Linear approximation in the kinetic Ising model on fractals

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We propose a scheme to study exactly the kinetic Ising model on fractals under the linear approximation. This enables us to obtain the critical slowing-down behavior of the longest relaxation time of the kinetic Ising model on a fractal exactly. As a result, we show that there are only two time scales in the relaxation process of the kinetic Ising model on a fractal cluster under the linear approximation; the critical dynamic exponent is always a constant that is unrelated to the temperature of the system and is equal to 2. The implication of our results to the linear approximation of the critical dynamics of the kinetic Ising model on fractals as well as to the time-dependent renormalization-group method is discussed.

Henley's theory[1-3] predicted a singular dynamic scaling in which $\tau \propto \xi^Z$ as in standard dynamic scaling but with $Z \sim 1/T$ instead of $Z=$const as predicted by the time-dependent renormalization-group (TDRG) method.[4] Henley noticed that in an Ising spin system of nonuniform geometry, which has only a zero-temperature phase transition, thermal activation over energy barriers became very important to the critical slowing down near the critical point. For a ferromagnetic ground state (the ground state of the nearest-neighbor ferromagnetic Hamiltonian $H_0=-J\sum\sigma_i\sigma_j$), the energy to flip a spin that must overcome the energy barrier is just $2J$. However, the number of spins that must overcome the energy barrier increases by 1 in each rescaling. So if we measure the maximum barrier energy $E_{\max}$ in units of $2J$, there is a simple relation

$$E_{\max}(L)/2J = E_{\max}(b^{-1}L)/2J + 1$$

under a scaling change. It is easy to obtain the result that the barrier energy $E_{\max}$ grows logarithmically as $E_{\max}(L)/2J = q\ln(L) + \text{const}$. Henley assumes that the relaxation between two degenerate ferromagnetic ground states of an Ising spin system satisfies the Arrhenius law, that is the relaxation time is

$$\tau(L,T) = \tau_0 \exp(E_{\max}/\kappa T) = \tau_0 L^{A/T} .$$

So the critical exponent Z is proportional to $1/T$. Further work[2,3] gave strong support to Henley's theory. Therefore the results[6-8] of the TDRG method become doubtful.

Because the basis of the TDRG method[4] is linear response, the aim of this paper is to make clear what can be learned from the linear approximation about the critical slowing down of the kinetic Ising model on fractals. Under the linear approximation of the kinetic Ising model,[4] $P(\{\sigma_j\},t)$, the probability of the spin configurations, takes the form

$$P(\{\sigma_j\},t) = P_e\left[1+\sum_i^N b_i(t)\sigma_i\right] . \tag{1}$$

Here, $P_e=\exp(-H_0/\kappa T)$ is its value at the equilibrium state. In studies using the TDRG method, $\{b_i(t)\}$ is only related to the number of nearest neighbors of the point i, and takes several particular values. Here, we do not give any constraint on $\{b_i(t)\}$. Defining

$$q_i(t) = \langle\sigma_i\rangle = \sum_i \sigma_i P(\{\sigma_j\},t) ,$$

we obtain the equation of motion of the magnetization from the Glauber master equation [see Eq. (1) in Ref. (7)]

$$\frac{dq_i(t)}{dt} = -2\langle\sigma_i w_i(\sigma_i)\rangle = -2\sum_i \sigma_i w_i(\sigma_i) P(\{\sigma_j\},t) . \tag{2}$$

