Exact solution of the Landau fixed point via bosonization

A. H. Castro Neto and Eduardo H. Fradkin
Loomis Laboratory of Physics, University of Illinois at Urbana-Champaign, 1110 West Green Street, Urbana, Illinois, 61801-3080
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We study, via bosonization, the Landau fixed point for the problem of interacting spinless fermions near the Fermi surface in dimensions higher than one. We rederive the bosonic representation of the Fermi operator and use it to find the general form of the fermion propagator for the Landau fixed point. Using a generalized Bogoliubov transformation we diagonalize exactly the bosonized Hamiltonian for the fixed point and calculate the fermion propagator (and the quasiparticle residue) for isotropic interactions (independently of their strength). We reexamine two well-known problems in this context: the screening of long-range potentials and the Landau damping of gauge fields. We also discuss the origin of the Luttinger fixed point in one dimension in contrast with the Landau fixed point in higher dimensions.

I. INTRODUCTION

During the last 50 years the Landau theory has been a paradigm used to explain the experimental behavior and electronic properties of quantum Fermi liquids.[1,2] Initially the theory appeared as a phenomenological framework with a few parameters fixed by experiments. The presence of unknown parameters reflected, at that time, the lack of a microscopic theory. However, it was an extraordinary and necessary first step. Landau himself also established the route for the microscopic explanation for the validity of the theory. The Landau theory became the main tool for the study of the effects of correlations in electronic systems and its foundation was eventually established on microscopic grounds using field theoretic methods.[3–5]

Although the main idea behind the Fermi-liquid theory is quite simple, that is, the idea of a quasiparticle, its realization in terms of microscopic calculations is far from that. The idea is that when an electron interacts with other electrons it polarizes its vicinity and a cloud is formed around it (in an electronic system a hole is formed around the electron due to the electron-electron repulsion[2]). In its motion the electron carries an extra inertia due to the existence of the other electrons. In more usual words, the quasiparticle is a dressed electron. In this picture the interaction between the electrons does not affect the intrinsic properties of the electron (that is, its quantum numbers) but only its dynamics. In field theoretic terms, the existence of a quasiparticle is related to the presence of an isolated singularity in the one-particle Green's function.[3] This singularity produces a Dirac δ peak in the spectral function of the Green's function. In the noninteracting case all the weight of the spectral function (or the quasiparticle residue, Z_F) is in the peak ($Z_F = 1$). The ground state in this case is a filled Fermi sea (due to Pauli's exclusion principle) with a sharp singularity at the Fermi momentum which defines the Fermi surface. When the interaction is turned on the strength of the peak is weakened ($0 < Z_F < 1$) but it is still infinitely sharp. The rest of the spectral weight is carried by an incoherent background which plays no essential role in Fermi-liquid theory. The most important consequence of the presence of this sharp peak is that the ground state is still a filled Fermi sea with a weakened singularity. The existence of quasiparticles is thus directly related with the existence of a well-defined Fermi surface.

Fermi-liquid theory has also been used as the starting point in order to understand the nature of condensed states of matter such as the superconducting state. The theory which explains the behavior of superconductivity of simple metals, the BCS theory, starts from the fact that the normal state is a quantum liquid described by the Landau theory.[6] The properties of the superconducting phase thus depend on the properties of the normal phase of the material.

The observed unusual properties of the normal phase of the cuprates[7,8] posed the question of the existence of new states of condensed matter which are not described by the Landau theory of Fermi liquids. In particular, the phenomenology of the normal states appears to indicate the vanishing of the singularity at the Fermi surface ($Z_F = 0$).[8] Many theoretical scenarios for such a non-Fermi-liquid behavior have been proposed.[9,10]

The apparent failure of the conventional Landau theory of the Fermi liquid in the context of the cuprates (and, perhaps, in more general strongly correlated systems) has motivated a wide search for alternative theoretical tools that, in principle, could handle the effects of strong correlations. In one space dimension, where the interactions are always strong as a result of the kinematical constraints, bosonization[11] has emerged as the main theoretical tool. Recently the problem of bosonizing a dense Fermi system in arbitrary dimensions has been the focus of intense research. The main ideas were introduced originally by Luther,[12] rediscovered recently by Haldane,[13] and developed in great detail by Houghton

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and Marston[14] and by us.[15] Here we will use the method of bosonization by coherent states that we have developed recently.[15]

In this paper we investigate a class of fixed-point Hamiltonians for interacting fermions which exhibit Landau-type behavior. The main motivation of this work is to see how the conventional behavior predicted by Fermi liquid theory arises in the bosonization approach. It is important to reexamine this well-known problem not just as a check on our methods but also since, unlike the conventional many-body perturbation theory approach to Fermi-liquid theory, bosonization is not based on self-consistent resummations of perturbation theory. In principle, it should yield exact results for the low-energy behavior of the system. Thus this is a necessary step if these methods are to be applied to more interesting physical systems in which the Landau theory is believed to fail, such as the problem of a dense system of fermions coupled to dynamical gauge fields. Such systems are central for the understanding of some of the most novel approaches to the problem of high-temperature superconductors[16] and to the compressible states of the fractional quantum Hall effect.[17,18]

In previous papers[15] we have examined the Landau fixed point. Here we show by an explicit calculation that systems of fermions (at finite density, relativistic or not, continuum or lattice) which interact via scalar potentials in the absence of nesting or gauge fields belong to this universality class. We investigate the properties of the operators that create the physical low-energy states of the system described by Fermi-liquid theory. We construct a Hilbert space of states which represents the physical states close to the Fermi energy. The fixed-point Hamiltonians contain only marginal operators acting on these states. We also characterize the relevant operators at these fixed points which are connected with low-energy instabilities of the system. We will not consider here the problem of spin and magnetic excitations.

We have shown in our earlier work that it is possible to bosonize an interacting spinless fermionic liquid, at long wavelengths, in terms of operators which create particle-hole pairs close to the Fermi surface.[15] Our results showed that the bosons of the theory are sound waves which propagate on the Fermi surface and have no resemblance to free nonrelativistic bosons. Essentially these bosons are topologically constrained to the Fermi surface and therefore they propagate in a nonflat metric. The dynamics of these bosons is related to the elastic properties of the Fermi surface, that is, the Fermi surface sustains a surface tension when the quasiparticle residue is nonzero ($Z_F \neq 0$).[15] When the surface tension vanishes ($Z_F = 0$) a phase transition occurs at the Fermi surface and many properties of the system change abruptly. We further showed explicitly that the bosonized theory yields the correct thermodynamic properties of Fermi liquids.

In this paper we begin by reviewing in some detail the bosonization procedure we have introduced before.[15] We use this approach to develop the generating functional for the bosonic fields in terms of coherent states which are coherent superpositions of particle-hole pairs and represent the distortions of the Fermi surface. This representation for the generating functional allows us to discuss the form of the fermion operator in terms of the bosons. Here the similarity with the bosonization in one-dimensional systems becomes immediately clear. We also show that bosonization and non-Fermi-liquid behavior (or rather, non-Landau behavior) are not one and the same thing. We argue that bosonization is a more general concept and that the non-Fermi-liquid behavior of one-dimensional systems is a product of the smallness of available phase space which enhances the interactions. In one dimension the constraints imposed by the conservation laws couple the oscillations of the Fermi surface (in this case only two Fermi points) in a manner which will be discussed below. In dimensions higher than one the number of degrees of freedom (or Fermi points) is infinite and conservation laws alone are not enough to drive the system away from the Fermi-liquid fixed point.

We rederive an explicit formula for the fermion operator written in terms of bosons and we use it to obtain the correct free fermion propagator in the limit of long wavelengths. This calculation reveals many important aspects of the bosonization procedure we have been developing and confirms, once more, the usefulness of this method.

Since the boson operator is a product of two particle operators in terms of the original fermions, we show that a two-particle interaction (which is written as the product of four particle operators) can be written only in four different forms which are bilinears in bosonic operators. Assuming an isotropic interaction for the fermions (that is, no dependence of the interaction on the position of the Fermi surface) we diagonalize exactly the problem in the thermodynamic limit. We calculate explicitly the fermion propagator and the quasiparticle residue for any strength of the potential. We show that, for local interactions, in dimensions greater than one, Fermi-liquid behavior is always expected. Recently, Houghton *et al.*[19] have examined this problem using bosonization. In their work, they were able to reproduce the expected behavior of the self-energy for Fermi liquids in two and three dimensions. The main difference between our approach and that of Houghton *et al.* is that we make use of the full diagonalization of the bosonic Hamiltonian by means of a generalized Bogoliubov transformation which mixes all pieces of the Fermi surface.

We reexamine the problem of dynamical screening of the fermion-fermion interactions. Unlike the conventional random-phase approximation (RPA) approach, we do not resum the bubble diagrams self-consistently since bosonization is a nonperturbative approach. Hence one has to work with the bare interaction and let the dynamics of the system decide. Thus, instead of assuming a "screening first" scenario, we approach the problem of dynamical screening by calculating directly the full fermion one-particle Green's function. We show that dynamical screening of long-range interactions is due to the excitation of particle-hole fluctuations with momentum transfer tangent to the Fermi surface. We further show that assuming screening first, as it is usually done in the conventional approach to Fermi-liquid theory, leads to physically incorrect results for one-dimensional systems.

We also discuss the problem of screening of external probes in the bosonic language. We show that while external scalar potentials are always screened, external gauge fields are not screened but get Landau damping instead (this is the case of a nonsuperconducting material). The reason is well known; liquids do not screen transverse oscillations at low frequency. We also show how bosonization can explain the differences between the screening of scalar (longitudinal) and vector (gauge) fields and in particular we obtain expressions for the susceptibilities (response functions) in both cases. We rederive the well-known fact that the RPA result[2] is exact in the limit of long wavelengths.

The paper is organized as follows: in Sec. II we review our bosonization procedure introducing some details which were not discussed in our previous works; using the coherent states defined by the boson annihilation operator we obtain the generating functional for the bosonic fields in Sec. III; in Sec. IV we discuss the form of the fermionic operator in terms of the bosons and its relationship with the coherent state path integral developed in the preceding section; with this machinery at hand, in Sec. V we obtain the noninteracting one-particle Green's function; in Sec. VI we study what kind of interactions the bosonic Hamiltonian can have if we start with a two-body interaction between the fermions, we show that we have only four kinds of terms which are possible in the bosonic language, and we are able to classify them; in Sec. VII using a generalized Bogoliubov transformation we diagonalize exactly the bosonized Hamiltonian for fermions interacting via isotropic interactions in dimensions higher than one; in Sec. VIII we calculate the one-particle propagator and the quasiparticle residue as a function of the potential strength for Fermi liquids; in Secs. IX and X we obtain the well-known results for the response functions for scalar and vector fields, respectively, and in Sec. XI we discuss the differences between one and higher dimensions in the context of bosonization. Section XII contains our conclusions.

II. BOSONIZATION

The bosonization of a fermionic system is based on the algebra obeyed by the densities and currents.[15] This algebra is obtained in a restricted Hilbert space which contains the states close to the Fermi surface. The relevant operator, which generates all the states in this Hilbert space is defined as

$$n_{\vec{q}}(\vec{k}) = c^{\dagger}_{\vec{k}-\frac{\vec{q}}{2}} c_{\vec{k}+\frac{\vec{q}}{2}}. \tag{2.1}$$

In general the commutation relations between the operators in (2.1) are written in terms of an expansion of operators. However, in the restricted Hilbert space, we expect that the substitution of the commutation relations by their expectation value on the state $|FS\rangle$ (representing the filled Fermi sea) will generate all the relevant dynamics of the interacting fermionic system. The commutation relation is obtained for the case of small momenta normal to the Fermi surface (there is no restriction for momenta tangent to the Fermi surface). We have shown that, in this Hilbert space, the commutation relation between the operators in (2.1) is written as[15]

$$\left[n_{\vec{q}}(\vec{k}), n_{-\vec{q}'}(\vec{k}')\right] = \delta_{\vec{k},\vec{k}'}\delta_{\vec{q},\vec{q}'}\vec{q}\cdot\vec{v}_{\vec{k}}\delta(\mu-\epsilon_{\vec{k}}) + \hat{\mathcal{O}}. \tag{2.2}$$

The operator $\hat{\mathcal{O}}$ in Eq. (2.2) represents additional operators whose effects become negligible as $\vec{q}\to 0$. These operators will drop from our formalism once properly smeared operators are defined (see below).

We begin by defining a complete set of one-particle states with spectrum $\epsilon_{\vec{k}}$ which is used to build the full Hilbert space. The velocity of the particles is defined in the usual way,

$$\vec{v}_{\vec{k}} = \nabla\epsilon_{\vec{k}}. \tag{2.3}$$

The Fermi surface is defined by the set of vectors $\{\vec{k}_F\}$ which obey the relation

$$\mu = \epsilon_{\vec{k}_F}, \tag{2.4}$$

where μ is the chemical potential of the system.

Although the particle-hole operators (2.1) have almost bosonic character, the operators do not annihilate the reference state $|FS\rangle$. We need to normal order these operators relative to this state. Also, canonical bosonic commutation relations are only obeyed by suitably smeared operators at each Fermi point. We define creation and annihilation operators,

$$a_{\vec{q}}(\vec{k}_F) = \sum_{\vec{k}} \Phi_\Lambda(|\vec{k}-\vec{k}_F|) \times[n_{\vec{q}}(\vec{k})\Theta(\vec{v}_{\vec{k}_F}\cdot\vec{q}) + n_{-\vec{q}}(\vec{k})\Theta(-\vec{v}_{\vec{k}_F}\cdot\vec{q})] \tag{2.5}$$

and

$$a^{\dagger}_{\vec{q}}(\vec{k}_F) = \sum_{\vec{k}} \Phi_\Lambda(|\vec{k}-\vec{k}_F|) \times[n_{-\vec{q}}(\vec{k})\Theta(\vec{v}_{\vec{k}_F}\cdot\vec{q}) + n_{\vec{q}}(\vec{k})\Theta(-\vec{v}_{\vec{k}_F}\cdot\vec{q})], \tag{2.6}$$

where $\Theta(x) = 1(-1)$ if $x > 0(< 0)$ and $\Phi_\Lambda(|\vec{k}-\vec{k}_F|)$ is a dimensionless smearing function which keeps the vectors $\vec{k}$ close to $\vec{k}_F$, that is,

$$\lim_{\Lambda\to 0}\Phi_\Lambda(|\vec{k}-\vec{k}_F|) = \delta_{\vec{k},\vec{k}_F}, \tag{2.7}$$

where Λ can be viewed as a cutoff in momentum space. The idea is to construct spheres of radius Λ which cover all the states on the Fermi surface.[13] We parametrize each one of these spheres by the Fermi momentum $\vec{k}_F$, with $\vec{q}$ at the center of the spheres (alternatively, we could also have constructed pill boxes of height Λ and length D instead of spheres,[14] the difference will be immaterial since none of the physical quantities will depend on the way we introduce the cutoff). This construction will give good results whenever the states in the problem have momentum close to $\vec{k}_F$ with fluctuations of order $\vec{q}$ such that $q \ll \Lambda \ll k_F$.

