Quantum Hall effect in three-dimensional field-induced spin-density-wave phases with a tilted magnetic field

Yasumasa Hasegawa
Faculty of Science, Himeji Institute of Technology, Kamigouri-chou, Akou-gun, Hyogo 678-12, Japan
(Received 25 April 1994)

The quantum Hall effect for three-dimensional anisotropic tight-binding electrons is investigated in the field-induced spin-density-wave phases with a magnetic field tilted in any direction. The Hall conductivities σ_{xy} and σ_{xz} are shown to be quantized as a function of the wave vector of the field-induced spin-density wave, while σ_{yz} remains zero, where x is the direction with the largest conductivity and y and z are perpendicular to x.

I. INTRODUCTION

Quantum Hall effect has been observed in organic conductors $(TMTSF)_2PF_6$ and $(TMTSF)_2ClO_4$ (Refs. 1–3) in the field-induced spin-density-wave (FISDW) phases. These organic conductors are quasi-one-dimensional conductors and described by a tight-binding Hamiltonian with anisotropic hopping matrix elements, $t_b/t_a \approx 0.1$ and $t_c/t_a \approx 0.003$. The Fermi surface consists of two warped planes in the absence of a magnetic field. If the nesting of the Fermi surface is perfect, the staggered susceptibility corresponding to the nesting vector which connects two planes of the Fermi surface diverges as temperature becomes low. In such a case the spin-density wave (SDW) is stabilized by the repulsive interaction of electrons. The nesting of the Fermi surface is imperfect in general due to transverse hoppings. For the imperfect nesting case the transition temperature of SDW depends on the imperfectness and the strength of the electron interaction. Since the transverse hoppings are enhanced by pressure, the SDW state is suppressed as the pressure is increased. The divergence of the staggered susceptibility recovers when the magnetic field is applied,[4] and the successive transitions to FISDWs occur as the magnetic field is increased.[5–9] One can explain the stability of the FISDW by the fact that the energy is lowered when a gap at the Fermi surface is opened due to the combined effect of the electron interactions and magnetic field.

A magnetic field required to be one flux quantum, $\phi_0 = hc_0/e$, per electron in each plane for these organic conductors is extremely large ($\sim 10^4$ T). In other words, the quantum Hall effect in organic conductors is observed in much smaller field than what is necessary to fill electrons only in the lowest few Landau levels. Moreover, the Hall conductivity is observed to change its sign in some region of the magnetic field.[10] The quantum Hall effect in the presence of a FISDW has been studied theoretically.[11–14] The quantization of the Hall conductivity is explained by using the Středa formula[11] or the general theory of the Hall effect in the periodic system.[12–14] These authors studied the case where the magnetic field is perpendicular to the conducting plane.

Due to the hoppings along the third direction new phenomena are expected when the magnetic field is tilted.[15] In a three-dimensional lattice there exist three fluxes, ϕ_a, ϕ_b, and ϕ_c, per unit area in each plane, while only one flux per unit cell is present in two dimensions. Indeed, many cusps of the magnetoresistance have been observed as a function of a direction of the magnetic field.[16,17] The FISDW in three dimensions in the tilted magnetic field or in a nonorthogonal lattice has been studied.[18–20] The Hall conductivities σ_{xy} and σ_{xz} are predicted to be quantized. These values are argued to be fractionally quantized.[18,21] These authors, however, studied the case where the magnetic field is perpendicular to the axis, say a, when two fluxes, ϕ_b and ϕ_c, exist instead of three in the general direction of the field.

On the other hand, the general formula for the quantum Hall effect in the periodic system in three dimensions was given by Halperin[22] and Kohmoto *et al.*[23] They have shown that the conductivity tensor is quantized when the Fermi energy is in the energy gap.

In this paper we study the Hall conductivity in the anisotropic three-dimensional lattice with the magnetic field tilted to any direction. In Sec. II the conductivity tensor is calculated for the noninteracting electrons in the orthorhombic lattice. The quantized value is given in the perturbation in t_b/t_a and t_c/t_a. The quantum Hall effect in the presence of a FISDW is given in Sec. III. The generalization to the triclinic lattice is done in Sec. IV.

II. NONINTERACTING CASE

In this section we study the anisotropic tight-binding electrons on the orthorhombic lattice in the magnetic field,

$$\mathcal{H}_0 = -t_a \sum_{(i,j)_a,\sigma} e^{i\phi_{ij}} c^\dagger_{i,\sigma} c_{j,\sigma} - t_b \sum_{(i,j)_b,\sigma} e^{i\phi_{ij}} c^\dagger_{i,\sigma} c_{j,\sigma} - t_c \sum_{(i,j)_c,\sigma} e^{i\phi_{ij}} c^\dagger_{i,\sigma} c_{j,\sigma} , \tag{2.1}$$

where $c^{\dagger}_{i,\sigma}$ and $c_{i,\sigma}$ are creation and annihilation operators of the electron with spin σ at site i, t_a, t_b, and t_c ($t_a \gg t_b$, t_c) are the hopping matrix elements along a, b, and c directions, respectively, and

$$\phi_{ij} = \frac{2\pi}{\phi_0}\int_i^j \mathbf{A}\cdot d\boldsymbol{l}, \tag{2.2}$$

where $\mathbf{A}$ is a vector potential. In this section and Sec. III the a, b, and c axes are assumed to be orthogonal to each other and parallel to the x, y, and z direction, respectively.

