Spontaneous interlayer coherence in double-layer quantum Hall systems: Charged vortices and Kosterlitz-Thouless phase transitions

K. Moon, H. Mori,* Kun Yang,† S.M. Girvin, and A.H. MacDonald
Department of Physics, Indiana University, Bloomington, Indiana 47405

L. Zheng
Department of Physics and Astronomy, University of Kentucky, Lexington, Kentucky 40506

D. Yoshioka
Institute of Physics, College of Arts and Sciences, University of Tokyo, Komaba, Meguroku, Tokyo 153, Japan

Shou-Cheng Zhang
Department of Physics, Stanford University, Palo Alto, California 94305
(Received 5 July 1994)

At strong magnetic fields, double-layer two-dimensional electron-gas systems can form an unusual broken-symmetry state with spontaneous interlayer phase coherence. In this paper we explore the rich variety of quantum and finite-temperature phase transitions associated with this broken symmetry. We describe the system using a pseudospin language in which the layer degree of freedom is mapped to a fictional spin 1/2 degree of freedom. With this mapping the spontaneous symmetry breaking is equivalent to that of a spin 1/2 easy-plane ferromagnet. In this language, spin textures can carry a charge. In particular, vortices carry $\pm e/2$ electrical charge and vortex-antivortex pairs can be neutral or carry charge $\pm e$. We derive an effective low-energy action and use it to discuss the charged and collective neutral excitations of the system. We have obtained the parameters of the Landau-Ginzburg functional from first-principles estimates and from finite-size exact diagonalization studies. We use these results to estimate the dependence of the critical temperature for the Kosterlitz-Thouless phase transition on layer separation.

I. INTRODUCTION

Technological progress has made it possible to produce double-layer two-dimensional (2D) electron-gas systems of extremely high mobility. As illustrated schematically in Fig. 1, these systems consist of a pair of 2D electron gases separated by a distance d so small ($d \sim 100$ Å) as to be comparable to the typical spacing between electrons in the same layer. In a large magnetic field, strong correlations between the layers have long been expected to lead to novel fractional quantum Hall effects. Correlations are especially important in the strong magnetic-field regime because all electrons can be accommodated within the lowest Landau level and execute cyclotron orbits with a common kinetic energy. The fractional quantum Hall effect occurs when the system has a gap for making charged excitations, i.e., when the system is incompressible, and theory has predicted[1–3] that at some Landau level filling factors, gaps occur in double-layer systems only if interlayer interactions are sufficiently strong. These theoretical predictions have recently been confirmed.[4] More recently work from several different points of view[5–9] has suggested that interlayer correlations can also lead to unusual broken-symmetry states with spontaneous phase coherence between layers which are isolated except for interlayer Coulomb interactions. We have recently argued[10] that it is spontaneous interlayer phase coherence that is responsible for the recently discovered[11] extreme sensitivity of the fractional quantum Hall effect at total Landau level filling factor $\nu = 1$ to small tilts of the magnetic field away from the normal to the layers. ($\nu \equiv N/N_\phi$, where N is the number of electrons and N_ϕ is the number of single-particle levels per Landau level.)

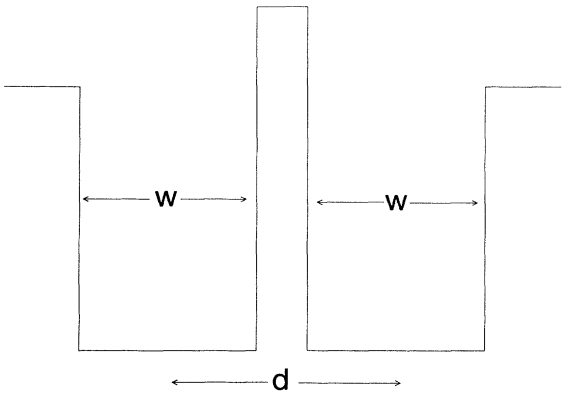

FIG. 1. Schematic conduction band edge profile for a double-layer two-dimensional electron gas system.

We have previously[10] presented a phenomenological theory of the rich zero-temperature phase diagram associated with spontaneous interlayer coherence. In the present paper we provide a detailed microscopic deriva-

0163-1829/95/51(8)/5138(33)/$06.00

tion of the effective action used in our phenomenological theory. We also discuss the low-energy neutral and charged excitations of the system in some detail. Much of our discussion of the properties of double-layer systems with spontaneous interlayer coherence will be couched in language based on a simple mapping[7] of the layer degree of freedom in a double-layer system to an artificial pseudospin degree of freedom. In this language the spontaneous-interlayer-coherence broken symmetry appears as easy-plane ferromagnetism. The mapping is convenient because the Hamiltonian of the system may be simply expressed in terms of pseudospin operators and because some aspects of the physics are familiar when expressed in this way. This mapping is discussed in detail in Sec. II. In Sec. III we present a microscopic derivation of the connection between spin textures and Coulomb charges in our model. This relationship was discussed previously by Sondhi *et al.*[12] in the context of a Chern-Simons effective field theory for the case of a single-layer system at $\nu = 1$ with weak Zeeman coupling (to the real spin). In Sec. IV we derive an effective action which describes the low-energy physics of the system whenever the system has spontaneous interlayer coherence *and* is incompressible. We use the spin-charge connection and our effective action in Sec. V to discuss the low-lying excitations of the system which are formed from vortices in the pseudospin configuration. We show that vortices carry charge $\pm e/2$ and that the vortices appear in four flavors corresponding to the independently available sign choices for vorticity and Coulomb charge. Neutral excitations of finite energy can be formed from vortex pairs with both opposite vorticity and opposite charge. Collective spin-wave-like excitations of the system also occur and dominate response functions at long wavelengths. These collective modes are discussed in Sec. VI. We believe that a double-layer system with spontaneous interlayer coherence will have a finite-temperature Kosterlitz-Thouless phase transition,[5,6] which we discuss in Sec. VII. In the low-temperature phase a kind of superconducting behavior will occur in which the linear resistivity vanishes when opposite currents are carried in the two layers. Fully microscopic calculations for the double-layer systems, using exact diagonalization studies of finite-size systems and many-body perturbation theory, are reported in Sec. VIII. These calculations allow us to estimate the parameters of the low-energy effective action and hence to provide quantitative estimates of the dependence of the Kosterlitz-Thouless temperature on the separation between the layers. Section IX gives a brief summary of the Chern-Simons effective field theory description of double-layer systems. Finally, in Sec. X we briefly summarize our findings. A companion paper[13] will discuss issues which arise when a weak symmetry-breaking tunneling term is added to the Hamiltonian of the double-layer system, particularly those issues which arise from the recent experiments of Murphy *et al.*[11]

II. SPIN ANALOGY

We wish to show that the double-layer system at certain total filling factors, particularly at $\nu = 1$, can be viewed as an *easy-plane quantum itinerant ferromagnet.* In this section we will give a qualitative introduction to the essential ideas of the physical picture. The mathematical details of the microscopic physics will be presented in the subsequent sections.

We will use a pseudospin magnetic language in which pseudospin up (down) refers to an electron in the upper (lower) layer.[14] Using this language and building upon recent progress in understanding the case of single-layer systems at $\nu = 1$ with real spin,[15,12] we will explore the consequences of the mixing of charge and pseudospin degrees of freedom and discuss the rich variety of phase transitions controlled by temperature, layer separation, layer charge imbalance, and in the companion paper[13] we explore the effects of tunneling between layers, and magnetic-field tilt angles. The present section will be devoted to development of a physical picture of the rather counterintuitive concept of spontaneous phase coherence between the layers, which in the magnetic analogy corresponds to spontaneous pseudospin magnetization. Technical details of the microscopic calculations on which this picture is based will be presented in the subsequent sections.

It is helpful to begin study of the pseudospin analogy by reminding ourselves of the unusual properties of a single-layer system at $\nu = 1$ in the limit of zero Zeeman splitting (for the real spins).[12,16] In the presence of Coulomb repulsion between the particles, Hund's rule would suggest that the system could lower its interaction energy by maximizing its total spin since states with maximum total spin are symmetric under spin exchange and hence the spatial wave function is necessarily fully antisymmetric. In an ordinary ferromagnet the Hund's rule tendency to maximize the total spin is partially counteracted by the increase in kinetic energy (due to the Pauli principle) that accompanies spin polarization. In the lowest Landau level, however, the kinetic energy has been quenched by the magnetic field and the system will spontaneously develop 100% polarization. An explicit microscopic wave function believed[17] to exactly describe the ground state of N electrons at $\nu = 1$ is[18]

$$\Psi = \Psi_V |\uparrow\uparrow\uparrow\uparrow\uparrow\uparrow\uparrow\uparrow\uparrow\uparrow\uparrow\uparrow\cdots\uparrow\rangle, \tag{1}$$

where Ψ_V is a Vandermonde determinant wave function[1] of the form

$$\Psi_V \equiv \prod_{i<j}(z_i - z_j)\prod_k \exp(-|z_k|^2/4\ell^2), \tag{2}$$

where $\ell \equiv (\hbar c/eB)^{1/2}$ is the quantized cyclotron orbit radius of the lowest Landau level. The first term in Eq. (1) is simply the Laughlin spatial wave function for the filled Landau level and the second term indicates that every spin is up. This state has total spin $S = N/2$ and $S^z\Psi = (N/2)\Psi$. Because Coulomb interactions do not directly affect the spins (magnetism is caused by Coulomb forces plus the Pauli principle, not magnetic forces), $[H, S^\mu] = 0$ and Ψ is simply one of a total of $2S + 1$ degenerate states, all with $S = N/2$. The other states are simply created using the total spin lowering

operator $S^- \equiv \sum_{j=1}^N s_j^-$, which is itself fully symmetric under spin exchange. Since the exact ground state (at $S^z = S = N/2$) is a single Slater (Vandermonde) determinant, the two-particle distribution function for these states is readily computed

$$g(|\mathbf{r}-\mathbf{r}'|) = 1 - e^{-|\mathbf{r}-\mathbf{r}'|^2/2}. \tag{3}$$

Here and in the rest of the paper we set the magnetic length ℓ to unity. One clearly sees, in this expression, the exchange hole which surrounds each particle and lowers its Coulomb energy by an amount

$$\begin{aligned} E_x &= \frac{1}{2\pi}\int d^2r\,\frac{e^2}{r}\,[g(r)-1] \\ &= e^2\sqrt{\pi/2} \sim 64\ \mathrm{K}\ \sqrt{B\ (\mathrm{T})}, \end{aligned} \tag{4}$$

where the dielectric function of the semiconductor environment is implicit, B (T) indicates the magnetic field in tesla, and the numerical estimate in kelvins is for the case of GaAs. For excited states, the spin wave function is not fully symmetric and thus the spatial wave function is not fully antisymmetric. There is then a finite amplitude for particles of opposite spin to approach each other closely. The low-lying excited states are magnons and just as for the ferromagnetic Heisenberg model on a lattice, the exact single magnon excitations for this (itinerant) magnet can be found. They are labeled by a conserved momentum $\mathbf{k}$ and have the form[19]

$$\Psi_{\mathbf{k}} = \overline{S_{\mathbf{k}}^-}\Psi, \tag{5}$$

where

$$\overline{S_{\mathbf{k}}^-} \equiv \sum_{j=1}^N \overline{e^{i\mathbf{k}\cdot\mathbf{r}_j}}\, s_j^- \tag{6}$$

is the Fourier transform of the local spin lowering operator and the overbar indicates projection onto the spatial wave functions of the lowest Landau level. The dispersion of these excitations is similar (at long wavelengths) to those of the Heisenberg model on a lattice and is given by[19,20]

$$\omega(k) = \int \frac{d^2q}{(2\pi)^2} V(q)\exp(-|q|^2/2)\,[1-\cos(\hat{\mathbf{z}}\cdot q\times k)], \tag{7}$$

where $V(q)$ is the Fourier transform of the electron-electron interaction. For large wave vectors one can show that this excitation crosses over from being a collective spin wave mode to a single-particle-type excitation consisting of a magnetic exciton which is a bound state of a spin-flipped particle-hole pair.[20]

In the discussion above, we have focused on the ground state which has definite total S and have examined a basis in which S^z is a good quantum number. It is convenient for later use, however, to recall that since the Hamiltonian is invariant under spin rotations, we could have chosen the spin quantization axis along any direction. Suppose, for example, we considered the ferromagnetic state $|\Psi_\varphi\rangle$ in which every electron has the spinor

$$\begin{pmatrix} 1 \\ e^{i\phi} \end{pmatrix}. \tag{8}$$

This state is an exact ground state of the Hamiltonian, but has indefinite S^z; that is, it is made up of a linear combination of all the degenerate S^z eigenstates. The mean value of S^z is zero

$$\langle\Psi_\varphi|S^z|\Psi_\varphi\rangle = 0, \tag{9}$$

but

$$\langle\Psi_\varphi|\mathbf{S}|\Psi_\varphi\rangle = \frac{N}{2}\,[\cos(\varphi)\hat{\mathbf{x}} + \sin(\varphi)\hat{\mathbf{y}}], \tag{10}$$

showing that the spin is fully aligned, but now lies in the xy plane. Fluctuations in S^z, while nonzero, are relatively small (in the limit of large N)

$$\left[\langle\Psi_\varphi|\,[S^z]^2\,|\Psi_\varphi\rangle\right]^{1/2} = \frac{1}{2}N^{1/2}. \tag{11}$$

This state can be represented as a coherent superposition of the eigenstates of S^z (obeying $S^z|m\rangle = m|m\rangle$). For large N, we have, to a good approximation,

$$|\Psi_\varphi\rangle = \sum_{m=-N/2}^{N/2} e^{-\frac{2}{N}m^2} e^{-im\varphi}|m\rangle. \tag{12}$$

Such a state has a coherence[21] which is analogous to that in the BCS state of a superconductor (with S^z playing the role of number operator). There is a definite phase relationship between states with different values of S^z. Notice that this is not a direct consequence of the dynamics, but merely a result of our choice of linear combination of the degenerate basis vectors. That is, there are no terms in the Coulomb Hamiltonian which flip spins and yet in the ground state there can be a definite phase relationship between amplitudes for different numbers of flipped spins.

We turn now from the application of these ideas to single-layer systems with real spin to the analogous ideas for double-layer systems described by pseudospin. We will ignore real spin, assuming it to be frozen out by the Zeeman energy, although this is not necessarily a valid assumption at low B fields. The spinors

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{13}$$

describe states in which the electron is in the upper or the lower layer, respectively. Thus the layer number difference operator is simply the z component of the total pseudospin

$$N_\uparrow - N_\downarrow = 2S^z. \tag{14}$$

One's first reaction upon thinking about pseudospin is that it is a perfectly sensible concept as long as it is an

Ising-like variable; i.e., each electron is either in the upper layer or in the lower, but not both. However, it is a fundamental feature of quantum mechanics that it is perfectly sensible to talk about states in which there is a coherent superposition of two amplitudes and the layer index is therefore uncertain. For certain filling factors[5–9] the ground state spontaneously develops interlayer coherence and the electron pseudospin condenses into a state

$$\alpha_\varphi = \begin{pmatrix} 1 \\ e^{+i\varphi} \end{pmatrix} \tag{15}$$

magnetized in the xy plane. Such a state has $\langle N_\uparrow - N_\downarrow \rangle = 0$, which reduces the charging energy of the double-layer system. Such a state also has good exchange energy because, if two electrons of the same pseudospin orientation (phase φ) approach each other, the spatial part of the wave function must vanish. It is this exchange effect that gives rise to the finite pseudospin stiffness which polarizes the pseudospins. Of course, in the absence of tunneling, $Q_- \equiv N_\uparrow - N_\downarrow$ is a good quantum number, while our variational wave function Ψ_φ has $N^{1/2}$ fluctuations in Q_-. In analogy with coherent BCS states, however, this is not important to the physics (usually); it is simply mathematically convenient not to project Ψ_φ onto a state of definite Q_-. Of course it is perfectly possible to do so using

$$\Psi_{Q_-} = \int d\varphi e^{-iQ_-\varphi} \Psi_\varphi, \tag{16}$$

without seriously modifying the good correlations built into the wave function. [For the SU(2) invariant case, this wave function is in fact exact. For the XY symmetry case, which applies when the intralayer and interlayer interactions are not identical, this wave function is the precise analog of the BCS wave function projected onto a state of definite particle number and is a good starting approximation.]

Even though the total number of electrons in a given layer is a good quantum number, the dynamics will enforce a definite phase relationship among states with different S^z (layer charge difference) due to a spontaneously broken U(1) symmetry[5,6,9] corresponding to rotations in pseudospin space about the $\hat{\mathbf{z}}$ axis. This is quite analogous to what happens in superconductors. For finite layer separation, the charging energy will limit the fluctuations in Q_- and modified correlations will have to be built into the wave function as discussed in Sec. VI.

The above discussion for $\nu = 1$ is, of course, somewhat oversimplified for general filling factors. Hund's rule, which suggests that the ground state should have the maximum total spin quantum number consistent with the Pauli exclusion principle, does not always apply to two-dimensional electrons in the strong magnetic-field limit. In particular it is known both from theoretical work[1,19,22] and from experimental work[23] that at some filling factors incompressible ground states can occur which are spin singlets.[24] We focus our discussion here on the case $\nu = 1$ where the consequences of spontaneous interlayer coherence are likely to be most easily observable. At this filling factor there is ample evidence that Hund's rule does give the correct answer for the spin quantum number of the ground state. We believe that much of the physics we discuss will occur at any filling factor for which the corresponding spin system has an incompressible ground state which *is not* a spin singlet. In particular, the ground state for $\nu = 1/m$ for any odd integer m is believed to have $S = N/2$ for any physically realistic interaction. The ground-state orbital wave functions at these filling factors are well approximated by the simple Jastrow wave functions discovered by Laughlin[25,26] and we will present some results for $m = 3$ and $m = 5$ based on these trial wave functions.

III. SPIN-CHARGE RELATION

A. Review of projection onto the lowest Landau level

A convenient formulation of quantum mechanics within the subspace of the lowest Landau level (LLL) was developed by Girvin and Jach[27] and was exploited by Girvin, MacDonald, and Platzman in the magnetoroton theory of collective excitations of the incompressible states responsible for the fractional quantum Hall effect.[28] Here we briefly discuss the part of this formalism that is most relevant to the present paper.

We first consider the one-body case and choose the symmetric gauge. The single-particle eigenfunctions of kinetic energy and angular momentum in the LLL are[26,27]

$$\phi_m(z) = \frac{1}{(2\pi 2^m m!)^{1/2}} z^m \exp\left(-\frac{|z|^2}{4}\right), \tag{17}$$

where m is a non-negative integer and $z = (x + iy)/\ell$. From (17) it is clear that any wave function in the LLL can be written in the form

$$\psi(z) = f(z)\, e^{-\frac{|z|^2}{4}}, \tag{18}$$

where $f(z)$ is an analytic function of z, so the subspace in the LLL is isomorphic to the Hilbert space of analytic functions.[29,27,30] Following Bargmann,[29,27] we define the inner product of two analytic functions as

$$(f, g) = \int d\mu(z)\, f^*(z)\, g(z), \tag{19}$$

where

$$d\mu(z) \equiv (2\pi)^{-1}\, dx dy\, e^{-\frac{|z|^2}{2}}. \tag{20}$$

Now we can define bosonic ladder operators that connect ϕ_m to $\phi_{m\pm1}$ (and which act on the polynomial part of ϕ_m only):

$$a^\dagger = \frac{z}{\sqrt{2}}, \tag{21a}$$

$$a = \sqrt{2}\,\frac{\partial}{\partial z}, \tag{21b}$$

so that

$$a^\dagger \varphi_m = \sqrt{m+1}\,\varphi_{m+1}, \tag{22a}$$

$$a\,\varphi_m = \sqrt{m}\,\varphi_{m-1}, \tag{22b}$$

$$(f, a^\dagger g) = (a\,f, g), \tag{22c}$$

$$(f, a\,g) = (a^\dagger f, g). \tag{22d}$$

All operators that have nonzero matrix elements only within the LLL can be expressed in terms of a and $a^\dagger$. It is essential to notice that the adjoint of $a^\dagger$ is not $z^*/\sqrt{2}$ but $a \equiv \sqrt{2}\partial/\partial z$, because z^* connects states in the LLL to higher Landau levels. Actually a is the projection of $z^*/\sqrt{2}$ onto the LLL as seen clearly in the following expression:

$$\left(f, \frac{z^*}{\sqrt{2}}\,g\right) = \left(\frac{z}{\sqrt{2}}\,f, g\right) = (a^\dagger f, g) = (f, a\,g).$$

So we find

$$\overline{z^*} = 2\frac{\partial}{\partial z}, \tag{23}$$

where the overbar indicates projecton onto the LLL. Since $\overline{z^*}$ and z do not commute, when we need to project an operator which is a combination of z^* and z, we must first normal order z^*'s to the left of z's and then replace z^* by $\overline{z^*}$. With this rule in mind and (23), we can easily project onto the LLL any operator that involves space coordinates only.

For example, the one-body density operator in momentum space is

$$\rho_{\mathbf{q}} = \frac{1}{\sqrt{A}}\,e^{-i\mathbf{q}\cdot\mathbf{r}} = \frac{1}{\sqrt{A}}\,e^{-\frac{i}{2}(q^*z+qz^*)} = \frac{1}{\sqrt{A}}\,e^{-\frac{i}{2}qz^*}\,e^{-\frac{i}{2}q^*z},$$

where A is the area of the system and $q = q_x + iq_y$. Hence

$$\overline{\rho_q} = \frac{1}{\sqrt{A}}\,e^{-iq\frac{\partial}{\partial z}}\,e^{-\frac{i}{2}q^*z} = \frac{1}{\sqrt{A}}\,e^{-\frac{|q|^2}{4}}\,\tau_q, \tag{24}$$

where

$$\tau_q = e^{-iq\frac{\partial}{\partial z}-\frac{i}{2}q^*z} \tag{25}$$

is a unitary operator satisfying the closed Lie algebra

$$\tau_q\tau_k = \tau_{q+k}\,e^{\frac{i}{2}q\wedge k}, \tag{26a}$$

$$[\tau_q, \tau_k] = 2i\,\tau_{q+k}\,\sin\frac{q\wedge k}{2}, \tag{26b}$$

where $q \wedge k \equiv \ell^2(\mathbf{q}\times\mathbf{k})\cdot\hat{\mathbf{z}}$. We also have $\tau_q\tau_k\,\tau_{-q}\tau_{-k} = e^{iq\wedge k}$. This is a familiar feature of the group of translations in a magnetic field, because $q \wedge k$ is exactly the phase generated by the flux in the parallelogram generated by $\mathbf{q}\ell^2$ and $\mathbf{k}\ell^2$. Hence the τ's form a representation of the magnetic translation group (see Fig. 2). In fact τ_q translates the particle a distance $\ell^2\hat{\mathbf{z}}\times\mathbf{q}$. This means that different wave-vector components of the charge density do not commute. It is from here that the nontrivial dynamics arises even though the kinetic energy is totally quenched in the LLL subspace.

This formalism is readily generalized to the case of many particles with spin, as we will show next. In a system with area A and N particles the projected charge and spin density operators are

$$\overline{\rho_q} = \frac{1}{\sqrt{A}}\sum_{i=1}^{N}\overline{e^{-i\mathbf{q}\cdot\mathbf{r}_i}} = \frac{1}{\sqrt{A}}\sum_{i=1}^{N}e^{-\frac{|q|^2}{4}}\,\tau_q(i)\,, \tag{27a}$$

$$\overline{S_q^\mu} = \frac{1}{\sqrt{A}}\sum_{i=1}^{N}\overline{e^{-i\mathbf{q}\cdot\mathbf{r}_i}}\,S_i^\mu = \frac{1}{\sqrt{A}}\sum_{i=1}^{N}e^{-\frac{|q|^2}{4}}\,\tau_q(i)\,S_i^\mu, \tag{27b}$$

where $\tau_q(i)$ is the magnetic translation operator for the ith particle and S_i^μ is the μth component of the spin operator for the ith particle. We immediately find that unlike the unprojected operators, the projected spin and charge density operators do not commute:

$$[\bar{\rho}_k, \bar{S}_q^\mu] = \frac{2i}{\sqrt{A}}\,e^{\frac{|k+q|^2-|k|^2-|q|^2}{4}} \times\overline{S_{k+q}^\mu}\,\sin\left(\frac{k\wedge q}{2}\right) \neq 0. \tag{28}$$

This implies that within the LLL, the dynamics of spin and charge are entangled, i.e., when you rotate spin, charge gets moved. As a consequence of that, spin textures carry charge,[12] as we will soon see.

