)$, which is dominated by one of them [Eq. (24)] at $t < t_*$ and by the other [Eq. (26)] at $t > t_*$.

It is sensible now to analyze whether the diffusion treatment is valid in the 1D and 2D cases. Using the solutions of Eqs. (12) and (17) we find the value of t_* such that for $t < t_*$ the derivative $ld\theta/dr$ is less than unity. This gives

$$t_* = \frac{\hbar}{\Delta} \begin{cases} e^{L/l}, & d = 1 \\ \left(\frac{R}{l} \right)^2, & d = 2. \end{cases} \quad (28)$$

Therefore, we can expect that the asymptotes (16) and (23) are valid for $t < t_*$. At longer times $t > t_*$ for all dimensions $d = 1, 2, 3$ the asymptote cannot be found

within the diffusion approximation. A detailed kinetic analysis of this problem will be done in a separate work.

We want now to speculate on the possible relation of this long-time tail to the AKL asymptote. AKL studied the coefficient growth rate in a power expansion of $G(\omega)$ in $\omega\tau$. The rate they found means that $G(t)$ has a universal logarithmically normal asymptote at very long-times. The presence of the mean-free time τ in this expansion makes it plausible that the AKL asymptote is related to a kind of kinetic problem. This is why we hope that at $t \gg t_*$ the tail will match the results of Ref. 1.

In the case of a tunnel barrier at the sample-lead interface, the time dispersion of the conductance can be considered in the same way with the usage of the generalized boundary condition (see Ref. 8).

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¹ B.L. Altshuler, V.E. Kravtsov, and I.V. Lerner, Pis'ma Zh. Eksp. Teor. Fiz. **45**, 160 (1987) [JETP Lett. **45**, 199 (1987)]; **67**, 799 (1988); in *Mesoscopic Phenomena in Solids*, edited by B.L. Altshuler, P.A. Lee, and R.A. Webb (North-Holland, Amsterdam, 1991), p. 449.

² Our results represent the time-dependent conductance $G(t)$ of a large set of mesoscopic junctions with identical macroscopic parameters. For a single junction the fluctuations of the conductance $G(t)$ are small compared to its averaged value, if $t\Delta \ll 1$, and are not small in the opposite limit.

³ The explicit form of prefactor $P\{Q\}$ is irrelevant within the exponential accuracy.

⁴ K.B. Efetov, Adv. Phys. **32**, 53 (1983).

⁵ J.J.M. Verbaarschot, H.A. Weidenmuller, and M. Zirnbauer, Phys. Rep. **129**, 367 (1985).

⁶ H. Eilenberger, Z. Phys. **44**, 1288 (1968); K.D. Usadell, Phys. Rev. Lett. **25**, 507 (1970); A.I. Larkin and Yu.N.

Ovchinnikov, Zh. Eksp. Teor. Fiz. **73**, 299 (1977) [Sov. Phys. JETP **46**, 155 (1977)]; A. Schmid, in *Nonequilibrium Superconductivity, Phonons and Kapitza Boundaries*, Vol. 65 of *NATO Advanced Study Institute, Series B: Physics*, edited by K.E. Gray (Plenum, New York, 1981).

⁷ To be specific, we consider the unitary ensemble. As it follows from the derivation below, to the exponential accuracy, the results do not depend on the ensemble.

⁸ For an arbitrary transparency of the sample-lead interface T the boundary condition has the form $lQ\nabla Q + 3/4[T/(1-T)][\Lambda, Q] = 0$. M.Yu. Kupriyanov and V.F. Lukichev, Zh. Eksp. Teor. Fiz. **94**, 139 (1988) [Sov. Phys. JETP **67**, 1163 (1988)].

⁹ The substitutions $z = e^{-\eta}$ and $\theta = \psi(\eta) + 2\eta$ transform Eq. (17) into the form $\psi'' + \gamma^2/4e^\psi = 0$ which has the first integral.