It is straightforward to see that, by construction, we have

$$a_{\vec{q}}(\vec{k}_F) \mid FS\rangle = 0 \tag{2.8}$$

and the smeared operators $a_{\vec{q}}(\vec{k}_F)$ are found to obey the commutation relations

$$\left[a_{\vec{q}}(\vec{k}_F), a^{\dagger}_{\vec{q}'}(\vec{k}'_F)\right] = N_\Lambda(\vec{k}_F)V \mid \vec{q}\cdot\vec{v}_{\vec{k}_F} \mid \times\delta_{\vec{k}_F,\vec{k}'_F}\left(\delta_{\vec{q},\vec{q}'} + \delta_{\vec{q},-\vec{q}'}\right), \tag{2.9}$$

where $N_\Lambda(\vec{k}_F)$ is a local density of states defined as follows (V is the volume of the system):

$$N_\Lambda(\vec{k}_F) = \frac{1}{V}\sum_{\vec{k}} |\Phi_\Lambda(|\vec{k}-\vec{k}_F|)|^2\delta(\mu-\epsilon_{\vec{k}}). \tag{2.10}$$

Equations (2.8) and (2.9) show that these operators have bosonic character and generate the restricted Hilbert space of interest.

The local density of states is a measure of the number of states per unit of energy per solid angle in the Fermi surface. In general, in the absence of Van Hove singularities, the local density of states is well behaved and independent of the cutoff. In this case we substitute it by its natural average which is the total density of states $N(0)$,

$$N(0) = \frac{1}{V}\sum_{\vec{k}} \delta(\mu-\epsilon_{\vec{k}}) \tag{2.11}$$

divided by the solid angle on the Fermi surface $S_d = \int d\Omega$. Indeed, in our previous papers where we have studied the transport and thermodynamic properties of Fermi liquids we have used $N_\Lambda(\vec{k}_F) = N(0)/S_d$.[15]

Although we could work directly with the operators defined above it is usual to rescale the operators in such a way as to absorb the density of states in its definition. We set

$$b_{\vec{q}}(\vec{k}_F) = \left[N_\Lambda(\vec{k}_F)V|\vec{q}\cdot\vec{v}_{\vec{k}_F}|\right]^{-1/2} a_{\vec{q}}(\vec{k}_F), \tag{2.12}$$

which obey the usual bosonic algebra (independent of the cutoff),

$$\left[b_{\vec{q}}(\vec{k}_F), b^{\dagger}_{\vec{q}'}(\vec{k}'_F)\right] = \delta_{\vec{k}_F,\vec{k}'_F}\left(\delta_{\vec{q},\vec{q}'} + \delta_{\vec{q},-\vec{q}'}\right). \tag{2.13}$$

These relations between bosonic operators, which have particle-hole character, will be the basis of our work.

III. COHERENT STATES AND GENERATING FUNCTIONAL

Although we have a closed algebra and a reference state to work out the physics, we do not yet have a clear physical interpretation for the bosonic operators. We know from Eq. (2.1) that they are related to particle-hole excitations. However, in order to develop more physical intuition about the excitations created by these operators, we use the coherent states associated with them.

The coherent states are defined via a unitary operator,

$$U([\phi]) = \exp\left(-\sum_{\vec{k}_F,\vec{q}} \frac{1}{N_\Lambda(\vec{k}_F)V\vec{q}\cdot\vec{v}_{\vec{k}_F}} \times\phi_{\vec{q}}(\vec{k}_F)n_{-\vec{q}}(\vec{k}_F)\right), \tag{3.1}$$

where $U^{-1} = U^{\dagger}$ implies $\phi_{-\vec{q}}(\vec{k}_F) = \phi^*_{\vec{q}}(\vec{k}_F)$. The operator U is a functional of the fields $\phi_{\vec{q}}(\vec{k}_F)$ which are defined on the Fermi surface.

We can also rewrite (3.1) as

$$U([\phi]) = \exp\left(-\sum_{\vec{k}_F,\vec{q},\vec{q}\cdot\vec{v}_{\vec{k}_F}>0} \frac{1}{N_\Lambda(\vec{k}_F)V|\vec{q}\cdot\vec{v}_{\vec{k}_F}|} \times[\phi_{\vec{q}}(\vec{k}_F)n_{-\vec{q}}(\vec{k}_F) -\phi^*_{\vec{q}}(\vec{k}_F)n_{\vec{q}}(\vec{k}_F)]\right) \tag{3.2}$$

and using the definitions (2.5) and (2.12) we have

$$U([\phi]) = \exp\left(-\sum_{\vec{k}_F,\vec{q},\vec{q}\cdot\vec{v}_{\vec{k}_F}>0}\left(\frac{1}{N_\Lambda(\vec{k}_F)V|\vec{q}\cdot\vec{v}_{\vec{k}_F}|}\right)^{1/2}\left[\phi_{\vec{q}}(\vec{k}_F)b^{\dagger}_{\vec{q}}(\vec{k}_F) - \phi^*_{\vec{q}}(\vec{k}_F)b_{\vec{q}}(\vec{k}_F)\right]\right). \tag{3.3}$$

The coherent state is defined as the evolution of the reference state via the operator U,

$$|[\phi]\rangle = U([\phi])|FS\rangle. \tag{3.4}$$

It is easy to show that this state is an eigenstate of the destruction operator,

$$b_{\vec{q}}(\vec{k}_F)|[\phi]\rangle = -\left(\frac{1}{N_\Lambda(\vec{k}_F)V|\vec{q}\cdot\vec{v}_{\vec{k}_F}|}\right)^{1/2}\phi_{\vec{q}}(\vec{k}_F)|[\phi]\rangle. \tag{3.5}$$

This last result has a clear physical meaning. It means that the coherent state represents a deformation of the Fermi surface at the point $\vec{k}_F$ in the direction of $\vec{q}$ due to a coherent (collective) superposition of particle-hole pairs. Therefore the bosonic field $\phi_{\vec{q}}(\vec{k}_F)$ is a measure of this deformation at that point. These bosonic fields are topologically constrained excitations which propagate on the Fermi surface.

As usual with coherent states, we can show that they are not orthogonal to each other and that they are overcomplete. The overcompleteness of these states[20] physically means that the bosons which propagate on the Fermi surface are wave packets.[15] This is fully consistent with the standard picture of the Landau theory.

The propagation of the many-body system in time, from time 0 to time t, can be obtained by a calculation of the S matrix with initial and final states of the kind considered above. The S matrix is defined in the following way:

$$\langle[\varphi],t|[\tilde{\varphi}],0\rangle = \langle[\varphi]|e^{-iHt}|[\tilde{\varphi}]\rangle, \tag{3.6}$$

where H is the Hamiltonian of the system.

Using the closure relation for these states it is easy to show that it can also be written in terms of path integrals,

$$\langle[\varphi],t|[\tilde{\varphi}],0\rangle = \exp\left\{-\sum_{\vec{k}_F,\vec{k}'_F}\sum_{\vec{q},\vec{q}\cdot\vec{v}_{\vec{k}_F}>0}\frac{1}{2N_\Lambda(\vec{k}_F)V}\left(\frac{1}{|\vec{q}\cdot\vec{v}_{\vec{k}_F}|}|\varphi_{\vec{q}}(\vec{k}_F)|^2+\frac{1}{|\vec{q}\cdot\vec{v}_{\vec{k}'_F}|}\tilde{\varphi}_{\vec{q}}(\vec{k}'_F)|^2\right)\right\} \times\int_{[\tilde{\varphi}]}^{[\varphi]}D^2[\phi]e^{iS[\phi]}, \tag{3.7}$$

where

$$D^2[\phi]=\prod_{\vec{k}_F,\tau,\vec{q},\vec{q}\cdot\vec{v}_{\vec{k}_F}>0}\frac{d\phi_{\vec{q}}(\vec{k}_F,\tau)d\phi^*_{\vec{q}}(\vec{k}_F,\tau)}{\pi N_\Lambda(\vec{k}_F)V|\vec{q}\cdot\vec{v}_{\vec{k}_F}|}$$

and

$$iS[\phi]=\sum_{\vec{k}_F,\vec{q},\vec{q}\cdot\vec{v}_{\vec{k}_F}>0}\frac{1}{N_\Lambda(\vec{k}_F)V|\vec{q}\cdot\vec{v}_{\vec{k}_F}|}\left[\frac{1}{2}\left[\tilde{\varphi}_{\vec{q}}(\vec{k}_F)\phi^*_{\vec{q}}(\vec{k}_F,0)+\varphi^*_{\vec{q}}(\vec{k}_F)\phi_{\vec{q}}(\vec{k}_F,t)\right]+\int_0^t d\tau L(\phi(\tau))\right] \tag{3.8}$$

is the classical action for the motion of the bosonic fields and

$$L(\phi)=\sum_{\vec{k}_F}\frac{1}{2}\left(\phi_{\vec{q}}(\vec{k}_F)\frac{d\phi^*_{\vec{q}}(\vec{k}_F)}{d\tau}-\phi^*_{\vec{q}}(\vec{k}_F)\frac{d\phi_{\vec{q}}(\vec{k}_F)}{d\tau}\right) -i\tilde{H}[\phi(\tau)] \tag{3.9}$$

is the Lagrangian density. The rescaled Hamiltonian $\tilde{H}$ is defined as

$$\tilde{H}[\phi(\tau)]=\sum_{\vec{k}_F}N_\Lambda(\vec{k}_F)V|\vec{q}\cdot\vec{v}_{\vec{k}_F}|\,\frac{\langle\vec{k}_F,[\phi]|H|\vec{k}_F,[\phi]\rangle}{\langle\vec{k}_F,[\phi]|\vec{k}_F,[\phi]\rangle}. \tag{3.10}$$

In the path integral the boundary conditions are defined by the S matrix (3.6),

$$\phi_{\vec{q}}(\vec{k}_F,0)=\tilde{\varphi}_{\vec{q}}(\vec{k}_F),\qquad \phi^*_{\vec{q}}(\vec{k}_F,t)=\varphi^*_{\vec{q}}(\vec{k}_F). \tag{3.11}$$

As in any field theory we can also write a generating functional Z which is the trace of the S matrix. In terms of path integrals it is written as

$$Z=\int D^2[\phi]e^{i\int dt\,\sum_{\vec{q},\vec{k}_F}L(\phi^*_{\vec{q}}(\vec{k}_F,t)\phi_{\vec{q}}(\vec{k}_F,t))}, \tag{3.12}$$

where the Lagrangian is the same as in (3.9) and (3.10). Now we have periodic boundary conditions due to the trace. As usual the Euler-Lagrange equations for the classical action will generate the semiclassical dynamics for the problem. We have shown that in the case of a Fermi liquid the semiclassical dynamics is represented by the Landau equation of sound waves. Therefore the bosonic fields represent these waves which are distortions of the Fermi surface evolving in time. Moreover, the same functional integral in imaginary time reproduces the thermodynamics of the Fermi liquids.[15]

As we will see in the next section these results, besides providing a physical insight into the physics of fermionic systems (and an interpretation for the existence of bosons in a fermionic theory), are also powerful tools which will allow us to calculate correlation functions of interest. To this end, however, we need a dictionary to translate from the language of fermions to bosons. The aim of the next section is to provide it.

IV. THE FERMION OPERATOR

The connection between bosons and fermions is inspired from the bosonization methods in one-dimensional systems. Luther[12] was the first to explore the analogy between one dimension and higher dimensions. Luther's idea was to define at each point of the Fermi surface a one-dimensional system using the radial directions. However, Luther worked only with noninteracting fermions. Here, we present an argument paralleling Luther's work.

The fermion operator $\psi(\vec{r})$ is written in momentum space as

$$\psi(\vec{r}) = \frac{1}{\sqrt{V}} \sum_{\vec{k}} e^{i\vec{k}\cdot\vec{r}} c_{\vec{k}}. \tag{4.1}$$

Inspired by the results in one-dimensional systems we follow Luther's construction of the fermionic operator $\Psi(\vec{r}, \vec{k}_F)$ as follows:

$$\psi(\vec{r}) = \sum_{\vec{k}_F} \Psi(\vec{r}, \vec{k}_F),$$

where

$$\Psi(\vec{r}, \vec{k}_F) = f(\vec{k}_F) e^{J(\vec{r}, \vec{k}_F)}, \tag{4.2}$$

for some operators $f(\vec{k}_F)$ and $J(\vec{r}, \vec{k}_F)$. The correct commutation relations for the $\Psi(\vec{r}, \vec{k}_F)$ can be obtained by imposing the commutation relation $[n_{\vec{q}}(\vec{k}), f(\vec{k}_F)] = 0$ and $[n_{\vec{q}}(\vec{k}), J(\vec{r}, \vec{k}_F)] = c$ number. This choice seems to be the simplest possible.