B. Spin-charge relation within the lowest Landau level

We have argued above that in the SU(2) invariant case (i.e., when $d/\ell = 0$), the ground state at $\nu = 1/m$ has

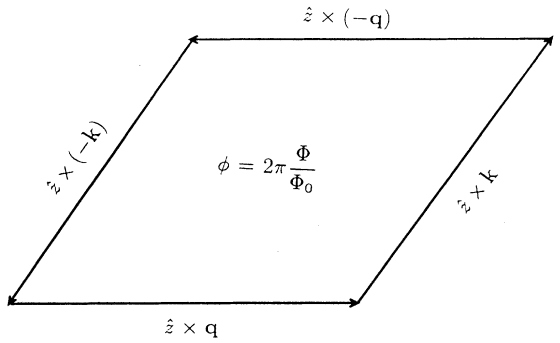

FIG. 2. Illustration of magnetic translations and phase factors. When an electron travels around a parallelogram (generated by $\tau_q\tau_k\tau_{-q}\tau_{-k}$) it picks up a phase $\phi = 2\pi\frac{\Phi}{\Phi_0} = q\wedge k$, where Φ is the flux enclosed in the parallelogram and Φ_0 is the magnetic flux quantum.

its pseudospin fully polarized spontaneously. At these filling factors the interaction energy is minimized in such a state. For the same reason, we might expect that the low-lying excited states will be spin textures in which the local spin alignment varies slowly with position. To be explicit, we define the following as a spin texture state:

$$|\tilde{\psi}[\mathbf{m}(\mathbf{r})]\rangle = e^{-i\bar{O}}\,|\psi_0\rangle. \tag{29}$$

Here $|\psi_0\rangle$ is the $S^z = N/2$ member of the ground-state spin multiplet given in Eq. (1) and the operator O is a nonuniform spin rotation which reorients the local spin direction from $\hat{\mathbf{z}}$ to $\mathbf{m}(\mathbf{r})$ ($\mathbf{m}$ is a unit vector). We limit ourselves to small tilts away from the $\hat{\mathbf{z}}$ direction so that

$$O = \sum_{j=1}^{N} \mathbf{\Omega}(\mathbf{r}_j)\cdot\mathbf{S}_j \equiv \sum_q e^{\frac{|q|^2}{4}}\,\Omega_q^\mu\,S_{-q}^\mu, \tag{30}$$

where $\mathbf{\Omega}(\mathbf{r}) = \hat{\mathbf{z}}\times\mathbf{m}(\mathbf{r})$ is the angle over which a spin is rotated. [Note that $\Omega^z(\mathbf{r})\equiv 0$, $\Omega^x(\mathbf{r}) = -m^y(\mathbf{r})$, and $\Omega^y(\mathbf{r}) = m^x(\mathbf{r})$]. We will later argue that our final result requires only that Ω is slowly varying in space and not that Ω is small. However, the assumption is convenient at present since it allows us to use a simple expression for Ω and also to expand $\bar{O}$ as a small quantity. Projecting O onto the LLL ensures that $|\tilde{\psi}\rangle$ has no projection on higher Landau levels,[31] as required in the strong perpendicular magnetic field limit. The extra factor $e^{\frac{|q|^2}{4}}$ in Eq. (30) implies a nonstandard definition for the Fourier components of $\Omega^\mu(\mathbf{r})$, which is adopted as a convenience.

We can now calculate the excess charge density in a spin texture state:

$$\delta\rho_k = \langle\psi_0|e^{i\bar{O}}\,\bar{\rho}_k\,e^{-i\bar{O}}|\psi_0\rangle - \langle\psi_0|\bar{\rho}_k|\psi_0\rangle. \tag{31}$$

Expanding in powers of $\bar{O}$ gives

$$\delta\rho_k = i\,\langle\psi_0|[\bar{O},\rho_k]|\psi_0\rangle - \frac{1}{2}\,\langle\psi_0|[\bar{O},[\bar{O},\rho_k]]|\psi_0\rangle + \cdots. \tag{32}$$

It is easy to check that the first term is zero. Using (28) we obtain

$$[\bar{O},\bar{\rho}_k] = \frac{2i}{\sqrt{A}}\sum_q e^{\frac{|k-q|^2-|k|^2}{4}}\,\Omega_q^\mu\,\overline{S_{k-q}^\mu}\,\sin\frac{k\wedge q}{2}. \tag{33}$$

Substituting this into (32) and keeping only the second-order term we obtain

$$\delta\rho_k = -\frac{i}{\sqrt{A}}\sum_{p,q} e^{\frac{|k-q|^2-|k|^2+|p|^2}{4}}\,\Omega_q^\mu\,\Omega_p^\nu \times\sin\frac{k\wedge q}{2}\,\langle\psi_0|[\overline{S_{-p}^\nu},\overline{S_{k-q}^\mu}]|\psi_0\rangle. \tag{34}$$

At this point we use our assumption that $|\psi_0\rangle$ is a state with uniform spin density so that the expectation value in Eq. (34) is nonzero only when $p = k-q$. Using

$$[\overline{S_{q-k}^\nu},\overline{S_{k-q}^\mu}] = -\frac{i}{A}\,e^{-\frac{|k-q|^2}{2}}\,\epsilon_{\mu\nu\lambda}\sum_{j=1}^{N} S_j^\lambda, \tag{35}$$

we obtain (using the fact that only the z component of spin is nonzero)

$$\delta\rho_k = -\frac{N}{2A^{3/2}}\,\epsilon_{\mu\nu}\,e^{-\frac{|k|^2}{4}}\sum_q \Omega_q^\mu\,\Omega_{k-q}^\nu\,\sin\frac{k\wedge q}{2}. \tag{36}$$

Since Ω is a slowly varying function and Ω_q^μ is negligible when q is large, we can make the approximation

$$e^{-\frac{|k|^2}{4}}\,\sin\frac{k\wedge q}{2} \simeq \frac{k\wedge q}{2}. \tag{37}$$

Substituting into (36) we obtain

$$\begin{aligned}\delta\rho_k &= -\frac{\nu}{8\pi}\,\frac{1}{\sqrt{A}}\,\epsilon_{\mu\nu}\sum_q \Omega_q^\mu\,\Omega_{k-q}^\nu\,(k\wedge q)\\ &= -\frac{1}{8\pi}\,\frac{\nu}{\sqrt{A}}\,\epsilon_{\mu\nu}\sum_q (iq\,\Omega_q^\mu)\wedge[i(k-q)\,\Omega_{k-q}^\nu]\\ &= -\frac{1}{8\pi}\,\frac{\nu}{\sqrt{A}}\,\epsilon_{\mu\nu}\sum_q (\nabla\Omega^\mu)_q\wedge(\nabla\Omega^\nu)_{k-q}\\ &= -\frac{\nu}{8\pi}\,\epsilon_{\mu\nu}\,\{(\nabla\Omega^\mu)\wedge(\nabla\Omega^\nu)\}_k.\end{aligned} \tag{38}$$

Here we have used the fact that $N/A = \nu/(2\pi)$. Fourier transforming back to real space, we obtain

$$\delta\rho(\mathbf{r}) = -\frac{\nu}{8\pi}\,\epsilon_{\mu\nu}\,[\nabla\Omega^\mu(\mathbf{r})\wedge\nabla\Omega^\nu(\mathbf{r})]. \tag{39}$$

Expressing $\delta\rho$ in terms of $\mathbf{m}$ instead of $\mathbf{\Omega}$, we finally obtain

$$\delta\rho(\mathbf{r}) = -\frac{\nu}{8\pi}\,\epsilon_{\mu\nu}\,\mathbf{m}(\mathbf{r})\cdot[\partial_\mu\mathbf{m}(\mathbf{r})\times\partial_\nu\mathbf{m}(\mathbf{r})]\ , \tag{40}$$

which is exactly ν times the Pontryagian index density, or topological charge density.[12,32] In Eq. (40) we have used the fact that Ω is small to replace $\hat{\mathbf{z}}$ by $\mathbf{m}$. The final result depends on $\nabla\mathbf{\Omega}$ instead of $\mathbf{\Omega}$ and it is clear[33] that the expansion in (32) is actually an expansion in terms of $\nabla\mathbf{\Omega}$ rather than Ω. Hence our final result is valid as long as $\mathbf{\Omega}$ is slowly varying so that $\nabla\mathbf{\Omega}$ is small (compared to ℓ^{-1}).[12,32]

The density in Eq. (40) can be viewed as the timelike component of a conserved (divergenceless) topological three current

$$j^\alpha = -\frac{\nu}{8\pi}\,\epsilon^{\alpha\beta\gamma}\,\epsilon_{abc}\,m^a(\mathbf{r})\partial_\beta m^b(\mathbf{r})\partial_\gamma m^c(\mathbf{r}). \tag{41}$$

Using the fact that $\mathbf{m}$ is a unit vector, it is straightforward to verify that $\partial_\mu j^\mu = 0$. We note that the fact that the expression for the topological current is not parity invariant is a direct reflection of the lack of parity symmetry in the presence of the external magnetic field.

Thus we have shown that for spin states with $S = N/2$ the physical charge density is ν times the topological charge density, in the long-wavelength limit. This remarkable result was first obtained by Sondhi *et al.* within the context of a Chern-Simons effective field theory description of spin textures.[12] The present derivation gives a microscopic proof of their result. The total extra charge carried by the spin texture is exactly the Pontryagian index:

$$\triangle N = -\frac{\nu}{8\pi}\int d^2\mathbf{r}\,\epsilon_{\mu\nu}\,\mathbf{m}(\mathbf{r})\cdot[\partial_\mu\mathbf{m}(\mathbf{r})\times\partial_\nu\mathbf{m}(\mathbf{r})], \tag{42}$$

which is an integer multiple of ν because it is the number of times a unit sphere is wrapped around by the order parameter, i.e., it is the winding number of the spin texture.[32] For $\nu = 1/m$ and $m = 3, 5$ elementary spin-textures carry the same fractional charge as the quasiparticles discovered by Laughlin[25] for spinless electrons. As we discuss below, the fact that the charges are the same follows from very general considerations. Actually the spin texture states we have defined must contain precisely the same number of particles as $|\psi_0\rangle$ since the spin-rotation operator does not change the total electron number. However, the spin density may contain a number of well-separated textures with well-defined nonzero topological charge densities and hence well localized charges; only the net charge in the spin-texture states defined above will be zero. The system clearly also has states with locally nonzero net charge in the spin textures.

The fact that spin textures carry charge can also be understood from the following very different point of view. In the Hartree-Fock picture the electrons "see" a very strong exchange field which locally aligns the spins ferromagnetically to produce the spin texture. As an electron propagates through this (slowly spatially varying) exchange field, its spin adiabatically follows the local orientation of the exchange field. A consequence of the tilting of the spin is that when an electron moves along a closed path that surrounds the area Γ, the spin contributes a Berry's phase to the path integral:[32]

$$\varphi = -\frac{1}{4}\int_\Gamma d^2\mathbf{r}\,\epsilon_{\mu\nu}\,\mathbf{m}(\mathbf{r})\cdot[\partial_\mu\mathbf{m}(\mathbf{r})\times\partial_\nu\mathbf{m}(\mathbf{r})]\,. \tag{43}$$

This extra phase is exactly equivalent to having an Aharonov-Bohm phase due to additional magnetic flux inside Γ. In our system, the Hall conductance is not zero:

$$\sigma_{xy} = \frac{\nu e^2}{h}, \tag{44}$$

which means that additional flux Φ gives additional charge $Q = e\nu\Phi/\Phi_0$.[34] This is the same mechanism that causes Laughlin quasiparticles to carry quantized fractional charge when σ_{xy} is quantized.[25] Combining (43) and (44) tells us that the additional charge density is given by Eq. (40).[34]

IV. EFFECTIVE ACTION

A. SU(2)-invariant interactions

In this section we calculate the effective action of a smooth spin texture for SU(2)-invariant (i.e., $d/\ell = 0$) electron-electron interactions. The considerations in this section apply to both a single-layer system with zero Zeeman energy and to a pseudospin-polarized double-layer system in which the two layers are spatially coincident so that the interactions between layers and within layers are the same. In the double-layer case this limit can of course never be achieved experimentally, but it is a convenient place to begin the analysis since the ground state is exactly soluble in this limit. We assume, for the sake of convenience, that the spins are almost aligned in the $\hat{\mathbf{z}}$ direction and that they vary slowly in space, i.e., Ω_q is negligible when $q\ell \geq 1$. The interaction, after projection onto the LLL, is

$$\overline{V} = \frac{1}{2}\sum_q V_q\,(\overline{\rho}_q\,\overline{\rho}_{-q} - Ne^{-\frac{q^2}{2}}), \tag{45}$$

where $V_k = \int d^2r\,V(r)e^{-i\mathbf{k}\cdot\mathbf{r}}$. The expectation value of the energy is

$$\begin{aligned}\delta E &= \langle\psi_0|e^{i\overline{O}}\,[\overline{V}, e^{-i\overline{O}}]|\psi_0\rangle \\ &= -i\langle\psi_0|[\overline{V},\,\overline{O}]|\psi_0\rangle - \frac{1}{2}\,\langle\psi_0|[\overline{O},\,[\overline{O},\,\overline{V}]]|\psi_0\rangle + \cdots .\end{aligned} \tag{46}$$

Since $\Omega^z = 0$, the leading term vanishes. The second-order term gives

$$\begin{aligned}\delta E &= -\frac{1}{2}\,\langle\psi_0|[\overline{O},\,[\overline{O},\,\overline{V}]]|\psi_0\rangle \\ &= -\frac{1}{4}\sum_k V_k\,\langle\psi_0|[\overline{O},\,[\overline{O},\,\overline{\rho}_k\,\overline{\rho}_{-k}]]|\psi_0\rangle.\end{aligned} \tag{47}$$

The commutators in Eq. (47) can be evaluated using

$$\begin{aligned}[\overline{\rho}_k,\,\overline{O}] &= \frac{1}{\sqrt{A}}\,e^{-\frac{|k|^2}{4}}\sum_j\left[\tau_k(j),\,\sum_q\Omega_q^\mu\,\overline{S}^\mu_{-q}\right] \\ &= \frac{2i}{A}\sum_{j,q}e^{-\frac{k^2}{4}}\,\Omega_q^\mu\,S_j^\mu\,\tau_{k-q}(j)\,\sin\frac{q\wedge k}{2}.\end{aligned} \tag{48}$$

Substituting this into the above expression, we obtain for small q

$$\begin{aligned}\delta E &= \frac{-N}{2A^2}\sum_k V_k\sum_q(\Omega_q^x\Omega_{-q}^x + \Omega_q^y\Omega_{-q}^y)\,\frac{1}{4}(q\wedge k)^2h(k) \\ &= \frac{\rho_s^0}{2}\sum_q[(iq)\Omega_q^x\,(-iq)\Omega_{-q}^x + (iq)\Omega_q^y\,(-iq)\Omega_{-q}^y] \\ &\equiv \frac{\rho_s^0}{2}\int d^2r\,[(\nabla\Omega^x)^2 + (\nabla\Omega^y)^2] = \frac{\rho_s^0}{2}\int d^2r(\nabla\mathbf{m})^2\,.\end{aligned} \tag{49}$$

The spin stiffness ρ_s^0, implicitly defined above, is related to the pair-correlation function of $|\psi_0\rangle$ by

$$\rho_s^0 = \frac{-\nu}{32\pi^2}\int dk k^3V_kh(k)\,, \tag{50}$$

where the pair-correlation function $h(k) \equiv (\nu/2\pi)\int d^2\mathbf{r}(g(r)-1)\exp(-i\mathbf{k}\cdot\mathbf{r})$. For $\nu = 1$, $|\psi_0\rangle$ is known analytically and the pair-correlation function can be evaluated analytically; $h(k) = -\exp(-|k|^2/2)$.

In this calculation we have kept the lowest-order gradient terms only. The physical origin of the stiffness is the loss of exchange and correlation energy when the spin orientation varies with position. For the Coulomb interaction, $\rho_s^0 = e^2/(16\sqrt{2\pi}\epsilon\ell) \sim (e^2/\epsilon\ell)2.49\times10^{-2}$ at

$\nu = 1$. For $\nu = 1/3$ and $\nu = 1/5$ we have evaluated ρ_s^0 numerically using hypernetted-chain-approximation[35] correlation functions and find $\rho_s^0 = (e^2/\epsilon\ell)\, 9.23 \times 10^{-4}$ and $\rho_s = (e^2/\epsilon\ell)\, 2.34 \times 10^{-4}$, respectively. For $\nu = 1$ this is exactly the coefficient of the gradient term which Sondhi *et al.*[12] obtained by fitting the known[20,19] long-wavelength spin-wave spectrum, but here we obtain it from a *first-principles* calculation. The classical model defined by Eq. (49) is called the $O(3)$ nonlinear sigma model and has been studied in great detail.[36] We note in passing that for the SU(2) invariant case, the spin stiffness ρ_s found here is exact. Quantum fluctuation corrections to the Hartree-Fock approximation affect only higher gradient terms in the action.

Now let us see what happens at higher order in Ω. The third term is

$$E^{(3)} = \frac{-i}{2}\sum_k V_k \frac{1}{3!}\langle\psi_0|[\overline{O},[\overline{O},[\overline{O},\bar\rho_k\bar\rho_{-k}]]]|\psi_0\rangle. \quad (51)$$

We have an odd number of powers of S^x or S^y combined together, but the state is polarized in the $\hat{\mathbf{z}}$ direction, so the expectation value must be zero. In general, odd order terms are zero by symmetry. The next nonzero term appears at fourth order:

$$E^{(4)} = \frac{1}{2}\sum_k V_k \frac{1}{4!}\langle\psi_0|[\overline{O},[\overline{O},[\overline{O},[\overline{O},\bar\rho_k\bar\rho_{-k}]]]]|\psi_0\rangle. \quad (52)$$

The nested commutators can be expanded as

$$[\overline{O},[\overline{O},[\overline{O},[\overline{O},\bar\rho_k\bar\rho_{-k}]]]] = \bar\rho_k[\overline{O},[\overline{O},[\overline{O},[\overline{O},\bar\rho_{-k}]]]] + 4[\overline{O},\bar\rho_k][\overline{O},[\overline{O},[\overline{O},\bar\rho_{-k}]]] + 6[\overline{O},[\overline{O},\bar\rho_k]][\overline{O},[\overline{O},\rho_{-k}]]$$
$$+4[\overline{O},[\overline{O},[\overline{O},\bar\rho_k]]][\overline{O},\bar\rho_{-k}] + [\overline{O},[\overline{O},[\overline{O},[\overline{O},\bar\rho_k]]]]\bar\rho_{-k}. \quad (53)$$

We remark that for $\nu = 1$ the first and last terms (with $k \neq 0$) give no contribution to $E^{(4)}$, since the ground state is annihilated by $\bar\rho_k$. (There are no nonuniform density, $S_z = N/2$ states since we have a full Landau level.) The third order term contains the Hartree interaction between the charges of the spin-textures as can be recognized after the following manipulations:

$$\frac{1}{8}\sum_k V_k\langle\psi_0|[\overline{O},[\overline{O},\bar\rho_k]][\overline{O},[\overline{O},\bar\rho_{-k}]]|\psi_0\rangle$$

$$\approx \frac{1}{8}\sum_k V_k\langle\psi_0|[\overline{O},[\overline{O},\bar\rho_k]]|\psi_0\rangle\langle\psi_0|[\overline{O},[\overline{O},\bar\rho_{-k}]]|\psi_0\rangle$$

$$= \frac{1}{2}\sum_k V_k\langle\delta\rho_k\rangle\langle\delta\rho_{-k}\rangle$$
$$= \frac{1}{2}\int\int d^2r d^2r'\, V(\mathbf{r}-\mathbf{r}')\delta\rho(\mathbf{r})\delta\rho(\mathbf{r}'), \quad (54)$$

where $\delta\rho = \frac{\nu}{8\pi}\epsilon_{ij}(\partial_i\mathbf{m}\times\partial_j\mathbf{m})\cdot\mathbf{m}$ is the topological charge density. In the first step above we have used the fact that $|\psi_0\rangle$ is an approximate eigenstate of $[\overline{O},[\overline{O},\bar\rho_{-k}]]$, since Ω^z is zero and even combinations of the spin operators S^x and S^y (i.e., S^xS^x, S^xS^y, etc.) commute with S^z. (For $\nu = 1$ this maneuver is exact.) Thus the third term in Eq. (53) contains the direct Coulomb energy term. If we had a short-range interaction, other terms would give contributions of the same importance; however, because of the long-range of the Coulomb interaction, i.e., because $V_k \sim 1/k$ as $k \to 0$, one can show that this term is the dominant fourth-order term in the spin-texture energy. We take this to be the next leading term in the spin-texture energy functional.

So far we have calculated the most relevant terms of the static energy of a spin texture. The dynamics can be obtained by studying the equation of motion. The quantum equation of motion is

$$\frac{dm_q^\mu}{dt} = \frac{4\pi}{\nu}e^{\frac{|q|^2}{4}}\left\langle\frac{dS_q^\mu}{dt}\right\rangle = -\frac{4\pi i}{\hbar\nu}\langle\tilde\psi|[e^{\frac{|q|^2}{4}}S_q^\mu,\overline{V}]|\tilde\psi\rangle$$
$$\simeq -\frac{4\pi}{\hbar\nu}\langle\psi_0|[\overline{O},[e^{\frac{|q|^2}{4}}S_q^\mu,\overline{V}]]|\psi_0\rangle$$
$$= -\frac{2\pi}{\hbar\nu}\frac{\delta}{\delta\Omega_{-q}^\mu}\langle\psi_0|[\overline{O},[\overline{O},\overline{V}]]|\psi_0\rangle$$
$$= \frac{4\pi}{\hbar\nu}\frac{\delta}{\delta\Omega_{-q}^\mu}E[\mathbf{m}], \quad (55)$$

where $E[\mathbf{m}]$ is the energy functional of the spin texture,

$$E[\mathbf{m}] = \frac{\rho_s^0}{2}\int d^2r(\nabla\mathbf{m})^2 + \frac{1}{2}\int\int d^2r d^2r'\, V(\mathbf{r}-\mathbf{r}')\delta\rho(\mathbf{r})\delta\rho(\mathbf{r}'). \quad (56)$$

If we include only the leading gradient term in the energy functional, an approximation which is always valid for sufficiently slowly varying spin textures, we obtain

$$\frac{d\mathbf{m_q}}{dt} = \frac{4\pi\rho_s^0 q^2}{\hbar\nu}\hat{\mathbf{z}}\times\mathbf{m_q}\,. \quad (57)$$

The equation of motion has spin-wave solutions in which the magnetization precesses around the $\hat{\mathbf{z}}$ direction with wave vector q and frequency $\hbar\omega = (4\pi\rho_s^0 q^2/\nu)$. This is precisely the energy of the long-wavelength spin waves of the system.[20,19] This equation of motion immediately leads to the following effective Lagrangian:

$$L = \frac{\nu}{4\pi}\int d^2r\mathbf{A}[\mathbf{m}(\mathbf{r})]\cdot\partial_t\mathbf{m}(\mathbf{r}) - E[\mathbf{m}], \quad (58)$$

where $\mathbf{A}$ is the vector potential of a unit magnetic monopole[37,32] in the spin space, i.e., $\nabla_{\mathbf{m}}\times\mathbf{A} = \mathbf{m}$. [For spins oriented close to the $\hat{\mathbf{z}}$ direction, as in the above discussions, $\mathbf{A} \approx (1/2)(-m_y, m_x, 0)$.] The first term simply contributes to the action a geometric phase proportional

to the solid angle traced out by the spin vector during its motion. This is exactly Berry's phase for the spin and appears at the adiabatic level as expected.[38]

We have now derived all the terms in the effective Lagrangian of Sondhi *et al.*[12] from a first-principles calculation.

