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Resonances in driven dynamical lattices

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We predict that the response of a weakly damped one-dimensional lattice to a localized ac driving force demonstrates a strong resonance when the driving frequency is close to the *upper* edge of the lattice phonon band. The response is considered in terms of the energy absorption rate vs the driving frequency, which is an experimentally observable characteristic. We consider the effect semianalytically for the linear lattice, and then display results of systematic direct numerical simulations of both linear and nonlinear cases, which clearly demonstrate the resonance predicted. In comparison with the known resonance for the case when the driving frequency is close to the *lower* edge of the phonon band, the new resonance is stronger, and also more robust when competing with dissipation and nonlinearity. Both resonances are closely related to the singularities of the lattice's density of states at the edges of the phonon band.

The dynamical properties of one-dimensional lattices, which have been a subject of studies for a long time,[1,2] have again attracted a great deal of attention recently (see, e.g., Ref. 3). If the dynamical lattice is placed into a substrate potential, its phonon frequencies ω are bounded from below, $\omega > \omega_{\min}$, $\omega_{\min}$ being the lower cutoff frequency produced by the substrate (on-site) potential. The existence of this cutoff (the lower forbidden gap) is not a peculiarity of discrete systems, as it survives in the continuum limit. However, discrete systems, contrary to their continuum counterparts, always have the upper cutoff frequency $\omega_{\max}$, so that the frequencies of the propagating modes (phonons) belong to the band

$$\omega_{\min} < \omega < \omega_{\max} \, . \tag{1}$$

The aim of the present work is to study the simplest dynamical characteristic of the lattice, viz., its response to a local ac driving force applied to a single particle. The ac drive can be necessary, e.g., to compensate dissipative losses,[4] and the consideration of the dynamical response is of a fundamental interest by itself. We will concentrate on the most interesting case, when the driving frequency is close to either of the cutoff frequencies $\omega_{\min}$ or $\omega_{\max}$. We will consider slightly damped lattices, and, generally, the driving amplitude will be taken to be small enough. As a characteristic of the response of the lattice to the applied ac drive, we will consider the rate of absorption of energy by the lattice. This is an experimentally observable characteristic which bears a fundamental information about the dynamical properties of the lattice.[2]

We will consider both the linear and nonlinear damped driven lattices. For the linear case, an analytical approach will be developed, the usefulness of which, however, proves to be limited. The main results will be obtained by means of direct numerical simulations of the lattice equations of motion. We will display a set of representative plots for the energy absorption rate vs the driving frequency. As one can expect from the analogy with known results for driven continuum systems,[5] the amplitude of the wave created in the lattice by the localized drive, and, accordingly, the energy dissipation rate, have a sharp maximum when the driving frequency is close to the lower cutoff frequency of the lattice. However, an essentially new effect is a stronger maximum observed near the *upper* cutoff. Unlike the maximum at ω close to $\omega_{\min}$, the new maximum at ω close to $\omega_{\max}$ is a distinctive property of the discrete systems. This maximum can be clearly interpreted as a special type of resonance in the discrete system driven by the ac force. This new resonance should be a generic property of the dynamical lattices. In particular, it must demonstrate itself not only in the case considered in the present work, when a single particle is driven, but also when the ac drive is distributed in the lattice.

To observe the effect experimentally, one can use a lattice doped with ions, which may be driven by an ac electric field. In this case of impurity-activated optical absorption, it is necessary to exclude a resonance with the possible localized modes created by the impurities (ions), which should not be a very hard problem[2] since that effect can be estimated. If the density of impurities is very small, the localized states would form a very narrow frequency band away from the region of interest. Alternatively, a narrow segment of a long overlayer on a substrate can be driven mechanically (e.g., acoustically), which completely excludes the impurity modes. At last, it looks plausible that the resonant response considered in this work should be easily observed in the dynamical lattices represented by electrical transmission lines, where the linear dispersion relation can be tailored with the appropriate choice of circuit design and nonlinear electronic elements.