The uniform magnetic field $\mathbf{B}$ is applied in any direction,

$$\mathbf{B} = (B_x, B_y, B_z). \tag{2.3}$$

The fluxes through the unit area perpendicular to the a, b, and c axes are given by

$$\phi_x = bcB_x, \tag{2.4a}$$

$$\phi_y = caB_y, \tag{2.4b}$$

$$\phi_z = abB_z, \tag{2.4c}$$

where a, b, and c are the lattice constants.

First we consider a rational flux case, i.e., the flux through a unit area in each plane is a rational number, $(\phi_x/\phi_0, \phi_y/\phi_0, \phi_z/\phi_0) = (p_x/q_x, p_y/q_y, p_z/q_z)$, with mutually prime integers p_α and q_α ($\alpha = x$, y, or z). Then the Hamiltonian in the presence of the magnetic field is described as a generalization of the problem in two dimensions studied by Azbel[24] and Hofstadter[25] to three dimensions.[21,26–29] The energy spectrum for the rational flux case can be obtained numerically as the eigenvalue of Harper equations. The size of the Harper equations is the least common multiple of q_x, q_y, and q_z, which we define as Q. The volume of the magnetic Brillouin zone is $1/Q$ of the Brillouin zone in the absence of a magnetic field. For a fixed momentum $\mathbf{k}$ in the magnetic Brillouin zone there exist Q eigenvalues and Q eigenstates. By varying the momentum we get Q bands. In two dimensions these bands do not overlap. Although the bands may overlap in three dimensions, it has been shown that the energy gaps due to a magnetic field exist in the wide range of parameters.[26,28] We can use the extended zone by unfolding the magnetic Brillouin zone to the Brillouin zone in the absence of a field. Then the energy is obtained uniquely for each momentum.

The quantum Hall effect is expected in the three-dimensional case when some of the energy bands are completely filled and the other bands are empty.[22,18,21,23] The conductivity tensor for a filled band is given as[30,22,23]

$$\sigma_{\alpha\beta} = 2\frac{e^2}{h}\frac{1}{4\pi^2 i}\int d^3k\left[\frac{\partial}{\partial k_\alpha}\left(\left\langle\Psi\left|\frac{\partial}{\partial k_\beta}\right|\Psi\right\rangle\right) - \frac{\partial}{\partial k_\beta}\left(\left\langle\Psi\left|\frac{\partial}{\partial k_\alpha}\right|\Psi\right\rangle\right)\right], \tag{2.5}$$

where the factor of 2 comes from the spin degrees of freedom, α and β are x, y, or z, $|\Psi\rangle$ is the wave function for the filled band and the integral is performed in the magnetic Brillouin zone. Kohmoto, Halperin, and Wu[23] have shown that if the Fermi energy lies in the energy gap, the conductivity tensor is described as

$$\sigma_{\alpha\beta} = 2\frac{e^2}{2\pi h}\varepsilon_{\alpha\beta\gamma}G_\gamma, \tag{2.6}$$

with the vector in the reciprocal lattice,

$$\mathbf{G} = -(l_{a'}\mathbf{G}_{a'} + l_{b'}\mathbf{G}_{b'} + l_{c'}\mathbf{G}_{c'}), \tag{2.7}$$

where $\mathbf{G}_{a'}$, $\mathbf{G}_{b'}$, and $\mathbf{G}_{c'}$ are the fundamental reciprocal vectors satisfying $\mathbf{B}\cdot\mathbf{G}_{a'} = \mathbf{B}\cdot\mathbf{G}_{b'} = 0$ and $l_{a'}$, $l_{b'}$, and $l_{c'}$ are integers. These integers are the first Chern numbers on the tori obtained by slicing the three-torus of the magnetic Brillouin zone. Since we can take any fundamental reciprocal vectors, $\mathbf{G}$ is written as

$$\mathbf{G} = -(l_a\mathbf{G}_a + l_b\mathbf{G}_b + l_c\mathbf{G}_c), \tag{2.8}$$

where $\mathbf{G}_a = (2\pi/a)\mathbf{x}/|\mathbf{x}|$, $\mathbf{G}_b = (2\pi/b)\mathbf{y}/|\mathbf{y}|$, and $\mathbf{G}_c = (2\pi/c)\mathbf{z}/|\mathbf{z}|$ for the orthorhombic lattice and l_a, l_b, and l_c are integers. If the direction of the magnetic field is changed infinitesimally, $\mathbf{G}$ stays constant as long as the Fermi energy lies in the energy gap. Although $\mathbf{G}_{a'}$, $\mathbf{G}_{b'}$, and $\mathbf{G}_{c'}$ depend on the direction of the magnetic field, $\mathbf{G}_a$, $\mathbf{G}_b$, and $\mathbf{G}_c$ do not depend on the magnetic field. As a result, while $l_{a'}$, $l_{b'}$, and $l_{c'}$ are not constant, l_a, l_b, and l_c are constant for the infinitesimal change of the magnetic field.