Since the operators $n_{\vec{q}}(\vec{k})$ generate the Hilbert space of interest, the form of the fermion operator $\Psi(\vec{r}, \vec{k}_F)$ will be defined by the commutation relation between the former and the latter. It is easy to show, using the fermionic algebra for the operators $c_{\vec{k}}$, that

$$\left[\psi(\vec{r}), \sum_{\vec{k}} n_{\vec{q}}(\vec{k})\right] = e^{-i\vec{q}\cdot\vec{r}} \psi(\vec{r}). \tag{4.3}$$

From (4.2) we have

$$[\Psi(\vec{r}, \vec{k}_F), n_{\vec{q}}(\vec{k}'_F)] = e^{-i\vec{q}\cdot\vec{r}} \Psi(\vec{r}, \vec{k}'_F)\, \delta_{\vec{k}_F, \vec{k}'_F}. \tag{4.4}$$

Using the choices for the commutation relations between the operators $f(\vec{k}_F)$ and $J(\vec{r}, \vec{k}_F)$ with $n_{\vec{q}}(\vec{k})$, we easily get

$$[\Psi(\vec{r}, \vec{k}_F), n_{\vec{q}}(\vec{k}_F)] = [n_{\vec{q}}(\vec{k}_F), J(\vec{r}, \vec{k}_F)] \Psi(\vec{r}, \vec{k}_F). \tag{4.5}$$

Therefore the fermion operator will be well defined if we ensure that

$$[n_{\vec{q}}(\vec{k}_F), J(\vec{r}, \vec{k}_F)] = e^{-i\vec{q}\cdot\vec{r}}. \tag{4.6}$$

Due to the commutation relations (2.2) it is easy to see that the correct choice is

$$J(\vec{r}, \vec{k}_F) = -\sum_{\vec{q}} \frac{e^{-i\vec{q}\cdot\vec{r}}}{N_\Lambda(\vec{k}_F) V \vec{q}\cdot\vec{v}_{\vec{k}_F}} n_{-\vec{q}}(\vec{k}_F). \tag{4.7}$$

Notice the similarity between the form (3.1) and (4.7). This is not a mere coincidence. The fermion operator is represented as a coherent state of bosons. Since, as we will see, the bosons diagonalize the problem of an interacting electronic system, the fermion operator is a nonperturbative object in the language of the bosons. Moreover, since the operator U is a functional of the fields, we observe that the fermion operator $\Psi(\vec{r}, \vec{k}_F)$ is given by U for the following choice of the fields:

$$\phi_{\vec{q}}(\vec{k}'_F) = e^{-i\vec{q}\cdot\vec{r}} \delta_{\vec{k}_F, \vec{k}'_F}, \tag{4.8}$$

which means that while the fermions interact among themselves the bosons are free excitations in the Fermi surface.

From this observation we conclude that the fermion propagator,

$$K(\vec{r} - \vec{r}', t - t') = \sum_{\vec{k}_F, \vec{k}'_F} \langle FS| \Psi^\dagger(\vec{r}, \vec{k}_F, t) \Psi(\vec{r}', \vec{k}_F, t') |FS\rangle, \tag{4.9}$$

can be written as

$$K(\vec{r} - \vec{r}', t - t') = \sum_{\vec{k}_F, \vec{k}'_F} f^*(\vec{k}_F) f(\vec{k}'_F) \langle \vec{k}_F, \vec{r}, t | \vec{k}'_F, \vec{r}', 0\rangle, \tag{4.10}$$

where $|\vec{k}_F, \vec{r}, t\rangle$ is the coherent state (3.4) with the prescription (4.8). The functions $f(\vec{k})$ are written in terms of the total number of fermions in the system.[12,21] Their role is to ensure that the fermion operators anticommute with each other.

The one-particle Green's function can be obtained directly from the propagator in the case when the actual ground state of the system is the filled Fermi sea:

$$\begin{aligned} G(\vec{r} - \vec{r}', t - t') = {} & K(\vec{r} - \vec{r}', t - t')\Theta(t - t') \\ & - K(\vec{r} - \vec{r}', t - t')^* \Theta(t' - t). \end{aligned} \tag{4.11}$$

As we will show in the next section this is the case of the noninteracting electronic system and Fermi liquids.

V. THE GREEN'S FUNCTION FOR THE FREE SYSTEM

The Hamiltonian for a noninteracting electronic system is just the kinetic term,

$$H_0 = \sum_{\vec{k}} \epsilon_{\vec{k}} c^\dagger_{\vec{k}} c_{\vec{k}}. \tag{5.1}$$

Since the operators (2.1) generate the Hilbert space of interest, the form of the Hamiltonian (5.1) in this space will depend on the commutation relations between the Hamiltonian and these operators. It is easy to show that[15]

$$\left[H_0, b_{\vec{q}}(\vec{k}_F)\right] = -|\vec{q}\cdot\vec{v}_{\vec{k}_F}| b_{\vec{q}}(\vec{k}_F), \tag{5.2}$$

therefore the noninteracting Hamiltonian can be written as

$$H_0 = \sum_{\vec{k}_F} \sum_{\vec{q},\vec{q}\cdot\vec{v}_{\vec{k}_F}>0} |\vec{q}\cdot\vec{v}_{\vec{k}_F}| b^\dagger_{\vec{q}}(\vec{k}_F) b_{\vec{q}}(\vec{k}_F). \tag{5.3}$$

This result is only valid in the restricted Hilbert space of states which lie close to the Fermi surface.

Therefore the free system, which is composed of only the continuum of particle-hole pairs, is described by a set of independent harmonic modes at each point of the Fermi surface. These modes oscillate with arbitrary phase difference between them. It is an incoherent oscillation of the Fermi surface which vanishes on average. It means that, on average, the shape of the Fermi surface is kept constant, as expected.

Given the Hamiltonian (5.3) we can use the methods of the last section in order to calculate the noninteracting Green's function. The rescaled Hamiltonian defined in (3.10) is written as

$$\tilde{H}[\phi(\tau)] = |\vec{q}\cdot\vec{v}_{\vec{k}_F}| \phi^*_{\vec{q}}(\vec{k}_F)\phi_{\vec{q}}(\vec{k}_F). \tag{5.4}$$

The functional integral we obtain is the same as for the harmonic oscillator and it is easily done,[20]

$$\langle \vec{k}_F, [\varphi], t|\vec{k}'_F, [\tilde{\varphi}], 0\rangle = \exp\Bigg\{ -\sum_{\vec{q},\vec{q}\cdot\vec{v}_{\vec{k}_F}>0} \left(\frac{1}{2N_\Lambda(\vec{k}_F)V}\right) \times \Bigg[\frac{1}{|\vec{q}\cdot\vec{v}_{\vec{k}_F}|}|\varphi_{\vec{q}}(\vec{k}_F)|^2 + \frac{1}{|\vec{q}\cdot\vec{v}_{\vec{k}'_F}|}|\tilde{\varphi}_{\vec{q}}(\vec{k}'_F)|^2 - \frac{2}{|\vec{q}\cdot\vec{v}_{\vec{k}_F}|}\delta_{\vec{k}_F,\vec{k}'_F} e^{-i|\vec{q}\cdot\vec{v}_{\vec{k}_F}|t} \tilde{\varphi}_{\vec{q}}(\vec{k}_F)\varphi^*_{\vec{q}}(\vec{k}_F) \Bigg]\Bigg\}. \tag{5.5}$$

We have seen that the propagator is obtained in this language by using the following choice (4.8):

$$\varphi_{\vec{q}}(\vec{k}_F) = e^{-i\vec{q}\cdot\vec{r} - \frac{\alpha|\vec{q}\cdot\vec{v}_{\vec{k}_F}|}{2|\vec{v}_{\vec{k}_F}|}}, \tag{5.6}$$

where α ($\sim \Lambda^{-1}$) is a cutoff in the normal direction to the Fermi surface which we introduce in order to regularize the radial integrals (exactly as in one dimension). Making the substitution above we find

$$\langle \vec{k}_F, \vec{r}, t|\vec{k}'_F, 0, 0\rangle_0 = \exp\Bigg\{ -\sum_{\vec{q},\vec{q}\cdot\vec{v}_{\vec{k}_F}>0} \left(\frac{1}{2N_\Lambda(\vec{k}_F)V}\right) \Bigg[\frac{1}{|\vec{q}\cdot\vec{v}_{\vec{k}_F}|} e^{-\frac{\alpha|\vec{q}\cdot\vec{v}_{\vec{k}_F}|}{|\vec{v}_{\vec{k}_F}|}} + \frac{1}{|\vec{q}\cdot\vec{v}_{\vec{k'}_F}|} e^{-\frac{\alpha|\vec{q}\cdot\vec{v}_{\vec{k}'_F}|}{|\vec{v}_{\vec{k}'_F}|}} - \frac{2\delta_{\vec{k}_F,\vec{k}'_F}}{|\vec{q}\cdot\vec{v}_{\vec{k}_F}|} e^{-\frac{\alpha|\vec{q}\cdot\vec{v}_{\vec{k}_F}|}{|\vec{v}_{\vec{k}_F}|}} e^{i(\vec{q}\cdot\vec{r} - |\vec{q}\cdot\vec{v}_{\vec{k}_F}|t)} \Bigg]\Bigg\}. \tag{5.7}$$

Notice that for different points of the Fermi surface the integral diverges logarithmically due to the factor $1/|\vec{q}\cdot\vec{v}_{\vec{k}'_F}|$ which diverges in the limit of $q \to 0$. Thus we can write

$$\langle \vec{k}_F, \vec{r}, t|\vec{k}'_F, 0, 0\rangle_0 = 0, \qquad \vec{k}_F \neq \vec{k}'_F. \tag{5.8}$$

If $\vec{k}_F = \vec{k}'_F$ we find the integral

$$\langle \vec{k}_F, \vec{r}, t|\vec{k}'_F, 0, 0\rangle_0 = \delta_{\vec{k}_F,\vec{k}'_F} \exp\left(-\sum_{\vec{q},\vec{q}\cdot\vec{v}_{\vec{k}_F}>0} \frac{1}{N_\Lambda(\vec{k}_F)V} \left(1 - e^{i(\vec{q}\cdot\vec{r} - |\vec{q}\cdot\vec{v}_{\vec{k}_F}|t)}\right) \frac{e^{-\frac{\alpha|\vec{q}\cdot\vec{v}_{\vec{k}_F}|}{|\vec{v}_{\vec{k}_F}|}}}{|\vec{q}\cdot\vec{v}_{\vec{k}_F}|} \right). \tag{5.9}$$

In order to evaluate this integral we observe that the component of $\vec{q}$ normal to the Fermi surface dominates the behavior of the integral. It is natural, therefore, to split $\vec{q} = \vec{q_N} + \vec{q_T}$ where $\vec{q_N} = (\vec{q}\cdot\vec{n}_{\vec{k}_F})\vec{n}_{\vec{k}_F}$ (with $\vec{n}_{\vec{k}_F} = \frac{\vec{v}_{\vec{k}_F}}{|\vec{v}_{\vec{k}_F}|}$) which is the normal component to the Fermi surface and the tangential component $\vec{q_T}$ which is defined as $\vec{q_T}\cdot\vec{n}_{\vec{k}_F} = 0$. This choice can be viewed as a Fresnel construction of differential geometry. We have built a local reference frame on the Fermi surface which follows its local geometry.

Observe that the integrals over the tangent component can be easily evaluated. They have the form

$$\int_{-\infty}^{\infty} dq_T q_T^{d-2} e^{iq_T x_T} = \frac{1}{i^{d-2}} \frac{\partial^{d-2}}{\partial x_T^{d-2}} [\delta(x_T)], \tag{5.10}$$

which gives a negligible contribution except for $x_T = 0$. We conclude, therefore, that the part tangent to the Fermi surface does not contribute to the long distance behavior of the Green's function and therefore can be neglected in this limit. However, it does contribute to the counting of states at the Fermi surface. Indeed, from (2.10), we see that the local density of states can be written as

$$N_\Lambda(\vec{k}_F) = \frac{1}{V}\int \frac{d\vec{S}}{|\vec{v}_{\vec{k}_F}|}|\Phi_\Lambda(\vec{k}_F)|^2, \tag{5.11}$$

where $d\vec{S}$ is the area element on the Fermi surface. However, the area on the surface is written as an integral over the tangential component of $\vec{q}$ alone. Therefore the tangential part of the integral contributes to the density of states and not to the long distance behavior of the correlation functions. Thus we make the following substitution:

$$\sum_{\vec{q},\vec{q}\cdot\vec{v}_{\vec{k}_F}>0} \to N_\Lambda(\vec{k}_F)V|\vec{v}_{\vec{k}_F}|\int_0^\infty dq_N. \tag{5.12}$$

The same type of substitution gives the correct thermodynamic properties for these systems, namely, a linear specific heat proportional to the density of states.[15]

Substituting (5.12) in (5.9) we finally get

$$\langle\vec{k}_F,\vec{r},t|\vec{k}'_F,0,0\rangle_0 = \delta_{\vec{k}_F,\vec{k}'_F}\exp\left(-\int_0^\infty dq_N\left(1-e^{iq_N(\vec{n}_{\vec{k}_F}\cdot\vec{r}-|\vec{v}_{\vec{k}_F}|t)}\right)\frac{e^{-\alpha q_N}}{q_N}\right). \tag{5.13}$$

Now we use the integral,

$$\int_0^\infty dx\left(1-e^{ixa}\right)\frac{e^{-\alpha x}}{x} = \ln\left(1-i\frac{a}{\alpha}\right), \tag{5.14}$$

and we finally conclude

$$\langle\vec{k}_F,\vec{r},t|\vec{k}'_F,0,0\rangle_0 = \frac{i\alpha\delta_{\vec{k}_F,\vec{k}'_F}}{\vec{n}_{\vec{k}_F}\cdot\vec{r}-|\vec{v}_{\vec{k}_F}|t+i\alpha}, \tag{5.15}$$

which is the correct asymptotic form for the noninteracting system in any number of dimensions.[12] It represents a fermion moving with velocity $\vec{v}_{\vec{k}_F}$, as expected.