In the fractal lattice[3] that we will study, there are three kinds of points with one, two, and three nearest neighbors, respectively. Corresponding to the three kinds of spin, the flip probabilities[4] take the forms

$$w^1(\sigma) = \tfrac{1}{2}\gamma[1-\sigma\sigma'\tanh(k)] ,$$

$$w^2(\sigma) = \tfrac{1}{2}\gamma[1-\tfrac{1}{2}\sigma(\sigma'+\sigma'')\tanh(2k)] ,$$

$$w^3(\sigma) = \tfrac{1}{2}\gamma[1-\sigma\{(\sigma'+\sigma''+\sigma''')a_1+\sigma'\sigma''\sigma'''a_2\}] ,$$

respectively, with $a_1=\frac{1}{4}[\tanh(3k)+\tanh(k)]$, $a_2=\frac{1}{4}[\tanh(3k)-3\tanh(k)]$. σ', σ'', and σ''' are the first, second, and third nearest neighbors, of the spin σ. $\langle\sigma_i w_i(\sigma_i)\rangle$ is related to the triplet spin correlation $\langle\sigma'\sigma''\sigma'''\rangle$ of the nearest neighbors of the spin σ_i because the flip probability $w_i(\sigma_i)$ is determined by the nearest neighbors of the spin σ_i. Each spin with three

nearest neighbors produces a triplet spin correlation term in Eq. (2). This makes Eq. (2) very difficult to solve.

In the following, we try to find the forms of $\langle\sigma'\sigma''\sigma'''\rangle$ under the exact linear approximation of Eq. (1). Under this approximation, the magnetization becomes $q_i=\sum_j\langle\sigma_i\sigma_j\rangle_0 b_j(t)$. $\langle\sigma_i\sigma_j\rangle_0=\sum_{\{\sigma_j\}}\sigma_i\sigma_j P_e$ is the spin correlation at the equilibrium state. It is well known that the two-body correlation[5] of Ising spins on a fractal is $\langle\sigma_i\sigma_j\rangle_0=g^{|x_i-x_j|+|y_i-y_j|}$ with $g=\tanh(k)$. x and y are the position coordinates of the points i,j. Defining $\mathbf{q}=(q_1,q_2,\ldots,q_N)$, $\mathbf{b}=(b_1,b_2,\ldots,b_N)$, we have $\mathbf{q}=\mathbf{b}A$ or $\mathbf{b}=\mathbf{q}A^{-1}$ with A a known $N\times N$ matrix of the function g. The triplet spin correlation $\langle\sigma'\sigma''\sigma'''\rangle$ can be expressed in terms of $\mathbf{q}$. We take $\langle\sigma_1\sigma_3\sigma_4\rangle$ as an example to explain how this can be done. In the approximation of Eq. (1), it can be written as

$$\langle\sigma_1\sigma_3\sigma_4\rangle=\sum_j\langle\sigma_1\sigma_3\sigma_4\sigma_j\rangle_0 b_j(t)=\mathbf{b}\cdot\mathbf{D}$$

with

$$\langle\sigma_1\sigma_3\sigma_4\sigma_j\rangle_0=\sum_{\{\sigma_j\}}\sigma_1\sigma_3\sigma_4\sigma_j P_e$$

and

$$\mathbf{D}=(\langle\sigma_1^2\sigma_3\sigma_4\rangle_0,\langle\sigma_1\sigma_2\sigma_3\sigma_4\rangle_0,\ldots,\langle\sigma_1\sigma_3\sigma_4\sigma_N\rangle_0)^T$$

(T means the transform). Because P_e is a configuration of Gaussian type, the four-body correlations can always be decomposed as multiplications of two-body correlations. For example,

$$\langle\sigma_1\sigma_3\sigma_4\sigma_6\rangle_0=\langle\sigma_1\sigma_3\rangle_0\langle\sigma_4\sigma_6\rangle_0=g^2\times g^2 .$$

So $\mathbf{D}$ is a known vector. It is not difficult to show that

$$\begin{aligned}\langle\sigma_1\sigma_3\sigma_4\rangle&=\mathbf{q}\cdot A^{-1}\mathbf{D}\\&=\tanh^2(k)[q_1+q_3+q_4-2q_2\tanh(k)] .\end{aligned}\tag{3}$$