If the Fermi energy lies between the rth and $(r+1)$th band from the bottom of the energy, the conductivity tensor is given as the summation of the contributions from filled r bands. The contributions from the filled bands cancel each other except that from the upper energy gap of the rth band.[31]

In order to calculate the Hall conductivity explicitly, we take the vector potential $\mathbf{A}$ as

$$\mathbf{A} = \left(0,\ (B_z x - B_x z),\ -\frac{B_y}{B_z}(B_z x - B_x z)\right). \tag{2.9}$$

In the above we have assumed $B_z \neq 0$ without loss of generality, since we may exchange y and z in the case of $B_z = 0$. With this vector potential the noninteracting Hamiltonian is written

$$\begin{aligned}\mathcal{H}_0 = &-t_a\sum_{\mathbf{k},\sigma} 2\cos(ak_x)c^{\dagger}_\sigma(\mathbf{k})c_\sigma(\mathbf{k}) \\ &-t_b\sum_{\mathbf{k},\sigma} e^{-ibk_y}c^{\dagger}_\sigma(\mathbf{k}-\mathbf{u})c_\sigma(\mathbf{k}) + \text{H.c.} \\ &-t_c e^{i\pi\frac{\phi_x\phi_y}{\phi_0\phi_z}}\sum_{\mathbf{k},\sigma} e^{-ick_z}c^{\dagger}_\sigma(\mathbf{k}+\mathbf{u}')c_\sigma(\mathbf{k}) + \text{H.c.},\end{aligned} \tag{2.10}$$

where H.c. means Hermitian conjugate,

$$\mathbf{u} = \left(\frac{2\pi}{a}\frac{\phi_z}{\phi_0}, 0, -\frac{2\pi}{c}\frac{\phi_x}{\phi_0}\right), \tag{2.11}$$

and

$$\mathbf{u}' = \frac{\phi_y}{\phi_z}\mathbf{u}. \tag{2.12}$$

We define the state as

$$\psi_\sigma(\mathbf{k}) = c_\sigma^\dagger(\mathbf{k})|0\rangle\ , \tag{2.13}$$

with vacuum $|0\rangle$. The Hamiltonian mixes $\psi_\sigma(\mathbf{k})$ with $\psi_\sigma(\mathbf{k}\pm\mathbf{u})$ and $\psi_\sigma(\mathbf{k}\pm\mathbf{u}')$. If $\phi_x = 0$, k_y and k_z are constants of motion. If $\phi_x \neq 0$, k_z is not a constant of motion but k_y and $(\phi_z k_z+\phi_x k_x)/\sqrt{\phi_x^2+\phi_z^2}$ are constants of motion.

The Fermi surface is given by $k_x = \pm k_F$ in the zeroth order perturbation in t_b/t_a and t_c/t_a. When the condition

$$2k_F = m_z u_x + m_y u'_x + s(2\pi/a) \tag{2.14}$$

is satisfied with integers m_y, m_z, and s, the degenerate states at $k_x = k_F$ and $k_x = -k_F$ are mixed, resulting in the energy gap at the Fermi energy in the $|m_z|$th perturbation in (t_b/t_a) and the $|m_y|$th perturbation in (t_c/t_a). The magnitude of the energy gap is of the order of $|\Gamma_0(\mathbf{k})|$, where

$$\begin{aligned}\Gamma_0(\mathbf{k}) &= t_a\left(\frac{t_b}{t_a}\right)^{|m_z|}\left(\frac{t_c}{t_a}\right)^{|m_y|} e^{-im_y\pi\frac{\phi_x\phi_y}{\phi_0\phi_z}} e^{-im_z bk_y}\left(e^{ick_z}e^{ic(k_z+\frac{2\pi}{c}\frac{\phi_x\phi_y}{\phi_0\phi_z})}\cdots e^{ic[k_z+(|m_y|-1)\frac{2\pi}{c}\frac{\phi_x\phi_y}{\phi_0\phi_z}]}\right)^{\mathrm{sgn}(m_y)}\\ &= t_a\left(\frac{t_b}{t_a}\right)^{|m_z|}\left(\frac{t_c}{t_a}\right)^{|m_y|} e^{-im_y^2\pi\frac{\phi_x\phi_y}{\phi_0\phi_z}} e^{i(-m_z bk_y+m_y ck_z)}.\end{aligned} \tag{2.15}$$