VI. THE HAMILTONIAN

We now discuss the possible forms of the Hamiltonian which can appear in the problem. We assume that the Hamiltonian is composed of two terms, the kinetic energy of the fermions, Eq. (5.1), and a two-particle interaction,

$$U = \frac{1}{2V}\sum_{\vec{p},\vec{p}',\vec{q}} U_{\vec{p},\vec{p}'}(q)\; c^\dagger_{\vec{p}+\frac{\vec{q}}{2}}c_{\vec{p}-\frac{\vec{q}}{2}}c^\dagger_{\vec{p}'-\frac{\vec{q}}{2}}c_{\vec{p}'+\frac{\vec{q}}{2}}. \tag{6.1}$$

The scattering processes described by the operators in (6.1) can be visualized by the pairing of the annihilation and creation operators which are depicted in Fig. 1(a) and Fig. 1(b). In Fig. 1(a), we have small momentum transfer $\vec{q}$ (forward scattering) between the pairs. In Fig. 1(b) the pairs exchange momentum $\vec{p}-\vec{p}'$ can be of order of $2k_F$ (backward scattering).

The form of the kinetic energy in terms of the bosonic operators was discussed in the preceding section and it is given by Eq. (5.3). In terms of the operators which generate the Hilbert space of interest, it is easy to see that the interaction can be written as

$$U = \frac{1}{2V}\sum_{\vec{p},\vec{p}',\vec{q}} U_{\vec{p},\vec{p}'}(q)\; n_{-\vec{q}}(\vec{p})n_{\vec{q}}(\vec{p}'). \tag{6.2}$$

We can now parametrize the form of the Hamiltonian in terms of processes involving particle-hole pairs close to the Fermi surface. From (2.5) we get four kinds of terms [these are the only types of terms (scattering processes) which are present in the restricted Hilbert space],

$$a^\dagger_{\vec{q}}(\vec{k}_F)a_{\vec{q}}(\vec{k}'_F) \quad \text{for } \vec{q}\cdot\vec{v}_{\vec{k}_F}>0\,,\ \vec{q}\cdot\vec{v}_{\vec{k}'_F}>0, \tag{6.3}$$

$$a_{\vec{q}}(\vec{k}_F)a^\dagger_{\vec{q}}(\vec{k}'_F) \quad \text{for } \vec{q}\cdot\vec{v}_{\vec{k}_F}<0\,,\ \vec{q}\cdot\vec{v}_{\vec{k}'_F}<0, \tag{6.4}$$

$$a_{\vec{q}}(\vec{k}_F)a_{\vec{q}}(\vec{k}'_F) \quad \text{for } \vec{q}\cdot\vec{v}_{\vec{k}_F}<0\,,\ \vec{q}\cdot\vec{v}_{\vec{k}'_F}>0, \tag{6.5}$$

$$a^\dagger_{\vec{q}}(\vec{k}_F)a^\dagger_{\vec{q}}(\vec{k}'_F) \quad \text{for } \vec{q}\cdot\vec{v}_{\vec{k}_F}>0\,,\ \vec{q}\cdot\vec{v}_{\vec{k}'_F}<0. \tag{6.6}$$

For a fixed direction of $\vec{q}$ the processes (6.3), (6.4) and (6.5), (6.6) are shown in Fig. 2 and Fig. 3, respectively.

Observe that the momentum $\vec{q}$ breaks the rotation symmetry O(3) of the Fermi surface (for the isotropic system, of course) and introduces an axis in the problem. We can now divide the Fermi sea into south and north hemispheres. While the interactions (6.3) and (6.4) occur within one of the hemispheres the interactions (6.5)

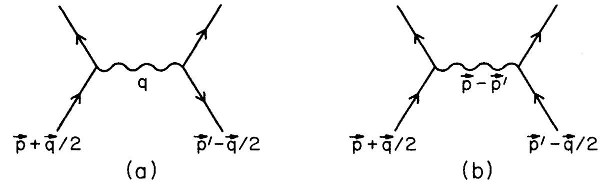

FIG. 1. (a) Forward scattering; (b) backward scattering.

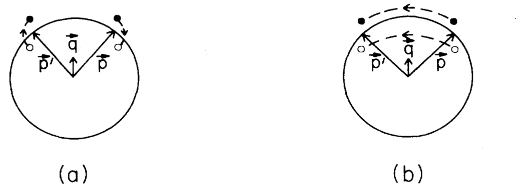

FIG. 2. Processes described by Eqs. (6.3) and (6.4); (a) forward scattering; (b) backward scattering. The dashed line indicates the evolution of the state.

and (6.6) link different hemispheres of the Fermi sea.

In Fermi-liquid theory only the interactions (6.3) and (6.4) have a physically relevant effect in the asymptotic low-energy regime.[2] The reason for that, in the absence of nesting and singular interactions, is very easy to understand. The scattering of a particle-hole pair initially located at vector $\vec{k}_F$ to a new position $\vec{k}'_F$ requires an amount of energy $|\vec{q}\cdot\left(\vec{v}_{\vec{k}_F}-\vec{v}_{\vec{k}'_F}\right)|$. Thus, at low energy, the most important contribution for fixed momentum transfer comes from regions of momentum space in which $\vec{v}_{\vec{k}_F}\sim\vec{v}_{\vec{k}'_F}$. Indeed, it is worth noticing that the corrections to the specific heat due to the interaction between particle-hole pairs is calculated in this limit[1,15,22] and they agree very well with the experimental results. The presence of this process will give rise to terms of the form $a^\dagger a$ in the Hamiltonian, that is, the same form as the noninteracting system (except for being nondiagonal in the index $\vec{k}$).[15] Therefore these interactions never mix the creation and annihilation operators for the bosons. As a consequence, the only effect that these interactions can produce is a different phase of oscillation for the fields or, in other words, to renormalize the Fermi velocity.

However, the terms (6.5) and (6.6) which link the south and north hemispheres are more interesting and their effects are potentially more devastating. They will appear in the Hamiltonian as a combination of $a^\dagger a^\dagger$ and $a\,a$. The consequence of these terms is dramatic. Besides changing the phase of oscillation, they also mix creation and annihilation operators. It means that they introduce a renormalization of the amplitude of the fields by a kind of Bogoliubov transformation. If this process survives in the thermodynamic limit it can be shown to give rise to anomalous dimensions in the correlation functions. We notice here that this is the same kind of interaction which produces anomalous dimensions in one dimension.

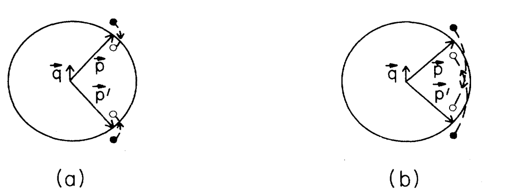

FIG. 3. Processes described by Eqs. (6.5) and (6.6); (a) forward scattering; (b) backward scattering. The dashed line indicates the evolution of the state.

Indeed, suppose we have the interacting part of the Hamiltonian written for a one-dimensional lattice as

$$H=U\sum_n \rho_n\,\rho_{n+1}, \tag{6.7}$$

where ρ_n is the charge density at the site n. The interacting term (6.7) gives rise to two different terms in the bosonized Hamiltonian,[11]

$$U\left[\rho_1(q)\rho_1(-q)+\rho_2(q)\rho_2(-q)\right], \tag{6.8}$$

$$2U\rho_1(q)\rho_2(-q), \tag{6.9}$$

where $\rho_1(q)$ and $\rho_2(q)$ are the Fourier component of the right and left movers, respectively (they are analogous to the south and north movers).

The interaction term (6.8) is associated with the processes described by (6.3) and (6.4) which are responsible for forward scattering. The process (6.9) is related to the terms (6.5) and (6.6). The same effect will occur in higher dimensions, for analogous reasons, if we have a nested Fermi surface. Observe that in the case of nesting the vectors $\vec{v}_{\vec{k}_F}$ at one side of the Fermi surface are the same and therefore processes which translate the particle-hole pair within one side of the Fermi surface cost no energy $[|\vec{q}\cdot(\vec{v}_{\vec{k}_F}-\vec{v}_{\vec{k}'_F})|=0]$. The only other available processes in the system transfer particle-hole pairs across the Fermi sea. They are described by (6.5) or (6.6) [or (6.9) in the one-dimensional case which is a special case of nesting]. This process will appear in the case of a nested Fermi surface even in the absence of singular matrix elements. We note here that the vanishing of the quasiparticle residue due to nesting was already obtained in numerical calculations in the two-dimensional Hubbard model close to half filling.[23]

Notice also that the processes in different hemispheres of the Fermi surface require an amount of energy $[|\vec{q}\cdot(\vec{v}_{\vec{k}_F}-\vec{v}_{\vec{k}'_F})|]$ which, for fixed momentum transfer, does not vanish because $\vec{v}_{\vec{k}_F}\sim-\vec{v}_{\vec{k}'_F}$. Therefore, the effects of processes will be kinematically suppressed at low energy and will not contribute to the physics of the states close to the Fermi surface. In contrast, in one dimension, these processes not only are not suppressed but are the origin of the non-Fermi-liquid behavior.

In the next section we diagonalize completely the Hamiltonian for the bosonized theory of this system. This Hamiltonian contains all four processes represented in Eqs. (6.3)–(6.6). It will become clear from our solution that in the thermodynamic limit and in the asymptotic low-energy regime, the anomalous dimensions generated by the processes which link opposite hemispheres of the Fermi surface effectively vanish. At the same time, and in the same limit, the effect of the remaining interaction turns out to become equivalent to a rotation of the bosonic states. We interpret this rotation as Landau's adiabatic principle.

Recently, by means of a perturbative renormalization group approach to Fermi liquids, Shankar[24] has shown explicitly that, apart from redundant interactions which merely change the shape of the Fermi surface, all interactions in the vicinity of a Fermi-liquid fixed point are actually marginally irrelevant (except for pairing processes and nesting). In other words, in normal conditions these processes will scale away in the thermodynamic limit in the bosonic form of the Hamiltonian. We will show this result explicitly in the next section with the exact diagonalization of the bosonic Hamiltonian. For these processes to become part of the bosonic Hamiltonian special physical interactions are needed. For instance, even in perturbation theory, these processes will become important if the matrix elements between the states is singular at low energies and long wavelengths.[9,25] It is clear that in this case the perturbative expansion fails. More importantly, it is possible to show that coupling a Fermi liquid to dynamical gauge fields generically leads to interactions of this type and, consequently, to a breakdown of Fermi-liquid theory.[26] We will study this problem using our methods in a forthcoming paper.[27]

VII. DIAGONALIZATION OF THE BOSONIC HAMILTONIAN

As it was explained in the preceding section, the Hamiltonian for interacting fermions close to the Fermi surface can be rewritten in bosonic form as

$$H = H_0 + \sum_{\vec{k}_F,\vec{k}_{F'},\vec{q}} \sqrt{\vec{q}\cdot\vec{v}_{\vec{k}_F}\vec{q}\cdot\vec{v}_{\vec{k}_{F'}} N_\Lambda(\vec{k}_F) N_\Lambda(\vec{k}_{F'})}\,U(q) \times[b^\dagger_{\vec{q}}(\vec{k}_F)b_{\vec{q}}(\vec{k}_{F'}) + b^\dagger_{\vec{q}}(-\vec{k}_F)b_{\vec{q}}(-\vec{k}_{F'}) + b_{\vec{q}}(\vec{k}_F)b_{\vec{q}}(-\vec{k}_{F'}) + b^\dagger_{\vec{q}}(-\vec{k}_{F'})b^\dagger_{\vec{q}}(\vec{k}_F)], \quad (7.1)$$

where the sum is restricted to vectors such that $\vec{q}\cdot\vec{v}_{\vec{k}_F} > 0$ and $\vec{q}\cdot\vec{v}_{\vec{k}_{F'}} > 0$. We also have used the definition (2.12) and (2.5) and changed the sums to be restricted to just one hemisphere of the Fermi surface. The free term H_0 is already defined in (5.3) and we assume that the Fermi surface is round and the interaction is isotropic (these assumptions are made in order to simplify the calculations, however, none of them are really important in what follows).

In order to define the problem unambiguously, we group the low-energy states in such a way that the Fermi sphere is divided into N patches[14] with area $S_{d-1}\Lambda^{d-1}$ such that the total area of the Fermi sphere is written as $S_d k_F^{d-1} = S_{d-1}\Lambda^{d-1}N$. This procedure constitutes a regularization of the theory of the fluctuations of the Fermi surface which, as such, contains a continuum of degrees of freedom. This discretization is in fact the operational definition of this problem. Inside each patch the operators are smeared following the prescription of Eq. (2.5). We will split the fluctuations into two subsets: (a) normal and (b) tangent to the Fermi surface. Since the operators are smeared inside each patch, we need to give a prescription for how the contributions both from inside one patch and from different patches are taken into account. Except for the fact that, for each patch, there is a narrow ray of directions out of the Fermi surface, the contributions from inside one patch have the same physics as one-dimensional systems, the only difference being that we have a finite density of states. Thus, inside each patch, we will integrate over normal fluctuations (which cost energy) whereas the effects of tangent fluctuations inside each patch will be replaced by an average over the patch. The dynamical effects associated with tangential fluctuations are due in fact to processes involving different nearby patches. At the end of our calculations we will take the limit in which the number of patches goes to infinity in such a way that the Fermi surface becomes smooth. This prescription gives us a controllable way to deal, at the same time, with both the physics of one-dimensional systems and with the extra phase space of higher dimensions.

In passing, we notice that, as soon as any physical dimension becomes finite, there is a natural discretization of the Fermi surface. This definition of the Hilbert space is, in fact, an alternative procedure to the method of covering the Fermi surface with patches, which is the one that we use in this work. Thus the limit of sending the number of patches to infinity (at fixed Fermi momentum) is, in this alternative procedure, the same as taking the thermodynamic limit in all directions. One example of this situation is a system with a finite number of chains N which are thermodynamically long. The results of this (and of the following) section imply that all anomalous dimensions scale as $1/N$ unless the effective two-dimensional (or three-dimensional) system has a nested Fermi surface. Thus, in the generic situation, the non-Fermi-liquid features at finite N disappear as $N\to\infty$ and should be viewed as finite size effects. This is an important issue in the analysis of numerical data of dense Fermi systems.