B. Effective action for symmetry-breaking interaction

In this section we derive an effective action suitable for double-layer systems using the pseudospin analogy discussed in Sec. II. We assume[39] that terms in the Hamiltonian where electrons are scattered from layer to layer by interactions can be neglected and that the two wells are identical. We define

$$V^0 \equiv \frac{1}{2}\left(V_k^A + V_k^E\right), \tag{59}$$

$$V_k^z \equiv \frac{1}{2}\left(V_k^A - V_k^E\right), \tag{60}$$

where V_k^A is the Fourier transform with respect to the planar coordinate of the (intralayer) interaction potential between a pair of electrons in the same layer and V_k^E is Fourier transform of the (interlayer) interaction potential between a pair of electrons in opposite layers. If we neglect the finite thickness[40] of the layers, $V_k^A = 2\pi e^2/k$ and $V_k^E = \exp(-kd)V_k^A$. The interaction Hamiltonian can then be separated into a pseudospin-independent part with interaction V^0 and a pseudospin-dependent part. The pseudospin dependent term in the Hamiltonian is

$$\overline{V}_{\rm SB} = 2\sum_k V_k^z \overline{S}_k^z \overline{S}_{-k}^z. \tag{61}$$

Since $V_k^A > V_k^E$, this term produces an easy-plane rather than an Ising anisotropy. The pseudospin symmetry of the Hamiltonian is reduced from SU(2) to U(1) by this term. In addition, this term changes the quantum fluctuations in the system since it does not commute with the order parameter

$$[\overline{V}_{\rm SB}, S^\mu] \neq 0, \tag{62}$$

where $\mu = x, y$.

The pseudospin-texture energy due to the pseudospin-independent term in the Hamiltonian can be calculated as discussed in Sec. IV A. In this section we calculate the contribution of the pseudospin-dependent term in the Hamiltonian to the spin-texture energy.

In calculating the contribution of the pseudospin-dependent term to the spin-texture energy we approximate the ground state by the $S = N/2$ total pseudospin eigenstate which *is* the ground state in the limit of zero layer separation. We argue that the form of the energy functional we derive must remain valid even when quantum fluctuations due to the pseudospin-dependent terms in the Hamiltonian are present. However, the coefficients which appear in the energy functional will be altered by quantum fluctuations and the explicit expressions we derive below are accurate only when the pseudospin-dependent interactions are weak, i.e., only when the layers are close together. In Sec. VIII we will discuss estimates obtained for the quantum fluctuation corrections to these coefficients from finite-size exact diagonalization and many-body perturbation theory calculations.

It is convenient here to take the ground state $|\psi_0\rangle$ to be spin polarized along the $\hat{\mathbf{x}}$ direction. We first calculate the energy change associated with small oscillations of the spin texture away from the $\hat{\mathbf{x}}$ direction. The leading term vanishes just as in the SU(2) invariant case. To understand qualitatively the physics contained in the pseudospin-dependent term in the Hamiltonian we focus first on the second-order term

$$\delta E_{\rm SB} = -\sum_k V_k^z \langle\psi_0|2[\overline{O}, \overline{S}_k^z][\overline{O}, \overline{S}_{-k}^z] + \overline{S}_{-k}^z[\overline{O}, [\overline{O}, \overline{S}_k^z]] + [\overline{O}, [\overline{O}, \overline{S}_{-k}^z]]\overline{S}_k^z|\psi_0\rangle. \tag{63}$$

The first term on the right-hand side of Eq. (63) yields exactly (for $\nu = 1$)

$$\begin{aligned} -2\sum_k V_k^z \langle\psi_0|[\overline{O}, \overline{S}_k^z]|\psi_0\rangle\langle\psi_0|[\overline{O}, \overline{S}_{-k}^z]|\psi_0\rangle &= \frac{\nu^2}{2A^2}\sum_k V_k^z e^{-\frac{k^2}{2}}\Omega_k^y\Omega_{-k}^y \\ &= \frac{\nu^2}{2A^2}\sum_k V_k^z e^{-\frac{k^2}{2}} m_k^z m_{-k}^z \\ &= \frac{N^2}{2A^2}\int d^2r d^2r' \tilde{V}^z(\mathbf{r}-\mathbf{r}')m^z(\mathbf{r})m^z(\mathbf{r}'), \end{aligned} \tag{64}$$

where $\tilde{V}^z(r)$ is the Fourier transform of $V_k^z e^{-\frac{k^2}{2}}$. This is exactly the Hartree-like charging energy. In the limit of a smooth spin texture, using the gradient expansion, this term becomes a local mass term, which is the capacitive charging energy. That is,

$$\beta_H \int d^2r (m^z)^2 \ , \tag{65}$$

where

$$\beta_H = \frac{\nu^2}{8\pi^2}\int d^2r \tilde{V}^z(r). \tag{66}$$

We see immediately from this term that the symmetry-breaking interactions favor equal population of the two layers, or in pseudospin language they favor spin textures where the pseudospin orientation is in the $\hat{\mathbf{x}}$-$\hat{\mathbf{y}}$ plane.

The right-hand side of Eq. (63) can, after a straightforward but lengthy and tedious calculation, be expressed in terms of the two-point correlations of $|\psi_0\rangle$. The calculations are similar to those in Ref. 7 and involve commutators of magnetic translation operators and of pseudospin operators. The following identity enters the calculation at several points:

$$[\overline{S}^{\mu}_{p}, \overline{S}^{\nu}_{q}] = \frac{i}{\sqrt{A}}\epsilon_{\lambda\mu\nu} \cos\frac{\mathbf{p}\wedge\mathbf{q}}{2}\, e^{\frac{1}{2}\mathbf{p}\cdot\mathbf{q}}\, \overline{S}^{\lambda}_{p+q} + \frac{i}{2\sqrt{A}}\delta_{\mu\nu} \sin\frac{\mathbf{p}\wedge\mathbf{q}}{2}\, e^{\frac{1}{2}\mathbf{p}\cdot\mathbf{q}}\, \overline{\rho}_{p+q}. \quad (67)$$

In the limit of slowly varying spin textures we obtain the following result for the contribution of the symmetry-breaking term to the energy of the spin texture:

$$E_{\rm SB}[\mathbf{m}] \simeq \int d^2r \Big\{ \beta_m (m^z)^2 + C[\mathbf{m}] + \frac{\rho^z_s}{2}(\nabla m^z)^2 - \frac{\rho^z_s}{2}\left[(\nabla m^x)^2 + (\nabla m^y)^2\right] \Big\}, \quad (68)$$

where

$$\rho^z_s = \frac{-\nu}{32\pi^2}\int_0^\infty dk V^z(k) h(k) k^3 \quad (69)$$

and $\beta_m = \beta_H + \beta_{\rm xc}$ with

$$\beta_{\rm xc} = \frac{\nu}{8\pi^2}\int_0^\infty dk k V^z_k h(k). \quad (70)$$

The functional $C[\mathbf{m}]$ is nonanalytic in the wave vector due to the long-range nature of the Coulomb interaction

$$C[\mathbf{m}] = -\frac{e^2d^2}{16\pi\epsilon}\int \frac{d^2q}{(2\pi)^2}\,|\mathbf{q}| m^z_{-\mathbf{q}} m^z_{\mathbf{q}}. \quad (71)$$

The total mass term is given by

$$\beta_m \equiv \beta_H + \beta_{\rm xc} = \frac{-\nu}{8\pi^2}\int_0^\infty dk \left[V^z(0) - V^z(k)\right] k\, h(k). \quad (72)$$

Notice that $V_A > V_E$, but that the intralayer interaction contains an exchange term which reduces the effect of V_A. Also note that in the limit where the symmetry-breaking interaction is local, i.e., the limit where V^z_k is independent of k, the exchange-correlation contribution to the coefficient of the mass term vanishes. We can understand this result in the pseudospin language by noting that in this limit the symmetry-breaking interaction is proportional to $\sum_i (S^z_i)^2$, which (for spin $\frac{1}{2}$) still commutes with all components of the total spin operator and does not, despite appearances, destroy the SU(2) symmetry of the Hamiltonian. That is to say, an (apparently) symmetry-breaking interaction that is purely local does not actually break the symmetry.

From another point of view we can understand this result by noting that in $|\psi_0\rangle$ no two particles can be at the same position and therefore they will not experience a local interaction. This property of $|\psi_0\rangle$ and the fact that β_m vanishes for local V^z_k can be confirmed from the following identity satisfied by $h(k)$:

$$\int_0^\infty dk k h(k) = -\nu. \quad (73)$$

It is important to observe that in the limit of small layer separations V^z_k approaches $\pi e^2 d$, which is local. Because there is no contribution to β_m from this local term, the mass coefficient ends up being proportional to the d^2 rather than proportional to d at small d, as would be expected based on naive considerations of the capacitance energy.

Including both the SU(2) invariant contribution defined in Eq. (49) and the symmetry-breaking contributions found above, the total (static) energy functional for a spin texture (ignoring for the moment the long-range Coulomb interaction terms among the charged defects) is given by

$$E_{\rm tot}[\mathbf{m}] \simeq \int d^2r \Big\{ \beta_m (m^z)^2 + C[\mathbf{m}] + \frac{\rho_A}{2}(\nabla m^z)^2 + \frac{\rho_E}{2}\left[(\nabla m^x)^2 + (\nabla m^y)^2\right] \Big\}, \quad (74)$$

where

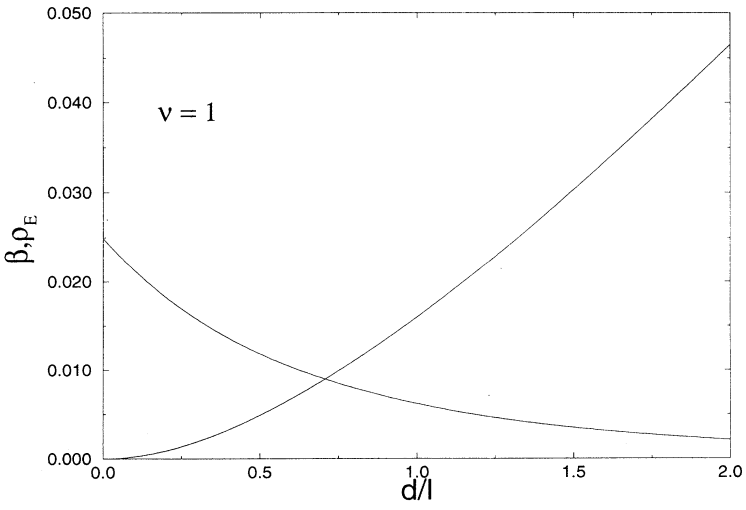

FIG. 3. Anisotropy mass (in unit of $e^2/\epsilon\ell^3$) and easy-plane spin stiffness (in unit of $e^2/\epsilon\ell$) as a function of layer separation for $\nu = 1$. These results do not include quantum fluctuations which are important at finite layer separation.

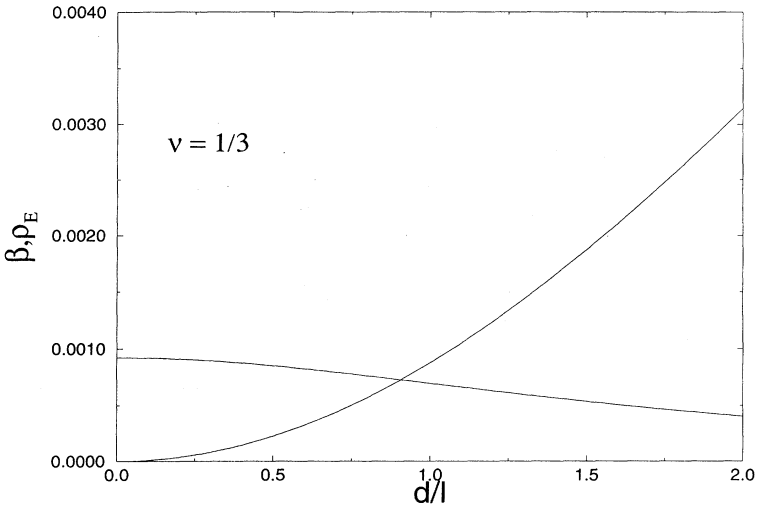

FIG. 4. Anisotropy mass and easy-plane spin stiffness as a function of layer separation for $\nu = 1/3$. These results do not include the dependence of quantum fluctuations on the layer separation.

$$\rho_A = \frac{-\nu}{32\pi^2}\int_0^\infty dk V_k^A h(k) k^3 \tag{75}$$

and

$$\rho_E = \frac{-\nu}{32\pi^2}\int_0^\infty dk V_k^E h(k) k^3 \ . \tag{76}$$

This result is easy to understand. The contribution to the exchange-correlation energy which is dependent on the $\hat{\mathbf{z}}$ polarization of the pseudospin includes the Hartree energy which favors $m_z = 0$ and the exchange correlation within the layers. The exchange-correlation energy within each layer increases superlinearly $\sim \rho^{3/2}$ with the layer density so this term favors $m_z \neq 0$. As discussed above, the Hartree energy is always larger for constant spin densities. The term proportional to $(\nabla m^z)^2$ in the energy density captures the reduction of the exchange-correlation energy from within each layer when the density in the layer is not constant and therefore ρ_A is dependent only on the intralayer interaction. ($\rho_A = \rho_s^0$ at all layer separations. Because of the presence of the mass term, neither this gradient term nor the $C[\mathbf{m}]$ term is important at long wavelengths and so will be neglected.) On the other hand, pseudospin order in the $\hat{\mathbf{x}}$-$\hat{\mathbf{y}}$ plane represents interlayer phase coherence. As discussed earlier, an interlayer phase relationship that changes as a function of position results in a loss of interlayer correlation energy so that ρ_E depends only on interlayer interactions. In Figs. 3 and 4 we illustrate the dependence of β and ρ_E on layer separation calculated from the above expressions for $\nu = 1$ and $\nu = 1/3$, respectively. We emphasize that these results are not expected to be accurate at large layer separations. We will compare these results with estimates from exact diagonalization calculations in Sec. VIII.

C. Hartree-Fock picture of spin textures and gradient expansion of the energy functional

In this section we develop a Hartree-Fock picture to describe spin textures and derive the corresponding energy functional. We also show that a gradient expansion of this energy functional gives exactly the result we obtained in Sec. IV B.

We work in the Landau gauge $\mathbf{A} = (0, Bx, 0)$. The one-body orbital wave functions in the LLL in this gauge are

$$\psi_X(\mathbf{r}) = \frac{1}{\sqrt{\pi^{1/2} L_y \ell}} e^{ik_y y} e^{-\frac{(x-X)^2}{2\ell^2}}, \tag{77}$$

where $X = k_y \ell^2$ is the guiding center.

A particular class of single Slater determinants at $\nu = 1$ in this gauge can be written in the form

$$|\tilde{\psi}\rangle = \prod_X \left(C^\dagger_{X\uparrow} \cos\frac{\theta(X)}{2} + C^\dagger_{X\downarrow} \sin\frac{\theta(X)}{2} e^{i\varphi(X)} \right) |0\rangle, \tag{78}$$

where $|0\rangle$ is the fermion vacuum, $C^\dagger_{X\uparrow,\downarrow}$ creates an electron in the upper (lower) layer in orbit ψ_X, respectively. In this state each Landau gauge orbital is occupied by a single electron whose pseudospin orientation is specified by the polar angles $\theta(X)$ and $\phi(X)$. Each Landau gauge orbital is localized within $\sim \ell$ of its guiding center. We are interested in states for which $\theta(X)$ and $\phi(X)$ vary slowly on the magnetic length energy scale so that $|\tilde{\psi}\rangle$ describes a spin texture in which

$$\begin{aligned} m^z(x) &= \cos\theta(x), \\ m^x(x) &= \sin\theta(x)\cos\varphi(x), \\ m^y(x) &= \sin\theta(x)\sin\varphi(x). \end{aligned} \tag{79}$$

$|\tilde{\psi}\rangle$ is not the most general spin texture, because $\mathbf{m}$ does not depend on y. As a consequence there is no spatial variation in charge density.

It is straightforward to evaluate the energy of $|\tilde{\psi}\rangle$. For the following discussion we include the term in the Hamiltonian which allows electrons to tunnel from layer to layer and whose consequences will be explored in detail in a subsequent paper.[13] The Hamiltonian in this representation is

$$\hat{H} = \hat{T} + \hat{V},$$
$$\hat{T} = -t \sum_X (C^\dagger_{X\uparrow} C_{X\downarrow} + C^\dagger_{X\downarrow} C_{X\uparrow}),$$
$$\hat{V} = \frac{1}{2} \sum_{\substack{X_1,X_2,X_3,X_4 \\ \sigma_1,\sigma_2}} V^{\sigma_1\sigma_2}_{X_1X_2X_3X_4} C^\dagger_{X_1\sigma_1} C^\dagger_{X_2\sigma_2} C_{X_4\sigma_2} C_{X_3\sigma_1}, \tag{80}$$

where $\sigma = \uparrow, \downarrow$ is the layer index,

$$V^{\uparrow\uparrow} = V^{\downarrow\downarrow} = V^A, \quad V^{\uparrow\downarrow} = V^{\downarrow\uparrow} = V^E,$$
$$V_{X_1X_2X_3X_4} = \int d\mathbf{r}_1 d\mathbf{r}_2 \, V(\mathbf{r}_1 - \mathbf{r}_2) \times \psi^*_{X_1}(\mathbf{r}_1)\psi^*_{X_2}(\mathbf{r}_2)\psi_{X_3}(\mathbf{r}_1)\psi_{X_4}(\mathbf{r}_2). \tag{81}$$

Hence we have

$$\langle\tilde{\psi}|\hat{T}|\tilde{\psi}\rangle = -2t \sum_X \cos\frac{\theta(X)}{2} \sin\frac{\theta(X)}{2} \cos\varphi(X) \tag{82}$$
$$= -t \sum_X m^x(X) \tag{83}$$
$$= -\frac{tL_y}{2\pi\ell^2} \int dx \, m^x(x)$$
$$= -\frac{t}{2\pi\ell^2} \int d^2\mathbf{r} \, m^x(\mathbf{r}). \tag{84}$$

In the last step we allow the possibility that m^x depends on y as well. We see this result agrees exactly with the tunneling energy we obtained previously. The contribution from interactions can also be evaluated using Wick's theorem:

$$\langle\tilde{\psi}|\hat{V}|\tilde{\psi}\rangle = \frac{1}{2} \sum_{\substack{X_1,X_2,X_3,X_4 \\ \sigma_1,\sigma_2}} V^{\sigma_1\sigma_2}_{X_1X_2X_3X_4} (\langle C^\dagger_{X_1\sigma_1} C_{X_3\sigma_1}\rangle\langle C^\dagger_{X_2\sigma_2} C_{X_4\sigma_2}\rangle - \langle C^\dagger_{X_1\sigma_1} C_{X_4\sigma_2}\rangle\langle C^\dagger_{X_2\sigma_2} C_{X_3\sigma_1}\rangle)$$
$$= \frac{1}{4} \sum_{X_1,X_2} \{-E^A(X_1 - X_2) + [D^A(X_1 - X_2) - D^E(X_1 - X_2) - E^A(X_1 - X_2)] \, m^z(X_1)m^z(X_2)$$
$$- E^E(X_1 - X_2)[m^x(X_1)m^x(X_2) + m^y(X_1)m^y(X_2)]\} . \tag{85}$$

In this equation we have absorbed the Hartree energy of the system for equal layer densities into the zero of energy. The quantities $D(X) \equiv V_{X+Y,Y,X+Y,Y}$ and $E(X) \equiv V_{X+Y,Y,Y,X+Y}$ are the direct and exchange two-body integrals for both intralayer and interlayer interactions. The above equation has clear physical content. The first term in the final form of the equation is the exchange energy in the absence of pseudospin polarization. The second term is the Hartree charging energy including an exchange correction. The third term is the exchange energy due to interlayer coherence, which is the source of the loss of exchange energy when m^x and m^y change. We now make a gradient expansion by writing

$$m^\nu(x_1)m^\nu(x_2) = [m^\nu(x_1)]^2 + m^\nu(x_1)(x_2 - x_1) \times \left(\frac{\partial}{\partial x} m^\nu(x_1)\right) + \frac{1}{2} m^\nu(x_1)(x_2 - x_1)^2 \times \left(\frac{\partial^2}{\partial x^2} m^\nu(x_1)\right) + \cdots . \tag{86}$$

Replacing the summation over guiding centers by integrations, we easily recover Eq. (74). [The coefficients of the gradient terms are proportional to the second moments of the exchange integrals $\sum_X X^2 E(X)$.] In this formulation we see explicitly that the leading gradient corrections are adequate as long as the pseudospin orientation changes slowly on the scale of ℓ.

D. Hubbard-Stratonovich transformation approach to the effective action

In Sec. IV B above, we derived the effective action by calculating the energy functional for spin textures. The dynamical term in the effective action was determined by requiring that the Lagrangian implies the correct equation of motion for the spin textures. In this section we briefly sketch an alternate route for deriving the same effective action. The basic idea is the familiar Hubbard-Stratonovich (HS) transformation. We introduce auxiliary fields to decouple the interaction and integrate out the fermionic degrees of freedom to obtain the effective action of the auxiliary fields. The auxiliary fields are essentially the order parameters.

It is inconvenient to project onto the lowest Landau level until the end of the calculation so we work with the full Hamiltonian which has the form

$$\hat{H} = \hat{H}_0 + \hat{V}, \tag{87}$$

where

$$\hat{H}_0 = \sum_j \frac{1}{2m} \left[\mathbf{p}_j - \frac{e}{c}\mathbf{A}(\mathbf{r}_j)\right]^2 \tag{88}$$

is the kinetic energy of the particles in the presence of the external magnetic field. This approach is more easily implemented if we have a delta-function-like interaction $V(\mathbf{r}) = V_0\delta(\mathbf{r})$ and so we discuss this case first:

$$\hat{V} = \frac{V_0}{2} \sum_q \rho_q\rho_{-q} = -V_0 \sum_q \mathbf{S}_q \cdot \mathbf{S}_{-q} - \frac{V_0}{4} \sum_q \rho_q\rho_{-q}. \tag{89}$$

The advantage of a δ-function interaction is that it can be expressed in terms of spin operators,[41] which makes it possible to decouple the interaction in terms of spin auxiliary fields. Written in the above particular form, we will find that the saddle point of the auxiliary fields corresponds to the Hartree-Fock mean-field Hamiltonian.[42] The partition function is[43]

$$Z(\beta) = \mathrm{Tr}\left\{e^{-\beta\hat{H}_0}\hat{S}(\beta)\right\}, \tag{90}$$

where

$$\hat{S}(\beta) = T_\tau e^{-\int_0^\beta \hat{V}(\tau)} \tag{91}$$

and

$$\hat{V}(\tau) = e^{\hat{H}_0\tau}\hat{V}e^{-\hat{H}_0\tau}. \tag{92}$$

T_τ is the (imaginary) time ordering operator. Now we can introduce a vector auxiliary field $\mathbf{h}(\tau,\mathbf{r})$ and a scalar auxiliary field $\phi(\tau,\mathbf{r})$ to decouple V:

$$Z(\beta) = \int D\mathbf{h}D\phi e^{-\int_0^\beta \int d^2r d\tau (\frac{1}{4V_0}\mathbf{h}^2 + \frac{1}{V_0}\phi^2)} \times \mathrm{Tr}\{e^{-\beta\hat{H}_0}T_\tau e^{-\int_0^\beta d\tau \int d^2r(-\mathbf{S}\cdot\mathbf{h}-\rho\phi)}\}. \tag{93}$$

After the HS decoupling, we find that the fermionic Hamiltonian becomes quadratic so we can (at least in principle) carry out the trace over the fermion degrees of freedom and hence obtain the effective action in terms of the auxiliary fields $\mathbf{h}$ and ϕ. In doing that, however, we still need to make approximations. We notice that the direction fluctuations of $\mathbf{h}$ are massless [due to broken SU(2) symmetry], while the fluctuations of ϕ are massive.[42] It is therefore a good approximation to assume that ϕ is frozen to be a constant in space and time so it only contributes a chemical potential-like term in the fermion Hamiltonian and is hence unimportant. Thus we will concentrate on the fluctuations of $\mathbf{h}$.