Since the flux is assumed to be rational, $\mathbf{u}$ and $\mathbf{u}'$ are written as the integer multiples of the vector $\mathbf{u}_0$ as

$$\mathbf{u} = P_z\mathbf{u}_0, \tag{2.16a}$$

$$\mathbf{u}' = P_y\mathbf{u}_0, \tag{2.16b}$$

where integers P_y and P_z are defined by

$$\frac{P_y}{P_z} = \frac{\phi_y}{\phi_z}. \tag{2.17}$$

The Harper equations are obtained by using the *one-dimensional* basis states

$$\psi_j = \psi(\mathbf{k}+j\mathbf{u}_0), \tag{2.18}$$

with integer j. The number of Harper equations obtained by this basis set is not always Q but the integer multiple of Q, because ψ_Q is not necessarily the same as ψ_0. Thus the basis ψ_j may be overcomplete, although the eigenenergy is obtained correctly by the overcomplete basis. In order to get the complete basis for the rational flux case a careful choice of the vector potential is necessary.[29] The overcomplete basis set corresponds to the larger unit cell in the real space or the folded magnetic Brillouin zone in the momentum space. The Hall conductivity is, however, obtained even if the integration in Eq. (2.5) is done in the folded magnetic Brillouin zone, since the degeneracy of the bands due to the overcomplete bands compensates the folded region of the integration.

The magnetic Brillouin zone has the volume $(2\pi)^3/(Qabc)$ and $2k_F aQ/(2\pi) = r$ is the number of filled bands below the Fermi energy. Then we get from Eq. (2.14) that m_y and m_z should fulfill the Diophantine equation,

$$r = sQ + \frac{\phi_z}{\phi_0}Qm_z + \frac{\phi_y}{\phi_0}Qm_y. \tag{2.19}$$

The wave function for the rth band gets the phase of $\Gamma_0(\mathbf{k})$ at $k_x \approx k_F$,[31] i.e., the phase of the wave function cannot be defined globally in the magnetic Brillouin zone.

On the other hand, the phase of the wave function can be defined globally in the torus of k_y and k_z for fixed k_x, which is the slice of three-torus by the plane with a fixed k_x. Therefore the second term in Eq. (2.5) is zero by partial integration with respect to k_y or k_z if $\beta = y$ or $\beta = z$. Then the conductivity tensor is written as[31,13]

$$\sigma_{xy} = 2\frac{1}{c}\frac{e^2}{h}\frac{1}{2\pi i}\int_{-\pi/b}^{\pi/b} dk_y\left(\left\langle\Psi\left|\frac{\partial}{\partial k_y}\right|\Psi\right\rangle\Big|_{k_x=k_F^-} - \left\langle\Psi\left|\frac{\partial}{\partial k_y}\right|\Psi\right\rangle\Big|_{k_x=k_F^+}\right), \tag{2.20a}$$

$$\sigma_{xz} = 2\frac{1}{b}\frac{e^2}{h}\frac{1}{2\pi i}\int_{-\pi/c}^{\pi/c} dk_z\left(\left\langle\Psi\left|\frac{\partial}{\partial k_z}\right|\Psi\right\rangle\Big|_{k_x=k_F^-} - \left\langle\Psi\left|\frac{\partial}{\partial k_z}\right|\Psi\right\rangle\Big|_{k_x=k_F^+}\right), \tag{2.20b}$$

$$\sigma_{yz} = 0, \tag{2.20c}$$

where k_F^- (k_F^+) is the momentum smaller (larger) than the Fermi momentum. Since the wave function changes only in the phase at the Fermi momentum, the integration with respect to k_y or k_z gives the winding number of $\Gamma_0(\mathbf{k})$ in the complex plane around zero when k_y or k_z is moved. Using Eqs. (2.15) and (2.20), we obtain

$$\sigma_{xy} = \frac{2}{c}\frac{e^2}{h}m_z, \tag{2.21a}$$

$$\sigma_{zx} = \frac{2}{b}\frac{e^2}{h}m_y. \tag{2.21b}$$

The conductivity per plane is quantized as $\tilde{\sigma}_{xy} = 2(e^2/h)m_z$ and $\tilde{\sigma}_{zx} = 2(e^2/h)m_y$.

Unfortunately, the energy gap will be very small if $|m_y| + |m_z| \gg 1$, which is the case for $|\phi_y|, |\phi_z| \ll \phi_0$.

For the quantum Hall effect, it is not necessary that all of ϕ_x/ϕ_0, ϕ_y/ϕ_0, and ϕ_z/ϕ_0 are integers. The only required condition is Eq. (2.14).