From this definition the local density of states at the Fermi points is simply $N_\Lambda(\vec{k}_F) = N(0)/N$ where the overall density of states is $N(0) = \frac{S_d k_F^{d-1}}{(2\pi)^d v_F}$. In this case the dimensionless coupling constant of the theory is given by

$$g(q) = N_\Lambda(\vec{k}_F)U(q) = \frac{N(0)U(q)}{N} = \frac{S_{d-1}\Lambda^{d-1}U(q)}{(2\pi)^d N}, \quad (7.2)$$

which vanishes in the thermodynamic limit for dimensions greater than one. This is the route for the stability of Fermi-liquid theory as already discussed by Haldane.[13–15,24]

From now on we consider the Fermi-liquid problem in two dimensions, partly because of its relevance to the cuprates and because of its general interest.[9,28] To each point j in the Fermi circle there is an angle $\theta_j = \frac{2\pi}{N} j$. Since the sum in (7.1) is restricted to $\vec{q} \cdot \vec{v}_{\vec{k}_F} > 0$, the allowed values of j are $-\frac{N}{4} + 1 \leq j \leq \frac{N}{4} - 1$. In the direction of $\vec{q}$ we have $j = 0$. In this notation we can rewrite

$$\vec{q} \cdot \vec{v}_{\vec{k}_j} = q v_F \cos\theta_j . \tag{7.3}$$

Moreover, we define the following notation for the bosonic operators:

$$\begin{aligned} b_{\vec{q}}(\vec{k}_j) &= b_j , \\ b_{\vec{q}}(-\vec{k}_j) &= a_j . \end{aligned} \tag{7.4}$$

Using the previous definitions it is easy to see that the Hamiltonian (7.1) can be rewritten in the form

$$H = \sum_{\vec{q}} q v_F \mathcal{H}(q), \tag{7.5}$$

where

$$\begin{aligned} \mathcal{H} = &\sum_i s_i \left(b_i^\dagger b_i + a_i^\dagger a_i \right) \\ &+ g \sum_{i,j} \sqrt{s_i s_j} \left(b_i^\dagger b_j + a_i^\dagger a_j + a_i b_j + b_i^\dagger a_j^\dagger \right), \end{aligned} \tag{7.6}$$

where $s_i = \cos\theta_i$. In the last expression we have dropped the index q since the original Hamiltonian is already diagonal in this index.

The Hamiltonian (7.6) describes a set of coupled harmonic oscillators. For a Fermi system in d space dimensions, the oscillators live on a $(d-1)$-dimensional manifold (plus time). We are interested in diagonalizing this Hamiltonian for finite N and to find the behavior of its eigenstates in the limit $N \to \infty$. We will refer to this limit as the thermodynamic limit of the system of oscillators. We should keep in mind that this limit does not necessarily coincide with the true thermodynamic limit of the Fermi system in d-dimensional space.

In order to diagonalize this Hamiltonian we define a generalized Bogoliubov transformation which mixes different points at the Fermi surface. We introduce two real orthogonal matrices $\mathcal{M}_{il}$ and $\mathcal{N}_{il}$ and two new bosonic operators β_l and α_l such that

$$\begin{aligned} b_i &= \sum_l \left(\mathcal{M}_{il} \beta_l + \mathcal{N}_{il} \alpha_l^\dagger \right), \\ a_i &= \sum_l \left(\mathcal{M}_{il} \alpha_l + \mathcal{N}_{il} \beta_l^\dagger \right). \end{aligned} \tag{7.7}$$

The commutation relation between the operators enforces that

$$\begin{aligned} \left[b_i, b_j^\dagger \right] &= \left[a_i, a_j^\dagger \right] = \delta_{i,j} = \sum_l \left(\mathcal{M}_{il} \mathcal{M}_{jl} - \mathcal{N}_{il} \mathcal{N}_{jl} \right), \\ [b_i, a_j] &= 0 = \sum_l \left(\mathcal{M}_{il} \mathcal{N}_{jl} - \mathcal{N}_{il} \mathcal{M}_{jl} \right), \end{aligned} \tag{7.8}$$

where the new bosonic operators obey the usual algebra, $\left[\beta_l, \beta_k^\dagger\right] = \left[\alpha_l, \alpha_k^\dagger\right] = \delta_{l,k}$ and $[\beta_l, \beta_k] = [\alpha_l, \alpha_k] = 0$.

It is assumed that these new operators diagonalize the problem completely, that is, the Hamiltonian is now written as

$$\mathcal{H} = \sum_l S_l \left(\beta_l^\dagger \beta_l + \alpha_l^\dagger \alpha_l \right), \tag{7.9}$$

where S_l is the eigenfrequency.

To find the equation which defines the above matrices we look at the commutation relation between the Hamiltonian and the operators. Using the Hamiltonian (7.6) and the definition (7.7) with the Hamiltonian (7.9) we find

$$\begin{aligned} [b_i, \mathcal{H}] &= s_i b_i + g \sum_j \sqrt{s_i s_j} \left(b_j + a_j^\dagger \right) \\ &= \sum_l S_l \left(\mathcal{M}_{il} \beta_l - \mathcal{N}_{il} \alpha_l^\dagger \right). \end{aligned} \tag{7.10}$$

Substituting (7.7) and taking the commutation relations with the operators α_l and β_l we find the equations

$$\begin{aligned} (S_l - s_i) \mathcal{M}_{il} &= g \sum_j \sqrt{s_i s_j} \left(\mathcal{M}_{jl} + \mathcal{N}_{jl} \right), \\ (S_l + s_i) \mathcal{N}_{il} &= -g \sum_j \sqrt{s_i s_j} \left(\mathcal{M}_{jl} + \mathcal{N}_{jl} \right). \end{aligned} \tag{7.11}$$

These equations define the Bogoliubov transformation.

Observe that, if $S_l \neq s_i$ for all l and i, then the solution would be written as

$$\begin{aligned} \mathcal{M}_{il} &= g \frac{\sqrt{s_i} C_l}{(S_l - s_i)}, \\ \mathcal{N}_{il} &= -g \frac{\sqrt{s_i} C_l}{(S_l + s_i)}, \end{aligned}$$

where

$$C_l = \sum_j \sqrt{s_j} \left(\mathcal{M}_{jl} + \mathcal{N}_{jl} \right)$$

and by direct substitution we find

$$1 = g \sum_j \frac{2 s_i^2}{S_l^2 - s_i^2},$$

which is the equation for the eigenenergies. This equation can be seen graphically as in Fig. 4. Notice that in principle we have two different kinds of solutions: the first one is related to the particle-hole continuum and it is defined for $S_l \leq 1$ and the second is related to the collec-

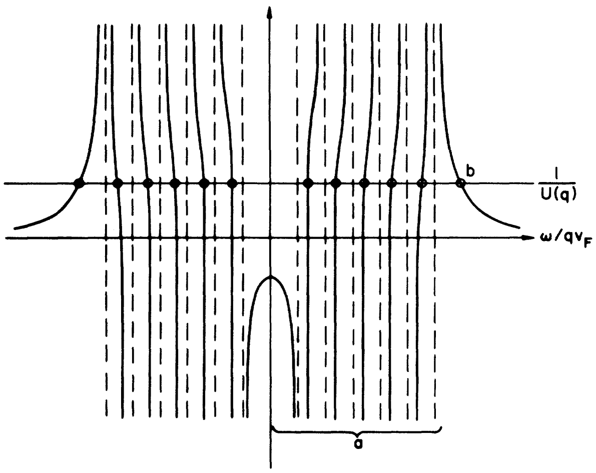

FIG. 4. Solution of Eq. (7.16). The dashed lines represent the eigenfrequencies of the modes for the noninteracting system, the circles show the eigenfrequencies for the interacting system. (a) Particle-hole continuum; (b) collective mode.

tive modes and is obtained for $S_l > 1$. All the solutions of the particle-hole continuum can be written in the form $S_l = \cos(\theta_l + \delta_l)$ where δ_l is a correction of order $\frac{1}{N}$ and they correspond to an angular shift of the positions of the points at the Fermi surface. In the thermodynamic limit (which is the limit of interest) they fill densely the Fermi surface and renormalize to the bare frequencies.[2] The collective modes cannot be written as a cosine of some angle. They detach from the particle-hole continuum. However, in principle, they are also present in the solution of the bosonic Hamiltonian. In the case of the Landau theory there is only one collective mode (the zero sound for neutral systems or the plasmon for charged systems) which is born at the direction of the momentum transfer $\vec{q}$, or in our notation, at $j = 0$. In some sense the Fermi-liquid problem resembles an impurity problem in which the states of the particle-hole continuum get only a phase shift and a bound state (the collective mode) appears in the spectrum. We therefore divide our solution in two cases as follows.

A. Particle-hole continuum $S_l = s_l$

Observe that in this case Eq. (7.11) can only be inverted if we dispose of the singularity which appears for $i = l$. This is done by writing (7.11) as

$$\mathcal{M}_{il} = Z_l g \sqrt{s_l} C_l \delta_{i,l} + \frac{g\sqrt{s_i} C_l}{s_l - s_i}, \qquad \mathcal{N}_{il} = -\frac{g\sqrt{s_i} C_l}{s_l + s_i}, \tag{7.12}$$

where

$$C_l = \sum_j \sqrt{s_j}\left(\mathcal{M}_{jl} + \mathcal{N}_{jl}\right). \tag{7.13}$$

The first term in the expression for $\mathcal{M}_{il}$ represents the solution for $i = l$ where we introduced an unknown factor Z_l which must be evaluated. The second term must be understood as the principal value of the function, that is, it vanishes for $i = l$. We must comment here that we also need to avoid the term $i = -l$ since $s_l = s_{-l}$. However, this only introduces simple modifications in the algebra and does not affect the content of the results. It is easy to show that with the introduction of these terms the final result will be given in terms of symmetric and antisymmetric combinations of the matrices that we obtain. We will come back to this point later in the paper.

By substituting the expression (7.12) in (7.13) we end up with an equation for Z_l which can be written as

$$Z_l = \frac{1}{g s_l}\left(1 - g\sum_j \frac{2 s_j^2}{s_l^2 - s_j^2}\right). \tag{7.14}$$

B. Collective mode $S_l = S_0$

In the case of the collective mode there is no divergence in Eq. (7.11) and the solution is just

$$\mathcal{M}_{i0} = \frac{g\sqrt{s_i} C_0}{S_0 - s_i}, \qquad \mathcal{N}_{i0} = -\frac{g\sqrt{s_i} C_0}{S_0 + s_i}, \tag{7.15}$$

where C_0 is defined as in (7.13). By substituting these equations there we find

$$1 = g\sum_j \frac{2 s_j^2}{S_0^2 - s_j^2}, \tag{7.16}$$

which defines S_0. The solution of this equation can be seen graphically in Fig. 4. For a finite number of points the solutions for the particle-hole continuum have a finite angular shift while the collective mode detaches from the continuum. In the thermodynamic limit the particle-hole continuum renormalizes to the bare frequencies while the collective mode gets a finite renormalization in the frequency.

The sum in (7.16) can be rewritten in a well-known form if we go back to our original notation and take the thermodynamic limit ($N \to \infty$). Using the definition of the beginning of the section we find

$$\begin{aligned} g\sum_j \frac{2 s_j^2}{s^2 - s_j^2} &= N(0)U(q)\int_{-\pi/2}^{\pi/2} \frac{d\theta}{2\pi}\frac{2\cos^2\theta}{s^2 - \cos^2\theta} \\ &= N(0)U(q)\int_0^{2\pi} \frac{d\theta}{2\pi}\frac{\cos\theta}{s - \cos\theta} \\ &= U(q)\Pi(s), \end{aligned} \tag{7.17}$$

where

$$\Pi\left(\frac{\omega}{q v_F}\right) = N(0)\int \frac{d\Omega}{S_d}\frac{\vec{q}\cdot\vec{v}_F}{\omega - \vec{q}\cdot\vec{v}_F} \tag{7.18}$$

is the RPA polarization function.[1] Therefore Eq. (7.16) is nothing but the equation for the collective modes, as expected. In two dimensions the form of the polarization function is given by[15]

$$\Pi(s) = N(0)\left(\frac{|s|}{\sqrt{s^2-1}}\Theta(|s|-1) + \frac{s}{i\sqrt{1-s^2}}\Theta(1-|s|) - 1\right), \qquad (7.19)$$

where we have introduced a small imaginary part in the denominator in (7.18). Notice that we can solve Eq. (7.16) for S_0 using (7.19) and (7.17). Using the real part of (7.19) (the principal value) one finds $S_0(q) = \frac{1+N(0)U(q)}{\sqrt{1+2N(0)U(q)}}$, which is always greater than 1.