In computing the trace we notice that the trace is over nothing but the propagator of a system governed by the time-dependent Hamiltonian

$$\hat{H}(\tau) = \hat{H}_0 - \int d^2r\, \mathbf{S}(\mathbf{r})\cdot\mathbf{h}(\tau,\mathbf{r}) + \mathrm{const} \tag{94}$$

in imaginary time. Since there is always a large one-body gap for $\hat{H}(\tau)$ while the collective modes of $\mathbf{h}$ are gapless [which means $\hat{H}(\tau)$ is slowly varying], we can use the adiabatic (or Born-Oppenheimer) approximation to evaluate the propagator[38]

$$\mathrm{Tr}\left\{e^{-\beta\hat{H}(\tau)}\right\} = e^{i\gamma[\Gamma]-\int_0^\beta E(\tau)d\tau}, \tag{95}$$

where $\gamma[\Gamma]$ is Berry's phase determined by[38]

$$i\frac{d\gamma}{d\tau} = -\left\langle \Psi(\tau)\left|\frac{d}{d\tau}\Psi(\tau)\right.\right\rangle, \tag{96}$$

where $|\Psi(\tau)\rangle$ is the ground state of $H(\tau)$ and $E(\tau)$ is the ground-state energy. In order to proceed, we need to find $|\Psi(\tau)\rangle$ and $E(\tau)$. It is now possible to take the strong magnetic-field limit. It is both plausible and easily checked that in this case $|\Psi(\tau)\rangle$ is nothing but our familiar spin-texture state

$$|\Psi(\tau)\rangle = e^{i\overline{O(\tau)}}|\Psi_0\rangle, \tag{97}$$

where $|\Psi_0\rangle$ is the fully polarized state in the $\hat{z}$ direction and

$$O(\tau) = \sum_j \mathbf{S}_j\cdot\mathbf{\Omega}(\tau,\mathbf{r_j}), \tag{98}$$

where $\mathbf{\Omega}(\tau,\mathbf{r}) = \hat{\mathbf{z}}\times\mathbf{m}(\tau,\mathbf{r})$ and $\mathbf{m}(\tau,\mathbf{r})$ is a unit vector in the direction of $\mathbf{h}$.

Now let us look at Berry's phase term

$$\begin{aligned} id\gamma &= 1 - \langle\Psi(\tau)|\Psi(\tau+d\tau)\rangle \\ &= 1 - \langle\Psi(\tau)|e^{i\overline{O}(\tau+d\tau)}e^{-i\overline{O}(\tau)}|\Psi(\tau)\rangle \\ &= 1 - \langle\Psi(\tau)|e^{i\partial_\tau\overline{O}(\tau)d\tau - \frac{1}{2}[\overline{O}(\tau),\partial_\tau\overline{O}(\tau)]d\tau}|\Psi(\tau)\rangle \\ &= -i\langle\Psi(\tau)|\partial_\tau\overline{O}(\tau) + \frac{i}{2}[\overline{O}(\tau),\partial_\tau\overline{O}(\tau)]|\Psi(\tau)\rangle d\tau \\ &= i\frac{\rho}{2}d\tau\int d^2\mathbf{r}\,\mathbf{A}(\mathbf{m}(\tau,\mathbf{r}))\cdot\partial_\tau\mathbf{m}(\tau,\mathbf{r}). \end{aligned} \tag{99}$$

Here $\rho = 1/(2\pi\ell^2)$ is the density of electrons. Hence we find that Berry's phase term is exactly the dynamical (time-dependent) term we obtained previously

$$i\gamma = i\frac{\rho}{2}\int d\tau\int d^2\mathbf{r}\,\mathbf{A}(\mathbf{m}(\tau,\mathbf{r}))\cdot\partial_\tau\mathbf{m}(\tau,\mathbf{r}). \tag{100}$$

Now let us calculate $E(\tau)$:

$$\begin{aligned} E(\tau) &= \langle\Psi(\tau)|H(\tau)|\Psi(\tau)\rangle = \langle\Psi_0|e^{-i\overline{O}(\tau)}H(\tau)e^{i\overline{O}(\tau)}|\Psi_0\rangle \\ &= \langle\Psi_0|H - i[\overline{O},H] - \frac{1}{2}[\overline{O},[\overline{O},H]] + \cdots|\Psi_0\rangle \\ &= -\frac{\rho}{2}\int d^2\mathbf{r}\,\mathbf{h}\cdot\mathbf{m} + \frac{1}{8}\rho\ell^2\int d^2\mathbf{r}\,(\nabla h^\mu)\cdot(\nabla m^\mu) \\ &\quad + \cdots. \end{aligned} \tag{101}$$

The first term is just the gain of energy by polarizing the spins in the direction of the external field. The second term comes from the nonlocality induced by the projection to the LLL, so the spins cannot take full advantage of the external field if its direction is changing in space. This is exactly the physics that is responsible for the stiffness. Hence the effective action is

$$S_E[\phi] = i\gamma[\Gamma] + \int_0^\beta d\tau\int d^2r\left\{\frac{1}{4V_0}h^2 - \frac{\rho}{2}\mathbf{h}\cdot\mathbf{m} + \frac{1}{8}\rho\ell^2(\nabla h^\mu)\cdot(\nabla m^\mu)\right\}. \tag{102}$$

Here $h = |\mathbf{h}|$. We find the action has its minimum when

$$\mathbf{h} = \rho V_0\hat{\mathbf{n}}, \tag{103}$$

where $\hat{\mathbf{n}}$ is a constant unit vector. At this point the one-

body Hamiltonian given by $\mathbf{h}$ is exactly the Hartree-Fock Hamiltonian. The fluctuations of the magnitude of $\mathbf{h}$ are massive and we can integrate them out to obtain the nonlinear sigma model

$$S_E[\mathbf{m}] = i\gamma[\Gamma] + \frac{1}{2}\rho_s \int_0^\beta d\tau \int d^2r\, (\nabla \mathbf{m})^2, \qquad (104)$$

where the stiffness $\rho_s = \rho^2\ell^2 V_0/4 = V_0/(16\pi^2\ell^2)$ is exactly what we obtained from the spin-texture calculations.

The above discussion was for the case of short-range interactions. To generalize to the finite-range-interaction case we separate a δ-function-like part from the interaction, follow the same procedures as illustrated above, and treat the remaining part of the interaction as a perturbation. Note that non-δ-function-like interactions are not necessarily SU(2) invariant. At the first order in this perturbation theory [where $|\Psi(\tau)\rangle$ is unaffected], we recover all the results obtained in previous subsections.

V. CHARGED OBJECTS IN THE SYSTEM

A. Skyrmions

We start by briefly reviewing and commenting on results for charged excitations obtained by Sondhi *et al.*[12] for a single-layer system of spin 1/2 particles with no Zeeman coupling. These results can be directly taken over to the double-layer system with $d/\ell = 0$ [which is therefore SU(2) invariant]. In this case one finds,[20,19] as discussed in Sec. VI and mentioned previously, that the double-layer system with spontaneous interlayer phase coherence supports *neutral* gapless spin-wave excitations which disperse quadratically in the long-wavelength limit. This property of spin waves is characteristic of isotropic ferromagnetism. However, in contrast to the case of quantum Heisenberg ferromagnets on a lattice, quantum Hall systems also possess charge degrees of freedom and are more analogous to itinerant electron ferromagnets. For example, after a spin is flipped, it can be scattered to other orbital states and carry charge throughout the system. In addition to the gapless spin-wave excitations, there are *gapful* charged excitations. Only charged excitations can contribute to the conductivity so that the low-temperature transport coefficients will be activated with an activation energy which is half the charge gap. Some of the low-energy charged excitations can be generated from the topologically nontrivial spin configurations discussed below.

We have seen in Sec. III that the physical charge density of the system at $\nu = 1/m$ is $1/m$ of the topological charge density. As a consequence of this the topological solitons, i.e., skyrmions, carry $1/m$ units of physical charge (see Fig. 5).[15,12] Inside a skyrmion the spins wrap around the unit order-parameter sphere exactly once.[36] In the SU(2) invariant case with Coulomb interactions, Sondhi *et al.* have shown that skyrmions are the lowest-energy charged excitations of the system at $\nu = 1$.[12] This is not surprising because for the skyrmion spin configuration, the spins are nearly parallel locally, so the exchange energy is only slightly reduced. In contrast, for ordinary single-particle excitations (see Fig. 6), an added electron has its spin opposite to the others and has no exchange energy. As pointed out earlier, in the SU(2) invariant limit we know the *exact* spin stiffness. Hence the exact energy of a single (large scale) skyrmion can be obtained:[36,12]

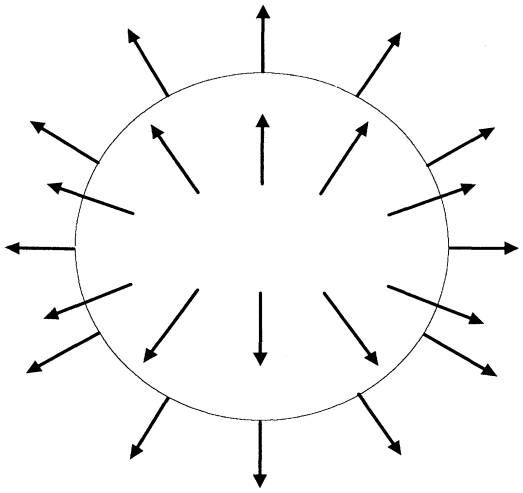

FIG. 5. Illustration of a skyrmion on a sphere.

$$E_s = 4\pi\rho_s. \qquad (105)$$

For the case of a system with Coulomb interactions at $\nu = 1$, we obtain from the nonlinear sigma model energy expression

$$E_s = \frac{1}{4}\sqrt{\frac{\pi}{2}}\frac{e^2}{\epsilon\ell}. \qquad (106)$$

It is important to realize that since the total particle number is fixed in our derivation of the nonlinear sigma model energy expression, E_s actually gives the energy to introduce an isolated skyrmion or antiskyrmion into the bulk of the system *and* maintain charge neutrality by adding or subtracting charge from the edge of the system. (For related careful discussions of quasiparticle energies in the fractional quantum Hall effect see Refs. 44 and 45.) This energy must be subtracted off if we wish to calculate $\epsilon_\pm$, the energy to add (+) or subtract (−) $1/m$ electrons from the system at $\nu = 1/m$ in the form of a skyrmion or antiskyrmion spin texture. It follows[45] that

$$\epsilon_\pm = \pm\nu\xi(\nu) + E_s\ , \qquad (107)$$

where $\xi(\nu)$ is the energy per electron in the incompressible ground state at $\nu = 1/m$. [For the Coulomb inter-

FIG. 6. Illustration of localized quasiparticle and quasihole excitations at $\nu = 1$.

action case $\xi(1) = -\sqrt{\pi/8}(e^2/\ell) \approx -0.6266(e^2/\ell)$ and[46] $\xi(1/3) \approx -0.4100(e^2/\ell)$.] The chemical potential for $\nu > 1/m$ is $m\epsilon_+$, while the chemical potential for $\nu < 1/m$ is $-m\epsilon_-$. For $\nu = 1$ it follows from the above that $\epsilon_+ = -(1/4)\sqrt{\pi/2}(e^2/\ell)$ and $\epsilon_- = (3/4)\sqrt{\pi/2}(e^2/\ell)$. The energies of the localized quasiparticle excitations of Fig. 6 cannot be reliably calculated from the nonlinear sigma model energy expression, but for $\nu = 1$ the microscopic energy can be calculated analytically. For these excitations $\epsilon_+ = 0$ (the spin-reversed added electron has no exchange energy) and $\epsilon_- = \sqrt{\pi/2}(e^2/\ell)$. In the absence of Zeeman coupling it follows that the lowest-energy particles and holes are both formed from skyrmion spin textures. The lowest-energy particle-hole excitation is a skyrmion-antiskyrmion pair which has energy $2E_s$. This is only one-half of that of the ordinary particle-hole pair in the case of the Coulomb interaction.[12] These results receive unequivocal support from numerical calculations.[47]

Using our result for the spin stiffness we can extend this analysis to the case of $\nu = 1/3$. At this filling factor $E_s \approx 0.0116(e^2/\ell)$, $\epsilon_+ \approx -0.1251(e^2/\ell)$, and $\epsilon_- \approx 0.1483(e^2/\ell)$. We find that the minimum particle-hole energy $2E_s \approx 0.0232(e^2/\ell)$ is close to the value of $0.024(e^2/\ell)$ estimated by Sondhi *et al.*, using the single-mode approximation for the spin-wave stiffness.[12] In the limit of large Zeeman energy at this filling factor both the localized quasihole and quasiparticle excitations will be completely spin polarized. The quasiparticle and quasihole energies in this limit have been estimated[45,44] to have the values $\epsilon_+ \approx -0.120(e^2/\ell)$ and $\epsilon_- = 0.2337$. Again the skyrmion and antiskyrmion states possible at zero Zeeman coupling have lower energy, although only barely so in the quasiparticle case. The particle-hole creation energy $2E_s$ for $\nu = 1/3$ is approximately four times larger than in the large Zeeman coupling limit. This is consistent with results from the finite-size exact-diagonalization study of Chakraborty *et al.* who found[48] that in the absence of Zeeman coupling, quasiparticle energies at $\nu = 1/3$ could be reduced by flipping a single spin.

Another consequence of skyrmions being the lowest-energy charged excitation is that in finite-size systems on a sphere, the total spin of the ground state changes suddenly from $\frac{N}{2}$ to zero or one-half (for odd and even N, respectively) when one changes the particle number from N to $N \pm 1$, where N is the Landau level degeneracy without spin.[47,12] (We have verified that the same property holds for $\nu = 1/3$ in agreement with the analysis of the preceding paragraph.) This is because when a skyrmion is put on a sphere, the spin configuration is like a hedgehog and it is plausible that the total spin is essentially zero.[12] (As we discuss below, the *total* angular momentum **J** is precisely zero.) Note, however, that the system is still ferromagnetic in the sense that it is not a *local* singlet[49] and its Zeeman susceptibility still diverges.

In this case the topological charge density is uniformly distributed on the sphere and the skyrmion is unfrustrated. The situation is very different for a system with rectangular geometry with periodic boundary conditions, i.e, the geometry of a torus. In this case the skyrmion is frustrated and hence squeezed by the finite-size effect, so its size is much smaller than the system size. This can be understood by looking at the effect of periodic boundary conditions on the energy of a skyrmion. We know the gradient energy term is scale invariant and for the ideal skyrmion solution, the energy is minimized to be $4\pi\rho_s$. However, an ideal solution does not satisfy the boundary conditions. In a rectangle a skyrmion has to be distorted near the boundary so the energy from the stiffness term will increase. It is obvious that the smaller the size of the skyrmion, the smaller the energy cost due to the boundary effect. However, the skyrmion cannot be too small because it costs too much Coulomb energy (which wants the skyrmion to be as large as possible so that the excess charge will be distributed as uniformly as possible). The optimal size of a skyrmion on a rectangle is determined by a competition between stiffness and Coulomb energies and as a consequence the total spin S of the ground state of the $N+1$ particle system is size dependent.

We can estimate the difference between $(N+1)/2$ and S (i.e., the number of spins flipped in the ground state) in the following way. Keeping the two leading terms, the energy of a spin texture at $\nu = 1$ is

$$E = \frac{1}{2}\rho_s \int d^2\mathbf{r}\,(\nabla \mathbf{m})^2 + \frac{1}{2}\int d^2\mathbf{r} d^2\mathbf{r}'\, V(\mathbf{r}-\mathbf{r}')\rho(\mathbf{r})\rho(\mathbf{r}'), \tag{108}$$

where ρ is the topological charge density and V is the Coulomb interaction. The energy of a skyrmion with linear size $\lambda \gg \ell$ in a system with linear size R is

$$E(\lambda) = 4\pi\rho_s + A\left(\frac{\lambda}{R}\right)^2 + B\frac{\ell}{\lambda}, \tag{109}$$

where A and B are positive constants with units of energy. The first term in Eq. (109) is the usual energy of an unfrustrated, infinite-size skyrmion and the last two are finite-size corrections from the two terms in Eq. (108), respectively. Minimizing E with respect to λ gives

$$\lambda \propto R^{\frac{2}{3}}\ell^{\frac{1}{3}}. \tag{110}$$

The number of spins flipped in a skyrmion with size λ and long distance cutoff R is[12]

$$\triangle S \propto \lambda^2 \ln\left(\frac{R}{\lambda}\right) \propto N^{\frac{2}{3}} \ln N, \tag{111}$$

where we used the fact that $N \propto R^2$. From Eq. (111) we see that as N becomes large, the number of flipped spins in the optimal skyrmion state gets large, but is always small compared to N so that the ground state is almost fully polarized.

Skyrmion spin textures produce an excitation energy which is independent of the texture size as long as the size is large compared to microscopic lengths ℓ and small compared to the system size. This leads to dramatic finite-size effects which are typified by the qualitative differences between ground-state spin quantum numbers for electrons in a rectangle and on a sphere, where the state is unpolarized. For the case of a square with peri-

odic boundary conditions (i.e., a torus), a crude estimate gives the coefficient in front of $N^{\frac{2}{3}} \ln N$ in Eq. (111) to be 0.26. In Fig. 7 we compare this estimate with results from finite-size exact diagonalization calculations for square boundary conditions[50] and find qualitative agreement.

Using the composite fermion theory, Jain and Wu[51] gave an alternative explanation of the fact that the total spin goes to zero for an $N+1$ electron system on a sphere; however, their approach does not shed any light on the nature of the spin correlations. We do not, at present, see how our results on a torus can be easily understood within the composite fermion formalism. This would, however, be an interesting avenue to pursue. Nevertheless, the quantitative agreement with the ground-state energy on the sphere and the qualitative agreement for the polarization on the torus lend strong weight to the skyrmion picture.

We note in passing that Jain and Wu[51] found for the case of higher Landau levels that the addition of a single electron does *not* destroy the ferromagnetic state by production of a skyrmion. Wu and Sondhi[52] have computed the skyrmion energy analytically and confirm this result as being due to a change in the spin stiffness in higher Landau levels.

We close this discussion by noting that it is possible to write down simple microscopic variational wave functions for the skyrmion, both in the plane and on the sphere. Consider the following state in the plane:

$$\psi_\lambda = \prod_m \left(\begin{matrix} z_m \\ \lambda \end{matrix} \right)_m \Psi_V, \tag{112}$$

where Ψ_V is defined in Eq. (2), $(\,)_m$ refers to the spinor for the mth particle, and λ is a fixed length scale. This is a skyrmion because it has its spin purely down at the origin (where $z_m = 0$) and has spin purely up at infinity (where $z_m \gg \lambda$). The parameter λ is simply the size scale of the skyrmion.[12,36] Notice that in the limit $\lambda \longrightarrow 0$ (where the continuum effective action is invalid, but this microscopic wave function is still sensible) we recover a fully spin-polarized filled Landau level with a charge-1 Laughlin quasihole at the origin. Hence the number of flipped spins interpolates continuously from zero to infinity as λ increases.

In order to analyze the skyrmion wave function in Eq. (112), we use the Laughlin plasma analogy. In this analogy the norm of ψ_λ, $\mathrm{Tr}_{\{\sigma\}} \int D[z] |\Psi[z]|^2$ is viewed as the partition function of a Coulomb gas. In order to compute the density distribution we simply need to take a trace over the spin

$$Z = \int D[z]\, e^{2\{\sum_{i>j} \ln|z_i - z_j| + \frac{1}{2}\sum_k \ln(|z_k|^2+\lambda^2) - \frac{1}{4}\sum_k |z_k|^2\}}. \tag{113}$$

This partition function describes the usual logarithmically interacting Coulomb gas with uniform background charge plus a spatially varying impurity background charge $\Delta\rho_b(r)$,

$$\Delta\rho_b(r) \equiv -\frac{1}{2\pi}\nabla^2 V(r) = -\frac{\lambda^2}{\pi(r^2+\lambda^2)^2}, \tag{114}$$

$$V(r) = \frac{1}{2}\ln(r^2+\lambda^2). \tag{115}$$

For large enough scale size $\lambda \gg \ell$, local neutrality of the plasma[53] implies that the excess electron number density is precisely $\Delta\rho_b(r)$, so that Eq. (115) is in agreement with the standard result for the topological density.[36]

For a complete microscopic analytic solution valid for arbitrary λ, we use the fact that the proposed many-body wave function is nothing but a Slater determinant of the single particle states $\phi_m(z)$,

$$\phi_m(z) = \frac{z^m}{\sqrt{2\pi 2^{m+1} m!(m+1+\frac{\lambda^2}{2})}} \left(\begin{matrix} z \\ \lambda \end{matrix} \right) e^{-\frac{|z^2|}{4}}. \tag{116}$$

The electron number density is then

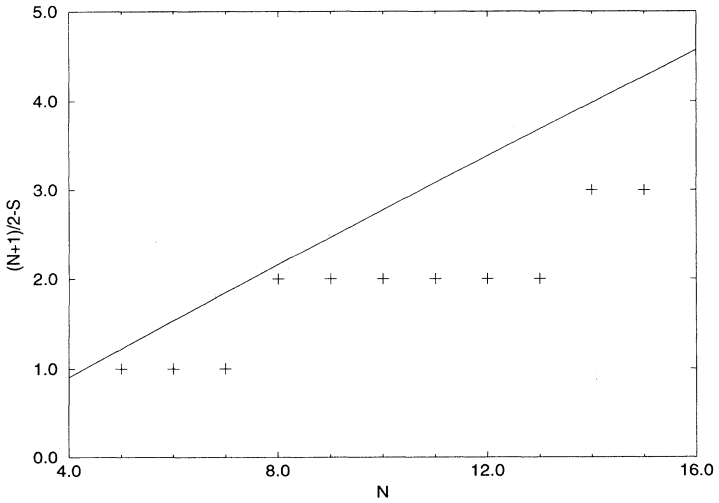

FIG. 7. The number of spins flipped in the ground state versus the system size (Landau level degeneracy) when a single electron is added to the $\nu = 1$ incompressible state on a torus. These results demonstrate that the ground state contains a single skyrmion spin texture whose size is determined by the competition between minimizing the frustration required by the boundary condition and the Coulomb energy.

$$\Delta n^{(1)}(z) \equiv \sum_{m=0}^{N-1} |\phi_m(z)|^2 - \frac{1}{2\pi}, \tag{117}$$

which yields

$$\Delta n^{(1)}(z) = \frac{1}{2\pi}\left(\frac{1}{2}\int_0^1 d\alpha\, \alpha^{\frac{\lambda^2}{2}} e^{-\frac{|z|^2}{2}(1-\alpha)} \times(|z|^2+\lambda^2) - 1\right). \tag{118}$$

Similarly, the spin density distribution $S^z(r)$ can be obtained and it also agrees with that for the standard skyrmion in[36] the large λ limit. We have also computed the skyrmion creation energy from the spin-dependent pair correlation functions of the plasma following the same procedure as in Ref. 45. Figure 8 shows a plot of this energy as a function of scale size λ and shows that the microscopic formula gives the correct asymptotic value of one-half the quasihole energy for the large-λ limit, in which the continuum field theoretic picture holds exactly. As the core size decreases, the skyrmion energy increases due to the increasing Coulomb charging energy. However, it does not diverge as the naive extrapolation of the field theoretic expression would. Finally, we note that by replacing $\binom{z}{\lambda}$ by $\binom{z^n}{\lambda^n}$, we can generate a skyrmion with a Pontryagin index n.