III. THREE-DIMENSIONAL QUANTUM HALL EFFECT IN THE PRESENCE OF FISDW

In this section we take account of the interaction

$$\mathcal{H}' = U \sum_i c^\dagger_{i,\uparrow} c_{i,\uparrow} c^\dagger_{i,\downarrow} c_{i,\downarrow}. \tag{3.1}$$

This interaction is written as

$$\mathcal{H}' = -U \left(\frac{abc}{(2\pi)^3}\right)^3 \int d\mathbf{k} \int d\mathbf{k}' \int d\mathbf{K}\; c^\dagger_\uparrow(\mathbf{k}) c_\downarrow(\mathbf{k}+\mathbf{K}) \times c^\dagger_\downarrow(\mathbf{k}'+\mathbf{K}) c_\uparrow(\mathbf{k}') \; . \tag{3.2}$$

The order parameter of the spin density is defined in the mean field approximation as

$$\Delta(\mathbf{K}) \equiv -U \frac{abc}{(2\pi)^3} \int d\mathbf{k}' \langle c^\dagger_\downarrow(\mathbf{k}'+\mathbf{K}) c_\uparrow(\mathbf{k}') \rangle \; . \tag{3.3}$$

The interaction term mixes the states $\psi_\uparrow(\mathbf{k})$ and $\psi_\downarrow(\mathbf{k}+\mathbf{K})$. As a result an energy gap at the Fermi surface is opened by the order parameter when $K_x = \pm 2k_F$. The y and z components of the wave vector should be determined to give the lowest energy, which can be calculated numerically with further approximation such as the linearization with respect to k_x.[4,5,7] Since we focus on the quantization of the Hall conductivity, we do not have to calculate the values of K_y and K_z explicitly. The Hall conductivity is zero in this case.

There are other possibilities for opening gaps in the presence of the magnetic field. The order parameter may have many components with respect to the wave vectors as in the two-dimensional case.[32–34] In the three-dimensional case the order parameter has the form

$$\Delta(\mathbf{K}) = \sum_{n_y,n_z} D_{n_y,n_z} \delta(\mathbf{K} - \mathbf{K}_{n_y,n_z}), \tag{3.4}$$

with the x component of the wave vector $\mathbf{K}_{n_y,n_z}$ given by

$$\begin{aligned}(\mathbf{K}_{n_y,n_z})_x &= 2k_F + n_y u'_x + n_z u_x \\ &= 2k_F + n_y \frac{2\pi}{a}\frac{\phi_y}{\phi_0} + n_z \frac{2\pi}{a}\frac{\phi_z}{\phi_0},\end{aligned} \tag{3.5}$$

where n_y and n_z are integers. The vectors $\mathbf{u}$ and $\mathbf{u}'$ are given in Eqs. (2.11) and (2.12). By this wave vector the state at $k_x = -k_F$ is mixed with that at $k_x = k_F + n_y u'_x + n_z u_x$. The state at $k_x = k_F + n_y u'_x + n_z u_x$ is mixed with the state at $k_x = k_F$ by $|n_y|$th perturbation in $t_b \exp(ibk_y)/t_a$ and $|n_z|$th perturbation in $t_c \exp(ick_z)/t_a$. As a result two states at the Fermi energy at $k_x = \pm k_F$ are mixed by a combined effect of SDW and the magnetic field. (See Fig. 1.) The gap function is given by

$$\Gamma(\mathbf{k}) = \sum_{n_y,n_z} \Gamma_{n_y,n_z} \exp\left[i \left(-n_z b k_y + n_y c k_z - \pi n_y^2 \frac{\phi_x \phi_y}{\phi_z \phi_0} \right) \right] , \tag{3.6}$$

where Γ_{n_y,n_z} is a constant of the order of $D_{n_y,n_z}(t_b/t_a)^{|n_z|}(t_c/t_a)^{|n_y|}$. The wave function near the Fermi momentum changes the phase due to this gap function. The Hall conductivity is calculated by Eq. (2.5) as in the noninteracting case.

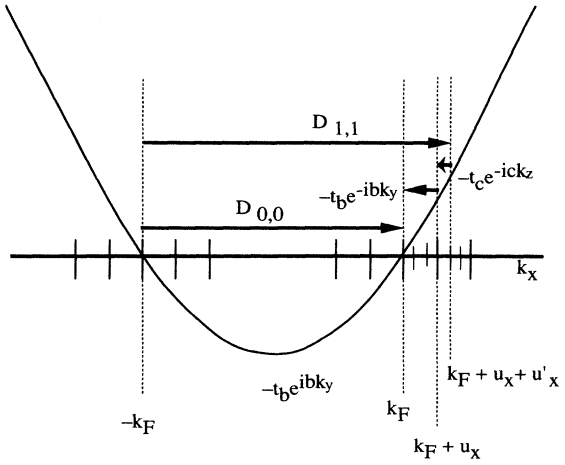

FIG. 1. Schematic diagram of opening the gap at the Fermi momentum. The right arrow labeled by $D_{0,0}$ indicates the order parameter of the SDW with $K_x = 2k_F$ by which two states at $k_x = \pm k_F$ are coupled. The right arrow labeled by $D_{1,1}$ indicates the SDW with wave vector $K_x = 2k_F + u_x + u'_x$. The left arrows indicate the coupling of the states caused by the magnetic field in the perturbation in t_b/t_a and t_c/t_a.