The normalization coefficients C_l in (7.12) are already undefined. In order to calculate these coefficients we go back to Eqs. (7.8). Using the results (7.12) the first of those equations gives

$$\delta_{i,j} = \delta_{i,j} Z_i^2 C_i^2 g^2 s_i + g^2\sqrt{s_i s_j}\,(s_i+s_j)\left(\frac{Z_i C_i^2 - Z_j C_j^2}{s_i^2 - s_j^2} + \sum_l \frac{2C_l^2 s_l}{(s_l^2-s_j^2)(s_l^2-s_i^2)} + \frac{2C_0^2 S_0}{(S_0^2-s_j^2)(S_0^2-s_i^2)}\right), \qquad (7.20)$$

while the second gives

$$\frac{Z_j C_j^2 - Z_i C_i^2}{s_i^2 - s_j^2} = \sum_l \frac{2C_l^2 s_l}{(s_l^2-s_j^2)(s_l^2-s_i^2)} + \frac{2C_0^2 S_0}{(S_0^2-s_j^2)(S_0^2-s_i^2)}. \qquad (7.21)$$

By substitution of (7.21) in (7.20) we find finally that

$$C_i = \frac{1}{Z_i g\sqrt{s_i}}. \qquad (7.22)$$

We can further simplify considerably our results in the $N \to \infty$ limit. The only sum which appears in the final form of the matrices $\mathcal{M}_{il}$ and $\mathcal{N}_{il}$ is the one related to the polarization function (7.19). It is easy to show that

$$\sum_j \frac{2s_j^2}{s_l^2 - s_j^2} = \frac{N}{2\pi}\left(s_l \int_0^{2\pi} \frac{d\theta}{s_l - \cos\theta} - 2\pi\right) = -N, \qquad (7.23)$$

since the principal value of the integral in (7.23) vanishes for $|s_l| < 1$ [see Eq. (7.19)]. If we substitute these results in (7.12) we get

$$\mathcal{M}_{il} = \delta_{i,l} + \frac{1}{N}\,\frac{Ng}{1+Ng}\,\frac{\sqrt{s_i s_l}}{s_l - s_i},$$
$$\mathcal{N}_{il} = -\frac{1}{N}\,\frac{Ng}{1+Ng}\,\frac{\sqrt{s_i s_l}}{s_l + s_i}, \qquad (7.24)$$

which is the final form of the solution for the particle-hole continuum. Observe that $\mathcal{M}_{il}$ has a diagonal term of order zero in N^{-1}. If we take into account the fact that $s_l = s_{-l}$ it is easy to show that this term has the form $\mathcal{M}^0_{il} = \frac{1}{\sqrt{2}}[\delta_{i,l} + \mathrm{sgn}(i-l)\delta_{i,-l}]$. This term corresponds to a rotation of the identity matrix in the bosonic representation. Any calculation of physical quantities with this matrix would give the same result as for the noninteracting system. In terms of the Landau theory the rotation of the bosonic eigenfunctions is essentially Landau's adiabatic principle which states that the states of the interacting system and the noninteracting system can be linked adiabatically.[1,2] The next term in $\mathcal{M}_{il}$ is order N^{-1} and represents the dressing of the particle by the interaction, that is, the formation of the polarization cloud. Although this is an effect of order N^{-1} in the matrix $\mathcal{M}_{il}$ it may be appreciable, since the number of points of the Fermi surface is N. We will show that this is the case for the calculation of the quasiparticle residue. Furthermore, observe that the matrix $\mathcal{N}_{il}$ has contributions of order N^{-1} only and therefore this matrix is null for the noninteracting system in the thermodynamic limit. Moreover, the matrix $\mathcal{N}_{il}$ carries no resonant terms in the denominator of (7.24). This can be understood easily from the discussion of the preceding section. From definition (7.7) we observe that while $\mathcal{M}_{il}$ links points on the same hemisphere of the Fermi surface the matrix $\mathcal{N}_{il}$ links points in opposite hemispheres. What we are showing here is that, since we choose a regular interaction, in the thermodynamic limit the contribution from opposite points of the Fermi surface vanishes completely. All the physics of the Fermi liquids is defined in just one hemisphere of the Fermi surface. We will show in the last section of this paper that in one dimension this is not the case because the Fermi surface has a finite number of points.

Another interesting feature of the solution (7.24) is its dependence on the interaction. Observe that in terms of our original variables the coefficient of the correction N^{-1} in the matrices can be written as $\frac{N(0)U(q)}{1+N(0)U(q)}$. Thus, if we have a long-range interaction, a Coulomb interaction for instance, this term is always finite and we can replace it for an effective interaction which is local at long wavelengths. In the next section we show that this term is closely related to the quasiparticle residue. Moreover, as we will show in Sec. IX, this term represents the screening of long-range interactions which arises naturally in the bosonic formalism.

VIII. THE FERMION PROPAGATOR FOR A FERMI LIQUID IN TWO DIMENSIONS

The fermion propagator can be evaluated by using Eq. (3.3) and the Bogoliubov transformation (7.7). It is a tedious, but trivial algebra to show that the fermion propagator for two points, i and j, at the same hemisphere of the Fermi surface can be written as

$$\langle i,\vec{r},t|j,0\rangle=\exp\left\{\sum_{\vec{q},l}\frac{1}{2N(0)Vqv_F}\left(\frac{\mathcal{M}_{il}\mathcal{M}_{il}+\mathcal{N}_{il}\mathcal{N}_{il}}{s_i}+\frac{\mathcal{M}_{jl}\mathcal{M}_{jl}+\mathcal{N}_{jl}\mathcal{N}_{jl}}{s_j}\right.\right.$$
$$\left.\left.-2\frac{\mathcal{M}_{il}\mathcal{M}_{jl}e^{i(qv_FS_lt-\vec{q}\cdot\vec{r})}+\mathcal{N}_{il}\mathcal{N}_{jl}e^{i(qv_FS_lt+\vec{q}\cdot\vec{r})}}{\sqrt{s_is_j}}\right)\right\}, \tag{8.1}$$

while for two points in opposite sides of the Fermi surface we find

$$\langle i,\vec{r},t|j,0\rangle=\exp\left\{\sum_{\vec{q},l}\frac{1}{2N(0)Vqv_F}\left(\frac{\mathcal{M}_{il}\mathcal{M}_{il}+\mathcal{N}_{il}\mathcal{N}_{il}}{s_i}+\frac{\mathcal{M}_{jl}\mathcal{M}_{jl}+\mathcal{N}_{jl}\mathcal{N}_{jl}}{s_j}\right.\right.$$
$$\left.\left.-2\frac{\mathcal{M}_{il}\mathcal{N}_{jl}e^{i(qv_FS_lt-\vec{q}\cdot\vec{r})}+\mathcal{N}_{il}\mathcal{M}_{jl}e^{i(qv_FS_lt+\vec{q}\cdot\vec{r})}}{\sqrt{s_is_j}}\right)\right\}. \tag{8.2}$$

Therefore our task is to calculate the sums in these two expressions. If we use the result (7.7) we find

$$\sum_l\mathcal{M}_{il}\mathcal{M}_{jl}e^{iv_FqS_lt}=\delta_{i,j}e^{iv_Fqs_it}+\frac{g}{1+Ng}\frac{\sqrt{s_is_j}}{s_i-s_j}\left(e^{is_iv_Fqt}-e^{is_jv_Fqt}\right)$$
$$+\left(\frac{g}{1+Ng}\right)^2\sqrt{s_is_j}\sum_l\frac{s_le^{iv_Fqs_lt}}{(s_l-s_i)\,(s_l-s_j)}. \tag{8.3}$$

Notice that the last term in the above expression has a double singularity at $s_l=s_i$ and $s_l=s_j$. We can extract this singularity using Poincaré's theorem,

$$\mathcal{P}\frac{1}{(s_l-s_i)\,(s_l-s_j)}=\mathcal{P}\frac{1}{s_i-s_j}\left(\frac{1}{s_l-s_i}-\frac{1}{s_l-s_j}\right)+\pi^2\delta\,(s_l-s_i)\,\delta\,(s_l-s_j)\,. \tag{8.4}$$

The double δ function can be rewritten in terms of the angles at the Fermi surface due to the definitions in (7.6). Indeed, it is trivial to show that

$$\delta\,(s_l-s_i)=\delta\,(\cos\theta_l-\cos\theta_i)=\frac{N}{2\pi}\frac{\delta_{i,l}}{|\sin\theta_l|}=\frac{N}{2\pi}\frac{\delta_{i,l}}{\sqrt{1-s_l^2}}, \tag{8.5}$$

where we have used $\theta_l=\frac{2\pi l}{N}$. If we substitute (8.4) in (8.3) we finally find

$$\sum_l\mathcal{M}_{il}\mathcal{M}_{jl}e^{iv_FqS_lt}=\left[1+\frac{1}{4}\left(\frac{gN}{1+Ng}\right)^2\frac{s_i^2}{1-s_i^2}\right]\delta_{i,j}e^{iv_Fqs_it}+\mathcal{O}\left(\frac{1}{N}\right), \tag{8.6}$$

where we have included only the terms of order zero in N^{-1}, that is, the only terms that survive in the $N\to\infty$ limit. By the same token we can show that

$$\sum_l\mathcal{N}_{il}\mathcal{N}_{jl}e^{iv_FqS_lt}=\mathcal{O}\left(\frac{1}{N}\right). \tag{8.7}$$

As we have explained before, this result shows that processes which involve particle-hole pairs at opposite sides of the Fermi surface disappear in the $N\to\infty$ limit.

If we substitute (8.6) and (8.7) in the expression for the Green's function (8.1) we end up with the expression

$$\langle i,\vec{r},t|j,0\rangle=\langle i,\vec{r},t|j,0\rangle_0\ \exp\left\{-\sum_{\vec{q}}\frac{1}{4N(0)v_FV}\left(\frac{N(0)U(q)}{1+N(0)U(q)}\right)^2\frac{q_N}{q_T^2}\left(1-e^{i(v_Fq_Nt-\vec{q}\cdot\vec{r})}\right)\right\}, \tag{8.8}$$

where $\langle i,\vec{r},t|j,0\rangle_0$ is the propagator for the noninteracting system calculated in (5.15). Observe again the form of the interaction coefficient which appears in the *screening* form as we have pointed out in previous sections. That is, long-range potentials will not change the quasiparticle residue because at long wavelengths the potential acting on the fermions is purely local.

Equation (8.8) gives the behavior of the fermion propagator in the asymptotic low-energy limit. We have only kept effects which survive the limit $N \to \infty$. In Fermi liquids, there are a variety of quantities of physical interest, such as quasiparticle lifetimes, which vanish faster than $1/N$ as required by Fermi-liquid theory. Such effects can be calculated with the method of bosonization but do not appear at this order in $1/N$. Recently, Houghton and Marston,[14] have used bosonization methods to compute the quasiparticle lifetime.

In what follows, and in order to simplify the calculation, we consider the case of local interactions from the beginning, $U(q) = U$. Nevertheless, we must keep in mind that screening is a natural consequence of bosonization in dimensions higher than one. Applying the same methods we have used in Secs. V and VII, we make the replacement

$$\int \frac{dq_T}{q_T^2} \to \frac{N(0)v_F}{\Lambda^2}, \tag{8.9}$$

and we are left with the integral

$$\int_0^\infty dq_N\, q_N\, e^{-\frac{q_N}{\Lambda}} \left(1 - e^{iq_N(v_F t - \vec{n}_j \cdot \vec{r})}\right)$$

$$= \Lambda^2 - \frac{\Lambda^2}{[1 - i\Lambda\,(v_F t - \vec{n}_j \cdot \vec{r})]^2} \approx \Lambda^2, \tag{8.10}$$

where we have disregarded the last term in (8.10) since we are looking at the asymptotic state of the system.

Substituting (8.10) and (8.9) in (8.8) we find

$$\langle i, \vec{r}, t | j, 0\rangle = Z_F \langle i, \vec{r}, t | j, 0\rangle_0, \tag{8.11}$$

where

$$Z_F = \exp\left\{-\frac{1}{4}\left(\frac{N(0)U}{1+N(0)U}\right)^2\right\} \tag{8.12}$$

is the quasiparticle residue. Observe that this form of the quasiparticle residue has clear nonperturbative character. In the limit of weak coupling, that is, $N(0)U \to 0$, we find

$$Z_F \approx 1 - \frac{[N(0)U]^2}{4}, \tag{8.13}$$

which is the expected form of the RPA solution for local interactions.[5,29] In the strong coupling limit, $N(0)U \to \infty$, it is easy to show that

$$Z_F \approx 0.78\left(1 + \frac{1}{2N(0)U}\right), \tag{8.14}$$

which, once more, has a nonperturbative character. Observe that the quasiparticle residue never vanishes and it actually saturates. That is, the Fermi-liquid state survives for any strength of the interaction. Also observe that the propagator for opposite hemispheres of the Fermi surface, (8.2), vanishes in the $N \to \infty$ limit.

We want to emphasize that the result (8.14) is not completely universal. The numerical factor in front of Eq. (8.14) can change due to geometric factors. We have used Haldane's construction[13] of a sphere with radius Λ at each point of the Fermi surface. In this construction the Fermi surface is locally flat. If we use another different construction, such as in Ref. 14, we would get a different numerical factor. This factor is irrelevant and leads to a small renormalization of the quasiparticle residue. But this is not the important point here, the most striking result is the survivor of the Fermi-liquid state for any strength of the interaction potential in the case of isotropic, local, interactions.

IX. SCREENING OF A SCALAR POTENTIAL

In the preceding section we have shown that screening appears naturally in the exact solution of the Landau fixed point if we use bosonization. In this section we reconsider the problem of screening of external probes by the interacting Fermi system in terms of the bosonized theory. This is a well-understood problem in the framework of Fermi-liquid theory and the results of our bosonized theory naturally agree with the Fermi-liquid theory results.

In this section we examine the behavior of the system under a scalar external field whose external Hamiltonian is written as

$$H_{\text{ext}} = \frac{1}{V}\sum_{\vec{q}} V_{\text{ext}}(q,t)\rho(\vec{q}), \tag{9.1}$$

where $\rho(\vec{q})$ is the Fourier component of the charge density of the system which is written in terms of the operators (2.1) as

$$\rho(\vec{q}) = \sum_{\vec{k}} n_{\vec{q}}(\vec{k}). \tag{9.2}$$

Therefore we consider external Hamiltonians of the form

$$H_{\text{ext}} = \frac{1}{V}\sum_{\vec{q},\vec{k}} V_{\text{ext}}(q,t) n_{\vec{q}}(\vec{k}), \tag{9.3}$$

which, in terms of bosons, is written as

$$H_{\text{ext}} = \sum_{\vec{q},\vec{k}_F,\vec{v}_{\vec{k}_F}\cdot\vec{q}>0} \left(\frac{N_\Lambda(\vec{k}_F)|\vec{q}\cdot\vec{v}_{\vec{k}_F}|}{V}\right)^{1/2} \times V_{\text{ext}}(q,t)[b^\dagger_{\vec{q}}(\vec{k}_F) + b_{\vec{q}}(\vec{k}_F)]. \tag{9.4}$$

It will be useful later to observe that the electronic density is given in terms of the bosons as

$$\rho(\vec{q}) = \sum_{\vec{k}_F,\vec{v}_{\vec{k}_F}\cdot\vec{q}>0} [N_\Lambda(\vec{k}_F)V|\vec{q}\cdot\vec{v}_{\vec{k}_F}|]^{1/2} \times[b^\dagger_{\vec{q}}(-\vec{k}_F) + b_{\vec{q}}(\vec{k}_F)]. \tag{9.5}$$

Here we consider the effect of the potential (9.4) in a Fermi liquid in order to see what kind of effect it can cause. The Hamiltonian of the system in this case is given by $H = H_{FL} + H_{\text{ext}}$ where H_{FL} is given in (7.1).