In fact, we can prove that Eq. (3) is a general result for the spin σ which has the three nearest neighbors $\sigma',\sigma'',\sigma'''$, on the fractal that we will study, that is,

$$\begin{aligned}\langle\sigma'\sigma''\sigma'''\rangle=\tanh^2(k)[&\langle\sigma'\rangle+\langle\sigma''\rangle+\langle\sigma'''\rangle\\&-2\langle\sigma\rangle\tanh(k)] .\end{aligned}$$

If we assume $q_i=q_0e^{-t/\tau}$, then the solution for the magnetization of Eq. (1) on the fractal cluster shown in Fig. 1 now becomes a problem of the eigenvalues of the equation $\mathbf{q}\cdot\mathbf{H}=0$, or

$$|H|=\begin{vmatrix}x&g&0&0&0&0&0&0&0\\q&a&q&q&0&0&0&0&0\\0&g&x&0&0&0&0&0&0\\0&p&0&x&p&0&0&0&0\\0&0&0&p&x&p&0&0&0\\0&0&0&0&p&x&0&p&0\\0&0&0&0&0&0&x&g&0\\0&0&0&0&0&q&q&a&q\\0&0&0&0&0&0&0&g&x\end{vmatrix}=0,\tag{4}$$

in which $p=\frac{1}{2}\tanh(2k)$, $q=a_1+g^2a_2$, $a=x-2g^3a_2$, and $x=1/\tau\gamma-1$. It can be further simplified as

$$|H|=|H_1|\times|H_2|\tag{5}$$

with

$$|H_1|=\begin{vmatrix}x&g&0&0&0\\q&a&q&q&0\\0&g&x&0&0\\0&p&0&x&p\\0&0&0&p&x\end{vmatrix}-p^2\times\begin{vmatrix}x&g&0\\q&a&q\\0&g&x\end{vmatrix}$$

and

$$|H_2|=\begin{vmatrix}x&0&p&0\\0&x&g&0\\q&q&a&q\\0&0&g&x\end{vmatrix} .$$

To obtain the above results, we utilize the Laplace theorem and the invariant property of a determinant under the rotation of its matrix by 180°. These procedures are universal for fractal clusters of any size, that is, an $N\times N$ determinant $|H|$ can always be reduced to the multiplication of two small determinants $|H_1|$ and $|H_2|$ which have completely similar forms, as in Eq. (5). One is of the order $(N+1)/2$ and the other of the order $(N-1)/2$ if we follow these procedures. In Eq. (5), $|H_2|$ can be solved exactly. Except for the two trivial solutions $\tau=\gamma$, the other two solutions are

$$\frac{1}{\gamma\tau^{\pm}}=1+g^3a_2\pm\sqrt{g^6a_2^2+q(2g+p)} .$$

The solution τ^- is interesting to us because τ^- is divergent as $k\to\infty$ ($T\to0$). It is a divergent relaxation time. We must emphasize that these two solutions can also be obtained analytically in any-order determinant of $|H|$ on a fractal cluster. In general, except for $\tau^{\pm}$ and the trivial solution $\tau=\gamma$, the eigenvalues of the matrix H cannot be obtained analytically. But this cannot prevent us from studying the behavior of the longest relaxation time analytically. In the following, we will adopt other methods to study the behavior of the longest relaxation time instead of solving the H matrix exactly.

First, we find out how many solutions of the relaxation time τ on a fractal cluster with N sites are divergent near the critical point $k^*=\infty$ under the linear approximation of Eq. (1), that is, how many solutions of an $N\times N$ deter-

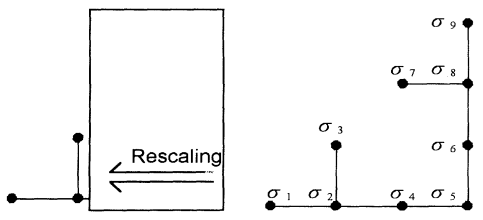

FIG. 1. First two stages of the fractal that has the fractal dimension $D_f=\ln2/\ln3$.