If only one component of the order parameter is dominant, i.e.,

$$\Delta(\mathbf{K}) \approx D_{n_{y0},n_{z0}} \delta(\mathbf{K} - \mathbf{K}_{n_{y0},n_{z0}}), \tag{3.7}$$

the k-dependent phase change is $\exp[i(-n_{z0}bk_y + n_{y0}ck_z)]$ and the Hall conductivity is obtained by Eqs. (2.20) as

$$\sigma_{xy} = \frac{2}{c}\frac{e^2}{h} n_{z0}, \tag{3.8a}$$

$$\sigma_{zx} = \frac{2}{b}\frac{e^2}{h} n_{y0}, \tag{3.8b}$$

and

$$\sigma_{yz} = 0. \tag{3.8c}$$

Comparing Eqs. (3.8a)–(3.8c) with Eqs. (2.6) and (2.8) we find that

$$l_a = 0, \tag{3.9a}$$

$$l_b = n_y, \tag{3.9b}$$

$$l_c = n_z. \tag{3.9c}$$

If the order parameter consists of many components with respect to the wave vectors K_{n_y,n_z}, the Hall conductivity σ_{xy} (σ_{xz}) is obtained as a winding number of the gap function $\Gamma(\mathbf{k})$ as k_y (k_z) is moved from $-\pi/b$ to π/b (from $-\pi/c$ to π/c). Note that ϕ_x appears as the relative phase between the components of the order parameter, so the Hall conductivity may depend on ϕ_x implicitly.

If the magnetic field is changed in magnitude or direction by a small amount, the wave vectors and the order parameter will change in order to keep the Fermi energy in the same energy gap. As a result the plateau of the

Hall conductivity is realized without the effect of a localization. When the energy gap is closed and opens again at the Fermi energy as the magnetic field is changed, the Hall conductivity changes. The sign change of the Hall conductivity may happen as the direction or the magnitude of the magnetic field is changed.

The condition for the rational flux is not necessary to get the quantum Hall effect. The wave vector of the order parameter is adjusted to satisfy Eq. (3.5) for any direction and amplitude of the magnetic field. If ϕ_y/ϕ_z is irrational, n_y and n_z are determined uniquely for a given wave vector $(\mathbf{K}_{n_y,n_z})_x$. On the other hand, n_y and n_z are not uniquely determined if ϕ_y/ϕ_z is rational.

IV. TRICLINIC LATTICE

In this section we consider the triclinic lattice with **a**, **b**, and **c** not being orthogonal to each other. The fundamental reciprocal vectors,

$$\mathbf{G}_a = 2\pi\frac{\mathbf{b}\times\mathbf{c}}{v}\ , \tag{4.1a}$$

$$\mathbf{G}_b = 2\pi\frac{\mathbf{c}\times\mathbf{a}}{v}\ , \tag{4.1b}$$

$$\mathbf{G}_c = 2\pi\frac{\mathbf{a}\times\mathbf{b}}{v}\ , \tag{4.1c}$$

are not orthogonal, where

$$v = |\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})|. \tag{4.2}$$

We take $\mathbf{a} \parallel \mathbf{x}$. Hopping matrix elements t_a, t_b, and t_c ($|t_a| \gg |t_b|, |t_c|$) are assumed between the nearest sites along a, b, and c axes. The vector potential is taken as Eq. (2.9) by using the orthogonal x, y, and z axes.

The noninteracting Hamiltonian is

$$\begin{aligned}\mathcal{H}_0 = &-t_a\sum_{\mathbf{k},\sigma}2\cos(\mathbf{a}\cdot\mathbf{k})c^\dagger(\mathbf{k})c(\mathbf{k})\\ &-t_b e^{i\theta_b}\sum_{\mathbf{k},\sigma}e^{-i\mathbf{b}\cdot\mathbf{k}}c_\sigma^\dagger(\mathbf{k}-\mathbf{w})c_\sigma(\mathbf{k})+\text{H.c.}\\ &-t_c e^{i\theta_c}\sum_{\mathbf{k},\sigma}e^{-i\mathbf{c}\cdot\mathbf{k}}c_\sigma^\dagger(\mathbf{k}+\mathbf{w}')c_\sigma(\mathbf{k})+\text{H.c.}\ ,\end{aligned} \tag{4.3}$$

where

$$\theta_b = \frac{\pi}{\phi_0}\left(b_y-\frac{B_y}{B_z}b_z\right)(B_z b_x - B_x b_z), \tag{4.4a}$$

$$\theta_c = \frac{\pi}{\phi_0}\left(c_y-\frac{B_y}{B_z}c_z\right)(B_z c_x - B_x c_z), \tag{4.4b}$$

and

$$\mathbf{w} = \frac{2\pi}{\phi_0}\left(b_y-\frac{B_y}{B_z}b_z\right)(B_z\hat{\mathbf{x}} - B_x\hat{\mathbf{z}}), \tag{4.5a}$$

$$\mathbf{w}' = -\frac{2\pi}{\phi_0}\left(c_y-\frac{B_y}{B_z}c_z\right)(B_z\hat{\mathbf{x}} - B_x\hat{\mathbf{z}}). \tag{4.5b}$$

Note that $\mathbf{w} \parallel \mathbf{w}'$.