As was mentioned above, the new resonance near the upper cutoff is conspicuously stronger than the known resonance at the lower cutoff. One should expect that any resonance is gradually attenuated with the increase

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of the dissipative constant, as well as with the increase of a coefficient in front of the nonlinear term.[5] We will demonstrate that the new resonance is more robust than the old one, being more slowly attenuated with the increase of the dissipation and nonlinearity. It seems relevant to stress that, although the idea of the resonant response when the driving frequency is close to the upper cutoff seems very simple and natural, we were not able to find in the literature any consideration of this effect. In this work, we will consider the standard dynamical lattice model with quartic nonlinearity along the chain and a symmetric substrate,

$$\frac{d^2x_n}{dt^2}+\lambda\frac{dx_n}{dt}=x_{n+1}+x_{n-1}-2x_n-\Omega^2x_n-\alpha x_n^3 +\beta(x_{n+1}-x_n)^3-\beta(x_n-x_{n-1})^3\,, \tag{2}$$

where x_n is the coordinate of the nth particle, λ is the dissipative coefficient, α and β are, respectively, the substrate and chain nonlinear coefficients, and the parameter Ω^2 represents the linear part of the on-site (substrate) potential. The particle with the coordinate x_0 is driven by the external force, so that, at $n=0$, Eq. (2) should be modified as follows:

$$\frac{d^2x_0}{dt^2}+\lambda\frac{dx_0}{dt}=x_1+x_{-1}-(2+\Omega^2)x_0-\alpha x_0^3 +\beta(x_1-x_0)^3-\beta(x_0-x_{-1})^3 +\epsilon\cos(\omega t)\,, \tag{3}$$

where ϵ and ω are the amplitude and frequency of the driving force.

We will consider analytically only the linear versions of Eqs. (2) and (3), i.e., $\alpha=0$ and $\beta=0$. In this case, our aim is, first of all, to find a wave supported in the weakly damped lattice (small λ) by the drive. As is well known, at $\epsilon=0$ and $\lambda=0$ the free traveling-wave solution to Eq. (2) is

$$x_n(t)=A\exp(ikn-i\omega t)\,, \tag{4}$$

where the wave number k takes the values $0<k<2\pi$, and the frequency ω is related to it by the dispersion equation

$$\omega^2=\Omega^2+4\sin^2\left(\frac{k}{2}\right)\,. \tag{5}$$

As follows from Eqs. (5) and (1), the upper cutoff frequency

$$\omega_{\max}=\sqrt{4+\Omega^2} \tag{6}$$

is attained at $k=k_0\equiv\pi$. The lower cutoff frequency determined by Eqs. (5) and (1) is $\omega_{\min}\equiv\Omega$.

Our analytical consideration will be confined to small vicinities of the cutoff frequencies, where the resonances are expected. Note, however, that for ω close to $\omega_{\min}$ the problem can be readily reduced to that solved earlier in Ref. 5 for the continuum model. Indeed, as follows from Eq. (5), ω is close to $\omega_{\min}$ for small wave numbers, i.e., large wavelengths. In this case, one can naturally approximate the discrete lattice by its continuum limit. In Ref. 5, the response of the semi-infinite continuum system to the ac drive applied at the edge was analyzed in detail. Evidently, the semi-infinite system driven at the edge is equivalent to the symmetric infinite system driven at the central point. It was demonstrated, in a fully analytical form, that the response of the semi-infinite system to the drive whose frequency is close to $\omega_{\min}$ is anomalously strong in comparison with the case when the driving frequency ω is far from $\omega_{\min}$. In particular, the energy absorption rate $W_{\rm diss}$, regarded as a function of ω, has a sharp resonant maximum at $\omega=\omega_{\min}$. Note that $W_{\rm diss}$ rapidly vanishes inside the forbidden gap, i.e., at $\omega<\omega_{\min}$, but, nevertheless, the dependence $W_{\rm diss}(\omega-\omega_{\min})$ is smooth; i.e., $W_{\rm diss}$ does not vanish by a jump at $\omega=\omega_{\min}$, and the smooth dependence is provided by influence of the dissipation. It is also noteworthy that, in the limit of the vanishing dissipation, the quantity $W_{\rm diss}$ does not vanish. When $\alpha=0$, the absorbed energy is not directly dissipated, but is emitted to infinity in the semi-infinite continuum system. The limit of $W_{\rm diss}$ at $\alpha\to 0$ was demonstrated in Ref. 5 to be exactly equal to the energy emission rate in the case $\alpha=0$.