The equation of motion for the fields is given by the saddle point equation for the action in (3.8),

$$i\frac{\partial\phi_{\vec{q}}(\vec{k}_F)}{\partial t} = -\vec{q}\cdot\vec{v}_{\vec{k}_F}\phi_{\vec{q}}(\vec{k}_F) - U(q)\sum_{\vec{k}'_F}[|\vec{q}\cdot\vec{v}_{\vec{k}_F}||\vec{q}\cdot\vec{v}_{\vec{k}'_F}|N_\Lambda(\vec{k}_F)N_\Lambda(\vec{k}'_F)]^{1/2}[\phi_{\vec{q}}(\vec{k}'_F)+\phi^*_{\vec{q}}(-\vec{k}'_F)] - \left(\frac{|\vec{q}\cdot\vec{v}_{\vec{k}_F}|N_\Lambda(\vec{k}_F)}{V}\right)^{1/2} V_{\rm ext}(q,t), \quad (9.6)$$

which, after a Fourier transform, gives

$$[|\vec{q}\cdot\vec{v}_{\vec{k}_F}|N_\Lambda(\vec{k}_F)V]^{1/2}\phi_{\vec{q}}(\vec{k}_F,\omega) = \frac{\vec{q}\cdot\vec{v}_{\vec{k}_F}}{\omega-\vec{q}\cdot\vec{v}_{\vec{k}_F}}N_\Lambda(\vec{k}_F)\left\{V_{\rm ext}(q,\omega)+U(q)\sum_{\vec{k}'_F}[|\vec{q}\cdot\vec{v}_{\vec{k}'_F}|N_\Lambda(\vec{k}'_F)V]^{1/2}[\phi_{\vec{q}}(\vec{k}'_F,\omega)+\phi^*_{\vec{q}}(-\vec{k}'_F,\omega)]\right\}. \quad (9.7)$$

Summing over $\vec{k}_F$ on both sides of (9.7) and using (9.5) we find

$$\langle\rho(\vec{q},\omega)\rangle = \Pi(\vec{q},\omega)\left[V_{\rm ext}(q,\omega)+U(q)\langle\rho(\vec{q},\omega)\rangle\right], \quad (9.8)$$

where

$$\Pi(\vec{q},\omega) = \sum_{\vec{k}_F}N_\Lambda(\vec{k}_F)\frac{\vec{q}\cdot\vec{v}_{\vec{k}_F}}{\omega-\vec{q}\cdot\vec{v}_{\vec{k}_F}} \quad (9.9)$$

is the polarization function we have found in (7.18).

The interpretation for (9.8) is very simple: the external potential produces a polarization of the fermionic gas which shields the interaction at long distances. Therefore, instead of the bare external potential $V_{\rm ext}(q,\omega)$, a new polarization potential appears, $U(q)\langle\rho(\vec{q},\omega)\rangle$, and the effective potential which is felt by the fermions is given by

$$U_{\rm eff}(\vec{q},\omega) = V_{\rm ext}(q,\omega)+U(q)\langle\rho(\vec{q},\omega)\rangle. \quad (9.10)$$

The effective potential can be obtained from the bare potential using (9.8) and (9.10). Solving (9.8) for the density one gets

$$\langle\rho(\vec{q},\omega)\rangle = \frac{\Pi(\vec{q},\omega)V_{\rm ext}(q,\omega)}{1-U(q)\Pi(\vec{q},\omega)}. \quad (9.11)$$

Substituting (9.11) in (9.10) we finally find

$$U_{\rm eff}(\vec{q},\omega) = \frac{V_{\rm ext}(q,\omega)}{\epsilon(\vec{q},\omega)}, \quad (9.12)$$

where $\epsilon(\vec{q},\omega)$ is the dielectric function of the fermionic system which is given by

$$\epsilon(\vec{q},\omega) = 1-U(q)\Pi(\vec{q},\omega). \quad (9.13)$$

The expressions (9.12) and (9.13) are the RPA results for the electronic gas.[1–5] Since we are dealing with the exact diagonalization of the fermionic system at long wavelengths, it is natural to recover the RPA approximation as the exact result in this limit since, as is well known, its long wavelength limit saturates the sum rules.

In the limit of interest, namely, low frequency and small momenta, the polarization function can be easily obtained. From (7.18) we have

$$\Pi(\vec{q},\omega) = \frac{N(0)}{S_d}\int d\Omega\frac{\cos\theta}{\frac{\omega}{v_Fq}-\cos\theta+i\eta}, \quad (9.14)$$

where we have included a small imaginary part for the frequency, Ω is the solid angle, and θ is the angle between the Fermi velocity and $\vec{q}$. Observe that the polarization function is only a function of $\frac{\omega}{v_Fq}$. We can also rewrite (9.14) as

$$\Pi(\vec{q},\omega) = N(0)\left[\mathcal{P}\int\frac{d\Omega}{S_d}\frac{\cos\theta}{\frac{\omega}{v_Fq}-\cos\theta} - i\pi\int\frac{d\Omega}{S_d}\cos\theta\,\delta\left(\frac{\omega}{v_Fq}-\cos\theta\right)\right], \quad (9.15)$$

where $\mathcal{P}$ means principal value.

In the limit of $\frac{\omega}{v_Fq}\ll 1$ we obtain

$$\Pi(\vec{q},\omega) = -N(0)\left(1+i\pi\gamma_d\frac{\omega}{v_Fq}\right)+\mathcal{O}\left(\frac{\omega^2}{v_F^2q^2}\right), \quad (9.16)$$

where γ_d is a numerical factor which depends on the spatial dimensionality d of the system ($\gamma_1 = 0, \gamma_2 = 1/\pi, \gamma_3 = 1/2$). The second term in the right-hand side of (9.16) is the Landau damping term due to the decay of the bosonic modes into the particle-hole continuum. Observe that in one-dimensional systems the bosonic modes never decay due to the fact that $\gamma_1 = 0$. It means that the true excitations of the one-dimensional system are collective modes. In higher dimensions this is not true because there is always a particle-hole continuum. In Sec. XI we will see that this result has important consequences for the calculation of the Green's function.

At this point it is tempting to follow the conventional approximation of Fermi-liquid theory and to argue that the fermion-fermion interaction should also get screened just as much as an external probe is. However, this is a delicate argument which is only justified by its success in explaining experiments. One should keep in mind that this is essentially a perturbative argument, motivated by a partial resummation of the perturbation theory series (namely, the sum of all the bubble diagrams or RPA). In earlier sections of this paper, we showed that since the tangential fluctuations do not change the energy, they mainly give rise to a density of states. In the conventional screening first argument of Fermi-liquid theory (in the RPA approximation[3]) one finds an effective electron-electron interaction of the form

$$U_{\text{eff}}(\vec{q},\omega) = \frac{1}{U^{-1}(q) - \Pi(\vec{q},\omega)}. \tag{9.17}$$

If we substitute the asymptotic form of the polarization Π the RPA expression for the effective potential becomes

$$U_{\text{eff}}(\vec{q},\omega) \approx \frac{U(q)}{1+U(q)N(0)}. \tag{9.18}$$

This expression shows that, within RPA, screening is an effect caused by the density of states of the Fermi system. Notice, however, that this is exactly the same expression that appears in Eq. (8.8) when we diagonalized exactly the bosonic problem. Thus bosonization in dimensions higher than one produces screening naturally and we do not have to put it in by hand as in the usual approach in condensed matter physics. Furthermore, notice that our arguments in this section do not depend on the dimensionality of the system, therefore, we conclude that in any number of dimensions the screening of an external potential has RPA character. However, as we will show in the last section of this paper, in one dimension the fermion-fermion interaction is *not* screened. In particular, this result leads to the vanishing of the Green's function for the Coulomb potential in one dimension. Once more, if we have screened the potential by hand we would get a wrong result.

X. COUPLING TO GAUGE FIELDS

In the previous two sections we showed how screening arises in bosonization of fermionic systems with long-range scalar interactions. The reason for that is that the fermionic system resembles a liquid which can sustain longitudinal oscillations. However, if transverse oscillations are present the physics of the system changes completely.

Let us consider a system of fermions in $D > 1$ space dimensions. In one space dimension, all gauge fields are purely longitudinal and, hence, the interactions they mediate are equivalent to Coulomb-like interactions of the form discussed in the preceding section. Suppose, exactly as in the preceding section, that we couple the fermionic system via minimal coupling with an external vector field. The Hamiltonian can be written as[30]

$$H = H_0 + H_{F\text{-}G}. \tag{10.1}$$

H_0 is the noninteracting Hamiltonian and

$$H_{F\text{-}G} = g\sum_{\vec{q},\vec{k}} \vec{v}_{\vec{k}-\frac{\vec{q}}{2}} \cdot \vec{A}_{-\vec{q}} n_{\vec{q}}(\vec{k}) + \frac{g^2}{2m^*}\sum_{\vec{q},\vec{q}',\vec{k}} \vec{A}_{\vec{q}}\cdot\vec{A}_{\vec{q}'} n_{\vec{q}+\vec{q}'}(\vec{k}) + H_G \tag{10.2}$$

is the fermion-gauge part of the Hamiltonian where m^* is the mass of the fermions. The first and the second terms on the right-hand side of (10.2) give the coupling between the gauge fields and the fermionic system (g is the coupling constant of the theory). The last term on the right-hand side of (10.2) is a pure gauge field term usually describing the energy of the gauge field or possible external current terms which generate the field in the system.

Since we are dealing with small momentum transfer only, we notice that the dominant contribution from the second term comes from $\vec{q} = -\vec{q}'$ (the terms with $\vec{q}=0$ or $\vec{q}' = 0$ are not present in order to preserve charge neutrality in the system). Using $\sum_{\vec{k}} n_0(\vec{k}) = N_f$ where N_f is the number of fermions we can rewrite (10.2) in terms of the boson operators approximately as

$$H_{F\text{-}G} \approx g\sum_{\vec{q},\vec{k}_F,\vec{v}_{\vec{k}_F}\cdot\vec{q}>0} [|\vec{q}\cdot\vec{v}_{\vec{k}_F}|N_\Lambda(\vec{k}_F)V]^{1/2} \times \vec{v}_{\vec{k}_F-\frac{\vec{q}}{2}}\cdot[\vec{A}_{-\vec{q}} b^\dagger_{\vec{q}}(\vec{k}_F) + \vec{A}_{\vec{q}} b_{\vec{q}}(\vec{k}_F)] + \frac{g^2 N_f}{2m^*}\sum_{\vec{q}} |\vec{A}_{\vec{q}}|^2 + H_G. \tag{10.3}$$

The generating functional is obtained exactly as before and the equations of motion for the bosonic fields are simply

$$i\frac{\partial\phi_{\vec{q}}(\vec{k}_F)}{\partial t} = -\vec{q}\cdot\vec{v}_{\vec{k}_F}\phi_{\vec{q}}(\vec{k}_F) - g[|\vec{q}\cdot\vec{v}_{\vec{k}_F}|N_\Lambda(\vec{k}_F)V]^{1/2}\vec{v}_{\vec{k}_F-\frac{\vec{q}}{2}}\cdot\vec{A}_{\vec{q}} \tag{10.4}$$

and for the gauge fields we find

$$\sum_j \left[D_0^{-1}\left(\frac{\partial}{\partial t},\vec{q}\right)\right]_{i,j}\left[\vec{A}_{\vec{q}}(t)\right]_j = g\sum_{\vec{k}_F}\left[\vec{v}_{\vec{k}_F-\frac{\vec{q}}{2}}\right]_i\left(\frac{|\vec{q}\cdot\vec{v}_{\vec{k}_F}|N_\Lambda(\vec{k}_F)}{V}\right)^{1/2}\phi_{\vec{q}}(\vec{k}_F,t) + \frac{g^2N_f}{m^*V}\left[\vec{A}_{\vec{q}}(t)\right]_i + \left[\vec{J}_{\text{ext}}(\vec{r},t)\right]_i, \tag{10.5}$$

where $D_0^{-1}\left(\frac{\partial}{\partial t},\vec{q}\right)$ is the bare propagator for the gauge field (which in general is a tensor with spatial components $i,j=1,2,\ldots,d$ where d is the number of spatial dimensions) and $\vec{J}_{\text{ext}}(\vec{r},t)$ is some external current. Equations (10.4) and (10.5) are coupled. They can be solved by a Fourier transform. Solving for (10.4) we find

$$\phi_{\vec{q}}(\vec{k}_F,\omega) = g\frac{[|\vec{q}\cdot\vec{v}_{\vec{k}_F}|N_\Lambda(\vec{k}_F)V]^{1/2}}{\omega-\vec{q}\cdot\vec{v}_{\vec{k}_F}}\vec{v}_{\vec{k}_F-\frac{\vec{q}}{2}}\cdot\vec{A}_{\vec{q}}(\omega). \tag{10.6}$$

Substituting this result in (10.5) one finds

$$\sum_j \left[D_0^{-1}(\omega,\vec{q})\right]_{i,j}\left[\vec{A}_{\vec{q}}(\omega)\right]_j = g^2\sum_{\vec{k}_F,l}N_\Lambda(\vec{k}_F)\frac{\vec{q}\cdot\vec{v}_{\vec{k}_F}}{\omega-\vec{q}\cdot\vec{v}_{\vec{k}_F}}\left[\vec{v}_{\vec{k}_F-\frac{\vec{q}}{2}}\right]_i\left[\vec{v}_{\vec{k}_F-\frac{\vec{q}}{2}}\right]_l\left[\vec{A}_{\vec{q}}(\omega)\right]_l + \frac{g^2 n_f}{m^*}\left[\vec{A}_{\vec{q}}(\omega)\right]_i + \left[\vec{J}_{\rm ext}(\vec{q},\omega)\right]_i, \tag{10.7}$$

where $n_f = N_f/V$ is the density of fermions. Equation (10.7) can be rewritten in a more appealing form as

$$\sum_j \left[D^{-1}(\omega,\vec{q})\right]_{i,j}\left[\vec{A}_{\vec{q}}(\omega)\right]_j = \left[\vec{J}_{\rm ext}(\vec{q},\omega)\right]_i, \tag{10.8}$$

where

$$\left[D^{-1}(\omega,\vec{q})\right]_{i,j} = \left[D_0^{-1}(\omega,\vec{q})\right]_{i,j} - g^2\left(N(0)v_F^2\delta_{i,j} + \sum_{\vec{k}_F}N_\Lambda(\vec{k}_F)\frac{\vec{q}\cdot\vec{v}_{\vec{k}_F}}{\omega-\vec{q}\cdot\vec{v}_{\vec{k}_F}}\left[\vec{v}_{\vec{k}_F-\frac{\vec{q}}{2}}\right]_i\left[\vec{v}_{\vec{k}_F-\frac{\vec{q}}{2}}\right]_j\right) \tag{10.9}$$

is the effective propagator for the gauge fields.[2] We have used $m^* = n_F/[v_F^2 N(0)]$. In general it is very difficult to calculate the correction to the bare propagator. However, for an isotropic system, due to the symmetry, only the diagonal terms survive. For long wavelengths ($q \ll k_F$) and small frequencies ($\omega \ll v_F k_F$) one finds

$$\left[D^{-1}(\omega,\vec{q})\right]_{i,j} = \left[D_0^{-1}(\omega,\vec{q})\right]_{i,j} - v_F^2 g^2\left[N(0)+\Pi(\vec{q},\omega)\right]\delta_{i,j}, \tag{10.10}$$

where $\Pi(\vec{q},\omega)$ is defined in (7.18).