The Fermi surface in the zeroth order in t_b/t_a and t_c/t_a is the plains given by $\mathbf{a}\cdot\mathbf{k} = \pm ak_F$. The order parameter is defined by Eq. (3.3) with the factor $v/(2\pi)^3$ instead of $abc/(2\pi)^3$. As in the preceding section, the energy gap opens at the Fermi momentum by the effects of the order parameter and the perturbation in t_b/t_a and t_c/t_a, if the order parameter has the form

$$\Delta(\mathbf{K}) = \sum_{n_b,n_c} D_{n_b,n_c}\delta(\mathbf{K}-\mathbf{K}_{n_b,n_c}), \tag{4.6}$$

where n_b and n_c are integers and the wave vector $\mathbf{K}_{n_b,n_c}$ satisfies the condition

$$\begin{aligned}\mathbf{K}_{n_b,n_c}\cdot\mathbf{a} &= 2ak_F + n_b\mathbf{w}'\cdot\mathbf{a} + n_c\mathbf{w}\cdot\mathbf{a}\\ &= 2ak_F + n_b 2\pi\frac{ac_yB_z - ac_zB_y}{\phi_0}\\ &\quad -n_c 2\pi\frac{ab_yB_z - ab_zB_y}{\phi_0}\\ &= 2ak_F + n_b 2\pi\frac{\phi_b}{\phi_0} + n_c 2\pi\frac{\phi_c}{\phi_0}.\end{aligned} \tag{4.7}$$

In the above ϕ_b and ϕ_c are the fluxes through a unit area in a-c and a-b planes, respectively,

$$\phi_c = \mathbf{a}\cdot(\mathbf{b}\times\mathbf{B})\ , \tag{4.8a}$$

$$\phi_b = -\mathbf{a}\cdot(\mathbf{c}\times\mathbf{B})\ . \tag{4.8b}$$

The phase of the wave function is not defined globally as in the previous sections. In the perturbation in t_b/t_a and t_c/t_a, the mismatch of the locally defined phase for the wave function of the highest occupied band is attributed to the phase of the gap at the Fermi momentum. The wave function changes the phase as the momentum is moved at the Fermi momentum as in the preceding section. We consider the case where one component is dominant in Eq. (4.6). Then the **k**-dependent part of the phase change is

$$\exp\left[i\left(-n_c\mathbf{b}\cdot\mathbf{k} + n_b\mathbf{c}\cdot\mathbf{k}\right)\right]. \tag{4.9}$$

The conductivity tensor is calculated by

$$\sigma_{xy} = 2\frac{e^2}{h}\frac{1}{2\pi^2 i}\int dk_y dk_z\left(\left\langle\Psi\left|\frac{\partial}{\partial k_y}\right|\Psi\right\rangle\Big|_{k_x=k_F^-} - \left\langle\Psi\left|\frac{\partial}{\partial k_y}\right|\Psi\right\rangle\Big|_{k_x=k_F^+}\right)\ , \tag{4.10a}$$

$$\sigma_{zx} = -2\frac{e^2}{h}\frac{1}{2\pi^2 i}\int dk_y dk_z\left(\left\langle\Psi\left|\frac{\partial}{\partial k_z}\right|\Psi\right\rangle\Big|_{k_x=k_F^-} - \left\langle\Psi\left|\frac{\partial}{\partial k_z}\right|\Psi\right\rangle\Big|_{k_x=k_F^+}\right)\ , \tag{4.10b}$$

$$\sigma_{yz} = 0\ . \tag{4.10c}$$

Because of the nonorthogonality of **a**, **b**, and **c**, the slice of the magnetic Brillouin zone by the plane perpendicular to k_y or k_z is not the torus. As a result, σ_{xy} and σ_{zx} are not quantized to be integer but they are quantized as

$$\sigma_{xy} = 2\frac{e^2}{h}(n_c b_y - n_b c_y)\frac{a}{v}$$
$$= 2\frac{e^2}{h}\frac{n_c b_y - n_b c_y}{|b_y c_z - b_z c_y|}\ , \tag{4.11}$$

$$\sigma_{zx} = 2\frac{e^2}{h}(-n_c b_z + n_b c_z)\frac{a}{v}$$
$$= 2\frac{e^2}{h}\frac{-n_c b_z + n_b c_z}{|b_y c_z - b_z c_y|}. \tag{4.12}$$

In the above we have used the facts that $\mathbf{a} \parallel \mathbf{x}$ and that the area of the Brillouin zone in the k_y-k_z plane is $(2\pi)^2 a/v$. Comparing these results with the general expression of the Hall conductivity in the three-dimensional periodic system [Eq. (2.8)], we get

$$l_a = 0\ , \tag{4.13a}$$
$$l_b = n_b\ , \tag{4.13b}$$
$$l_c = n_c\ . \tag{4.13c}$$

V. CONCLUSION

When the magnetic field is tilted to any direction, there is a possibility for the new wave vector for the FISDW. We have shown that the Hall conductivities σ_{xy}, σ_{zx}, and σ_{yz} are quantized as a function of the wave number of the FISDW. When the lattice is orthogonal, both σ_{xy} and σ_{xz} per plane are quantized as integers and σ_{yz} stays zero as long as t_b/t_a and t_c/t_a are treated in perturbation. Since the quantization is realized with respect to the fundamental reciprocal lattice, σ_{xy} and σ_{zx} are not quantized as integers if the **a**, **b**, and **c** axes are not orthogonal to each other. Even in that case the quantum values of σ_{xy} and σ_{zx} are given as a function of the wave vector of the FISDW, which is characterized by integers.