So, in the present work, we will consider only the case of ω close to $\omega_{\max}$. Note, first of all, that $\exp(ik_0n)\equiv(-1)^n$; i.e., in this case the solution (4) describes the so-called staggered state[6] in which the particles with even and odd numbers are oscillating π out of phase relative to each other. Therefore, in the case when λ is small but finite a solution to the linearized equation (2) corresponding to a small deviation

$$\delta\equiv\omega^2-\omega_{\max}^2 \tag{7}$$

of the frequency from the cutoff value can be naturally sought for in the form

$$x_n=A(-1)^ne^{-i\omega t+i\theta}\exp(-\mu_1n+i\mu_2n)\,, \tag{8}$$

where A is an arbitrary amplitude, θ is a phase shift, and μ_1 and μ_2 are, respectively, a small decay constant and a small wave number shift. Due to the smallness of these parameters, Eq. (2), rewritten for the slowly varying amplitude in front of the rapidly varying function $(-1)^n\exp(-i\omega_{\max}t)$ [see Eq. (8)], may be replaced by its continuum counterpart, and thus one finds

$$\mu_1-i\mu_2=\sqrt{\delta-i\lambda\,\omega_{\max}}\,. \tag{9}$$

Finally, one can obtain from Eq. (9)

$$\mu_1^2=\frac{1}{2}(\sqrt{\delta^2+\lambda^2\omega_{\max}^2}+\delta)\,. \tag{10}$$

The solution given by Eqs. (8)–(10) describes, at $n>0$, a driven wave with the local amplitude slowly decaying at $n\to\infty$ (the actual values of the displacements x_n are given by the real part of the complex solution). At $n<0$, we have the same solution with n replaced by $-n$.

Treating the driving force in Eq. (2) perturbatively, we assume that, in the lowest approximation, it does not alter the form of the driven wave as given by Eqs. (8) and (9). Then, the rate $W_{\rm dr}$ at which energy is supplied to the lattice by the ac drive can be calculated as the mean *power*, i.e., the time average of the product of the force and the velocity of the driven particle [according to what was said above, one should take the real part of the formally complex velocity following from Eq. (8)]. Thus we obtain

$$W_{\rm dr} \equiv \left\langle \epsilon\cos(\omega t)\frac{dx_0}{dt}\right\rangle = \frac{1}{2}\epsilon\,\omega A\,\sin\theta\,, \qquad (11)$$

where the angular brackets stand for the time average, and θ is a phase shift between the ac driving force and the oscillations of the driven particle [$n=0$; see Eq. (3)]. The equilibrium value of the amplitude A is determined by the *energy balance* condition (see, e.g., Ref. 4), i.e., by equating the energy input rate (11) to the rate of dissipation of energy $W_{\rm diss}$. As it follows from Eqs. (2), the mean value of the energy dissipation rate, summed up over all the particles, is

$$W_{\rm diss} \equiv \left\langle \sum_{n=-\infty}^{+\infty} \lambda\left(\frac{dx_n}{dt}\right)^2\right\rangle = \left[2\left(\sqrt{\delta^2+\lambda^2\omega_{\rm max}^2}+\delta\right)\right]^{-1/2}\lambda\omega_{\rm max}^2A^2\,, \qquad (12)$$

where we have inserted the real part of Eq. (8) (actually, with n replaced by $|n|$, according to what was said above about the solution at $n<0$), and everywhere except for the resonance detuning δ [Eq. (7)], ω was replaced by $\omega_{\rm max}$. Now, equating the expressions (11) and (12), we find the equilibrium value of the driven wave's amplitude,

$$A_{\rm eq} = \epsilon\,\sin\theta\,\omega_{\rm max}^{-1}\lambda^{-1}\left[\frac{1}{2}\left(\sqrt{\delta^2+\lambda^2\omega_{\rm max}^2}+\delta\right)\right]^{1/2}. \qquad (13)$$