The skyrmion wave function has a particularly simple form on a sphere. On a sphere with radius $R = S^{1/2}\ell$ where S is an integer or a half integer, the number of a flux quanta is $N_s = 2S$. The single-particle kinetic energy is[54]

$$T = \frac{1}{2}\omega_c|\mathbf{\Lambda}|^2/S, \tag{119}$$

where

$$\mathbf{\Lambda} = \mathbf{r}\times[-i\nabla + e\mathbf{A}(\mathbf{r})] \tag{120}$$

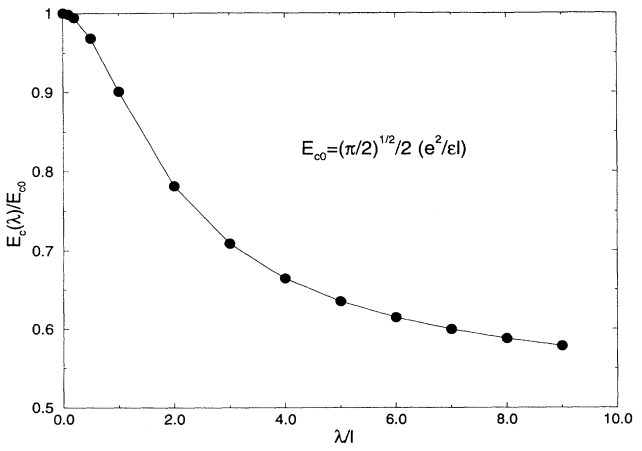

FIG. 8. Microscopic skyrmion energy vs the scale size λ. The trial wave function interpolates continuously between a spin-polarized quasihole at $\lambda = 0$ and a smooth skyrmion spin texture for $\lambda \to \infty$.

is the kinetic angular momentum. One can show that $\mathbf{L} = \mathbf{\Lambda} + S\mathbf{\Omega}$ (where $\mathbf{\Omega} = \frac{\mathbf{r}}{|\mathbf{r}|}$) is the generator of rotations for the system, i.e.,

$$[L^\alpha, X^\beta] = i\epsilon_{\alpha\beta\gamma}X^\gamma, \tag{121}$$

where $\mathbf{X}$ is any vector. We also have

$$|\mathbf{\Lambda}|^2 = |\mathbf{L}|^2 - S^2, \tag{122}$$

so the eigenvalues of $|\mathbf{\Lambda}|^2$ have the form $(n+S)(n+S+1) - S^2$, where n is an integer. For $n = 0$, one obtains the LLL energy $T = \frac{1}{2}\omega_c$ and the degeneracy is $2S+1 = N_s+1$. If we use the Dirac gauge $\mathbf{A} = \frac{S}{eR}\hat{\phi}\cot\theta$, everything can be easily expressed in terms of spinor coordinates $u = \cos\frac{\theta}{2}\exp(i\frac{\phi}{2})$, $v = \sin\frac{\theta}{2}\exp(-i\frac{\phi}{2})$. In this representation[54]

$$L^+ = u\partial_v, \tag{123}$$

$$L^- = v\partial_u, \tag{124}$$

$$L^z = \frac{1}{2}(u\partial_u - v\partial_v). \tag{125}$$

The LLL wave functions are simply homogeneous polynomials of u and v of degree $2S$. The filled LLL single Slater determinant is just

$$\prod_{i<j}^{N}(u_iv_j - u_jv_i), \tag{126}$$

where $N = N_s+1$. The single antiskyrmion (that carries charge -1) wave function is simply

$$\psi_{as} = \prod_{k=1}^{N-1}\begin{pmatrix} v_k \\ -u_k\end{pmatrix}\prod_{i<j}^{N-1}(u_iv_j - u_jv_i), \tag{127}$$

where we have explicitly put in the fact that the total number of particles is now $N-1$. The spin configuration of ψ_{as} is that of a hedgehog (with spins pointing inside toward the center of the sphere) since the ratio of $|u_k|$ to $|v_k|$ varies as $\cot(\theta/2)$. This state is neither an eigenstate of $\mathbf{S}_{\rm tot}$ nor an eigenstate of $\mathbf{L}_{\rm tot}$. It is, however, a singlet of the total angular momentum $\mathbf{J}$:

$$\mathbf{J} = \mathbf{L}_{\rm tot} + \mathbf{S}_{\rm tot}. \tag{128}$$

Physically this means the state $|\psi_{as}\rangle$ is invariant under a spin rotation followed by an identical space rotation, which is clear from the uniform nature of the hedgehog spin configuration.

If we project ψ_{as} onto the subspace of $S_{\rm tot} = 0$ (or $\frac{1}{2}$ if we have an odd number of particles), we automatically get $L_{\rm tot} = 0$ (or $\frac{1}{2}$). So the projected state will be invariant under both spin and space rotation. This state should have good overlap with the exact ground state.

The skyrmion (that carries charge $+1$) wave function has a similar form:

$$\psi_s = \prod_{k=1}^{N+1}\begin{pmatrix} \partial_{u_k} \\ \partial_{v_k}\end{pmatrix}\prod_{i<j}^{N+1}(u_iv_j - u_jv_i). \tag{129}$$

The spin configuration of this state is exactly the opposite of ψ_{as}, i.e., it is like a hedgehog with all spins pointing outward.

B. Merons

When $d/\ell \neq 0$, the $\hat{\mathbf{z}}$ component of the order parameter is massive and the system has U(1) symmetry. In this case, there is another class of topologically stable charged objects, merons.[55,56,16] As shown in the following, merons (see Fig. 9) carry one half unit of topological charge and hence[57] $1/2m$ units of electron charge. Far away from the core of a meron the order parameter lies in the (massless) XY plane and forms a vortex configuration with $\pm$ vorticity, while inside the core region the order parameter smoothly rotates either up or down out of the XY plane. Hence there are four types of merons. The energy of a single meron diverges logarithmically with the system size with a coefficient proportional to the interlayer spin stiffness. The interaction between merons has a contribution from the stiffness energy which is also logarithmic, attractive for opposite vorticity pairs and repulsive for same vorticity pairs. These properties are exactly the same as the vortices in the classical XY model. In order to determine the sign of the charge carried by a meron, one has to specify both its vorticity and the spin configuration in the core region. Merons will also have a long-range $1/r$ interaction due to their charges which is attractive for oppositely charged merons and repulsive for like-charged merons.

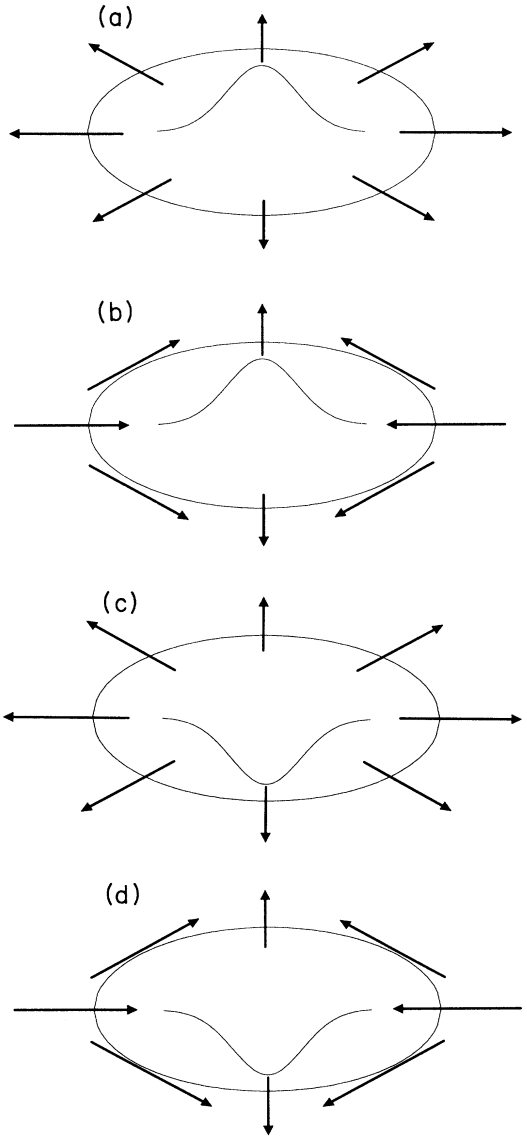

FIG. 9. Illustration of merons (vortices). The spin configuration in the core region tips smoothly out of the XY plane making this object essentially one-half of a skyrmion.

The fact that merons carry topological charge one-half can be seen by the following argument. Imagine a vortex in the spin system. If an electron circles the vortex at a large distance, its spin rotates through 2π. This induces a Berry phase of $\exp(i2\pi S) = -1$, which is equivalent to that induced by a charge moving around one-half of a flux quantum. Since $\sigma_{xy} = e^2/mh$, the vortex picks up charge $1/2m$. The topological charge of a meron can also be understood by considering a variational function for the meron spin texture

$$\mathbf{m} = \left\{ \sqrt{1-[m^z(r)]^2}\cos\varphi, \sqrt{1-[m^z(r)]^2}\sin\varphi, m^z(r) \right\}. \tag{130}$$

The local topological charge density calculated from $\delta\rho = -\frac{1}{8\pi}\epsilon_{ij}(\partial_i\mathbf{m}\times\partial_j\mathbf{m})\cdot\mathbf{m}$ can be expressed in the form

$$\delta\rho(r) = \frac{1}{4\pi r}\frac{dm^z}{dr} \tag{131}$$

and the total charge is

$$Q = \int d^2r\,\delta\rho(r) = \frac{1}{2}\left[m^z(\infty) - m^z(0)\right]. \tag{132}$$

For a meron, the spin points up or down at the core center and tilts away from the $\hat{\mathbf{z}}$ direction as the distance from the core center increases. The $\hat{\mathbf{x}}$ and $\hat{\mathbf{y}}$ components of $\mathbf{m}$ must vanish at the location of the vortex to prevent a divergence of the gradient energy. Hence we must have $m^z(0) = \pm 1$. At asymptotically large distances from the origin, the spins point purely radially in the $\hat{\mathbf{x}}$-$\hat{\mathbf{y}}$ plane. Thus the topological charge is $\pm\frac{1}{2}$ depending on the polarity of core spin. The variational function mentioned above corresponds to a vortex with positive vorticity. In order to make a vortex with negative vorticity (antivortex), we need to apply the space-inversion operation to the vortex solution. Since topological charge is a pseudoscalar quantity, it is odd with respect to parity. Hence the general result for the topological charge of the four meron types may be summarized by the following formula:

$$Q = \frac{1}{2}\left[m^z(\infty) - m^z(0)\right] n_v, \tag{133}$$

where n_v is the vortex winding number. The formu-

las derived above for the meron charge do not rely for their validity on the variational ansatz assumed in Eq. (130). They are quite general and follow from the fact that a meron topologically has half the spin winding of a skyrmion. The meron charge of $\pm 1/2$ is a topological invariant.

Finite-energy excitations can be formed by pairs of merons with opposite vorticity. It seems likely that under appropriate circumstances the lowest-energy charged excitations of the system will consist of a bound pair of merons. (A skyrmion can be viewed as a closely bound pair of merons with the same charge and opposite vorticity and a meron can be viewed as half a skyrmion.) The energy of a pair of merons with opposite vorticity (but like charge) separated by a distance R is given by

$$E_{mp} = 2E_{mc} + \frac{e^2}{4R} + 2\pi\rho_E \ln(R/R_{mc}) \tag{134}$$

where E_{mc} is the core energy of an isolated meron R_{mc} is the core radius of an isolated meron, and the expression should be applicable only when $R \gg R_{mc}$. Minimizing this expression with respect to R gives a meron separation $R^* = e^2/8\pi\rho_E$. Using the expression for the ρ_E derived in Sec. IV B, this gives $R^* \approx 6\ell$ for $d/\ell \sim 1$. Quantum corrections are expected to reduce ρ_E so this expression should give a lower bound on the meron separation. This very attractive picture of the lowest-energy charge carriers in the system (see Fig. 10) will only apply when the meron separation is larger than the meron core size (which is expected to be $\sim \ell$) and its energy is lower than the energy of a conventional quasiparticle excitation where a charge is added with pseudospin directed in opposition to the local pseudospin order. It is clear that we should expect the meron core size to increase as d/ℓ approaches zero and the mass term in the energy expression becomes small since there is only a small energy cost for pseudospins to point out of the xy plane. Hence the picture is not likely to apply for very small d/ℓ. Further work which estimates meron core energies and radii will be necessary to substantiate this picture and is currently in progress. We note in passing that the above description of charged vortex antivortex pairs can also be used *mutatis mutandis* for neutral vortex antivortex pairs. These will have a conserved momentum and the neutral collective mode (discussed in Sec. VI) will cross over from spin waves to such quasiexcitons at large wave vectors in analogy to what occurs in the single-layer case.[20]

As in the case of skyrmions, we can write down explicit microscopic variational wave functions for vortices (merons). We start with the simplest example. A meron

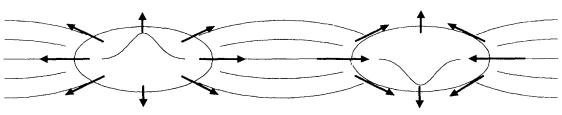

FIG. 10. Illustration of a meron pair with opposite vorticity and like charge. We propose that under appropriate circumstances these objects can form the lowest-energy charged excitations in the system.

with vorticity $+1$ and charge $-\frac{1}{2}$ that has the smallest possible core size:

$$|\Psi_{+1,-\frac{1}{2}}\rangle = \prod_{m=0}^{M} \left(\frac{1}{\sqrt{2}} c^\dagger_{m\uparrow} + \frac{1}{\sqrt{2}} c^\dagger_{m+1\downarrow} \right) |0\rangle. \tag{135}$$

Here $|0\rangle$ is the fermion vacuum, $c^\dagger_{m\uparrow,\downarrow}$ creates an electron in the upper (lower) layer in the angular momentum m state in the LLL, and M is the angular momentum quantum number corresponding to the edge. The vorticity is $+1$ because far away the spin wave function is essentially

$$\chi(\phi) = \frac{1}{\sqrt{2}} \begin{pmatrix} e^{-i\phi} \\ 1 \end{pmatrix}, \tag{136}$$

where ϕ is the polar angle. The charge is $-\frac{1}{2}$ because we have created a hole in the center of the lower layer ($m = 0 \downarrow$ is unoccupied). Since the spin is pointing up at the center, this agrees with the spin-charge relation derived earlier. From the spin-charge relation we know we can change the sign of the charge of a meron by changing the direction of spins in the core region without changing the vorticity. This can be seen explicitly from the wave function

$$|\Psi_{+1,+\frac{1}{2}}\rangle = c^\dagger_{0\downarrow} \prod_{m=0}^{M} \left(\frac{1}{\sqrt{2}} c^\dagger_{m\uparrow} + \frac{1}{\sqrt{2}} c^\dagger_{m+1\downarrow} \right) |0\rangle. \tag{137}$$

This state has charge $+\frac{1}{2}$ because we have put in an electron in the $m = 0$ state in the lower layer. Obviously what we did (in terms of the spin-texture language) is to flip the spins in the core region to the down direction without changing the vorticity of the meron at long distances. In this construction, one sees that in a sense, the merons are like fractionally charged mid-gap states, which can be empty or occupied.

A meron with vorticity -1 is readily obtained by simply interchanging the labels m and $m+1$ in the subscripts in Eqs. (135) and (137). Invariance under pseudospin reversal guarantees the equality of the energies of corresponding $\pm$ vorticity merons. However, the two charge states for a given vorticity are not necessarily degenerate, just as Laughlin quasiholes and quasiparticles are in general nondegenerate. (However, for the special case $\nu = 1$ particle-hole symmetry guarantees degeneracy.)

A general wave function that describes a meron with vorticity k has the form

$$|\Psi_{+k,-\frac{k}{2}}\rangle = \prod_{m=0}^{M} (a_m c^\dagger_{m\uparrow} + b_m c^\dagger_{m+k\downarrow}) |0\rangle. \tag{138}$$

This meron has charge $-\frac{k}{2}$ and we have assumed $k > 0$. Here a_m and b_m are parameters that satisfy

$$|a_m|^2 + |b_m|^2 = 1\ , \quad \lim_{m\to\infty} \frac{a_m}{b_m} = e^{i\phi_0}\ , \tag{139}$$

where ϕ_0 is a constant. By adjusting the asymptotic behavior of a_m and b_m (in particular their ratio), Eq. (138) can also describe skyrmions and other charged objects, while adjusting these coefficients in the core region one can modify the spin configuration there. Since (138)

is a single Slater determinant, we can calculate its energy using the original Hamiltonian directly without using the effective energy functional derived earlier. After subtracting the energy of the fully spin-polarized state, we can show that the vortex energy diverges logarithmically with the system size as would be expected for an XY system:

$$\Delta E_k \sim \gamma k^2 \ln M \ , \tag{140}$$

where

$$\gamma = \sum_{l=0}^{\infty} (-1)^l \frac{2l+1}{8} v_l^{\mathrm{E}}. \tag{141}$$

Here the v_l^E are Haldane's pseudopotential parameters for the interlayer interaction[58] and we have neglected terms that are finite. If we use the energy functional, we find the divergent part of the vortex energy is

$$\Delta E_k \sim \frac{\pi}{2} \rho_E k^2 \ln M \ , \tag{142}$$

implying

$$\rho_E = \sum_{l=0}^{\infty} (-1)^l \frac{2l+1}{4\pi} v_l^E. \tag{143}$$

One can easily show that this expression of ρ_E agrees with previous expression exactly. The present expression in terms of pseudopotential parameters may be more useful in finite-size numerical calculations.[59]

VI. COLLECTIVE MODES AND PSEUDOSPIN RESPONSE FUNCTIONS

In this section we combine the equation of motion for the spin textures [Eq. (55)] and the spin-texture energy functional [Eq. (74)] to calculate the pseudospin linear response[60,61] functions for $d/\ell \neq 0$. We take the pseudospin of the system to be polarized in the $\hat{\mathbf{x}}$ direction and consider the linear response to a time- and space-dependent Zeeman field in the $\hat{\mathbf{y}}$-$\hat{\mathbf{z}}$ plane. Fourier transforming with respect to both time and space we find that

$$\begin{pmatrix} -i\omega & -\frac{4\pi}{\nu}(2\beta + q^2\rho_A) \\ \frac{4\pi}{\nu}(q^2\rho_E) & -i\omega \end{pmatrix} \begin{pmatrix} m_y \\ m_z \end{pmatrix} = \begin{pmatrix} -h_z \\ h_y \end{pmatrix} , \tag{144}$$

where h_y and h_z are the Fourier coefficients of the pseudospin magnetic field at frequency ω and wave vector q. Physically h_z corresponds to a time- and space-dependent bias potential between the two wells, while h_y could arise from a space- and time-dependent interlayer tunneling amplitude. We see immediately that the response is singular when

$$\omega^2 = \omega_{\mathrm{CM}}^2 \equiv \left(\frac{4\pi}{\nu}\right)^2 \left[2\beta + q^2\rho_A\right] q^2 \rho_E. \tag{145}$$

Here ω_{CM} is the frequency of a long-wavelength collective mode of the system. For the $d/\ell = 0$ case, $\beta = 0$, $\rho_A = \rho_E = \rho_s^0$, and the collective mode frequency reduces to the result obtained previously for the spin-wave collective mode of isotropic ferromagnets. ($\omega_{\mathrm{CM}} = 4\pi q^2 \rho_s^0/\nu$.) The collective mode corresponds to a spin precession whose ellipticity increases as the long-wavelength limit is approached. The presence of the mass term ($\beta \neq 0$) changes the collective mode dispersion at long wavelengths from quadratic to linear. In the limit of small q

$$\omega_q = \frac{4\pi}{\nu} \sqrt{2\beta \rho_E}\ q. \tag{146}$$

We can solve Eq. (144) for the frequency and wave-vector-dependent linear response to bias potential between the two wells. The result is

$$\chi_{zz}(q,\omega) = \frac{4\pi q^2 \rho_E/\nu}{\omega^2 - \omega_{\mathrm{CM}}^2}. \tag{147}$$

It is interesting to compare this with formally exact relations for this response function, which can be obtained from the microscopic Hamiltonian of double-layer systems. We first note that χ_{zz} is related to the dynamic structure factor

$$s_{zz}(q,\omega) \equiv \sum_n |\langle \Psi_n | S_q^z | \Psi_0 \rangle|^2 \delta(\hbar\omega - (E_n - E_0)), \tag{148}$$

by

$$s_{zz}(q,\omega) = -\frac{1}{\pi} \mathrm{Im} \chi_{zz}(q, \omega + i\eta). \tag{149}$$

Here $|\Psi_n\rangle$ is an exact eigenstate of the double-layer system. Our result for $\chi_{zz}(q,\omega)$ thus implies that at long wavelengths

$$s_{zz}(q,\omega) = \frac{q}{2} \sqrt{\frac{\rho_E}{(2\beta + q^2\rho_A)}} \delta(\omega - \omega_{\mathrm{CM}}(q)). \tag{150}$$

Some frequency moments of this structure factor can be related to ground-state correlation functions of the double-layer system. The first moment gives the oscillator strength. From Eq. (150) we find that

$$f_{zz}(q,\omega) \equiv \int_0^{\infty} d\omega \omega s_{zz}(q,\omega) = \frac{2\pi}{\nu} q^2 \rho_E. \tag{151}$$

This result agrees with results for this moment calculated previously[62,63] directly from the microscopic Hamiltonian as we can confirm using Eq. (76). We see that the oscillator strength vanishes like q^2 as expected for an ordinary superfluid; however, the coefficient is proportional to ρ_E and hence is nonuniversal. [The remaining oscillator strength is found in a high-frequency collective mode lying below ω_c by an amount proportional to ρ_E, as shown below in Eq. (191).]

The zeroth moment of the dynamic structure factor gives the static structure factor

$$\begin{aligned} s_{zz}(q) &\equiv \int_0^{\infty} d\omega s_{zz}(q,\omega) = \langle \Psi_0 | S_{-q}^z S_q^z | \Psi_0 \rangle \\ &= \frac{q}{2} \sqrt{\frac{\rho_E}{(2\beta + q^2\rho_A)}}. \end{aligned} \tag{152}$$

We note that the static structure factor vanishes linearly with q as q goes to zero. This property illustrates a qualitative difference[64] between the ground state of the double-layer system at $d/\ell \neq 0$ and the ground state in the $d/\ell = 0$ limit for which $s_{zz}(q,\omega)$ approaches a constant as q goes to zero. This property is analogous to what happens in a repulsively interacting Bose gas such as ^{4}He in which the structure factor vanishes linearly with q. According to Feynman's single-mode approximation picture, this is required in order to achieve a linearly dispersing Goldstone mode.[65] The proper structure factor can be included to improve the variational ground-state wave function by means of the Jastrow ansatz

$$\Psi = \exp\left\{-\sum_q \frac{\lambda}{q}\bar{S}^z_{-q}\bar{S}^z_q\right\}\prod_j \begin{pmatrix} e^{i\varphi/2} \\ e^{-i\varphi/2}\end{pmatrix}_j \Psi_V, \quad (153)$$

where λ is a variational parameter. The incompressible mmn states with $n \neq m$, discussed in Sec. IX, have the property[63] that $s_{zz}(q) \sim q^2$ at long wavelengths. This property holds for the ground state whenever there is an excitation gap and additional Jastrow factors are not required to capture the correct long length scale fluctuations.

For $d/\ell \neq 0$ the mass term suppresses long length scale fluctuations in S^z, which measures the difference of the density in the two layers. The -1 moment of $s_{zz}(q,\omega)$ is proportional to the static response function

$$\chi_{zz}(q,\omega=0) = \frac{\nu}{4\pi(2\beta+\rho_A q^2)}\,. \quad (154)$$

Note that $\chi_{zz}(q,\omega=0)$ diverges in the small d/ℓ limit because of the broken SU(2) symmetry in the ground state of the $d/\ell = 0$ system.

The collective mode dispersion can also be obtained from a Lagrangian formulation, which may be useful in describing the thermodynamics of the system. The partition function can be expressed by

$$Z = \int D\mathbf{m}\, e^{-S^E[\mathbf{m}]}\,, \quad (155)$$

where the Euclidean action is

$$S^E[\mathbf{m}] = \int_0^\beta d\tau\left\{\int d^2r\left[-i\frac{\nu}{4\pi}\mathbf{A}(\mathbf{m})\cdot\partial_\tau\mathbf{m}\right] + E[\mathbf{m}]\right\}. \quad (156)$$

The massive m^z field is coupled to the massless field through the time derivative term and the constraint $|\mathbf{m}| = 1$. For simplicity, we assume that the pseudospins are aligned along the $\hat{\mathbf{x}}$ direction *or* we concentrate on a local patch of the spin texture, where the pseudospins are almost aligned along the $\hat{\mathbf{x}}$ direction. Using the constraint, we can express m^x in terms of m^y and m^z. We only keep terms quadratic in m^y and m^z. The monopole vector potential is given in a convenient gauge by $\mathbf{A} \simeq (0,-m^z,m^y)/2$. Previously we took the long-wavelength limit to obtain the local action in real space since we were interested in results which would become exact in the limit of extremely smooth spin textures. Here we generalize our discussion in order to obtain an approximation for the full spectrum of collective modes. The calculations are identical to those detailed in Sec. IV, except that we do not take the long-wavelength limit. Here we report results only for the case of experimental interest $\nu = 1$, where the expressions take a somewhat simpler form. For small m_z and m_y it is straightforward to integrate out the massive m^z field since the different momentum components decouple at the Gaussian level

$$e^{-S^E_{\rm eff}[m^y]} = \int Dm^z \exp\Big\{-\sum_{\omega_n,q} m^z(-\omega_n,-q) \times\omega_n m^y(\omega_n,q)/4\pi + D_z(q)|m^z(\omega_n,q)|^2 + D_y(q)|m^y(\omega_n,q)|^2\Big\}\,, \quad (157)$$

yielding

$$S^E_{\rm eff}[m^y] = \sum_{\omega_n,q}\left(\frac{\omega_n^2}{64\pi^2 D_z(q)} + D_y(q)\right)|m^y(\omega_n,q)|^2. \quad (158)$$

We can read off the collective mode frequency from this expression:

$$\omega_k = 8\pi\sqrt{D_z(k)D_y(k)}, \quad (159)$$

where

$$D_z(k) = \frac{1}{8\pi}\left\{\frac{1}{\pi\ell^2}V_z(k)e^{-\frac{k^2}{2}} + \int\frac{d^2q}{(2\pi)^2}V_E(q)\exp(-|q|^2/2) - \int\frac{d^2q}{(2\pi)^2}V_A(q)\exp(-|q|^2/2)e^{ik\wedge q}\right\} \quad (160)$$

and

$$D_y(k) = \frac{1}{8\pi}\int\frac{d^2k}{(2\pi)^2}V_E(q)\exp(-|q|^2/2)(1-e^{ik\wedge q})\,. \quad (161)$$

This spectrum agrees exactly with the dispersion relation in Refs. 8 and 7, where the excitation energy for the single magnon state was obtained using the time-dependent Hartree-Fock approximation and using a single-mode approximation combined with the Bogoliubov transformation. In the long-wavelength limit the dispersion relation reduces to the result discussed above.