In the limit of interest, namely, $\frac{\omega}{v_F q} \ll 1$, we can use the result (10.16) and rewrite the renormalized propagator as

$$\left[D^{-1}(\omega,\vec{q})\right]_{i,j} = \left[D_0^{-1}(\omega,\vec{q})\right]_{i,j} + i\omega_{pG}^2\frac{\omega}{v_F q}\delta_{i,j}, \tag{10.11}$$

where

$$\omega_{pG}^2 = \frac{\pi g^2 n_f\gamma_d}{m^*} = \pi\gamma_d g^2 N(0)v_F^2 \tag{10.12}$$

is the plasma frequency associated with the oscillations of the electronic system due to the coupling to gauge field.

Observe that, contrary to the scalar case of Sec. IX, the zero frequency gauge fields are not affected by fluctuations of the Fermi system, that is,

$$\left[D(0,\vec{q})\right]_{i,j} = \left[D_0(0,\vec{q})\right]_{i,j}. \tag{10.13}$$

The cancellation of the density of states term in Eq. (10.11) implies that the only effect of electron correlations at low frequencies and small wave vectors is a damping (not screening) of the transverse gauge fields. This is the well-known phenomenon of Landau damping of transverse gauge fields in metals. A noncanceling density of states would imply a gap in the spectrum of fluctuations of the transverse gauge fields and the expulsion of static gauge fields, namely, a Meissner effect. This is what happens in a superconducting state.

Let us assume, for the sake of the argument, that the bare propagator has the form

$$\left[D_0(\omega,\vec{q})\right]_{i,j} = \frac{1}{\omega^2 - v_G^2 q^2}\left[\bar{D}_0(\omega,\vec{q})\right]_{i,j}, \tag{10.14}$$

where v_G is the velocity of propagation of the gauge fields. $[\bar{D}_0(\omega,\vec{q})]_{i,j}$ is a tensor whose form depends on the choice of gauge and it has an analytic dependence in ω and $\vec{q}$. The renormalized propagator has the form (up to analytic structure tensors)

$$D(\omega,\vec{q}) = \frac{1}{\omega^2 - v_G^2 q^2 + i\omega_{pG}^2\frac{\omega}{v_F q}}. \tag{10.15}$$

The Landau damping introduces a new physical scale in the problem. Observe that the interaction is screened with a frequency dependent screening length of order

$$l_s(\omega) \sim \left(\frac{v_G^2 v_F}{\omega_{pG}^2}\right)^{1/3}\omega^{-1/3}, \tag{10.16}$$

which defines a new scale in the problem. In the strong coupling limit, that is, $g \to \infty$, the plasma frequency is much larger than the characteristic frequencies of the system ($\omega_{pG} \gg \omega$). In this limit the characteristic momentum of the system will be cut off by the plasma frequency, $\frac{\omega_{pG}}{\sqrt{2v_F v_G}} > q$, and the asymptotic form of the propagator is dominated by the Landau term, namely,

$$D(\omega,\vec{q}) = \frac{-1}{v_G^2 q^2 - i\omega_{pG}^2\frac{\omega}{v_F q}}. \tag{10.17}$$

In real space and time this form of the propagator implies that the gauge fields behave diffusively. The same type of propagator is found in the RPA approach.[2,30]

XI. THE ONE-DIMENSIONAL CASE

In the previous sections we have shown that in terms of response functions bosonization gives the same result as the RPA approximation. This is due to the fact that RPA fulfills all the sum rules at long wavelengths and low energies, which is exactly the limit where bosonization can be applied. Moreover, RPA is valid for high densities which in our language means that k_F is large compared with the fluctuations in the system. This would lead us to conclude, erroneously, that RPA and bosonization are one and the same thing. Actually, the RPA results are expected from the bosonization point of view. We have already shown that bosonization gives rise to screening naturally, something that you have to do by hand in the RPA approach. Moreover, RPA is only valid in the weak coupling limit, $N(0)U \ll 1$,[2] and, as we have shown before, this is not the case of bosonization. It is indeed well known that in one dimension RPA works fine for correlation functions while it cannot explain the absence of isolated singularities in the Green's function. In this section we try to explain the reason for this behavior by comparison with our results in higher dimensions.

In one dimension all the calculations simplify enormously. The matrices $\mathcal{M}_{il}$ and $\mathcal{N}_{il}$ reduce to numbers. The condition (7.8) can be rewritten in terms of a variable $\zeta(q)$ such that

$$\mathcal{M}(q) = \cosh\zeta(q), \qquad \mathcal{N}(q) = \sinh\zeta(q). \tag{11.1}$$

The other condition in (7.8) is automatically fulfilled. In the one-dimensional case there is no particle-hole continuum since the Fermi surface reduces to two points, that is, $s_1 = 1$. The collective mode equation (7.16) defines the eigenvalue for the collective mode,

$$g(q)\frac{2}{S_0^2(q)-1} = 1, \tag{11.2}$$

which is easily solved as

$$S_0(q) = \sqrt{1+2g(q)} = \sqrt{1+N(0)U(q)}, \tag{11.3}$$

where we used (7.2) with $N = 2$. The last expression can be put in a more standard form if we use the notation of Sec. VII and rewrite the frequency of oscillation of the collective mode as

$$E_q = v_F q\sqrt{1+N(0)U(q)}, \tag{11.4}$$

which is the well-known result for one-dimensional systems.[31] The variable $\zeta(q)$ is defined by the solution (7.15), that is,

$$\tanh[\zeta(q)] = \frac{N(q)}{M(q)} = -\frac{S_0-1}{S_0+1} = -\frac{\sqrt{1+2g(q)}-1}{\sqrt{1+2g(q)}+1}, \tag{11.5}$$

which can be rewritten in a more standard form,

$$\tanh[2\zeta(q)] = \frac{g(q)}{1+g(q)}, \tag{11.6}$$

which is the expected result.[31]

Observe that $\mathcal{N}$ is finite, contrary to higher dimensions. This is a result of the finite number of Fermi points. We will now see that this has deep consequences for the one-particle propagator. Indeed, using Eq. (11.1) we find

$$\langle i,x,t|j,0\rangle = \langle i,x,t|j,0\rangle_0 \exp\left\{\sum_q \frac{\sinh^2\zeta(q)}{VN(0)qv_F}[1-\cos(qx)e^{iqv_F S_0(q)t}]\right\} \tag{11.7}$$

at the same Fermi point (right movers, for instance) and

$$\langle i,x,t|j,0\rangle = 0 \tag{11.8}$$

for opposite Fermi points. This is the well-known result for the one-dimensional Luttinger model.[31]

The first consequence of the finite value of $\mathcal{N}$ is the presence of an anomalous dimension. Indeed, suppose the potential is local. Then the only effect of the interaction on the spectrum is a renormalization of the Fermi velocity from v_F to $\tilde{v}_F = v_F\sqrt{1+N(0)U}$. In this case the integral (11.7) is easily done [see (5.14)] and the result is

$$\langle i,x,t|j,0\rangle = \frac{\alpha}{x-\tilde{v}_F t+i\alpha}\left(\frac{\alpha^2}{x^2-(\tilde{v}_F t-i\alpha)^2}\right)^{\gamma}, \tag{11.9}$$

where

$$\gamma = \frac{1+N(0)U}{\sqrt{1+N(0)U}} - 1. \tag{11.10}$$

Observe that the above Green's function has an anomalous dimension given by γ and a branch cut in the spectrum instead of an isolated singularity. As we discussed before, this is a result of the process (a forward scattering) that links opposite sides of the Fermi surface, a process which is suppressed by the presence of the particle-hole continuum in higher dimensions (in the absence of nesting, singular interactions, or gauge fields).

But this is not the only difference between one and higher dimensions. In the form of the fermion propagator (11.7) the interactions are not screened. Suppose, for instance, that we have a Coulomb interaction, that is, $U(q) = e^2/q^2$. This is the case of the Schwinger model[32,33] which was studied via bosonization by Kogut and Susskind.[34] In real space the Fourier transform of the Coulomb potential gives rise to a linear potential and therefore to confinement. If we calculate the spectrum from (11.4) we get a massive relativist bosonic theory (the system has a gap at $q = 0$). It is easy to verify that the propagator in (11.7) has an infrared divergence at $t \neq 0$ and it vanishes in this limit, that is, the fermions decay and disappear completely from the spectrum, only the collective mode is left. However, as we have shown in Sec. IX the one-dimensional fermion gas screens external probes as expected, that is, in the RPA form. Of course, if we have screened the potential first, as in the RPA approach, we would never get this result.

Therefore the existence of the Luttinger fixed point in one dimension and the presence of anomalous dimensions is just a result of the lack of phase space (or due to a finite number of points in the Fermi surface).

XII. CONCLUDING REMARKS

In our previous papers[15] we studied the transport and thermodynamic properties of Fermi liquids. In this paper we have studied the Landau fixed point of a Fermi liquid and the associated one-particle propagator using the method of bosonization in arbitrary dimensions.

We have shown that it is possible to bosonize a theory of interacting fermions in the restricted Hilbert space of states close to the Fermi surface. The bosonization is based on the algebra of the particle-hole operators in this Hilbert space and in the introduction of bosonic coherent states which generate deformation of the Fermi surface. From the coherent states it is possible to define a generating functional which is a path integral over the histories of the Fermi surface. The fields which propagate on the Fermi surface are sound waves which can be viewed as coherent superposition of particle-hole pairs.

We have shown that from the construction of the fermion operator via coherent states we obtain the correct one-particle propagator which represents a fermion moving with the Fermi velocity. We also discuss the terms which appear in the interacting Hamiltonian and their relevance for the fixed point. It is obtained that at the fixed point, and for systems with nonsingular pair interactions, the only remnant effect of the interaction is an effective rotation of the bosonic eigenstates. Processes which in one-dimensional systems give rise to anomalous dimensions, such as those that couple points on opposite hemispheres of the Fermi surface, are found to have an effect which vanishes in the low-energy limit. Thus bosonization recovers in a natural way Landau's adiabatic principle. It is important to stress here that we are including forward scattering only and therefore our results are valid for Gaussian fixed points. This simple behavior at the fixed point is a consequence of the kinematics of d-dimensional systems. In contrast, in one dimension as well as for the cases of nested Fermi surfaces, singular interactions, and whenever dynamical gauge fields are present, the system may be controlled by infrared stable fixed points which have richer behavior. We will discuss these issues elsewhere.

We show that for the simple case of isotropic interactions the bosonic Hamiltonian can be diagonalized by a generalized Bogoliubov transformation which mixes different points of the Fermi surface. We obtain two different types of solutions which represent the particle-hole continuum and collective mode. We obtain the fermion propagator in the thermodynamic limit and in the low-energy regime, for local interactions. We show that in dimensions higher than one the fermion propagator has isolated singularities and the only difference between the noninteracting system and the interacting one is the presence of the quasiparticle residue. We evaluate the quasiparticle residue for any strength of the interaction (and thus showing the nonperturbative character of the bosonization approach and its difference from the perturbative approaches based on resummation of diagrams, such as RPA) and we show that the quasiparticle residue is always finite, that is, there is no possibility of breakdown of Fermi-liquid theory for local interactions, exactly as expected. Furthermore, our results agree with the perturbative ones in the limit of weak coupling. Moreover, dynamical screening is a natural result of the bosonization method in dimensions higher than one. That is, we obtain that long-range interactions are screened and we do not have to assume it as is usual in perturbative theories in condensed matter physics.

We also study the problem of the response of the fermionic gas to scalar external probes and obtain the RPA result for screening. This confirms that the bosonization is getting the correct physics at long wavelengths and low energies since the RPA fulfills the sum rules in this limit. And we stress once more that it does not mean that RPA and bosonization are the same thing because bosonization is a nonperturbative method which is valid for any strength of the interaction. We also show that when fermions are coupled to gauge fields the RPA result is also valid, the interactions are not screened but there is Landau damping (except in one dimension). We calculate the form of the effective propagator for the gauge fields and find the expected form for RPA.

Finally we compare the one-dimensional problem, related to a Luttinger fixed point, with the Landau fixed point. We show that due to the finite number of Fermi points in one dimension the mixing of points across the Fermi surface exists and leads to the appearance of anomalous dimensions in the Green's function. In one dimension there is only a collective excitation while in higher dimensions there is a particle-hole continuum which shares spectral weight with the collective mode. We also show that the perturbative approach of screening the fermion-fermion interaction by hand would lead to wrong results in one dimension. While the one-dimensional fermionic system screens external probes in the usual RPA form it does not screen the fermion-fermion interaction. The bosonization method leads to the vanishing of the Green's function in the case of Coulomb interactions in one dimension. In higher dimensions, due to screening, the Coulomb interaction leads to an effective local interaction which does no harm to the Fermi-liquid behavior.

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