Finally, the energy dissipation rate can be found inserting Eq. (13) back into Eq. (12):

$$W_{\rm diss} = \frac{1}{2}\epsilon^2\sin^2\theta\,\lambda^{-1}\left[\frac{1}{2}\left(\sqrt{\delta^2+\lambda^2\omega_{\rm max}^2}+\delta\right)\right]^{1/2}. \qquad (14)$$

Note that the phase shift θ remains an unknown parameter, which will be found below numerically.

A remarkable property of Eqs. (13) and (14) is that, due to the regularizing role of the weak dissipation,[5] they provide for a smooth transition from the phonon band at $\delta<0$ [see Eq. (5)] to the forbidden band at $\delta>0$. In particular, sufficiently deep inside the phonon band, i.e., at $-\delta \gg \lambda\omega_{\rm max}$, both expressions simplify and become independent of the friction coefficient λ:

$$A_{\rm eq} \approx \frac{1}{2}\epsilon\sin\theta\,|\delta|^{-1/2}, \quad W_{\rm diss} \approx \frac{1}{4}\omega_{\rm max}\epsilon^2\sin^2\theta\,|\delta|^{-1/2}\,. \qquad (15)$$

The independence of these expressions of λ has a simple physical meaning:[5] In the dissipationless limit, the expression for the rate of dissipative losses goes over into the expression which gives the rate of *emission* of energy to infinity. Indeed, the energy emission rate is

$$W_{\rm em} = 2EV_{\rm gr}\,, \qquad (16)$$

where E is the density of energy in the excited wave, $V_{\rm gr}$ is its group velocity, and the multiplier 2 takes account of the fact that the energy is emitted in both directions. In the present case, it is easy to find that the time-averaged energy density is $E \approx \frac{1}{2}\omega_{\rm max}^2A_{\rm eq}^2$, and the group velocity can be obtained from the dispersion law (5):

$$V_{\rm gr} \equiv \frac{d\omega}{dk} \approx \omega_{\rm max}^{-1}(k-k_0) \approx \omega_{\rm max}^{-1}\sqrt{|\delta|}$$

(recall $k_0\equiv\pi$). Inserting this into Eq. (16) and using the simplified expression for $A_{\rm eq}$ from Eq. (15), it is straightforward to obtain for $W_{\rm em}$ exactly the same expression which is given by Eq. (15).

The main idea pursued in this work is that the new resonance in the driven dynamical lattices should take place when the driving frequency is close to the upper cutoff frequency. For direct verification of this idea, we have performed detailed numerical simulations of the dynamical equations (2) and (3). The equations were solved for the symmetric configurations, $x_n = x_{-n}$, with the periodic boundary conditions $x_N \equiv x_{-N}$. In the simulations, we took $N \geq 500$. Looking at the numerical data, we could conclude that this length of the lattice was sufficient to simulate the infinite one; i.e., the local amplitude of the established wave was practically equal to zero deep inside the lattice. In all the runs, we took the fixed value of the drive's amplitude $\epsilon = 0.01$, while the dissipative and nonlinear constants λ, α, and β were varied, as well as the driving frequency ω [obviously, one parameter, e.g., ϵ, can always be fixed due to the scaling properties of Eqs. (2) and (3)].