minant $|H|$ are $-1(x=1/\gamma\tau-1)$. At $k^*=\infty$, it is easy to show that $g=1$, $p=\frac{1}{2}$, $g=0$, and $a=x+1$. The $N\times N$ H matrix becomes soluble. It only has four kinds of solutions, which have the values -1, 0, $\pm 1/\sqrt{2}$, respectively, and the last two are not degenerate. These results are very important to our discussion in the following. The solutions with the value -1 are interesting to us here. Obviously a spin with three nearest neighbors, which is represented by a row of the form $(0,\ldots,0,q,q,a,q,0,\ldots 0)$ or $(0,\ldots,0,q,a,q,q,0,\ldots 0)$ in the H matrix [see Eq. (4)], produces a solution $x=-1$. As a result, the number of divergent relaxation times is equal to the number of spins with three nearest neighbors. For a fractal cluster with N sites, there are $(N/3-1)$ sites that have three nearest neighbors. This means that the number of divergent relaxation times of the Ising spin system on that fractal cluster is $(N/3-1)$. Moreover, our reduction procedure to an H matrix distributes the spins with three neighbors homogeneously between the two subdeterminants $|H_1|$ and $|H_2|$. So each subdeterminant has $\frac{1}{2}(N/3-1)$ solutions -1. Furthermore, we must emphasize that the solutions $\pm 1/\sqrt{2}$ are always in the determinant $|H_1|$.

Secondly, we study the behavior of the longest relaxation time under the rescaling. Because all the quantities like p,q,a_1, and a_2 can be expressed in a variant g, the relaxation time τ is a function of g. However, $g=\tanh(k)$ can be rewritten as

$$g=\frac{e^k-e^k}{e^k+e^k}=\frac{1-e^{-2k}}{1+e^{-2k}}=\frac{1-y}{1+y}$$

with $y=e^{-2k}$ $(k=J/\kappa T)$. $g=1$ is equivalent to $y=0$. So τ is a function of the single variant y. You will understand immediately in the following why we express τ in terms of y. If we denote the inverse of the longest relaxation time $\tau_{\max}$ as $\tau'_{\min}$ $(\tau'_{\min}=1/\gamma\tau_{\max})$ and expand it at the critical point $y=0$, since $\tau'_{\max}$ $(y=0)=0$, $\tau'_{\min}$ has the form $\tau'_{\min}=y^S(c_1+c_2y+\cdots)$. According to Henley's theory,[1-3] the energy to flip a spin is just $2J$. So e^{2k} is a natured unit of the relaxation time from the Arrhenius law. Obviously, the inverse of the relaxation time is characterized by y. The above expression makes the physical meaning of the divergent relaxation time very clear. The value of S is a measure of the long-time behavior of the divergent relaxation time. The bigger S is, the longer is the relaxation time. It also provides us a direct and exact method to study the behavior of the longest relaxation time under the rescaling of a fractal. The reason is that the value of S for the longest relaxation time must increase under a recurrence of the fractal cluster because Henley's theory predicted an increase of $2J$ in the barrier energy under a rescaling of the fractal. If the value of S for the longest relaxation time is unchanged under the rescaling of the fractal cluster, this shows that the critical dynamic exponent of the system on the fractal is a constant as in the TDRG prediction. So our task in the following is to calculate the value of S analytically.

If τ_0 denotes the multiplication of all eigenvalues of the H matrix, it is easy to show that τ_0 is equal to $|H(x=-1)|$, the value of the determinant $|H|$ at $x=-1$. Expanding τ_0 in y at $y=0$, we have

$$\begin{aligned}\tau_0^1&=|H_1(x=-1)|=y^{S_1^1+S_2^1+\cdots S_m^1}(b_1^1+b_2^1y^{l_1}+\cdots),\\ \tau_0^2&=|H_2(x=-1)|=y^{S_1^2+S_2^2+\cdots S_m^2}(b_1^2+b_2^2y^{l_2}+\cdots),\end{aligned}\tag{6}$$