VII. KOSTERLITZ-THOULESS PHASE TRANSITION AND SPIN-CHANNEL SUPERFLUIDITY

The linearly dispersing gapless mode discussed in the preceding section and the absence of gapless charged excitations suggest that the system should show superfluid behavior in the pseudospin channel as has been noted

previously.[6,5] To make this suggestion more concrete we evaluate the linear response of the system to opposing electric fields in the two layers. Combining Eq. (147) with the continuity equation for the $\hat{\mathbf{z}}$ component of pseudospin we find that

$$\sigma_{zz}(q,\omega) = \frac{e^2\omega\chi_{zz}(q,\omega)}{iq^2} = \frac{4\pi e^2\rho_E\omega/i\hbar^2\nu}{\omega^2-\omega_{\rm CM}^2}. \quad (162)$$

This conductivity gives the difference between oppositely directed charge currents which flow in the two layers in response to oppositely directed electric fields. (Note that no net current will be induced by such electric fields.) In the long-wavelength limit this leads to a frequency-dependent conductivity equivalent to that of a superfluid with superfluid density proportional to ρ_E:

$$\sigma_{zz}(\omega) = \frac{4\pi e^2\rho_E}{i\nu\hbar^2\omega}. \quad (163)$$

Note that by the Kronig-Kramers relation this conductivity must have a real part which is proportional to a δ function at zero frequency. We remark that the superfluid property requires not only a gap for charged excitations and the linearly dispersing collective mode but also a total oscillator strength which vanishes as q^2 at long wavelengths. For the conductivity associated with the response to electric fields in the same direction in the two layers, the total oscillator strength associated with intraLandau-level excitations vanishes[63] as q^4 and the collective mode has a gap at long wavelengths. These properties lead to a $\sigma(q,\omega)$ which vanishes as q^2 in the long-wavelength limit leading to the quantum Hall effect rather than to superfluidity.

The above analysis is dependent on our analysis of the response functions of the double-layer system, which does not include thermal fluctuations. At finite temperatures both meron and pseudospin collective mode thermal fluctuations have to be accounted for. As in other two-dimensional superfluids, the linear response conductivity is still expected to vanish below a finite temperature. As we have discussed previously the low-energy excitations of the double-layer system consist of highly elliptical precessions of the spin about the direction of the local order parameter. It follows from a continuity equation and the equation of motion for the spin textures that the conserved number current corresponding to the $\hat{z}$ component of the pseudospin is related to the pseudospin by the usual minimal coupling prescription (see the Appendix for further discussion of this result)

$$J_{zz} = \frac{2\rho_E}{\hbar}\nabla\phi\ , \quad (164)$$

where the factor of 2 arises from the definition of J_{zz} as the difference of the number currents in the two layers and ϕ is the projection of the pseudospin orientation onto the $\hat{x}$-$\hat{y}$ plane. ϕ plays the same role as the phase of the superconducting order parameter in a two-dimensional superfluid and pseudospin channel superfluidity will be coincident with pseudospin easy-plane ferromagnetism. Likewise, as we discuss further below and in the Appendix, the divergent superfluid conductivity implies zero pseudospin Hall resistivity below the Kosterlitz-Thouless (KT) temperature.

In this system the KT phase transition,[66] which separates the superfluid and normal states, is expected to be[67] associated with the unbinding of meron pairs of opposite charge and opposite vorticity. In order to analyze the KT transition, it is better to work in real space, since the KT transition is controlled by the large-distance physics and the short-distance details can be effectively taken into account by the vortex core energy. In the limit of a large mass term ($\beta\ell^2 \gg \rho_E$) fluctuations in m_z will be small. After integrating out the massive field and finite frequency fluctuations in a Gaussian approximation, we obtain an effective XY model

$$S_{\rm eff}^E = \frac{\beta}{2}\rho_E \int d^2r\,(\nabla\varphi)^2, \quad (165)$$

where φ is an angle denoting the direction of the spin in the XY plane. We know that this model undergoes a Kosterlitz-Thouless phase transition associated with the unbinding of bound vortex pairs[66] at approximately

$$T_c = \frac{\pi}{2}\rho_E. \quad (166)$$

For a 2D XY model, corrections to this expression for T_c arise from finite-temperature spin-wave and vortex-antivortex polarization renormalizations of the effective spin stiffness at long distances. The magnitude of the corrections depends on details of the short-distance physics. For the 2D nearest-neighbor-coupling XY model on a square lattice[68] whose short-distance physics we believe to be similar to that of double-layer systems

$$T_c \sim 0.90\rho_E. \quad (167)$$

(We note however that the double-layer system possesses charged excitations whose energies are larger but of the same order as the meron core energies.) In the present case T_c should be further reduced, especially as d/ℓ goes to zero, because of fluctuations out of the $\hat{\mathbf{x}}$-$\hat{\mathbf{y}}$ plane. However, numerical studies of the anisotropic $O(3)$ model on a square lattice[69] demonstrate that these corrections are not very important except in the limit of extremely weak anisotropy. For example, T_c is reduced by a less than a factor of 2 compared to Eq. (167) even for parameters which correspond to $\rho_E\ell^2/\beta \sim 30$. Comparing with Fig. 3 we see that such weak anisotropies occur in double-layer systems only for $d/\ell < 0.3$, a regime which is not experimentally accessible. (Quantum fluctuations only increase the anisotropy by decreasing ρ_E.) In the following section we provide a quantitative estimate of the temperature scale expected for T_c by combining Eq. (167) with estimates of ρ_E, which include the effect of quantum fluctuations. As we discuss there, the principal source of uncertainty in the T_c estimate comes from attempting to estimate ρ_E.

It should be noted that the U(1) symmetry responsible for the XY model physics and the KT transition is robust under application of a bias voltage, which induces a charge imbalance between the layers. Within the pseu-

dospin picture, the spins simply tilt slightly out of the XY plane in the positive or negative z direction. This will reduce the component of the spin in the XY plane and hence may lower the KT temperature slightly, but it will not (for weak imbalance) destroy the ordering transition. This is in sharp contrast to the behavior for mmn states with $m > n$, which require precise charge balance for their existence. This point is discussed in more detail in Sec. VIII below.

In an ordinary superconducting film, the linear response conductivity is infinite below the KT transition temperature. More precisely the voltage-current relationship obeys

$$V \sim I^{p(T)} , \tag{168}$$

where the exponent $p(T) = 1$ for $T > T_{\rm KT}$ and $p(T) > 3$ for $T < T_{\rm KT}$. The exponent has a universal jump and changes discontinuously from 1 to 3 at the transition. The critical current is zero, i.e., there is always a finite voltage for any finite current. At any finite temperature Bogoliubov quasiparticles are present; however, these do not produce dissipation because the current is carried by the superfluid and the voltage is (essentially) zero. The superfluid shorts out the normal fluid and so there is no electric field to produce motion of the normal fluid. The present problem is richer because of the existence of two layers in which the current can flow and because of the possible presence of a Hall voltage in the presence of currents. It is instructive to define phenomenological[60] transport coefficients (for the case of two identical layers) as follows:

$$\begin{pmatrix} E_1^x \\ E_1^y \\ E_2^x \\ E_2^y \end{pmatrix} = \begin{pmatrix} \rho_{xx}^A & \rho_{xy}^A & \rho_{xx}^E & \rho_{xy}^E \\ -\rho_{xy}^A & \rho_{xx}^A & -\rho_{xy}^E & \rho_{xx}^E \\ \rho_{xx}^E & \rho_{xy}^E & \rho_{xx}^A & \rho_{xy}^A \\ -\rho_{xy}^E & \rho_{xx}^E & -\rho_{xy}^A & \rho_{xx}^A \end{pmatrix} \begin{pmatrix} j_1^x \\ j_1^y \\ j_2^x \\ j_2^y \end{pmatrix} . \tag{169}$$

Here the numerical subscripts label the two layers of the double-layer system. (Note that this phenomenology reflects the fact that the transport coefficients for the sum and differences of the currents or electric fields decouple.)

In general there are four independent transport coefficients, allowing for contributions to both dissipative and Hall electric fields due to currents flowing in the same layer, ρ_{xx}^A and ρ_{xy}^A, and the interlayer or drag[70] resistivities coming from currents flowing in the opposite layer, ρ_{xx}^E and ρ_{xy}^E. The ratios of the electric-field sums and differences to the current sums and differences are given by $\rho^A + \rho^E$ and $\rho^A - \rho^E$, respectively, for both dissipative and Hall fields. The matrix of conductivity coefficients may be related to the matrix of resistivity coefficients by inverting Eq. (169). We have argued above that the dissipative dc conductivity coefficient for current differences is infinite when $T < T_{\rm KT}$. This implies that both the Hall and dissipative resistivities are zero for $T < T_{\rm KT}$, i.e., that $\rho_{xx}^E = \rho_{xx}^A$ and $\rho_{xy}^E = \rho_{xy}^A$. Below the Kosterlitz-Thouless temperature there are only two independent dc linear transport coefficients. Moreover we know a great deal about these two transport coefficients because of the quantum Hall effect. When identical electric fields exist in each layer the total Hall conductance is nearly exactly quantized at low temperatures ($\sigma_{xy}^A + \sigma_{xy}^E \approx \nu e^2/h$) and the dissipative conductance is activated [$\sigma_{xx}^A + \sigma_{xx}^E \sim \sigma_0 \exp(-\Delta/2k_BT)$]. Here σ_0 is a nonuniversal constant and Δ is the gap for making charged particle-hole pairs; these charged objects are probably the meron pairs discussed previously. In terms of the linear resistivity matrix of Eq. (169) we conclude that for low temperatures

$$\rho_{xx}^A = \rho_{xx}^E = \frac{\sigma_0 \exp(-\Delta/2k_BT)}{2\nu^2(e^2/h)^2} \tag{170}$$

and

$$\rho_{xy}^A = \rho_{xy}^E = \frac{-h}{2\nu e^2} . \tag{171}$$

Note that this implies the occurrence of remarkable cross-talk phenomena.[60,71] For example, if current is injected in the $\hat{\mathbf{x}}$ direction in one layer but no current flows in the other layer, a quantized Hall field whose value corresponds to the density *per layer* will appear in both layers.

Ho[53] has recently considered the question of the stability of current flow in the spin channel. He finds that the existence of oppositely directed electric fields in each layer induces phase twists which can be relieved only by steady nucleation of topological defects in close analogy with textures induced in superfluid ^{3}He. We agree that such effects will occur at finite current densities; however, these results disagree with our findings above that the linear response σ_{xx} is infinite and ρ_{xy} is zero. We believe that this discrepancy has two origins: the finite easy-plane anisotropy is essential to the analysis of this problem and the fact that there are subtleties associated with the question of the existence of uniform current carrying states in the lowest Landau level. (This latter point is discussed in detail in the Appendix.) Without easy plane anisotropy, the SU(2) symmetry prevents the XY order necessary for superfluidity. However, since even *with* easy plane anisotropy, the XY order is only algebraic, the critical current density is zero as discussed above. For any finite current density there will indeed be dissipation due to generation of topological defects.

VIII. EXACT DIAGONALIZATION STUDIES

In this section we discuss some microscopically based investigations of the properties of double-layer systems at $\nu = 1$. We start by discussing some studies using the finite-size exact diagonalization in the spherical geometry.[54] As we have emphasized previously at $d/\ell = 0$, where intralayer and interlayer Coulomb potentials are the same, the electron-electron interaction term in the Hamiltonian is invariant under rotations in pseudospin space. Eigenstates of the Hamiltonian H can be simultaneous eigenstates of any one component of the total pseudospin operator. States with pseudospin quantum number S will have degeneracy $2S+1$. (As discussed

in Sec. II the ground state at $\nu = 1$ is a pseudospin eigenstate with $S = N/2$ where N is the number of electrons.) At finite layer separations, the interlayer interactions will be weaker than intralayer interactions and the broken symmetry in the ground state is reduced from SU(2) to U(1). Microscopically this corresponds to the fact that in the ground state, electrons within the same layer will be more strongly correlated than electrons in different layers in order to minimize the total Coulomb energy. For sufficiently widely separated layers we will provide evidence that the ground state no longer has broken symmetry. This problem has been studied before through examination of the ground-state wave function,[2,72] and a ground-state level crossing was found to occur in the vicinity of $d/\ell = 1.5$. In this section we will discuss another attempt to estimate the layer separation at which this quantum phase transition takes place. In addition we will discuss rough estimates obtained for the charge gap and for the parameters of the spin-texture energy functional.

The model we consider consists of two two-dimensional electron systems separated by a distance d. For the sake of definiteness the spread of the electron wave function in the perpendicular direction in each layer is neglected. (Such effects are easily accounted for if the geometry of a particular sample is known; we assume that it will normally be possible to define an effective layer separation for each sample.) We start by considering the case of greatest interest where each layer has the same number of electrons and define the zero of energy by placing neutralizing nonresponding background charge backgrounds on each layer. We parametrize the intralayer and interlayer interactions in terms of the Haldane pseudopotential parameters[54] for this model. Calculations were performed for N =4,6,8,10 at a variety of layer separations.

We first discuss finite-size estimates of the chemical potential dependence at $\nu = 1$. For a finite system of electrons on the surface of a sphere the chemical potential jump is expected to occur when $N = N_{\rm orb} = N_\phi + 1$. Here $N_{\rm orb}$ is the number of orbitals per Landau level when N_ϕ flux quanta pass through the surface of the sphere. The chemical potential discontinuity is given by the limit as $N \to \infty$ of

$$\Delta\mu = E(N_\phi, N+1) + E(N_\phi, N-1) - 2E(N_\phi, N), \tag{172}$$

where $N_\phi = N - 1$. In Fig. 11, the finite-size estimate of $\Delta\mu$ for the $\nu = 1$ state is shown as a function of layer separation d for $N = 8$ and $N = 10$. $\Delta\mu$ decreases continuously as d increases. The finite-size corrections can be crudely inferred from comparisons of results for $N = 8$ and $N = 10$. For a system size increase from $N = 8$ to $N = 10$, $\Delta\mu$ increases slightly for $d < 1.5\ell$ and decreases substantially for $d > 1.5\ell$. In all likelihood this result indicates that in the thermodynamic limit ($N = \infty$), there is no chemical potential discontinuity for $d \gtrsim 1.5\ell$. For layer separations larger than this, it seems likely that the properties of the double-layer system should be similar to those of two isolated layers with Landau level filling factor $\nu = 1/2$, which are of great interest in their own right.[73] The existence of an intermediate state, best viewed as a quantum disordered easy-plane ferromagnet, is an interesting speculative possibility.

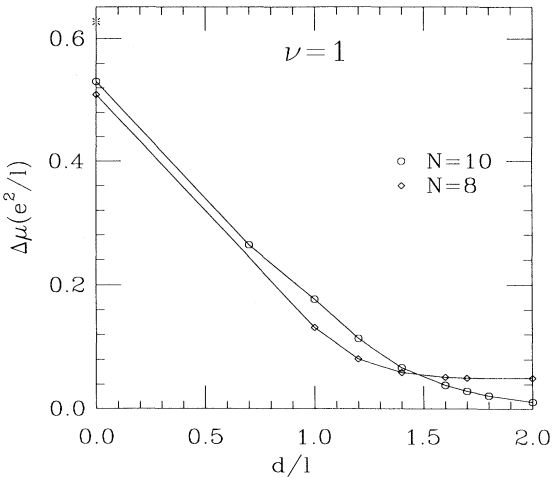

FIG. 11. Chemical potential discontinuity $\Delta\mu$ as a function of layer separation d for $\nu = 1$. The results are for system sizes of eight and ten electrons. The $*$ mark shows the value of $\Delta\mu$ at $d = 0$ and $N = \infty$ according to the skyrmion theory.

We have argued that the charged objects, at least away from $d/\ell = 0$, consist of meron pairs with opposite vorticity. Since we expect the meron core energy and the pseudospin stiffness to both vanish when the order parameter vanishes, we expect that the chemical potential discontinuity should vanish at the critical layer separation where the zero-temperature phase transition to a disordered state occurs. Thus these results also provide an estimate of the layer separation at which the quantum phase transition takes place. We can get an estimate of the finite-size errors in these results at small $d/\ell = 0$, where the value of the chemical potential discontinuity is known[12] exactly for $\nu = 1$. At $\nu = 1$ and $d/\ell = 0$, the ground-state many-body wave function *is* a single Slater determinant so that there are no quantum fluctuation corrections to the Hartree-Fock value for the spin stiffness given in Sec. IV. The chemical potential discontinuity is therefore given exactly for $N = \infty$ by $\Delta\mu = (e^2/\ell)(\pi/8)^{1/2}$. This value is marked by an asterisk in Fig. 11 and compares well with the finite-size estimates.

To get further insight into the system, we have attempted to estimate the dependence of the ground-state order parameter M and the easy-plane pseudospin magnetic susceptibility χ on d/ℓ for $\nu = 1$. We normalize the order parameter so that it has the value 1 at d/ℓ =0. The order parameter is defined by

$$M \equiv \lim_{t\to 0}\left[\lim_{N\to\infty} -(1/N)\frac{d}{dt}E(N_\phi, N)|\right] \tag{173}$$

and the magnetic susceptibility is defined by

$$\chi \equiv \lim_{t\to 0}\left[\lim_{N\to\infty} -(1/N)\frac{d^2}{dt^2}E(N_\phi, N)|\right], \tag{174}$$

where t is the interlayer tunneling amplitude (which acts like a Zeeman field on the pseudospins). Some typical[74] results for the dependence of the ground-state energy on tunneling amplitude are shown in Figs. 12 and 13. The solid lines in these figures are interpolations between energies calculated at a series of equally spaced t values *not* including $t = 0$. In a finite-size system it is clear that the ground-state energy always will have a quadratic dependence on tunneling amplitude until the total Zeeman coupling becomes comparable to the smallest excitation energy of the system so that the order of limits in Eqs. (173) and (174) is essential. The lowest-energy excitations of the system will be long-wavelength collective modes and we can expect to estimate the dependence of the zero-temperature order parameter and susceptibility on d/ℓ from the dependence of the ground-state energy on t in the regime where the t is small compared to e^2/ℓ, but the Zeeman energy is large compared to the minimum excitation energy of the system. Using our result for the collective mode energy and the correspondence between linear momenta and angular momenta on a sphere ($k \sim L/R$, where R is the sphere radius), we estimate that the minimum value of t we can consider is

$$t_{\min} \sim \frac{4\pi\sqrt{\rho_E \beta}}{\nu \ell N^{3/2}}. \tag{175}$$

For $\nu = 1$, ρ_E is strongly renormalized downward by quantum fluctuations and we expect it to drop to zero as d/ℓ approaches ~ 1.5, while β starts from zero at $d/\ell = 0$, rises quadratically and is less affected by quantum fluctuations. Comparing with Fig. 3 we estimate that for $d/\ell \sim 1$ and $N = 10$, $t_{\min} \sim 10^{-3}(e^2/\ell)$; $t_{\min}$ will approach zero for $d/\ell \to 0$. These rough estimates are consistent with the crossovers from quadratic to linear dependence on d seen in Figs. 12 and 13. We have estimated the magnetization and susceptibility by interpolating between calculated energy values and evaluating the derivatives at $d \sim 2.5 \times 10^{-3}$. The results are shown in Figs. 14 and 15. The magnetization decreases continuously with d and appears to vanish in the limit of large N

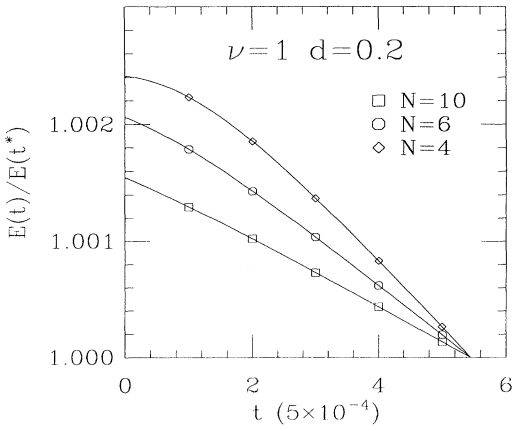

FIG. 12. Ground-state energy versus tunneling amplitude at small tunneling amplitudes at $d/\ell = 0.2$ for $N = 4$, $N = 6$, and $N = 10$.

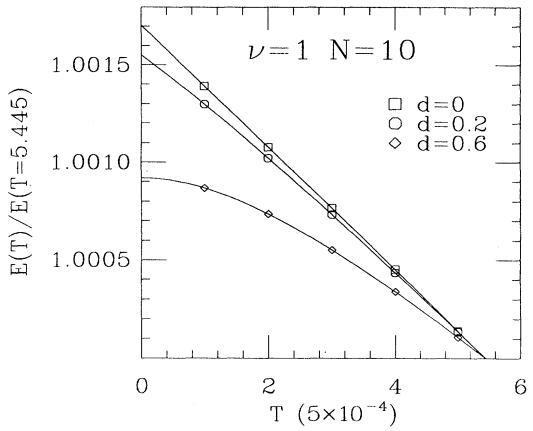

FIG. 13. Ground-state energy versus tunneling amplitude at small tunneling amplitude for $N = 10$ at $d/\ell = 0$, $d/\ell = 0.2$, and $d/\ell = 0.6$.

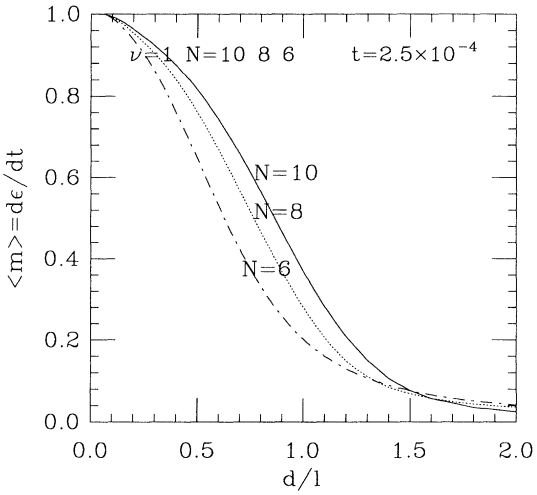

FIG. 14. Finite-size estimate for the ground-state order parameter as a function of layer separation for $\nu = 1$. The results are for system sizes of six, eight, and ten electrons.