The mode of the simulations was chosen as follows: At fixed values of all the parameters but ω, we directly measured the total energy dissipation rate according to the definition (12), and plotted the dependence $W_{\rm diss}(\omega)$. In Fig. 1, we display a set of these plots obtained for the linear system ($\alpha=0$, $\beta=0$) at different values of the dissipative constant. The resonant peaks at ω close to both edges of the phonon band are clearly seen. Moreover, the

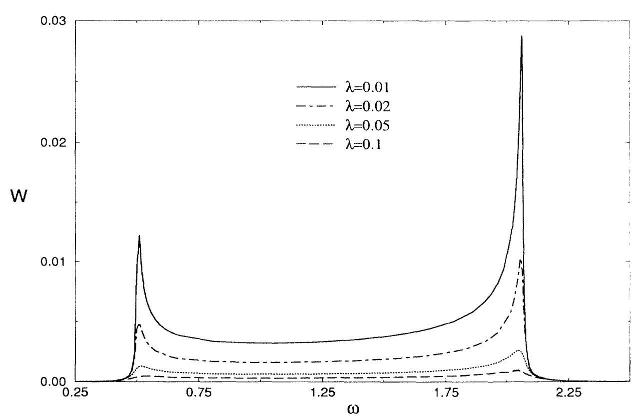

FIG. 1. The energy absorption rate vs the driving frequency for the linear lattice at different values of the dissipation constant λ.

newly predicted peak at ω close to $\omega_{\max}$ is conspicuously stronger than the one known at $\omega = \omega_{\min}$. Naturally, increase of the dissipation attenuates the resonances, but the high-frequency peak seems more stable against the action of the dissipation. In Fig. 2, we have displayed a set of the plots for different values of the nonlinearity parameter α (both positive and negative) at fixed $\lambda = 0.01$ and $\beta = 0$. We notice that the nonlinearity affects the low-frequency resonance (as is shown in the inset, the resonant peak is shifted to the left at positive α and to the right at α negative), while the new high-frequency peak is practically unaffected. It is not difficult to understand this since the low-frequency driving excites acoustic waves, which displace the atoms with respect to the substrate, so that the nonlinear part of the lattice-substrate interaction becomes important. That is why the low-frequency peak is affected by the substrate nonlinearity. On the contrary, when the driving frequency is near the upper edge, it excites optical vibrations (i.e., the so-called staggered state[6]), so that W_{diss} is affected by the chain nonlinearity (see Fig. 3), but it is insensitive to the substrate nonlinearity. Generally, the robustness of the high-frequency peak is natural as the same amount of the dissipation and nonlinearity in the system are relatively less important at larger frequencies. However, the situation may change if we add to Eqs. (2) terms which give rise to dispersion of the dissipation and nonlinearity. In the insets to Figs. 2 and 3, we show the above-mentioned effects on an expanded scale. Another noteworthy effect is that, for a strong nonlinearity, we find bistable behavior outside the phonon band.

In the analysis presented above, we were dealing with the unknown phase shift θ. To complete the analytical consideration, this parameter can now be taken from the numerical data. A typical plot of θ vs ω is displayed, for $\alpha = 0$, in Fig. 4. According to this picture, the quantity $\sin^2\theta$ very quickly changes its value from 1 inside the phonon band to 0 in the forbidden band. For all the other values of the parameters, including $\alpha \neq 0$, we have obtained very similar dependences $\theta(\omega)$. Looking at the expression (14), we notice that the factor $\sqrt{\delta^2 + \lambda^2\omega_{\max}^2}+\delta$ rapidly *increases* when the resonance detuning $\omega - \omega_{\max}$ changes from negative values inside the phonon band to positive ones outside of it [see Eq. (7)], while the factor $\sin^2\theta$ simultaneously rapidly *decreases*. The product of the rapidly increasing and decreasing factors in Eq. (14) can readily produce the sharp high-frequency resonant peaks.