where m is the number of divergent relaxation times in the determinant $|H_1|$ or $|H_2|$. Because of the inverses of the nondivergent relaxation time, at $y=0$ two are equal to $(1+1/\sqrt{2})$ and the others are equal to 1, the coefficients b_1^1 and b_1^2 are just multiplications of the coefficients c of each divergent relaxation time, that is,

$$\begin{aligned}b_1^1&=(-1)^l\tfrac{1}{2}c_1^1c_1^2\cdots c_1^m,\\ b_1^2&=(-1)^Ic_2^1c_2^2\cdots c_2^m,\end{aligned}\tag{7}$$

where l and I are the number of solutions in the determinants $|H_1|$ and $|H_2|$, respectively. So we can infer the S value as well as the c value of the longest relaxation time from τ_0^1 and τ_0^2. By use of the analytical language of the computer, the value of a determinant can be calculated analytically. This avoids the difficulties of solving the eigenvalue equation. We have calculated the $|H|$ determinants on fractal clusters with sizes 9, 27, and 81, which are the three stages of the fractal.

(1) For the cluster with nine sites, the τ_0^1 and τ_0^2 that we obtain analytically are

$$\begin{aligned}\tau_0^1(k)&=y^2(3+y-5y^2-2y^3+\cdots),\\ \tau_0^2(k)&=y(-\tfrac{3}{2}-\tfrac{1}{2}y+y^2+\tfrac{1}{2}y^3+\cdots).\end{aligned}\tag{8}$$

It is very clear that the longest relaxation time is characterized by τ_0^1 and has the S value 2 and the c value $c_0=6$. τ_0^2 describes the relaxation behavior of the solution τ^{-1}; it has the S value 1 and the c value $c_1=\frac{3}{2}$. All these show that there are two time scales under the linear approximation. To obtain the critical dynamic exponent, we must calculate Eq. (8) under the transformation[2] of finite-size scaling. Under the rescaling from 27 sites of nine sites, the effective parameter[3] of the interaction becomes $k'=\tanh^{-1}[\tanh^2(k)]$, that is, $g=\tanh^2(k)$ instead of $\tanh(k)$, and Eqs. (8) are changed to

$$\begin{aligned}\tau_0^{1'}(k')&=y^2(12+8y-104y^2-88y^3+\cdots),\\ \tau_0^{2'}(k')&=y(-3-2y+11y^2+12y^3+\cdots).\end{aligned}\tag{9}$$

From them, we know that the longest relaxation time has the form with $S=2$ and the c value $c'_0=24$, and the other divergent relaxation time has the form with $S=1$ and the c value $c'_1=3$ under the change of scaling.

(2) For the fractal cluster with 27 sites, we obtain

$$\begin{aligned}\tau_0^1(k)&=y^2(-\tfrac{27}{4}-\tfrac{27}{4}y+18y^2+\tfrac{89}{4}y^3+\cdots),\\ \tau_0^2(k)&=y^3(\tfrac{27}{8}+\tfrac{27}{8}y-\tfrac{45}{8}y^2-\tfrac{31}{4}y^3+\cdots).\end{aligned}\tag{10}$$

From Eq. (7), we know

$$b_1^1=(-1)^9\tfrac{1}{2}c_1^1c_1^2c_1^3=\tfrac{27}{4}=(-1)^9\tfrac{1}{2}c_0(c_1)^2,$$

$$b_1^2=(-1)^8c_2^1c_2^2c_2^3=\tfrac{27}{8}=(-1)^8(c_1)^3,$$

$S_1^1+S_2^1+S_3^1=4$, and $S_1^2+S_2^2+S_3^2=3$. So the only possibility is that the longest relaxation time has the form

$$\frac{1}{\gamma\tau_{\max}(k)}=c_0y^2+O(y^3)\ , \tag{11}$$

and the other divergent relaxation times have the same S value 1 as well as the c value $c_1=\frac{3}{2}$. Under the transformation of the scaling from 81 to 27 sites, Eq. (10) takes the form

$$\tau_0^{1'}(k')=y^4(-108-216y+1584y^2+3928y^3+\cdots)\ ,$$
$$\tau_0^{2'}(k')=y^3(27+54y-261y^2-712y^3+\cdots)\ . \tag{12}$$