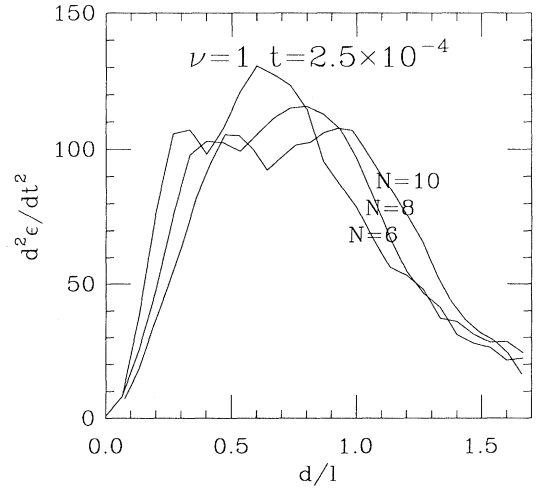

FIG. 15. Finite-size estimate for the ground-state magnetic susceptibility as a function of layer separation for $\nu = 1$. The results are for system sizes of six, eight, and ten electrons.

at a critical layer separation consistent with the estimate obtained by looking at the charge gap. The finite-size susceptibility estimate becomes large near where the order parameter vanishes as expected.[75]

The order parameter is an important parameter which will appear in the spin-texture energy functional in the presence of interlayer tunneling as discussed in Ref. 13. We discuss here attempts to estimate parameters β and ρ_E. Figure 16 shows the dependence on the layer separation of the increase in the ground-state energy when one and two electrons are transferred from one layer to another. According to the spin-texture energy functional this energy should be given, in the limit of large systems, by

$$\Delta E = \beta(d/\ell)\ell^2 2\pi \left[\frac{(\Delta N)^2}{N}\right]. \tag{176}$$

The finite-size exact diagonalization calculations were performed for $d/\ell = 0.5, d/\ell = 1.0$, and $d/\ell = 1.5$ and the values of $\beta\ell^2/(e^2/\ell)$ inferred by comparing with Eq. (176) were 5.1×10^{-3}, 1.7×10^{-2}, and 4.0×10^{-2}, respectively. These values compared to 2.0×10^{-2}, 4.0×10^{-2}, and 6.0×10^{-2}, respectively, for the Hartree approximation and 7.3×10^{-3}, 2.2×10^{-2}, and 3.8×10^{-2} for the Hartree-Fock approximation results derived earlier. The quadratic dependence derived in the Hartree-Fock approximation is apparent in the exact diagonalization results. The exact diagonalization estimates for β demonstrate that quantum fluctuation corrections to this quantity are quite small and that the Hartree-Fock results shown in Fig. 3 are quite accurate. An important aspect associated with the broken symmetry in the ground state at $\nu = 1$ is the fact that the chemical potential discontinuity is not strongly influenced by the transfer of charge from one layer to the other. In the presence of a bias potential the ground-state pseudospin is tilted out of the $\hat{x}$-$\hat{y}$ plane, but the system still has a broken U(1) symmetry. This situation contrasts with the case of the double-layer quantum Hall effect which occurs at total Landau level filling factor $\nu_T = 1/2$ where the chemical potential discontinuity occurs only near equal layer densities.[76]

In Figs. 17 and 18 we show the dependence of the lowest excitation energy on wave vector for a finite system of electrons on a torus with periodic boundary conditions[50] for $d/\ell = 0.5$ and $d/\ell = 1.0$, respectively. At $d/\ell = 0.5$ clear indications of the linearly dispersing collective mode are evident. The velocity of the linearly dispersing collective mode is related to the parameters of the spin-texture energy by

$$E(k) = k\ell\frac{4\pi}{\nu}\sqrt{2\beta\rho_E}. \tag{177}$$

Reading off the velocity from Fig. 17 and using the value of β obtained above from the exact diagonalization calculation, we estimate that for $d/\ell = 0.5$, $\rho_E \approx 1.5\times10^{-2}(e^2/\ell)$ compared to the Hartree-Fock value $\rho_E = 1.22\times10^{-2}$. For $d/\ell = 1.0$ the linear dispersion of the collective mode is less evident and it is more difficult to estimate the velocity accurately. We have chosen to estimate the velocity from the energy at the smallest finite-size system wave vector as indicated in Fig. 18, from which we obtain $\rho_E \approx 3.3\times10^{-3}(e^2/\ell)$ compared to the Hartree-Fock value $\rho_E = 4.22\times10^{-3}$. At this value it is evident that quantum fluctuations are decreasing the value of ρ_E as the critical layer separation is approached.

To obtain a rough quantitative estimate of the layer separation dependence of the Kosterlitz-Thouless transition temperature, we calculate the leading order corrections to the Hartree-Fock ground-state energy as a function of parallel magnetic field $B_{||}$. The renormalized spin stiffness ρ_E^R is related to the dependence of the ground-state energy on field by the following equation:[13]

$$\rho_E^R = \frac{d^2}{dQ^2}E_A(Q)|_{Q\to0}\ , \tag{178}$$

where E_A is ground-state energy per area and $Q =$

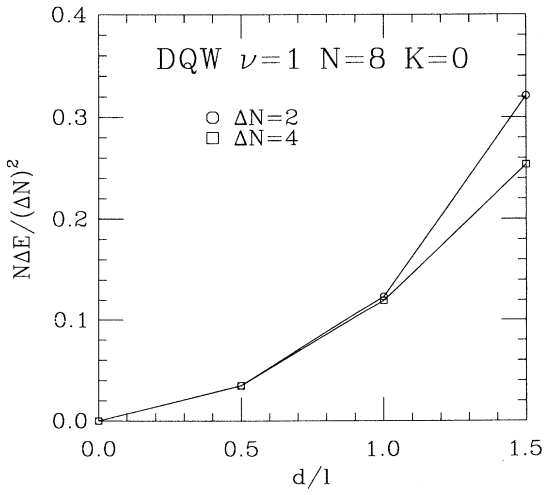

FIG. 16. Dependence of the ground-state energy level on d/ℓ for one and two electrons transferred between wells.

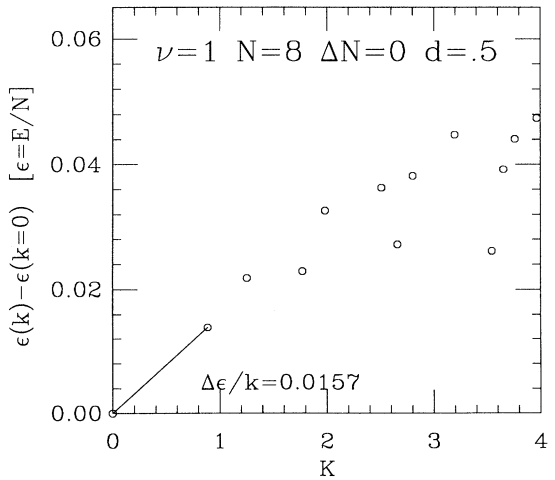

FIG. 17. Wave-vector dependence of low-energy excited states for $\nu = 1$, $d/\ell = 0.5$, and $N = 10$. These results can be used to estimate the quantum renormalized spin stiffness.

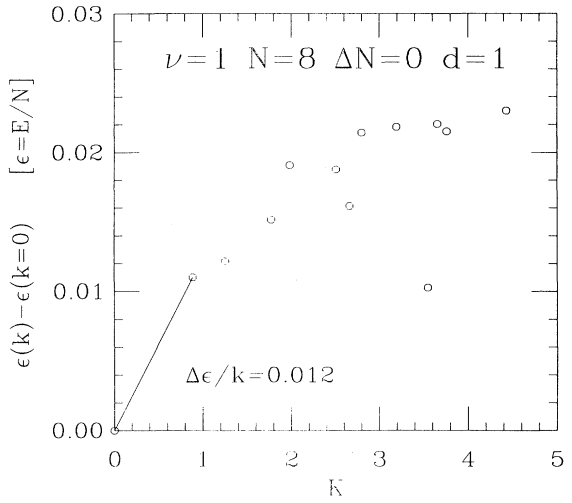

FIG. 18. Wave-vector dependence of low-energy excited states for $\nu = 1$, $d/\ell = 1.0$, and $N = 10$. These results can be used to estimate the quantum renormalized spin stiffness.

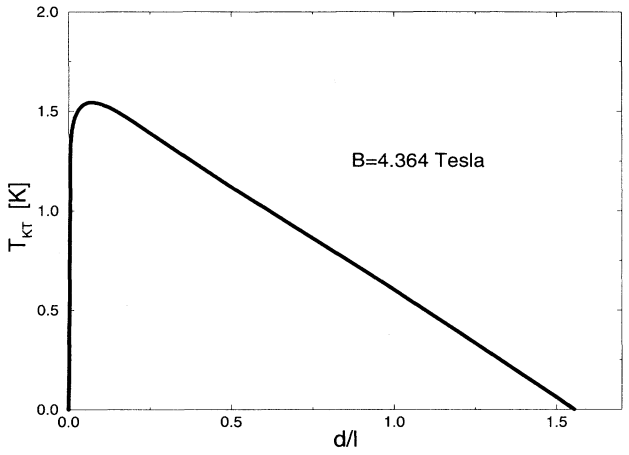

FIG. 19. Estimate of Kosterlitz-Thouless transition temperature for $B = 4.364$ T so that $e^2/\ell k_B \sim 106$ K. These estimates include quantum renormalizations of the spin stiffness and corrections to the XY model due to finite anisotropy strength.

$dB_{||}/\ell^2 Bp$. Given ρ_E^R and the fact that the mass term is not strongly influenced by quantum fluctuations, we can estimate the effect of fluctuations in m^z on the transition temperature by using results from numerical simulations[77] of the classical $O(3)$ easy-plane nonlinear sigma model. We find[78] the dependence of $T_{\rm KT}$ on d/ℓ, which is illustrated in Fig. 19. As the layer separation approaches zero, U(1) symmetry is enhanced to $O(3)$ and fluctuations of m^z eventually become sufficiently important that $T_{\rm KT}$ decreases and approaches zero. On the other hand, for layer separation close to the critical one, phase coherence between the layers is destroyed only by quantum fluctuations. Our results for $T_{\rm KT}$ are in good agreement with the d/ℓ dependence of the temperatures at which features in the dissipative resistance of double-layer systems have been seen[79] by Lay *et al.* These features might be associated with the vanishing charge gap expected to occur in parallel with the KT transition.

IX. CHERN-SIMONS-LANDAU-GINZBURG THEORY

A useful alternative way of understanding the physics of the quantized Hall effect is based on the concept of composite bosons.[80–82] Much of the physics of the single-layer quantum Hall effect can be described in terms of the Chern-Simons-Landau-Ginzburg (CSLG) theory.[81,83] One starts with the observation that the problem at odd denominator filling fraction can be mapped exactly to a problem of bosons with an odd number of statistical flux tubes attached to them. At the mean-field level, the statistical flux cancels the external magnetic field and one obtains a boson superfluid. Treating the fluctuations above this mean field within the random-phase approximation restores the gap and one obtains an incompressible bosonic liquid. Both Laughlin's wave function[25] and the long-wavelength algebraic off-diagonal long-order correlation function[80] can be derived explicitly from the CSLG theory.[83,84]

The CSLG formulation of the single-layer quantum Hall system has been extended both to the case involving electron spin[15,12] and to the double-layer case[60,5,6,85] by a number of authors. The Chern-Simons theory of the double-layer system based on the fermionic representation has also been constructed.[64] In the case of double-layer systems, there is both a statistical gauge interaction of the composite bosons within the layers and between the layers. The action for this problem is given by

$$\mathcal{L}_\phi = \phi_\uparrow^\dagger (i\partial_t + A^0 - a_\uparrow^0)\phi_\uparrow - \frac{1}{2m^*}\left|\left(\frac{1}{i}\nabla + \mathbf{A} - m\mathbf{a}_\uparrow - n\mathbf{a}_\downarrow\right)\phi_\uparrow\right|^2 + \frac{1}{4\pi}\,\varepsilon^{\mu\nu\rho}\, a_\uparrow^\mu \partial_\nu\, a_\uparrow^\rho$$
$$- \frac{1}{2}\int d^2y\; \delta\rho_\uparrow(x)\; V_A(x-y)\; \delta\rho_\uparrow(y) + (\uparrow\!\to\!\downarrow) - \int d^2y\; \delta\rho_\uparrow(x)\; V_E(x-y)\; \delta\rho_\downarrow(y), \tag{179}$$

where $(\uparrow\!\to\!\downarrow)$ indicates the first four corresponding terms with up and down labels interchanged. Here m is an odd integer. In the absence of tunneling, the particles in the two different layers are distinguishable, the relative phase winding between them can be either 0 or π, therefore, the integer n can be either even or odd. At the mean field level, the equations of motion are given by

$$\nabla \times \mathbf{a}_\sigma(x) = 2\pi\rho_\sigma(x)\ , \quad \mathbf{A} = m\mathbf{a}_\uparrow + n\mathbf{a}_\downarrow. \tag{180}$$

When the electron densities of both layers are equal, we see that the the filling factor has to be $\nu = 2/(m+n)$ for

these equations to be satisfied.

From this CSLG formulation Halperin's wave function[1] for the (mmn) state can be derived in a fashion similar to the single-layer case.[83,84,6] One can decompose the complex boson field in terms of the amplitude and the phase part

$$\phi_\sigma(x) = \sqrt{\bar{\rho} + \delta\rho_\sigma}\, e^{i\theta_\sigma(x)}. \tag{181}$$

The Chern-Simons gauge field induces a long-ranged logarithmic density-density interaction, giving rise to the following effective Hamiltonian at the quadratic level:

$$H = \frac{1}{2}\frac{\bar{\rho}}{m^*}\sum_q \left[\left(\frac{2\pi}{q}\right)^2 [(m^2+n^2)\pi_\sigma(q)\,\pi_\sigma(-q) + 4mn\pi_\uparrow(q)\,\pi_\downarrow(-q)] + q^2\,\theta_\sigma(q)\,\theta_\sigma(-q)\right], \tag{182}$$

where π_q^σ is proportional to the density $\delta\rho_\sigma(x)$ and is the conjugate variable of the phase, i.e.,

$$[\theta_\sigma(q),\ \pi_{\sigma'}(q')] = -i\,\delta_{\sigma\sigma'}\delta(q+q'). \tag{183}$$

This Hamiltonian is a direct sum of harmonic oscillator Hamiltonians for each wave vector q. By forming the sum and the differences of these operators, one can easily diagonalize the Hamiltonian and find the ground-state wave function in terms of its dependence on the generalized coordinates (densities) π_q^σ:

$$\Psi_0[\pi_q] = \exp\left\{\frac{1}{4}\sum_q \frac{2\pi(m+n)}{q^2}\,\pi_q^+\,\pi_{-q}^+ + \frac{2\pi(m-n)}{q^2}\,\pi_q^-\,\pi_{-q}^-\right\}. \tag{184}$$

One can express the density operators in terms of the ordinary first quantized coordinates of the particles

$$\pi_q^+ = \frac{1}{\sqrt{2}}\sum_i \left\{e^{iqx_i^\uparrow} + e^{iqx_i^\downarrow} - 2\bar{\rho}\right\},$$
$$\pi_q^- = \frac{1}{\sqrt{2}}\sum_i \left\{e^{iqx_i^\uparrow} - e^{iqx_i^\downarrow}\right\} \tag{185}$$

and obtain the first quantized wave function

$$\Psi_0(x_i^\uparrow, x_j^\downarrow) = \prod_{i<j} |x_i^\uparrow - x_j^\uparrow|^m\,|x_i^\downarrow - x_j^\downarrow|^m \times |x_i^\uparrow - x_j^\downarrow|^n e^{-\frac{1}{4}\sum_{\sigma,i}|x_i^\sigma|^2}. \tag{186}$$

This is nothing but Halperin's wave function for the (mmn) state expressed in terms of the composite boson variables. One can easily perform a singular gauge transformation back to the original electron coordinates and find explicitly Halperin's wave function expressed in terms of the original electrons.

From the effective Hamiltonian (182) one can also derive all static correlation functions in the long-wavelength limit. One example is the analog of the off-diagonal long-range order correlation function first introduced by Girvin and MacDonald[80] for the single-layer case. From (182) one easily obtains

$$\langle e^{i\theta_+(x)-i\theta_+(y)}\rangle = |x-y|^{-(m+n)},$$
$$\langle e^{i\theta_-(x)-i\theta_-(y)}\rangle = |x-y|^{-(m-n)}. \tag{187}$$

Similarly, one can obtain the static density correlation function in the long-wavelength limit (for $m \neq n$)

$$\langle \pi_q^+\,\pi_{-q}^+\rangle = \frac{q^2}{2\pi(m+n)},\ \langle \pi_q^-\,\pi_{-q}^-\rangle = \frac{q^2}{2\pi(m-n)}. \tag{188}$$

We therefore conclude that the CSLG theory correctly gives the static properties of the double-layer systems in the long-wavelength limit and they agree extremely well with the results of the microscopic calculations based on Halperin's wave function. At the level of the static correlation functions, the degree of agreement is similar to the single-layer case.[81,83] However, there is some discrepancy in the collective mode spectrum[63] between the CSLG theory and the microscopic theory obtained using the projected single-mode approximation (SMA). Within the CSLG theory, the collective mode spectrum can be obtained by studying the Gaussian fluctuations of the bose order parameter and one obtains the following spectrum for the in-phase and the out-of-phase collective modes:

$$\omega_+ = \hbar\omega_c,\ \omega_- = \hbar\omega_c\frac{m-n}{m+n}, \tag{189}$$

in the long-wavelength limit. One sees that the in-phase mode agrees exactly with the prediction of Kohn's theorem, since the in-phase magnetic translation is a good symmetry. The out-of-phase mode agrees exactly with the SMA spectrum obtained using the *full density operator* ρ_q

$$\frac{\langle\rho_q[H,\rho_{-q}]\rangle}{\langle\rho_q\rho_{-q}\rangle} = \hbar\omega_c\frac{m-n}{m+n}. \tag{190}$$

This result is not surprising, since this SMA formula involves only the static correlation functions of the full density operator and these are obtained correctly within the CSLG theory, as shown above. The drawback of the CSLG theory lies in the fact that there is no sensible way of obtaining the projected density operators. In the limit of large Landau level spacing, the SMA using the full density operator is not adequate and the projected operators must be used. Explicitly projecting onto transitions from the $N=0$ to the $N=1$ Landau levels, one obtains[63]

$$\omega_+ = \hbar\omega_c,\ \omega_- = \hbar\omega_c - \int\frac{d^2q}{(2\pi)^2}q^2V_q^E h^E(q) \tag{191}$$

for the inter-Landau-level transition modes. From a microscopic point of view, this discrepancy should be resolved. However, one can also take a phenomenological point of view and fix the mode frequencies obtained

within the CSLG theory as parameters fitted to the projected SMA calculations.

It is clear for the case $m = n$ that the above formulation breaks down. The true wave function is not simply given by Eq. (184), which lacks all correlations in the π^- channel. Instead one must build in a Jastrow correlation of the form discussed in Sec. VI in order to obtain the correct linear dispersion of the long-wavelength spin fluctuations.

For the special case of the (mmm) states, the SU(2) symmetry of the problem has been exploited by Lee and Kane,[15] and they formulated a slightly different version of the Chern-Simons theory that is very useful for uncovering the spin-charge connection. Their original motivation was to understand the spin unpolarized quantum Hall effect. However, with a simple change from spin to pseudospin, one can easily apply this formalism to double-layer systems [in the zero-separation limit where the SU(2) symmetry is preserved]. In the case of the (mmm) states, one only needs to introduce a single gauge field, and the action is given by

$$\mathcal{L}_\phi = \phi_\sigma^\dagger (i\partial_t + A^0 - a^0)\phi_\sigma - \frac{1}{2m^*}\left|\left(\frac{1}{i}\nabla + \mathbf{A} - m\mathbf{a}\right)\phi_\sigma\right|^2 + \frac{1}{4\pi}\varepsilon^{\mu\nu\rho}\, a^\mu \partial_\nu\, a^\rho - \frac{1}{2}\int d^2y\, \delta\rho_\sigma(x)\, V_0(x-y)\, \delta\rho_\sigma(y). \tag{192}$$

In this case, the mean-field equations are given by

$$\nabla \times \mathbf{a}(\mathbf{x}) = 2\pi\rho_\sigma(\mathbf{x})\ ,\ \mathbf{A} = m\mathbf{a} \tag{193}$$

and one sees easily that these equations are satisfied when the filling fraction is $\nu = 1/m$. We now decompose the bose fields in the form

$$\phi_\sigma = \sqrt{\bar\rho + \delta\rho_\sigma}\ \phi\ z_\sigma\ \ ,\ \bar\phi\phi = \bar z_\sigma z_\sigma = 1. \tag{194}$$

Here ϕ is a complex number of unit magnitude and z_σ represents the spinor variable and is related to the unit vector order parameter $\mathbf{m}$ defined previously by $\mathbf{m} = \bar z_\mu\, \sigma_{\mu\nu}\, z_\nu$.

Performing the standard duality transformation,[15] one obtains the following effective action in the dual representation

$$\mathcal{L} = 2\pi\, b_\alpha\, (J^v_\alpha + J^s_\alpha) + \frac{m^*}{2\bar\rho}\,(\partial_0\, b_\alpha)^2 - \frac{1}{4\pi}\frac{\bar\rho}{m^*}\left[2\pi(J^v_0 + J^s_0) - \frac{2\pi}{\nu}\,\varepsilon^{\alpha\beta}\,\partial_\alpha\,\delta b_\beta\right]\ln|x-y|\left[2\pi(J^v_0 + J^s_0) - \frac{2\pi}{\nu}\,\varepsilon^{\alpha'\beta'}\,\partial_{\alpha'}\,\delta b_{\beta'}\right] - \frac{1}{2}\,(\varepsilon^{\alpha\beta}\,\partial_\alpha\,\delta b_\beta)\,V_0(x-y)(\varepsilon^{\alpha'\beta'}\,\partial_{\alpha'}\,\delta b_{\beta'}) - \frac{\bar\rho}{m^*}[|\partial\mathbf{z}_\sigma|^2 + (\bar{\mathbf{z}}_\sigma\partial\mathbf{z}_\sigma)^2], \tag{195}$$

where $\delta b_\alpha \equiv b_\alpha + \frac{\nu}{2\pi}A_\alpha$ and the gauge field b_μ is defined by

$$J_\mu = \epsilon_{\mu\nu\rho}\partial_\nu b_\rho \tag{196}$$

with J_μ being the three current of the fluid,

$$J^v_\mu = \frac{1}{2\pi i}\epsilon_{\mu\nu\rho}\partial_\nu(\bar\phi\partial_\rho\phi) \tag{197}$$

is the vortex three current, and

$$J^s_\mu = \frac{1}{2\pi i}\epsilon_{\mu\nu\rho}\partial_\nu(\bar z_\sigma\partial_\rho z_\sigma) \tag{198}$$

is the skyrmion three current. Note that the word vortex in this context refers to vortices in the bosonic Chern-Simons field and should not be confused with vortices in the spin field discussed in the other sections in terms of the pseudospin XY ferromagnet analogy.

From this dual action, several important results follow. First of all, one sees that there is a long-ranged logarithmic interaction of the topological density with itself $\rho_{\rm top}(r) = J^v_0 + J^s_0 - \frac{1}{\nu}\varepsilon^{\alpha\beta}\,\partial_\alpha\,\delta b_\beta$. In the low-energy sector, the only excitations which can be created are those which have no net topological charge, i.e., $\int d^2r\rho_{\rm top}(r) = 0$. Therefore, these elementary excitations can be classified into three categories. A vortex excitation has

$$J^v_0 = \frac{1}{\nu}\varepsilon^{\alpha\beta}\,\partial_\alpha\,\delta b_\beta. \tag{199}$$

From (196) and (199) we see that it carries charge $\pm\nu e$, depending on the sign of the vorticity. A skyrmion excitation has

$$J^s_0 = \frac{1}{\nu}\varepsilon^{\alpha\beta}\,\partial_\alpha\,\delta b_\beta \tag{200}$$

and it also has charge $\pm\nu e$, depending on the sign of the skyrmion number. This is the equation in the Chern-Simons theory relating the spin and charge, which was first noted by Sondhi *et al.*[12] We see that this relation is exactly the same as the one obtained in Eq. (40) from a microscopic calculation. One can also form a bound state between these objects so that the net charge is zero

$$J^s_0 = -J^v_0. \tag{201}$$

In this case, the vorticity cancels the skyrmion number exactly and these objects are neutral.

The statistics of various spin textures can also be worked out explicitly from the dual action, following an approach used by Lee and Zhang[86] in the case of the CSLG theory for the single-layer quantum Hall effect. From Eq. (195), one sees that the first term $2\pi b_\alpha(J^v_\alpha + J^s_\alpha)$ couples the skyrmion density J^s_0 to the dual gauge field b_α. The coupling to the averaged flux $\langle b_\alpha\rangle$ gives the dynamics of the spin degrees of freedom discussed below. There is also a coupling to the fluctuating part δb_β, which is given by Eq. (200). Therefore, this gives rise to a statistical interaction between the skyrmion density

and the skyrmion current of the usual form

$$\nu \int d^2x\, d^2y J_0^s \epsilon^{\alpha\beta} \frac{x^\alpha - y^\alpha}{|\mathbf{x}-\mathbf{y}|^2} J_\beta^s(\mathbf{y}). \tag{202}$$

From this equation we see that a skyrmion with one unit of skyrmion number carries statistics ν and a meron, which carries a half unit of skyrmion number and a half unit of the skyrmion current, has fractional statistics $\nu/4$. In particular for $\nu = 1$, skyrmions are fermions, while the merons are quartons having quarter statistics. This can also be seen from the fact that a skyrmion is a bound state of two merons, the bound state of two quartons is a fermion, while the bound state of two semions is a boson.