ACKNOWLEDGMENT

The author would like to thank M. Kohmoto for helpful discussions.

[1] M.J. Naughton, R.V. Chamberlin, X. Yan, S.Y. Hsu, L.Y. Chiang, M.Ya. Azbel, and P.M. Chaikin, Phys. Rev. Lett. **61**, 621 (1988).
[2] J.R. Cooper, W. Kang, P. Auban, G. Montambaux, D. Jerome, and K. Bechgaard, Phys. Rev. Lett. **63**, 1984 (1989).
[3] S.T. Hannahs, J.S. Brooks, W. Kang, L.Y. Chiang, and P.M. Chaikin, Phys. Rev. Lett. **63**, 1988 (1989).
[4] L.P. Gor'kov and A.G. Lebed', J. Phys. Lett. **45**, 433 (1984).
[5] G. Montambaux, M. Héritier, and P. Lederer, Phys. Rev. Lett. **55**, 2078 (1985).
[6] K. Yamaji, J. Phys. Soc. Jpn. **54**, 1034 (1985).
[7] A. Virosztek, L. Chen, and K. Maki, Phys. Rev. B **34**, 3371 (1986).
[8] K. Yamaji, J. Phys. Soc. Jpn. **56**, 1841 (1987).
[9] T. Osada, S. Kagoshima, and N. Miura, Phys. Rev. Lett. **69**, 1117 (1992).
[10] M. Ribault, J. Cooper, D. Jerome, D. Mailly, A. Moradpour, and K. Bechgaard, J. Phys. (Paris) Lett. **45**, L935 (1984).
[11] D. Poilblanc, G. Montambaux, M. Héritier, and P. Lederer, Phys. Rev. Lett. **58**, 270 (1987).
[12] M. Kohmoto, J. Phys. Soc. Jpn. **59**, 1537 (1990).
[13] V.M. Yakovenko, Phys. Rev. B **43**, 11353 (1991).
[14] Y. Hasegawa, K. Machida, M. Kohmoto, and V.M. Yakovenko, J. Superconductivity **7**, 757 (1994).
[15] A.G. Lebed', Pis'ma Zh. Eksp. Teor. Fiz. **43**, 137 (1986) [JETP Lett. **43**, 174 (1986)].
[16] T. Osada, K. Kawasumi, S. Kagoshima, M. Miura, and G. Saito, Phys. Rev. Lett. **66**, 1525 (1991).
[17] M.J. Naughton, O.H. Chang, M. Chapara, X. Bu, and D. Coopers, Phys. Rev. Lett. **67**, 3712 (1991).
[18] G. Montambaux and P.B. Littlewood, Phys. Rev. Lett. **62**, 953 (1989).
[19] L. Chen and K. Maki, Synth. Met. **29**, F493 (1989).
[20] Y. Sun and K. Maki, J. Phys. (France) IV **3**, C2-327 (1993).
[21] G. Montambaux and M. Kohmoto, Phys. Rev. B **41**, 11 417 (1990).
[22] B.I. Halperin, Jpn. J. Appl. Phys. **26**, 1913 (1987).
[23] M. Kohmoto, B. Halperin, and Y. Wu, Phys. Rev. B **45**, 13488 (1992).
[24] M.Ya. Azbel, Zh. Eksp. Teor. Fiz. **46**, 929 (1964) [Sov. Phys. JETP **19**, 634 (1964)].
[25] D.R. Hofstadter, Phys. Rev. B **14**, 2239 (1976).
[26] Y. Hasegawa, J. Phys. Soc. Jpn. **59**, 4384 (1990).
[27] Z. Kunszt and A. Zee, Phys. Rev. B **44**, 6842 (1991).
[28] Y. Hasegawa, J. Phys. Soc. Jpn. **61**, 1657 (1992).
[29] Y. Hasegawa, M. Kohmoto, and G. Montambaux, Phys. Rev. B **48**, 1119 (1993).
[30] D.J. Thouless, M. Kohmoto, M.P. Nightingale, and M. den Nijs, Phys. Rev. Lett. **49**, 405 (1982).
[31] M. Kohmoto, Phys. Rev. B **39**, 11943 (1989).
[32] K. Machida, Y. Hori, and M. Nakono, Phys. Rev. Lett. **70**, 61 (1993).
[33] Y. Hori, K. Kishigi, and K. Machida, J. Phys. Soc. Jpn. **62**, 3598 (1993).
[34] K. Machida, Y. Hasegawa, M. Kohmoto, V.M. Yakovenko, Y. Hori, and K. Kishigi, Phys. Rev. B **50**, 921 (1994).