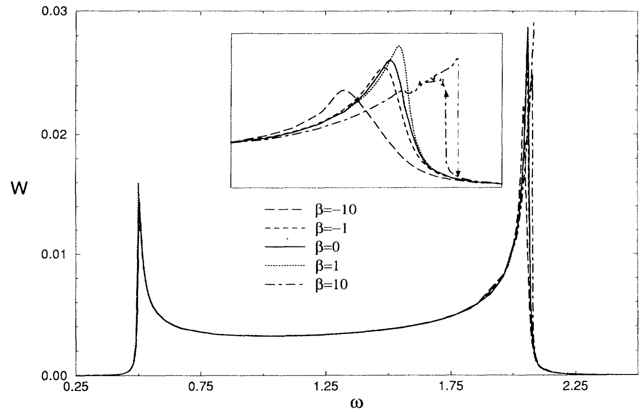

FIG. 3. The energy absorption rate vs the driving frequency at different values of the nonlinearity coefficient β ($\alpha = 0.0$). The inset shows the structure of the high-frequency peak in more detail.

In conclusion, in this work we have predicted semianalytically and found numerically a new type of resonance in driven one-dimensional lattices, when the driving frequency is close to the upper edge of the phonon band. This resonance proves to be stronger and more robust than the known low-frequency resonance at the lower edge. It is relevant to note that the resonant peaks are closely linked to the singularities of the phonon density of states at both edges of the band,[2] which provides a natural relation between static and dynamical properties of the lattice. It should be remarked, however, that even for the weak amplitude ($\epsilon = 0.01$) of the driving used in

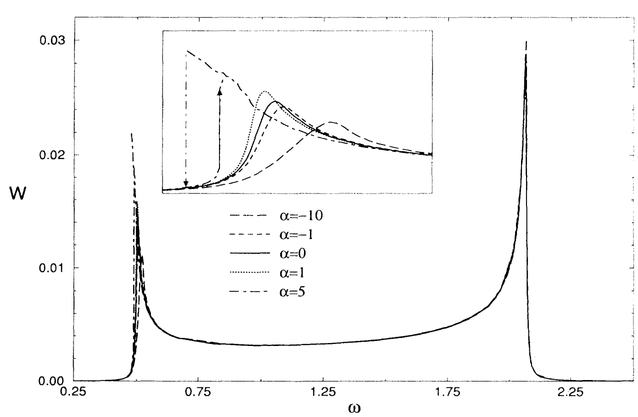

FIG. 2. The energy absorption rate vs the driving frequency at different values of the nonlinearity coefficient α ($\beta = 0.0$). The inset shows the structure of the low-frequency peak in more detail.

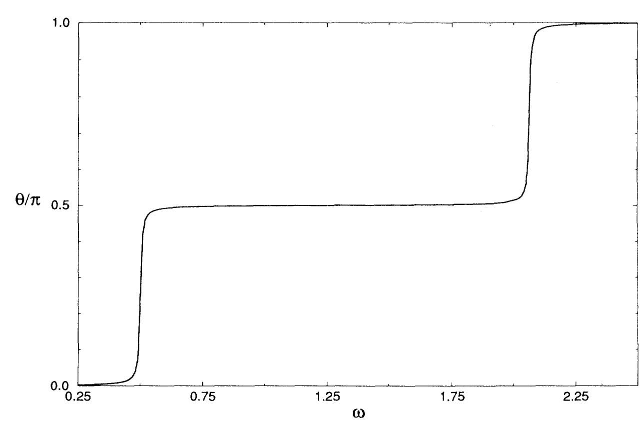

FIG. 4. A typical dependence of the phase shift θ between the driving force and oscillations of the driven particle upon the driving frequency ($\alpha = 0, \beta = 0, \lambda = 0.01$).

the simulations we see strong bistability behavior near the two edges of the linear phonon band. As it can also be seen in the insets in Figs. 2 and 3, for strong nonlinear coefficients (α or β) but still weak driving the dissipated energy along the chain becomes significant outside the frequency band where linear waves cannot propagate. Thus the effect of nonlinearity is essential.

The problem considered in this work can be generalized in different directions. First of all, the drive can be applied not to the single particle but to a group of them. A more complicated situation is then expected if different particles are driven at the same frequency but at different phases. One can also consider the case when the driven particles are located periodically along the lattice.[4] However, it seems more challenging to consider the same problem for a two- and three-dimensional lattice, although the relation between the effect considered and the density-of-states singularities[2] suggests that the resonant peaks should be less salient in higher dimensions.

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