As above, we obtain the form of the longest relaxation time under the rescaling as

$$\frac{1}{\gamma\tau'_{\max}(k')}=c_0'y^2+O(y^3)\ , \tag{13}$$

and the other divergent relaxation times have the same S value 1 as well as the c value $c_1'=3$.

(3) For the fractal cluster with 81 sites, we have

$$\tau_0^1(k)=y^{13}(-\tfrac{531441}{2048}-\tfrac{531441}{512}y+\tfrac{885735}{2048}y^2+\tfrac{3877551}{512}y^3+\cdots)\ ,$$
$$\tau_0^2(k)=y^{12}(\tfrac{531441}{4096}+\tfrac{531441}{1024}y-\tfrac{177147}{2048}y^2-\tfrac{1673055}{512}y^3+\cdots)\ . \tag{14}$$

The results are

$$b_1^1=(-1)^{33}\tfrac{1}{2}c_1^1c_1^2\cdots c_1^{12}=\tfrac{531441}{2048}=(-1)^{33}\tfrac{1}{2}c_0(c_1)^{11}\ ,$$
$$b_1^2=(-1)^{32}c_2^1c_2^2\cdots c_2^{12}=\tfrac{531441}{4096}=(-1)^{32}(c_1)^{12}\ ,$$

$S_1^1+S_2^1+\cdots+S_{12}^1=13$, and $S_1^2+S_2^2+\cdots+S_{12}^2=12$. The only possibility is that the longest relaxation time takes the form of Eq. (11), and the other divergent relaxation times have the same S value 1 as well as the c value $c_1=\frac{3}{2}$.

According to the finite-size scaling theory[2] of critical dynamics, $\tau(k)/\tau'(k')=b^Z$ (b is the scaling factor and equal to 2 here), we can calculate the critical dynamic exponent exactly from Eqs. (11) and (13) by $c_0'/c_0=b^Z$. So in the linear approximation the critical dynamic exponent of the kinetic Ising model on a fractal is a constant, 2.

We propose a method to study the critical dynamics of the kinetic Ising model on fractals. The advantage of our method is that we are able to obtain all the information of the longest relaxation time of the kinetic Ising model on a fractal under the linear approximation exactly without solving the master equation or equation of motion. Using this method, we study exactly the critical dynamics of the kinetic Ising model on a typical fractal shown in Fig. 1. We obtain the critical slowing-down behavior of the longest relaxation time of the kinetic Ising model exactly on fractal clusters with 9, 27, and 81 sites under the linear approximation. Our results show that there are only two time scales in the relaxation process of the kinetic Ising model on a fractal cluster under the linear approximation, instead of Henley's hierarchical time scales. So the critical dynamic exponent of the kinetic Ising model on the fractal under the linear approximation is always a constant that does not have a relation with the temperature of the system. This means that the linear approximation of the kinetic Ising model on a fractal cannot predict Henley's results. Moreover, we calculate the critical dynamic exponent of the kinetic Ising model on the fractal under the linear approximation exactly by the finite-size scaling theory. We show that it is different from the value[7,8] $D_f+\frac{3}{2}$ predicted by the TDRG method and equal to 2. This exact result is a great challenge for the application of the TDRG method to the critical dynamics of the kinetic Ising model on fractals. It shows that the TDRG method cannot even produce correctly the results of the linear approximation of the kinetic Ising model on the fractal.

In summary, our results show that any methods that are based on the linear approximation are not suitable to be used to study the critical dynamics of the kinetic Ising model on fractals, and the RG method to study the critical dynamics of spin systems on a fractal must be revised in the future.

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