Another important consequence of the dual action in Eq. (195) is the form of the effective spin action. In terms of the CP^1 fields, it is given by

$$\mathcal{L} = i\bar{\rho}\,(\bar{z}_\sigma \partial_t z_\sigma) - \frac{K}{2}[|\partial z_\sigma|^2 + (\bar{z}_\sigma \partial z_\sigma)^2] - \mathbf{i}\mathbf{J}_\mu(\bar{\mathbf{z}}_\sigma \partial_\mu \mathbf{z}_\sigma). \tag{203}$$

Without the last term, which couples the spin and charge degrees of freedom, this action can be transformed into the angular variables $\mathbf{m} = \bar{z}\sigma z$ and its form agrees exactly with the effective spin action in Eq. (58) derived earlier from the microscopic calculation. The microscopic calculation was carried out with the assumption that the charge degrees of freedom is massive. Under this assumption, integrating out the charge degrees of freedom in Eq. (203) will only produce a long-ranged Coulomb interaction between the topological density. However, strictly speaking, this assumption is true only at zero temperature. Eq. (203) is more generally valid even at finite temperature where the charge degrees of freedom is gapless. The effect of the gapless charge degrees of freedom on the spin dynamics and the Kosterlitz-Thouless transition is still to be explored. At this level, the coefficient $K = \bar{\rho}/m^*$ derived from the Chern-Simons theory depends on the mass of the electron rather than the Coulomb interaction as it should. This is a general feature encountered in all Chern-Simons theories. One can view this coefficient as a parameter and argue that higher-order corrections will bring it into agreement with microscopic theories. The coefficient of the time-dependent term is independent of the mass and agrees exactly with the result of microscopic calculations. At zero temperature, the charge degrees of freedom have a gap; integrating them out would only give rise to a higher derivative coupling between the spin variables. Therefore, at least for zero temperature, the effective spin action given here is sufficient.

Finally we note that, as the layer separation exceeds the critical value d^*, quantum disordering will cause merons to proliferate. The path integral configurations for the meron world lines are very similar to those of the vortices which disorder the 3D XY model. However, the universality class of the transition is different because of the coupling to the Chern-Simons field.[87]

X. SUMMARY

We have presented here a theory of the spontaneous development of interlayer phase coherence in double-layer quantum Hall systems at various filling factors. Using a pseudospin language we have shown that the system is equivalent to an easy plane itinerant ferromagnet with an unusual spin-charge connection. There is a zero-temperature phase transition to a quantum disordered phase if the layer separation exceeds a critical value $d > d^*$. For $0 < d < d^*$, the system is predicted to exhibit a finite-temperature Kosterlitz-Thouless transition, in the absence of interlayer tunneling. Our theory is expected to apply to any filling factor at which there is an incompressible state which is not a pseudospin singlet. Here, however, we have concentrated primarily on the case of filling factor $\nu = 1$.

In a companion paper[13] we will discuss the phase transitions that occur in the presence of tunneling and parallel magnetic field. We will also make contact with the recent experiments of Murphy *et al.*,[11] which appear to have observed one of these phase transitions.

ACKNOWLEDGMENTS

It is a pleasure to acknowledge useful conversations with Daniel Arovas, Greg Boebinger, Nick Bonesteel, Luis Brey, René Côté, Jim Eisenstein, Herb Fertig, Matthew Fisher, Jason Ho, Jun Hu, David Huse, Sheena Murphy, Phil Platzman, Scott Renn, Nick Read, Ed Rezayi, Mansour Shayegan, Shivaji Sondhi, and Mats Wallin. The work at Indiana University was supported by NSF Grants Nos. DMR91-13911 and DMR94-16906. The work at the University of Kentucky was supported by NSF Grant No. DMR92-02255. We are pleased to acknowledge the Aspen Center for Physics where part of this work was performed.

APPENDIX: GAUGE INVARIANCE AND UNIFORM CURRENTS IN LANDAU LEVELS

The (static) effective action we have derived for the case of easy plane anisotropy has as its leading gradient term

$$S = \frac{1}{2}\rho_E |\nabla\varphi|^2. \tag{A1}$$

The phase φ describes the local spinor orientation

$$\begin{pmatrix} e^{-i\varphi/2} \\ e^{+i\varphi/2} \end{pmatrix} \tag{A2}$$

and the charge conjugate to $\varphi(\mathbf{r})$ is $S^z(\mathbf{r})$, which gives the local (physical) charge difference between the layers. In order to study the gauge symmetry in this problem, it is convenient to introduce charge and pseudospin gauge fields

$$\mathbf{A}_\pm = \mathbf{A}_\uparrow \pm \mathbf{A}_\downarrow, \tag{A3}$$

where $\mathbf{A}_\uparrow$ and $\mathbf{A}_\downarrow$ are the electromagnetic vector potentials in each of the layers. The order parameter φ is gauge neutral with respect to $\mathbf{A}_+$ since it corresponds to the condensation of a (physical) charge-neutral operator

$$e^{i\varphi(\mathbf{r})} \propto \left\langle \psi_\uparrow^\dagger(\mathbf{r})\psi_\downarrow(\mathbf{r}) \right\rangle . \tag{A4}$$

However, a gauge change

$$\mathbf{A}_- \longrightarrow \mathbf{A}_- + 2\frac{e}{\hbar c}\nabla\chi_- \tag{A5}$$

modifies the wave function

$$\begin{pmatrix} \psi_\uparrow \\ \psi_\downarrow \end{pmatrix} \longrightarrow \exp\left(i2\chi_- S^z\right) \begin{pmatrix} \psi_\uparrow \\ \psi_\downarrow \end{pmatrix}, \tag{A6}$$

which means that the action has the usual minimal coupling form

$$S = \frac{1}{2}\rho_E \left| \nabla\varphi + \frac{e}{\hbar c}\mathbf{A}_- \right|^2 \tag{A7}$$

and the pseudospin current is given (for $\mathbf{A}_- = \mathbf{0}$) by

$$J_{zz} = \frac{2\rho_E}{\hbar}\nabla\phi , \tag{A8}$$

which is identical to the result derived in Eq. (164) using the equation of motion for the projected density. As mentioned previously in Eq. (151), the superfluid mode has oscillator strength proportional to q^2 as expected for an ordinary superfluid. However, the coefficient is proportional to ρ_E and hence nonuniversal.

The minimal coupling argument given above is quite correct; however, there are remarkably confusing subtleties lurking just beneath the surface. For $\nabla\phi$ equal to a constant, Eq. (A8) implies the existence of a uniform zero wave-vector current in the LLL. This is paradoxical, as we now discuss.

The question of the form of the current operator in the LLL is a subtle one which has been considered by several authors.[28,88–91] The difficulty lies in the fact that the true unprojected current operator has no matrix elements within the LLL. It is purely off diagonal, taking, for example, the $N = 0$ Landau level into the $N = 1$ Landau level. Hence it would appear to be impossible to have low-energy currents at zero wave vector, despite our previous derivation using the fact that the projected densities do not commute. The resolution of this paradox can be seen in the following simple example.

Consider the SU(2) invariant $\nu = 1$ case and let the ground state be fully polarized up. Restricting H to the LLL, this state is an exact eigenstate. The exact single-magnon excited states

$$\Psi_{\mathbf{k}} = \overline{S_{\mathbf{k}}^-}\Psi \tag{A9}$$

are labeled by a conserved momentum[20] due to the fact that they are gauge neutral (with respect to $\mathbf{A}_+$). They correspond to spin-flip magnetoexcitons and have a velocity

$$\mathbf{v} = \frac{1}{\hbar}\nabla_{\mathbf{k}}\epsilon_{\mathbf{k}} \times \hat{z} = \frac{1}{\hbar}\nabla_{\mathbf{k}}V(\ell^2\mathbf{k}) \times \hat{\mathbf{z}} \tag{A10}$$

[where $V(\mathbf{r_1} - \mathbf{r_2})$ is the particle interaction] and hence would seem to have a finite pseudospin current at $q = \omega = 0$. If we use the true unprojected current operator, then we must take care to include LL mixing in $\Psi_{\mathbf{k}}$. First-order perturbation theory shows that such mixing is small and proportional to $\ell|\nabla V|/\hbar\omega_c$. However, the current

$$J_{zz}^\mu = \frac{1}{m}\left(\mathbf{\Pi}_\uparrow^\mu - \mathbf{\Pi}_\downarrow^\mu\right) \tag{A11}$$

has matrix elements connecting adjacent Landau levels proportional to $\ell\omega_c$, so that Landau level mixing, though small, gives a crucial contribution to the current, independent of the smallness of the mass m. Carrying out the perturbation theory in detail yields results in complete agreement (to first order in $\ell|\nabla V|/\hbar\omega_c$) with the minimal coupling considerations and the equations of motion methods discussed above.

---

* On leave from Department of Materials Science, Faculty of Science, Hiroshima University, Higashi-Hiroshima 724, Japan.

† Present address: Dept. of Electrical Engineering, Princeton University, Princeton, NJ 08544.

[1] B. I. Halperin, Helv. Phys. Acta **56**, 75 (1983).

[2] T. Chakraborty and P. Pietiläinen, Phys. Rev. Lett. **59**, 2784 (1987); E. H. Rezayi and F. D. M. Haldane, Bull. Am. Phys. Soc. **32**, 892 (1987); Song He, S. Das Sarma, and X. C. Xie, Phys. Rev. B **47**, 4394 (1993); D. Yoshioka, A. H. MacDonald, and S. M. Girvin, *ibid.* **39**, 1932 (1989).

[3] For a brief review of the fractional quantum Hall effect in double-layer systems see A. H. MacDonald, Surf. Sci. **229**, 1 (1990).

[4] Y. W. Suen *et al.*, Phys. Rev. Lett. **68**, 1379 (1992); J. P. Eisenstein *et al.*, *ibid.* **68**, 1383 (1992).

[5] X. G. Wen and A. Zee, Phys. Rev. Lett. **69**, 1811 (1992); Phys. Rev. B **47**, 2265 (1993).

[6] Z. F. Ezawa and A. Iwazaki, Int. J. Mod. Phys. B **19**, 3205 (1992); Phys. Rev. B **47**, 7295 (1993); **48**, 15 189 (1993).

[7] A. H. MacDonald, P. M. Platzman, and G. S. Boebinger, Phys. Rev. Lett. **65**, 775 (1990).

[8] Luis Brey, Phys. Rev. Lett. **65**, 903 (1990); H. A. Fertig, Phys. Rev. B **40**, 1087 (1989); O. Narikiyo and D. Yoshioka, J. Phys. Soc. Jpn. **62**, 1612 (1993).

[9] R. Côté, L. Brey, and A. H. MacDonald, Phys. Rev. B **46**, 10 239 (1992); X. M. Chen and J. J. Quinn, *ibid.* **45**, 11 054 (1992).

[10] Kun Yang, K. Moon, L. Zheng, A. H. MacDonald, S. M. Girvin, D. Yoshioka, and Shou-Cheng Zhang, Phys. Rev. Lett. **72**, 732 (1994).

[11] S. Q. Murphy, J. P. Eisenstein, G. S. Boebinger, L. N. Pfeiffer, and K. W. West, Phys. Rev. Lett. **72**, 728 (1994).

[12] S. L. Sondhi, A. Karlhede, S. A. Kivelson, and E. H. Rezayi, Phys. Rev. B **47**, 16 419 (1993).

[13] K. Moon, H. Mori, Kun Yang, Lotfi Belkhir, S. M. Girvin,

A. H. MacDonald, L. Zheng, D. Yoshioka, and Shou-Cheng Zhang (unpublished).
[14] S. Datta, Phys. Lett. **103A**, 381 (1984).
[15] D.-H. Lee and C. L. Kane, Phys. Rev. Lett. **64**, 1313 (1990).
[16] An early discussion of the quantum Hall effect at $\nu = 1$ in the limit of weak Zeeman coupling was provided by F. D. M. Haldane, in *The Quantum Hall Effect*, 2nd ed., edited by Richard E. Prange and Steven M. Girvin (Springer-Verlag, New York, 1990).
[17] At $\nu = 1$ there is only one $S = N/2$ multiplet in the Hilbert space. The orbital wave function for this multiplet is the single-Slater determinant wave function of a full Landau level of spinless electrons. The $S = N/2$ state is expected on physical grounds to be the ground state for the case of Coulomb interactions between electrons, although we are not aware of any mathematically rigorous proof. Numerical finite-size exact diagonalization calculations provide unequivocal support for this expectation.
[18] See, for example, D. Yoshioka, A. H. MacDonald, and S. M. Girvin, Phys. Rev. B **38**, 3636 (1988).
[19] Mark Rasolt, F. Perrot, and A. H. MacDonald, Phys. Rev. Lett. **55**, 433 (1985); Mark Rasolt and A. H. MacDonald, Phys. Rev. B **34**, 5530 (1986); M. Rasolt, B. I. Halperin, and D. Vanderbilt, Phys. Rev. Lett. **57**, 126 (1986).
[20] C. Kallin and B. I. Halperin, Phys. Rev. B **31**, 3635 (1985).
[21] Z. F. Ezawa and A. Iwazaki, Int. J. Mod. Phys. B **8**, 2111 (1994).
[22] Tapash Chakraborty and F. C. Zhang, Phys. Rev. B **29**, 7032 (1984); F. C. Zhang and Tapash Chakraborty, *ibid.* **30**, 7320 (1984).
[23] R. G. Clark, S. R. Haynes, A. M. Suckling, J. R. Mallett, P. A. Wright, J. J. Harris, and C. T. Foxon, Phys. Rev. Lett. **62**, 1536 (1989); J. P. Eisenstein, H. L. Stormer, L. Pfeiffer, and K. W. West, *ibid.* **62**, 1540 (1989).
[24] Singlet states occur at some filling factors because the constraints imposed on correlations by the requirement of staying in the lowest Landau level can be more deleterious for spin-polarized states than for singlet states.
[25] R. B. Laughlin, in *The Quantum Hall Effect* (Ref. 16), Chap. 7.
[26] R. B. Laughlin, Phys. Rev. Lett. **50**, 1395 (1983).
[27] S. M. Girvin and T. Jach, Phys. Rev. B **29**, 5617 (1984).
[28] S. M. Girvin, A. H. MacDonald, and P. M. Platzman, Phys. Rev. Lett. **54**, 581 (1985); Phys. Rev. B **33**, 2481 (1986).
[29] V. Bargmann, Rev. Mod. Phys. **34**, 829 (1962).
[30] *Quantum Hall Effect*, edited by Michael Stone (World Scientific, Singapore, 1992).
[31] An alternate way to define the spin texture state is to project the entire rotation operator to get $\overline{e^{-i\hat{O}}}$. It turns out these two definitions of the spin texture give identical results for the quantities we calculate. Our present approach has the advantage that $e^{-i\hat{O}}$ is unitary.
[32] Eduardo Fradkin, *Field Theories of Condensed Matter Systems* (Addison-Wesley, Reading, MA, 1990).
[33] Consider the real space version of Eq. (32). $\delta\rho(\mathbf{r})$ can only depend on $\Omega(\mathbf{r}')$ for $\mathbf{r}'$ in the vicinity of $\mathbf{r}$. Perform a uniform global spin rotation so that $m^z(\mathbf{r}) = 1$. $\Omega(\mathbf{r}')$ is now small for $\mathbf{r}'$ near $\mathbf{r}$ and a Taylor series expansion can be performed.
[34] Kun Yang, L. K. Warman, and S. M. Girvin, Phys. Rev. Lett. **70**, 2641 (1993).
[35] A. H. MacDonald, G. C. Aers, and M. W. C. Dharmawardana, Phys. Rev. B **31**, 5529 (1985).
[36] R. Rajaraman, *Solitons and Instantons* (North-Holland, Amsterdam, 1982).
[37] F. D. M. Haldane, Phys. Lett. **93A**, 464 (1983); Phys. Rev. Lett. **50**, 1153 (1983).
[38] Michael Stone, Phys. Rev. D **33**, 1191 (1986).
[39] These terms are certainly present, but should be much smaller than the interaction terms we retain as long as the envelope functions are small in the barriers separating the two quantum wells. Such terms *cannot* be neglected in single wide quantum wells such as those studied by Suen *et al.* (Ref. 4).
[40] The finite thickness of a layer can be accounted for by including form factors corresponding to the isolated subband envelope functions. See, for example, Ref. 7.
[41] This is very similar to the Hubbard model, whose on-site interaction is the lattice version of the δ-function interaction. A by-product of the fact of that the interaction is short ranged is that it must be SU(2) invariant. See E. Fradkin, *Field Theories in Condensed Matter Systems* (Addison-Wesley, Reading, MA, 1991).
[42] H. J. Schulz, Phys. Rev. Lett. **65**, 2462 (1990).
[43] One can also express the partition function as a functional integral over Grassman variables. See J. W. Negele and H. Orland, *Quantum Many-Particle Systems* (Addison-Wesley, Reading, MA, 1987).
[44] R. Morf and B. I. Halperin, Phys. Rev. B **33**, 2221 (1986).
[45] A. H. MacDonald and S. M. Girvin, Phys. Rev. B **34**, 5639 (1986).
[46] D. Levesque, J. J. Weis, and A. H. MacDonald, Phys. Rev. B **30**, 1056 (1984).
[47] E. H. Rezayi, Phys. Rev. B **36**, 5454 (1987); **43**, 5944 (1991).
[48] Tapash Chakraborty, Pekka Pietiläinen, and F. C. Zhang, Phys. Rev. Lett. **57**, 130 (1986).
[49] S. M. Girvin, in *The Quantum Hall Effect* (Ref. 16), p. 401.
[50] F. D. M. Haldane, Phys. Rev. Lett. **55**, 2095 (1985).
[51] J. Jain and X. G. Wu, Phys. Rev. B **49**, 5085 (1994).
[52] X. G. Wu and S. L. Sondhi (unpublished).
[53] Tin-Lun Ho, Phys. Rev. Lett. **73**, 874 (1994).
[54] F. D. M. Haldane, Phys. Rev. Lett. **51**, 605 (1983); G. Fano, F. Ortolani, and E. Colombo, Phys. Rev. B **34**, 2670 (1986).
[55] D. J. Gross, Nucl. Phys. B **132**, 439 (1978).
[56] Ian Affleck, Phys. Rev. Lett. **56**, 408 (1986).
[57] Although we have not been able to construct a rigorous proof, we believe that the connection between topological charge and electron charge, which we have derived for the case of SU(2) invariant interactions, is not altered by the quantum fluctuations which reduce the magnitude of the order parameter for $d/\ell \neq 0$.
[58] F. D. M. Haldane, in *The Quantum Hall Effect* (Ref. 16), p. 303. Notice that his definition of $V(q)$ differs from ours by a factor of 2π.
[59] Our expressions of the spin stiffness provide a criterion for the stability of the ferromagnetic states since the stiffness must be positive. For example, in the SU(2) invariant case in a V_0,V_1 model we would have to have $V_0 > 3V_1$ in order to get a stable ferromagnetic state.
[60] S. R. Renn has recently considered mutual drag effects in *mmn* states of double-well systems: Phys. Rev. Lett. **68**, 658 (1992).
[61] Jason Ho has recently considered oscillatory effects due to nonlinear response to finite electric fields, oppositely directed in each layer. See Ref. 53.

[62] S. R. Renn and B.W. Roberts, Phys. Rev. B **48**, 10 926 (1993).
[63] A. H. MacDonald and Shou-Cheng Zhang, Phys. Rev. B **49**, 17 208 (1994).
[64] This observation has been made from another point of view by Ana Lopez and Eduardo Fradkin, Bull. Am. Phys. Soc. **39**, 400 (1994); and (unpublished).
[65] R. P. Feynman, *Statistical Mechanics* (Benjamin, Reading, MA, 1972).
[66] J. M. Kosterlitz and D. J. Thouless, J. Phys. C **6**, 1181 (1973); J. M. Kosterlitz, *ibid.* **7**, 1046 (1974).
[67] The present system has charged vortex-antivortex excitations as well as neutral vortex-antivortex excitations. We have not yet fully analyzed the possible importance of these excitations for the finite-temperature Kosterlitz-Thouless transition when disorder is present in the system.
[68] J. Tobochnik and G. V. Chester, Phys. Rev. B **20**, 3761 (1979); J. Fernandez, M. F. Ferreira, and J. Stankiewicz, *ibid.* **34**, 292 (1986); Rajan Gupta, Jerry DeLapp, George G. Batrouni, Geoffrey C. Fox, Clive F. Baillie, and John Apostolakis, Phys. Rev. Lett. **61**, 1996 (1988).
[69] Markus Klomfass, Urs M. Heller, and Henrik Flyvbjerg, Nucl. Phys. B **360**, 264 (1991).
[70] M. B. Pogrebinskii, Fiz. Tekh. Poluprovodn. **11**, 637 (1977) [Sov. Phys. Semicond. **11**, 372 (1977)]; P. M. Price, Physica (Amsterdam) **117B**, 750 (1983); P. M. Solomon, P. J. Price, D. J. Frank, and D. C. La Tulipe, Phys. Rev. Lett. **63**, 2508 (1989); T. J. Gramila, J. P. Eisenstein, A. H. MacDonald, L. N. Pfeiffer, and K. W. West, *ibid.* **66**, 1216 (1991), and references cited therein.
[71] A similar suggestion was made recently, although the cross-talk Hall voltage predicted is much weaker than suggested by our arguments. Ji-Min Duan (unpublished).
[72] D. Yoshioka and A. H. MacDonald, J. Phys. Soc. Jpn. **59**, 4211 (1990).
[73] A single-layer system at Landau level filling factor $\nu = 1/2$ has no charge gap but does show interesting anomalies which may indicate that it forms a liquid of composite fermions. For a discussion of recent work see B. I. Halperin, Patrick A. Lee, and Nicholas Read, Phys. Rev. B **47**, 7312 (1993), and references cited therein; and V. Kalmeyer and S. C. Zhang, *ibid.* **46**, 9889 (1992).
[74] Further numerical calculations performed with a finite tunneling amplitude will be detailed in Ref. 13.
[75] We have attempted to do a finite-size scaling analysis of our results with limited success.
[76] We have performed extensive finite-size exact diagonalization studies which demonstrate this important difference between the double-layer fractional quantum Hall effect at $\nu_T = 1$ and at $\nu_T = 1/2$. L. Zheng (unpublished).
[77] Markus Klomfass, Urs M. Heller, and Henrik Flyvbjerg, Nucl. Phys. B **360**, 264 (1991).
[78] Kyungsun Moon and Hiro Mori (unpublished).
[79] T. S. Lay, Y. W. Suen, H. C. Manoharan, X. Ying, M. B. Santos, and M. Shayegan (unpublished).
[80] S. M. Girvin and A. H. MacDonald, Phys. Rev. Lett. **58**, 1252 (1987).
[81] S. C. Zhang, T. H. Hansson, and S. Kivelson, Phys. Rev. Lett. **62**, 82 (1989).
[82] N. Read, Phys. Rev. Lett. **62**, 86 (1989).
[83] S.C. Zhang, Int. J. Mod. Phys. B **6**, 25 (1992).
[84] Z. F. Ezawa, M. Hotta, and A. Iwazaki, Phys. Rev. B **46**, 7765 (1992).
[85] J. Zang, D. Schmeltzer, and J. L. Birman, Phys. Rev. Lett. **71**, 773 (1993).
[86] D. H. Lee and S. C. Zhang, Phys. Rev. Lett. **66**, 1220 (1991).
[87] S. C. Zhang, D. P. Arovas, and S. M. Girvin (unpublished).
[88] S. L. Sondhi and S. A. Kivelson, Phys. Rev. B **46**, 13 319 (1992).
[89] J. Martinez and M. Stone, Int. J. Mod. Phys. B **7**, 4389 (1993).
[90] R. Rajaraman, Int. J. Mod. Phys. B **8**, 777 (1994).
[91] R. Rajaraman and S. L. Sondhi (unpublished).